



Zeros of Riemann's Zeta Functions in the Line $z=1/2+it_0$

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Abstract

It was found that, in addition to trivial zeros in points ($z = -2N$, $N = 1, 2, \dots$, natural numbers), the Riemann's zeta function $\zeta(z)$ has zeros only on the line $\{z = \frac{1}{2} + it_0, t_0 \text{ is real}\}$. All zeros are numerated, and for each number, N , the positions of the non-overlap intervals with one zero inside are found. The simple equation for the determination of centers of intervals is obtained. The analytical function $\eta(z)$, leading to the possibility fix the zeros of the zeta function $\zeta(z)$, was estimated. To perform the analysis, the well-known phenomenon, phase-slip events, is used. This phenomenon is the key ingredient for the investigation of dynamical processes in solid-state physics, for example, if we are trying to solve the TDGLE (time-dependent Ginzburg-Landau equation).

Keywords Superconductivity · phase-slip events · time-dependent Ginzburg-Landau equation · Riemann's zeta-function

1 Introduction

Investigations of Josephson effect, current flow in narrow superconducting strips [1], and dynamical states [2] in superconductors with use of TDGLE (time-dependent Ginzburg-Landau equation) lead to the necessity to deal with an important phenomenon: phase slip events. It is interesting that the study of the distribution of zeros for Riemann's ζ function (see below Eqs. (15–17)) also requires an analysis of the same phenomenon. It means that there exists a deep internal connection between all these problems. It turns out that the Euler Γ function is also the essential ingredient of the scenario.

Riemann's ζ function and Euler Γ function appear in many physical problems. For example, the study of the spin-orbit interaction in inhomogeneous superconductors state reveals the existence of the infinite set of nontrivial exact relations for Euler $\{\psi\}$ function [3]; these relations are essential for the evaluation of the critical temperature. These equations contain the Bernoulli numbers B_n . In general, these problems appear as the consequence of mentioned internal ties and arise

while the temperature technique and analytical continuation are used in the theory of superconductivity. We are trying to analyze these ties. We hope that the investigation of phase-slip events with the use of the distribution of zeros for Riemann's ζ function will lead to new results, which are important not only in mathematics. They will provide a deep understanding of the phenomena mentioned above, especially for the case of strong suppression of superconductivity by an external field.

In order to investigate the zeros of $\zeta(z)$, it is more convenient to use the function $\Xi(z)$, which can be presented in the symmetrical form as follows:

$$\Xi(z) = \frac{z(z-1)}{4} \left[\zeta(z)\Gamma(z/2)\pi^{-z/2} + \zeta(1-z)\Gamma((1-z)/2)\pi^{-(1-z)/2} \right] \quad (1)$$

where $\Gamma(z)$ is Euler gamma function. The function $\Xi(z)$ is the entire function of z . As for the zeros of $\zeta(z)$ and $\Xi(z)$, they coincide on the stripe. If we put $z = 1/2 + it$, than $\Xi(z)$ is even a function of t with real coefficients in its Taylor expansion in powers of t^2 [4]. The function of Ξ is real on the lines $\{z = \frac{1}{2} + it_0; t_0 \text{ is real}; z = \nu; \nu \text{ is real}\}$. The line $\{z = 1/2 + it_0\}$ is Stock's line for Ξ function.

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2 Main Equations

Each analytical function $f(z)$ can be presented in the form as follows:

$$f(x) = |f(x)|\exp(i\chi) \tag{2}$$

If $f(x) \neq 0$ on some close counter C , then we are dealing with the following quantization rule:

$$\oint_C \frac{\partial \chi}{\partial \mathbf{r} d\mathbf{r}} = 2\pi(N_z - N_p) \tag{3}$$

where N_z is a number of zeros, N_p is a number of poles inside the counter.

