Journal of Scientific Computing (2023) 95:7
https://doi.org/10.1007/s10915-023-02114-8

Second-Order Symmetric Duality for Multiple Objectives Nonlinear Programming Under Generalizations of Cone-Preinvexity Functions

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Received: 17 December 2021 / Revised: 9 January 2023 / Accepted: 12 January 2023 /
Published online: 15 February 2023

Abstract
In this article, we present generalizations of the cone-preinvexity functions and study a pair of second-order symmetric solutions for multiple objective nonlinear programming problems under these generalizations of the cone-preinvexity functions. In addition, we establish and prove the theorems of weak duality, strong duality, strict converse duality, and self-duality by assuming the skew-symmetric functions under these generalizations of the cone-preinvexity functions. Finally, we provide four nontrivial numerical examples to demonstrate that the results of the weak and strong duality theorems are true.

Keyword Multiple objective programming · Second-order symmetric duality · Nonlinear programming · Duality theorems · Cones · Preinvexity functions

Mathematical Subject Classifications 90-XX · 90CXX · 90C30 · 90C46 · 65K05 · 49M37 · 49N15

1 Introduction

Weir and Mond [21] introduced symmetric and self-dual for multiple objective nonlinear programming (MONLP) problems. Mond and Weir [17] introduced duality to the MONLP problems with pseudo-convexity and pseudo-concavity. Kim [13, 14] presented symmetric duality for these (MONLP) problems under pseudo-invexity functions. Devi [1] introduced symmetric programs involving η-bonvexity functions for these problems. Mishra [15] studied these programming problems under F-convexity. Yang [22] introduced converse duality for these programming problems under cone constraints. Seema [18] presented symmetric duality for these programming problems under cone-invex. Kassem [8–11] studied these programming problems under cone-invexity and (K, F)-pseudo-convexity functions. Suneja et al. [19] studied these programming problems with cone constraints. Dubey et al. [3–7] reported

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higher-order symmetric duality for non-differentiable fractional programming under cone constraints. Kassem [12] studied the second-order symmetric duality in vector optimization involving (K, η)-pseudobonvexity functions.

The article presents generalizations of the cone-preinvexity functions and studies a pair of second-order symmetric multiple objective nonlinear programming (MONLP) problems under these generalizations of the cone-preinvexity functions. In addition, we establish and prove duality theorems and a self-duality theorem for these (MONLP) problems under these generalizations of the cone-preinvexity functions. Finally, four nontrivial numerical examples are given to show that the results of the weak and strong duality theorems are true.

2 Notations and Preliminaries

Definition 2.1 [14] The set X is called an invex at the point $u \in X$ with respect to (w. r. t.)$\eta : X \times X \to R^n$ if $\forall\ x \in X$ we have $u + \lambda\, \eta(x, u) \in X,\ 0 \leqq \lambda \leqq 1$.

If it X is invex for all $u \in X$, the set X is invex w. r. t. $\eta : X \times X \to R^n$.

Definition 2.2 [14] The invex set X is called an invex cone, if $x \in X,\ \ \lambda \geq 0\ \Rightarrow \lambda\, x \in X$.

Definition 2.3 The function $f(x)$ is said to have a K-preinvexity w. r. t.$\eta : X \times X \to R^n$, if $\forall\ x,\ y \in R^n\ \ and\ \ \lambda \in (0, 1)$ we have.

$\lambda\, f(x) + (1 - \lambda)\, f(y) - f(y + \lambda\, \eta(x, y)) \in K.$

Where K is define as a closing invex cone.

Let us consider the following general (MONLP) problems:

$$(MONLP)^1 \qquad K - \min \quad f_i(x),\ \ i = 1, 2, ..p$$

$$\text{subject to}\ \ x \in X = \{x \in C :\ \ -g_j(x) \in Q,\ \ j = 1, 2, ...m\}$$

where $C \subset R^n,\ \ K\ \ and\ \ Q$ are closed invex cones and $f_i : R^n \to R^p,\ \ g_j : R^n \to R^m$.

