DYNAMICAL SYMMETRY GROUP $O(4,2)$ OF HYDROGEN ATOM ENERGY SPECTRUM AND WEIGHT DIAGRAM OF THE GROUP IRREDUCIBLE REPRESENTATION

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In this comment, we describe graphically the irreducible representation of the dynamical symmetry group $O(4,2)$, which contains complete information on the wave function (state vector $|n\ell m\rangle$) and discrete energy levels of hydrogen atom as well as yields natural connection of the representation with all quantum numbers ($n = 1, 2, \dots$, $\ell = 0, 1, \dots, n-1$, $m = -\ell, -\ell+1, \dots, \ell$) of the hydrogen-atom states.

Fock found the $O(4)$ compact symmetry group (called the hidden symmetry group) of the Schrödinger equation for hydrogen atom that explains the quantum degeneracy of discrete energy levels of hydrogen atom $E_n = -1/2n^2$ (in dimensionless units) with respect to the orbital quantum number ℓ , in addition to the natural rotation symmetry $O(3)$ of the Coulomb potential $-1/r = U(r)$, explaining the degeneracy of the energy levels with respect to the magnetic quantum number m .

In [1] was shown that all the states of discrete spectrum of hydrogen atom (or corresponding solutions of the Schrödinger equation) realize an infinite-dimension irreducible representation of noncompact symmetry group of hydrogen atom $O(4,2)$. The finite-dimension representations of the hidden symmetry describe the states belonging to the n th energy levels. The degeneracy of these levels is equal to n^2 . So the ground state ($n = 1$) is not degenerated, the first excited state ($n = 2$) has the degeneracy of the energy level equal to 4, the second excited state ($n = 3$) has degeneracy 9, and so on.

All the discrete levels are united as basis vectors of irreducible representation of $O(4,2)$. The history of this result can be found in books [2,3], review article [4]; it is developed in [5,6]. In this comment, we describe the weight diagram of the discrete energy levels of hydrogen atom by a triangle pyramid, being naturally connected with quantum numbers n , ℓ , and m . Since $\sum_{\ell=1}^n \sum_{m=-\ell}^{\ell} 1 = n^2$, these states of the n th energy levels can be presented not only as squares [7] but also as triangles, as shown in the figure.

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