For $z = 1/2 + \nu + it_0$ and $\nu \gg 1$, we have the following:

$$\zeta(z) = 1 + \frac{1}{\sqrt{2} \cdot 2^\nu} (\cos(t_0 \ln 2) - i \sin(t_0 \ln 2)) \tag{4}$$

From Eqs. (1, 4), we obtain that the integral of type (3) (see Fig. 1) for Ξ function is equal to two integrals type (3) on the counter $\{DABD_1\}$. Points $\{A, F\}$ and $\{B, E\}$ are symmetrically relative to the reflection over the $\{D, D_1\}$ axis. The pair of zeros of zeta function placed in point $(t_0^{(k)}, \nu; t_0^{(k)}, -\nu)$ is not giving any contribution to the phase change during the integration over the line $\{L, L_1\}$; the asymptotics (4) is canceling such a contribution by integration over the line $\{A, B\}$. Both these circumstances make impossible an appearance of pair zeros $(t_0^{(k)}, \nu; t_0^{(k)}, -\nu)$ with $\nu \neq 0$. As a result, the total number of zeros of function Ξ inside counters C_1 and C_0 coincides. As for the interval $\{B, D_1\}$, the essential corrections to the phase change appear only for the regional order of one near D_1 point. Below, we will use the following expression for the $\zeta(z)$ function as follows:

$$\zeta(z) = \frac{1}{1 - 2^{1-z}} \{D_1 + iD_2\}, \quad z = 1/2 + \nu + it_0 \tag{5}$$

where

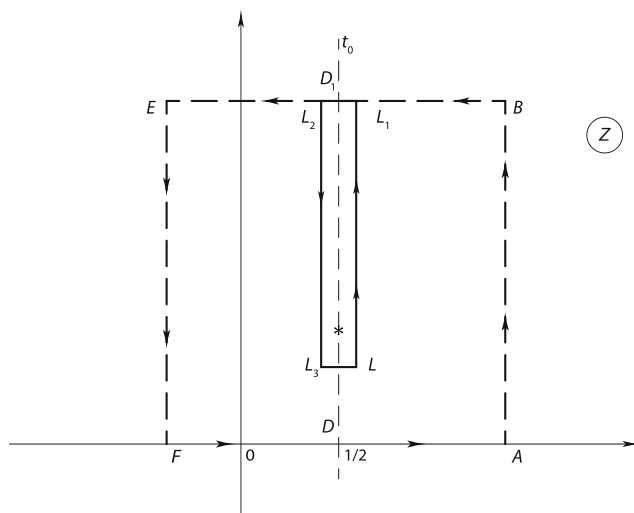


Fig. 1 A single asterisk indicates a first zero of ζ function, counter C_0 (L, L_1, L_2, L_3), counter C_1 (F, D, A, B, D_1, E)

$$D_1 = \left(1 - 2^{-1/2-\nu} \cos(t_0 \ln 2)\right) + \sum_{k=2}^{\infty} \left(\frac{\cos(t_0 \ln(2k-1))}{(2k-1)^{1/2+\nu}} - \frac{\cos(t_0 \ln(2k))}{(2k)^{1/2+\nu}}\right)$$

$$D_2 = 2^{-1/2-\nu} \sin(t_0 \ln 2) - \sum_{k=2}^{\infty} \left(\frac{\sin(t_0 \ln(2k-1))}{(2k-1)^{1/2+\nu}} - \frac{\sin(t_0 \ln(2k))}{(2k)^{1/2+\nu}}\right) \tag{6}$$

Note that in the points $\{t_0 \ln 2 = 2\pi N, N \neq 0, \nu = 1/2\}$, we have $D_1 = D_2 = 0$ [5].

Let us consider the region $t_0 \gg 1$ for the calculation of the distribution of zeros of the Ξ function. In this region, we obtain the following:

$$\Xi(z) = \frac{(t^2 + 1/4)\pi^{1/4} 2^{3/4} e^{-\pi t/4}}{4t^{1/4}} \{\zeta(z)\exp(i\phi(t)) + \zeta(1-z)\exp(-i\phi(t))\} \tag{7}$$

where $z = 1/2 + it = 1/2 + \nu + it_0$, ϕ is an analytical function with the following asymptotic expansion:

$$\phi(t) = \frac{t}{2} \left(\ln\left(\frac{t}{2}\right) - \ln \pi - 1\right) - \frac{\pi}{8} + \frac{1}{48t} + \dots \tag{8}$$