Definition 2.4 [15] The point $\overline{x} \in X$ is a weak minimum for the $(MONLP)^1$ problem, if $f(\overline{x}) - f(x) \notin \operatorname{int} K\ \ \forall\ x \in X$.

Definition 2.5 [18] The positive cone K^* is.

$$K^* = \{z \in R^p :\ x^T z \geq 0\ \ \forall\ \ x \in K\}.$$

Let us consider a pair of second-order symmetric duality $(MONLP)^1$ problems in the following form:

$$\text{(MONLP)} : \quad K - \min\ \ f(x, y) - [y^T \nabla_y (\lambda^T f)(x, y)]e - [y^T \nabla_{yy} (\lambda^T f)(x, y) p]e$$

$$\text{subject to}\ \ (x, y) \in C_1 \times C_2,$$

$$-\nabla_y (\lambda^T f)(x, y) - p^T \nabla_{yy} (\lambda^t f)(x, y) \in C_2^*,$$

$$\lambda,\ \ p \in K^*,\ \ e \in \operatorname{int} K,\ \ \lambda^T e = 1,$$

$$(MONLD) : \quad K - \max\ \ f(u, v) - [u^T \nabla_u (\lambda^T f)(u, v)]e - [u^T \nabla_{uu} (\lambda^T f)(u, v) r]e$$

$$\text{subject to } (u, v) \in C_1 \times C_2,$$

$$\nabla_u(\lambda^T f)(u, v) + r^T \nabla_{uu}(\lambda^T f)(u, v) \in C_1^*,$$

$$\lambda,\ r \in K^*,\ \ e \in \text{int } K,\ \ \lambda^T e = 1,$$

where $f : R^n \times R^m \to R^p$ is a twice-differentiable function, these $C_1^*,\ C_2^*$ are positive cones for closed invex cones $C_1\ ,\ C_2$, respectively, $\nabla_x(\lambda^T f)(x, y)\ ,\ \nabla_y(\lambda^T f)(x, y)$ and $\nabla_{xx}(\lambda^T f)(x, y),\ \nabla_{yy}(\lambda^T f)(x, y)$ the gradients and Hessian matrices for $(\lambda^T f)(x, y)$ w. r. t.$x\ ,\ y$, respectively.

3 Study the Second-Order Symmetric Duality Theorems

Theorem 3.1 (Weak Duality) *Assume the points* (x, y, λ, p) *and* (u, v, λ, r)*are feasible for the (MONLP) and (MONLD) problems, respectively, and.*

(i) *The function* $f(., y)$ *is a* $K-$*Preinvexity w. r. t. xfor a fixed* y.
(ii) *For a fixed* x*, the function* $-f(x, .)$ *is a* $K-$*Preinvexity w. r. t.y, and.*
(iii) $\begin{pmatrix} \eta(x, u) & (v - y) \\ -(x - u) & -\eta(v, y) \end{pmatrix} \begin{pmatrix} \nabla_u f(u, v) \\ \nabla_y f(x, y) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$
Then

$$f(u, v) - [u^T \nabla_u(\lambda^T f)(u, v)]e - [u^T \nabla_{uu}(\lambda^T f)(u, v)r]e - f(x, y) + [y^T \nabla_y(\lambda^T f)(x, y)]e + [y^T \nabla_{yy}(\lambda^T f)(x, y)p]e \notin \text{ int } K.$$

Proof Assume the inverse. That is,

$$f(u, v) - [u^T \nabla_u(\lambda^T f)(u, v)]e - [u^T \nabla_{uu}(\lambda^T f)(u, v)r]e - f(x, y) + [y^T \nabla_y(\lambda^T f)(x, y)]e + [y^T \nabla_{yy}(\lambda^T f)(x, y)p]e \in \text{ int } K$$

Because it $\lambda \in K^*$ implies

$$\begin{aligned} &\lambda^T f(u, v) - \lambda^T [u^T \nabla_u(\lambda^T f)(u, v)]e - \lambda^T [u^T \nabla_{uu}(\lambda^T f)(u, v)r]e - \lambda^T f(x, y) + \lambda^T [y^T \nabla_y(\lambda^T f)(x, y)]e \\ &\quad + \lambda^T [y^T \nabla_{yy}(\lambda^T f)(x, y)p]e \geq 0 \ \Rightarrow \\ &\lambda^T f(u, v) - \lambda^T f(x, y) - [u^T \nabla_u(\lambda^T f)(u, v)] - [u^T \nabla_{uu}(\lambda^T f)(u, v)r] + [y^T \nabla_y(\lambda^T f)(x, y)] \\ &\quad + [y^T \nabla_{yy}(\lambda^T f)(x, y)p] \geq 0 \ \ as \ \ \lambda^T e = 1. \qquad (*) \end{aligned}$$

Because the function $f(., y)$ is a $K-$ preinvexity w. r. t.$x\ \ for\ \ fixed\ \ y = v$, we get

$$\lambda\, f(x, v) + (1 - \lambda)\, f(u, v) - f(u + \lambda\eta(x, u), v) \in K, \quad \lambda \in (0, 1) \quad \Rightarrow$$
$$(f(x, v) - f(u, v)) - \left(\frac{f(u + \lambda\eta(x, u), v) - f(u, v)}{\lambda} \right) \in K$$

Using the mean-valued theorem, we get

$$\begin{aligned} &f(x, v) - f(u, v) - \eta(x, u)\, \nabla_u f(u, v) \in K \ \ \Rightarrow \ \ for \ \ \lambda \in K^* \\ &\lambda^T\ f(x, v) - \lambda^T f(u, v) \geq \lambda^T \eta(x, u)\, \nabla_u f(u, v), \end{aligned} \qquad (1)$$

Since function $-f(x, .)$ is a $K-$ preinvexity w. r. t.$y\ \ for\ \ fixed\ \ x$,

$$\lambda^T\ f(x, y) - \lambda^T f(x, v) \geq -\lambda^T \eta(v, y)\, \nabla_y f(x, y) \qquad (2)$$

When we add the inequalities (1) and (2) together, we get a

$$\lambda^T f(x, y) - \lambda^T f(u, v) \geq \lambda^T \eta(x, u) \nabla_u f(u, v) - \lambda\eta(v, y)\nabla_y f(x, y) \quad (3)$$

Because of assumption (iii), we have

$$\eta(x, u) \nabla_u f(u, v) \geq -(v - y)\nabla_y f(x, y); \qquad -(x - u)\nabla_u f(u, v) \geq \eta(v, y)\nabla_y f(x, y)$$

By adding the above inequalities,

$$\eta(x, u) \nabla_u f(u, v) - \eta(v, y)\nabla_y f(x, y) \geq (x - u)\nabla_u f(u, v) - (v - y)\nabla_y f(x, y).$$

Then a relationship (3) is formed.

$$\begin{aligned} &\lambda f(x, y) - \lambda f(u, v) - x^T \nabla_u(\lambda^T f)(u, v) + u^T \nabla_u(\lambda^T f)(u, v) \\ &\quad - y^T \nabla_y(\lambda^T f)(x, y) + v^T \nabla_y(\lambda^T f)(x, y) \geq 0. \end{aligned} \quad (4)$$

From the feasibility of the points (x, y, λ, p) *and* (u, v, λ, r) for the (MONLP) and (MONLD) problems, respectively, we have

$$-\nabla_y(\lambda^T f)(x, y) \geq p^T \nabla_{yy}(\lambda^T f)(x, y), \quad \nabla_u(\lambda^T f)(u, v) \geq -r^T \nabla_{uu}(\lambda^T f)(u, v) \quad (5)$$

Substituting (5) into (4), we get the following:

$$\begin{aligned} &\lambda^T f(x, y) - y^T \nabla_y(\lambda^T f)(x, y) - y^T \nabla_{yy}(\lambda^T f)(x, y)p - \lambda^T f(u, v) \\ &\quad + u^T \nabla_u(\lambda^T f)(u, v) + u^T \nabla_{uu}(\lambda^T f)(u, v)r \geq 0, \end{aligned}$$

That contradicts (*). Then the proof is complete.