All zeros of function Ξ coincide with zeros of expression in brackets of Eq. (7). Here we use the ideas from the consideration of phase-slip events in solid-state physics [6]. More specifically, we define an analytical function $\eta(t_0, \nu)$ as follows:

$$\eta(t_0, \nu) = \eta_1(t_0, \nu) + i\eta_2(t_0, \nu); \quad \eta(t_0, \nu) = -\eta_1(-t_0, \nu) + i\eta_2(-t_0, \nu) \tag{9}$$

Then on the Stock's line, we obtain the following:

$$2e^{-\eta_2(t_0)} \cos(\phi(t_0) + \eta_1(t_0)) = e^{i\phi(t_0)} \frac{D_1 + iD_2}{1 - \sqrt{2} \cos(t_0 \ln 2) + i\sqrt{2} \sin(t_0 \ln 2)} + e^{-i\phi(t_0)} \frac{D_1 - iD_2}{1 - \sqrt{2} \cos(t_0 \ln 2) - i\sqrt{2} \sin(t_0 \ln 2)} \tag{10}$$

On the Stock's line, the function $\eta_1(t_0, \nu)$ satisfies the condition as follows:

$$|\eta_1| < \pi/2 \tag{11}$$

Inside the range $\nu \gg 1$, we obtain the following equations for $\{\eta_1, \eta_2\}$:

$$\eta_1 = -\sin(t_0 \ln 2)/2^{1/2+\nu}, \quad \eta_2 = -\cos(t_0 \ln 2)/2^{1/2+\nu} \tag{12}$$

In addition, at the zero point with number N , we have two equations on the Stock's line as follows:

$$\frac{1}{\pi} \left(\phi\left(t_0^{(N)}\right) + \eta_1\left(t_0^{(N)}\right)\right) = N - 3/2, \tag{13}$$

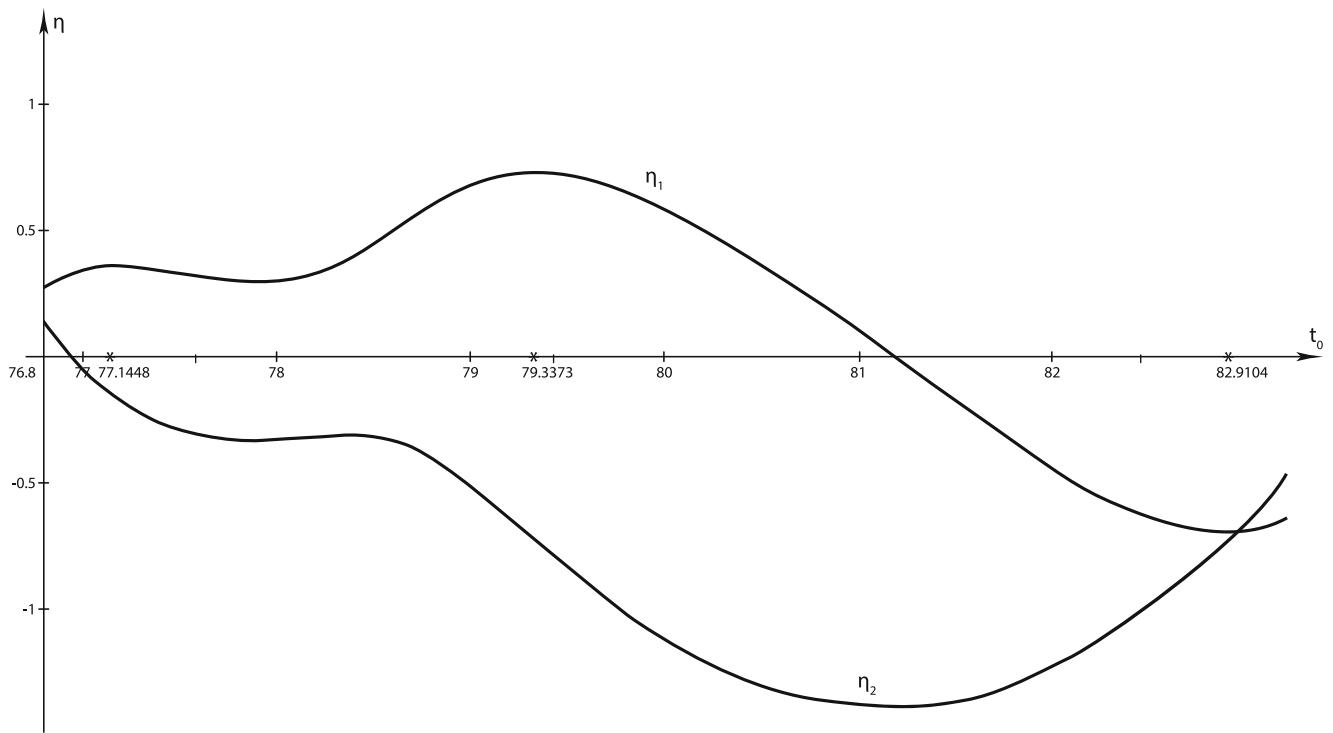


Fig. 2 Functions $\{\eta_1, \eta_2\}$ in the interval $76.8 < t_0 < 84.2$; point with a single asterisk indicates zeros of ζ function with $\{N=20,21,22; t_0 = 77.1448, 79.3373, 82.9104\}$

$$\left. \frac{\partial \eta_1(t_0)}{\partial t_0} \right|_{t_0^{(N)}} = 0 \tag{14}$$

Equations (8, 11, 13, 14) allow us to estimate the location of the interval, where the zero point with number N is placed.