To illustrate the results of this weak duality theorem, we introduce the following example:

Example 3.1.1 Consider the following:

$K = \{(x, y) \in R^2 : |y| \leq x, \ \ x \geq 0\}, f(x, y) = (x^3 - y^3, x^3), C_1 = C_2 = R_+,$

$\lambda f(x, v) + (1 - \lambda) f(u, v) - f(x + \lambda\eta(x, u), v) = (\lambda x^3 - \lambda u^3 + u^3 - (x + \lambda\eta(x, u))^3,$

$\lambda x^3 - \lambda u^3 + u^3 - (x + \lambda\eta(x, u))^3).$

There is $f(., y)$ a $K-$ preinvexity function w. r. t.x *for fixed* y. In addition, there is $-f(x, .)$ a $K-$ preinvexity function w. r. t.y *for fixed* x

Furthermore, the (MONLP) and (MONLD) problems take the form:

$$(MONLP)\,1: \qquad K - \min\ (x^3 - y^3, x^3) + 3\lambda_1 y^3(e_1, e_2) + 6\lambda_1 y^2 p\,(e_1, e_2)$$

$$subject\ \ to\ \ x \geq 0, \ \ y \geq 0,$$

$$3\lambda_1 y^2 + 6\lambda_1 y\, p \geq 0$$

,

$$\lambda = (\lambda_1, \lambda_2) \in K^* = K, \ \ (e_1, e_2) \in \text{int}\, K,$$

$$\lambda_1 e_1 + \lambda_2 e_2 = 1,$$

and

$$(\text{MONLD})\,1: \qquad K - \max\ (u^3 - v^3, u^3) - 3(\lambda_1 + \lambda_2)u^3(e_1, e_2) - 6(\lambda_1 + \lambda_2)u^2 r(e_1, e_2)$$

$$subject\ \ to\ u \geq 0, \ \ v \geq 0,$$

$$3(\lambda_1 + \lambda_2)u^2 + 6(\lambda_1 + \lambda_2)u\, r \geq 0,$$

$$\lambda = (\lambda_1, \lambda_2) \in K^* = K, \ \ (e_1, e_2) \in \text{int}\, K,$$

$$\lambda_1 e_1 + \lambda_2 e_2 = 1.$$

Assume the points (x, y, λ, p) *and* (u, v, λ, r) are feasible for (MONLP)1 and (MONLD)1, problems, respectively.

Because $f(., y)$ is a $K-$ preinvexity function w. r. t.x, we obtain

$$f(x, v) - f(u, v) - \eta(x, u)\nabla_u f(u, v) \in K \ \ \Rightarrow$$

$(x^3 - u^3 - 3u^2\eta(x, u),\ x^3 - u^3 - 3u^2\eta(x, u)) \in K.$
Furthermore, because $-f(x, .)$ is a $K-$ preinvexity function w. r. t.y, we get

$$(v^3 - y^3 - 3y^2\eta(v, y), 0) \in K.$$

where $\lambda = (\lambda_1, \lambda_2) \in K^* = K,\ (e_1, e_2) \in \text{int}\, K$ we have

$$(\lambda_1 + \lambda_2)(x^3 - u^3 - 3u^2\eta(x, u)) \geq 0 \ and \ \lambda_1(v^3 - y^3 - 3y^2\eta(v, y)) \geq 0.$$

Add the above two inequalities together, and we get

$$(\lambda_1 + \lambda_2)(x^3 - u^3) + \lambda_1(v^3 - y^3) \geq 3(\lambda_1 + \lambda_2)u^2\eta(x, u) + 3\lambda_1 y^2\eta(v, y)) \geq 0.$$