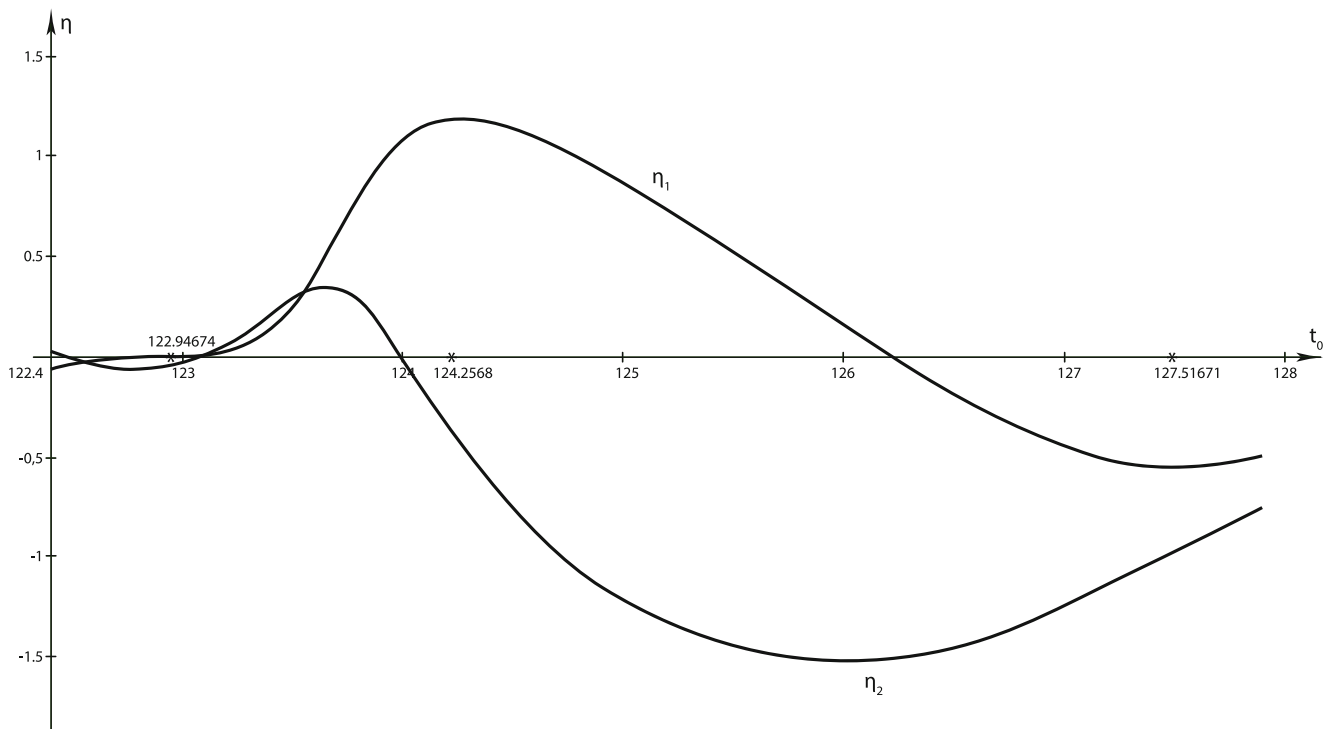


Fig. 3 Functions $\{\eta_1, \eta_2\}$ in the interval $122.4 < t_0 < 127.9$; point with a single asterisk indicates zeros of ζ function with $\{N=40,41,42; t_0 = 122.94674, 124.2568, 127.51671\}$

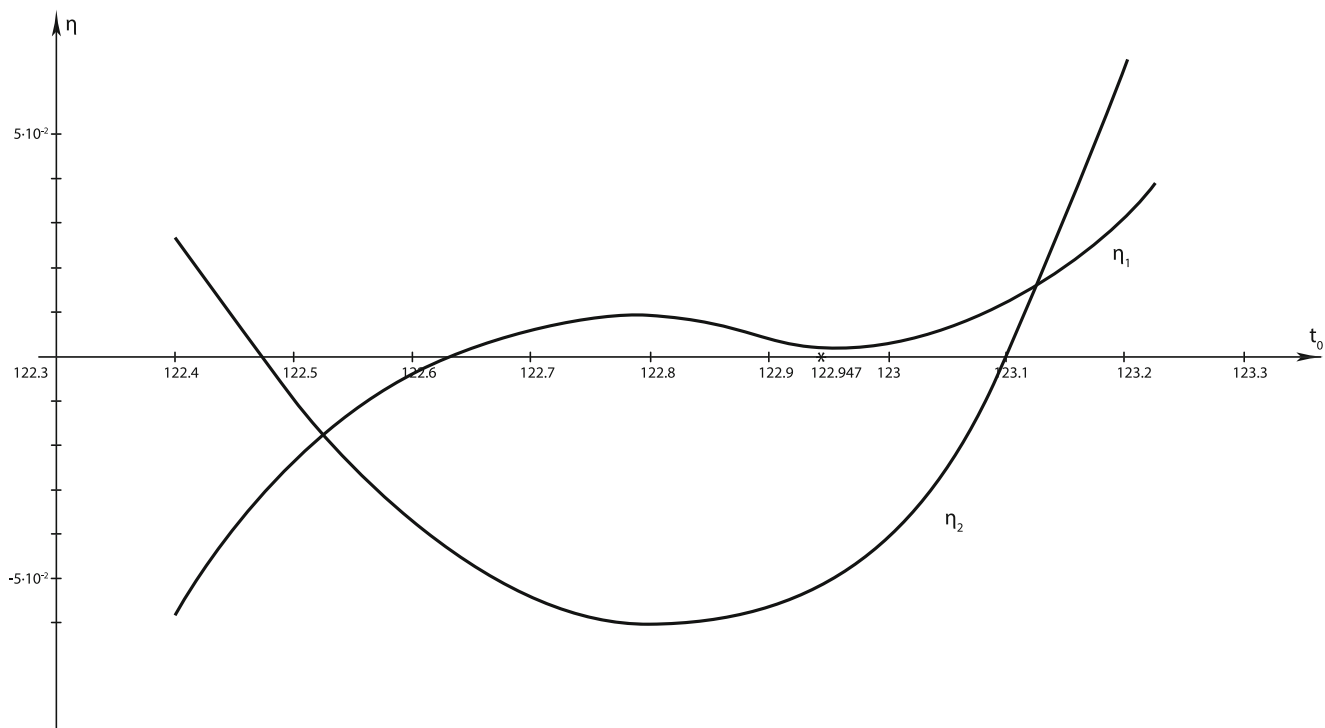


Fig. 4 Functions $\{\eta_1, \eta_2\}$ in the interval $122.4 < t_0 < 123.3$; point with a single asterisk indicates zeros of ζ function with $\{N = 40; t_0 = 122.94674\}$

Note that there exists a large parameter $t_0 \gg 1$ in the limit $\ln(t_0)$. It leads to the drastic increase of the first term in Eq. (10) relative the second term with an increase in ν ; as a result,

the existence of a pair of zeros $(\nu, -\nu)$ with $\nu \neq 0$ is impossible. The ratio of these two terms is $\exp(\nu \ln(t_0/2\pi))$. Equations (6, 13) allow us get the exact expression for the zeros of the Ξ

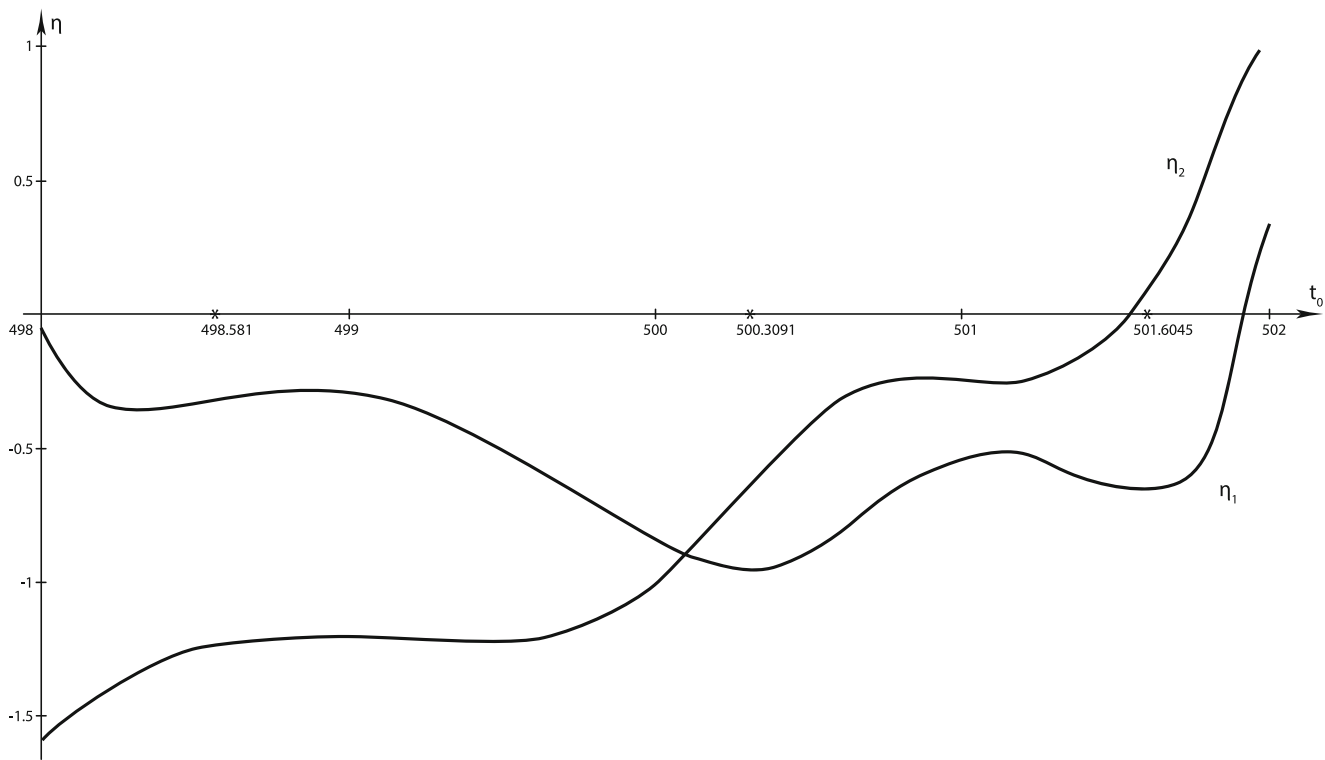


Fig. 5 Functions $\{\eta_1, \eta_2\}$ in the interval $498 < t_0 < 502$; point with a single asterisk indicates zeros of ζ function with $\{N = 269, 270, 271; t_0 = 498.5809, 500.30905, 501.6045\}$