If we have the functions $\eta(x, u) = x - u, \quad \eta(v, y) = v - y,$

$$(\lambda_1 + \lambda_2)(x^3 - u^3) + \lambda_1(v^3 - y^3) + 3(\lambda_1 + \lambda_2)u^3 + 3\lambda_1 y^3 - (\lambda_1 + \lambda_2)u^2 x - 3\lambda_1 y^2 v \geq 0$$

Since the points (x, y, λ, p) *and* (u, v, λ, r) are feasible for (MONLP)1 and (MONLD)1 problems, respectively, we get

$$-3(\lambda_1 + \lambda_2)u^2 \leq 6(\lambda_1 + \lambda_2)u\, r \ \ and \ \ -3\lambda_1 y^2 \leq 6\lambda_1 y\, p \Rightarrow$$

$-3(\lambda_1 + \lambda_2)u^2 x \leq 6(\lambda_1 + \lambda_2)u^2\, r \ \ and \ \ -3\lambda_1 y^2 v \leq 6\lambda_1 y^2\, p.$
Then,

$$(\lambda_1 + \lambda_2)(x^3 - u^3) + \lambda_1(v^3 - y^3) + 3(\lambda_1 + \lambda_2)u^3 + 3\lambda_1 y^3 + 6(\lambda_1 + \lambda_2)u^2 r + 6\lambda_1 y^2 p \geq 0 \tag{**}$$

Because $e = (e_1, e_2) \in \text{int}\, K,\ \lambda_1 e_1 + \lambda_2 e_2 = 1$, we get

$$(u^3 - v^3, u^3) - 3(\lambda_1 + \lambda_2)u^3(e_1, e_2) + 6(\lambda_1 + \lambda_2)u^2 r(e_1, e_2) - (x^3 - y^3, x^3) + 3\lambda_1 y^3(e_1, e_2) + 6\lambda_1 y^2 p(e_1, e_2) \notin \text{int}\, K. \tag{***}$$

Because $\lambda = (\lambda_1, \lambda_2)$ we get

$$\lambda_1(u^3 - x^3 + y^3 - v^3 - 3(\lambda_1 + \lambda_2)u^3 e_1 - 6(\lambda_1 + \lambda_2)u^2 r e_1 - 3\lambda_1 y^3 e_1 - 6\lambda_1 y^2 p e_1) + \lambda_2(u^3 - x^3 - 3(\lambda_1 + \lambda_2)u^3 e_2 - 6(\lambda_1 + \lambda_2)u^2 r e_2 - 3\lambda_1 y^3 e_2 - 6\lambda_1 y^2 p e_2) \geq 0$$

That is

$$(u^3 - x^3)(\lambda_1 + \lambda_2) + \lambda_1(y^3 - v^3) - 3(\lambda_1 + \lambda_2)u^3(\lambda_1 e_1 + \lambda_2 e_2) - 6(\lambda_1 + \lambda_2)u^2 r(\lambda_1 e_1 + \lambda_2 e_2) - 3\lambda_1 y^3(\lambda_1 e_1 + \lambda_2 e_2) - 6\lambda_1 y^2 p(\lambda_1 e_1 + \lambda_2 e_2) \geq 0 \ \Rightarrow$$

$$(\lambda_1+\lambda_2)(u^3-x^3)+\lambda_1(y^3-v^3)-3(\lambda_1+\lambda_2)u^3-6(\lambda_1+\lambda_2)u^2r-3\lambda_1y^3-6\lambda_1y^2p\geq 0$$

Or

$$(\lambda_1+\lambda_2)(x^3-u^3)+\lambda_1(v^3-y^3)+3(\lambda_1+\lambda_2)u^3+6(\lambda_1+\lambda_2)u^2r+3\lambda_1y^3+6\lambda_1y^2p<0$$

That contradicts (**) and shows that the results of the duality theorem are weak.