Table 1 Column 1, number of zeros; column 2, table data (ref. [7]), value of zero N ; column 3, calculated value of zero N with help of Eqs. (5, 6); columns 4–5, value of functions $\{\eta_1, \eta_2\}$ in the point $(t_0 = t_0^{(N)})$

N	$\alpha(J_a, E)$	α_{cal}	η_1	η_2
1	14.134725	14.134725	0.157875	- 0.67088
2	21.022040	21.022047	- 0.2205789	- 0.63322
3	25.010856	25.010856	0.3342749	- 0.68525
4	30.424878	30.424878	- 0.5374147	- 0.50314
5	32.935057	32.935065	0.5740572	- 0.51096
6	37.586176	37.586175	- 0.2939078	- 0.77282
7	40.918720	40.918725	- 0.2046136	- 0.46588
8	43.327073	43.327073	0.6459731	- 0.6406
9	48.005150	48.005155	- 0.8508028	- 0.4339
10	49.773832	49.773825	0.4764723	- 0.31471
11	52.970	52.97031	0.2600957	- 0.82039
12	56.446	56.446238	- 0.3591892	- .076845
13	59.347	59.34704	- 0.4384483	- 0.21453
14	60.833	60.83178	1.026932	- 0.37641
15	65.113	65.11254	- 0.7642258	- 0.67143
16	67.080	67.079783	$6.272686 \cdot 10^{-2}$	- 0.41039
17	69.546	69.54639	0.2614599	- 0.59757
18	72.067	72.06715	0.3503908	- 0.88480
19	75.705	75.7042	- 0.9904536	- 0.3584
20	77.145	77.14482	0.3521394	- 0.1489
21	79.337	79.337345	0.7290875	- 0.73164
22	82.910	82.91038	- 0.6992546	- 0.71322
23	84.734	84.7355	$7.80726 \cdot 10^{-2}$	- 0.50791
24	87.426	87.425253	- 0.3003618	- 0.31313
25	88.809	88.809105	1.014014	- 0.49443
26	92.494	92.49189	- 0.7591807	- 0.803829
27	94.651	94.65131	- 0.5337027	- $8.5412 \cdot 10^{-2}$
28	95.871	95.87063	0.9503964	- 0.21802
29	98.831	98.83123	$3.5379 \cdot 10^{-2}$	- 0.9336
30		101.31786	- 0.2645489	- 0.81867
40		122.94674	$3.15446 \cdot 10^{-3}$	- $4.6178 \cdot 10^{-2}$
41		124.2568	1.193301	- 0.4390
42		127.51671	- 0.5508639	- 0.9550723
269		498.5809	- 0.30956	- 1.25627
270		500.30905	- 0.948835	- 0.666625
271		501.6045	- 0.64338	0.116255

function with a given number N and an asymptotic exact expression for $\eta_1(t_0^{(N)})$ and $\eta_2(t_0^{(N)})$ on a Stock’s line.

The Table contains the values of 29 first zeros of the zeta function from Ref. [7] and additional seven points with numbers 30, 40, 41, 42, 269, 270, and 271, which follow from Eqs. (10, 13, 14). For all these points, we added the values of

functions $\eta_1(t_0^{(N)})$ and $\eta_2(t_0^{(N)})$.

The existence of the large parameter, $\ln t_0$, allows to analyze Eqs. (10, 13, 14) with the use of the perturbation theory and estimate the functions $\{\eta_1(t_0^{(N)}), \eta_2(t_0^{(N)})\}$.

To perform this estimation, we present Eq. (10) in the form of the following:

$$e^{-\eta_2(t_0)} \cos(\phi(t_0) + \eta_1(t_0)) = \mu(t_0) \tag{15}$$

where $\mu(t_0)$ is given by equation as follows:

$$\mu(t_0) = \frac{1}{2} \left\{ e^{i\phi(t_0)} \frac{D_1 + iD_2}{1 - \sqrt{2} \cos(t_0 \ln 2) + i\sqrt{2} \sin(t_0 \ln 2)} + e^{-i\phi(t_0)} \frac{D_1 - iD_2}{1 - \sqrt{2} \cos(t_0 \ln 2) - i\sqrt{2} \sin(t_0 \ln 2)} \right\} \tag{16}$$

The value of functions $(\eta_1, \eta_2)|_{t_0 \neq t_0^{(N)}}$ can be estimated from the equations as follows:

$$\begin{aligned} \operatorname{tg}(\phi(t_0) + \eta_1(t_0)) &= - \frac{\mu'(t_0)}{\phi'(t_0)\mu(t_0)} - \\ &\frac{1}{\phi'(t_0)} \left[\eta_2'(t_0) + \eta_1'(t_0) \exp(-\eta_2(t_0)) \frac{\sin(\phi(t_0) + \eta_1(t_0))}{\mu(t_0)} \right] \end{aligned} \tag{17}$$

with the use of the perturbation theory relative to the second term on the right side of Eq. (17) and Eq. (15). Here $f'(t_0) = \partial f(t_0) / \partial t_0$.

As an example, we made this procedure in the first order of the perturbation theory for three intervals $76.8 \leq t_0 \leq 83.2$, $112.4 \leq t_0 \leq 127.9$, and $498 \leq t_0 \leq 502$. Inside these intervals, the zeros $\{N : 20, 21, 22\}$, $\{N : 40, 41, 42\}$, and $\{N : 269, 270, 271\}$ are placed. Results are presented in Figs. 2, 3, 4, and 5 and in the Table 1.

There is a very interesting interval $\{122.4 \leq t_0 \leq 123.3\}$ with very small values of both functions $\{\eta_1, \eta_2\}$. Inside this interval, a zeta function zero with number $N = 40$ is located at the point $\{t_0 = 122.9467\}$. The value of functions $\{\eta_1, \eta_2\}$ at this point is $\eta_1(122.9467) = 3.154 \cdot 10^{-3}$; $\eta_2(122.9467) = - 4.618 \cdot 10^{-2}$.

The values of the function $\{\eta_1, \eta_2\}$ in the interval $122.4 \leq t_0 \leq 123.3$ are presented in Fig. 4 on a large scale.

The similar situation takes place in the third interval in the vicinity of a zero of the zeta function with the number $\{N = 269, t_0 = 498.5809\}$ $498.2 < t < 499.2$. Functions $\{\eta_1, \eta_2\}$ display fast oscillations with small amplitude in this interval near the values $\{- 0.31; - 1.28\}$. It looks as some special “zeros” are “attractive resonant centers.” Definitely, there exists the final concentration of such “centers.”

3 Conclusions

The key results of the paper are presented by the equations (10), (13), and (14). Equation (13) represents a strong improvement of the well-known result described in the book [5]. Equation (10) is the proof of the Riemann's hypothesis. We introduce two analytical functions $\{\phi$ and $\eta\}$. The exact asymptotical expression for function ϕ on the stripe $0 < \text{Re } z < 1$ for $t_0 \gg 1$ is obtained. With the use of the functions $\{\phi, \eta\}$, one can create additional relation between the functions $\{\Gamma(z), \zeta(z)\}$, and this allows us to produce a numeration of zeros of the Riemann's ζ function. Equation (10) was solved for the large values of the parameter $\ln\left(\frac{t_0}{2\pi}\right)$ for functions $\{\eta_1, \eta_2; \eta = \eta_1 + i\eta_2\}$. We also obtained the solution (in the first order of the perturbation theory) for three intervals $76.8 \leq t_0 \leq 83.2$, $122.4 \leq t_0 \leq 127.9$, and $498 \leq t_0 \leq 502$.

The condition (14) connected with symmetry of the Riemann's zeta function and corresponding simultaneous equality to zero of both terms in Eq. (7) appears to be very essential.

The study of the phase-slip events is connected with the zeros, which are discussed above. Unlike all previous studies based on the use of the perturbation theory, the present analysis allows us to obtain an exact solution, without invoking any small parameter.

The connection between the numbers N of zeros of the Riemann's ζ function inside of the abovementioned "resonant centers" and the primes is rather obvious.

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