Example 3.1.2 Let $\{(x,y)\in R^2: |y|\geq x,\ x\geq 0\},\ C_1,\ C_2=R_+, f(x,y)=(x^2,y^2-x^2)$. In addition, the (MONLP) and (MONLD) problems become the forms:

$$\begin{aligned}(MONLP)2: \quad & K-\min(x^2,y^2-x^2)-2\lambda_2y(e_1,e_2)-2\lambda_2p(e_1,e_2)\\ & \text{subject to}\quad -2\lambda_2y-2\lambda_2p\geq 0\\ & x\geq 0,\ \ y\geq 0\end{aligned}$$

and

$$\begin{aligned}(MONLD)2: \quad & K-\max\ (u^2,v^2-u^2)-2u^2(\lambda_1+\lambda_2)(e_1,e_2)-2u(\lambda_1+\lambda_2)\,r\,(e_1,e_2)\\ & \text{subject to}\quad 2(1-2\lambda_2)(u+r)\geq 0\\ & u\geq 0,\ \ v\geq 0\end{aligned}$$

Assume the inverse of the weak duality theorem, that is,

$$\begin{aligned}&\lambda^Tf(u,v)-\lambda^Tf(x,y)-[u^T\nabla_u(\lambda^Tf)(u,v)]-[u^T\nabla_{uu}(\lambda^Tf)(u,v)r]\\ &+[y^T\nabla_y(\lambda^Tf)(x,y)]+[y^T\nabla_{yy}(\lambda^Tf)(x,y)p]\geq 0\ \Rightarrow\end{aligned}$$

$$\begin{aligned}&\lambda^T(u^2,v^2-u^2)-\lambda^T(x^2,y^2-x^2)-\lambda^Tu^T(2u,-2u)-\lambda^Tu^T(2,-2)\,r+\lambda^Ty^T(0,2y)\\ &+\lambda^Ty^T(0,2)p\geq 0 \qquad (*")\end{aligned}$$

Since $f(0,y)$ is a $K-$ preinvexity w. r. t. x for fixed y = v and $-f(x,0)$ is a $K-$ preinvexity w. r. t. y for fixed x, we get

$$\lambda^Tf(x,y)-\lambda^Tf(u,v)\geq\lambda^T\eta(x,u)\nabla_uf(u,v)-\lambda^T\eta(v,y)\nabla_yf(x,y)$$

That indicates.

$$\lambda^T(x^2,y^2-x^2)-\lambda^T(u^2,v^2-u^2)\geq\lambda^T\eta(x,u)(2u,-2u)-\lambda^T\eta(v,y)(0,2y) \qquad (I")$$

By using assumption (ii), we have

$$\eta(x,u)\nabla_uf(u,v)-\eta(v,y)\nabla_yf(x,y)\geq(x-u)\nabla_uf(u,v)-(v-y)\nabla_yf(x,y)$$

Therefore, the relationship (I") gives

$$\begin{aligned}&\lambda^T(x^2,y^2-x^2)-\lambda^T(u^2,v^2-u^2)-\lambda^Tx^T(2u,-2u)+\lambda^Tu^T(2u,-2u)\\ &-\lambda^Ty^T(0,2y)+\lambda^Tv^T(0,2y)\geq 0\end{aligned}$$

From the feasibility points (x,y,λ,p), (u,v,λ,r) for the (MONLP) and (MONLD) problems, respectively, we have

$$-\lambda^T(0,2y)\geq\lambda^Tp^T(0,2),\qquad \lambda^T(2u,-2u)\geq-\lambda^Tr^T(2,-2)$$

We substitute into (3) to get

$$\lambda^T(x^2,y^2-x^2)-\lambda^Ty^T(0,2y)-\lambda^Ty^T(0,2)p-\lambda^T(u^2,v^2-u^2)$$

$$+\lambda^T u^T(2u,-2u)+\lambda^T u^T(2,-2)r \geq 0$$

This contradicts ($*^{''}$), then the proof is complete.

Lemma 3.1 *Yang [22]If there is x^*a weak minimum for the (MONLP) problem, then there $\alpha^* \in K^*,\ \ \beta^* \in Q^*$cannot be both zeros such that.*
$(\alpha^{*^T}\nabla f(x^*)^T+\beta^{*^T}\nabla g(x^*)^T)(x-x^*)\geq 0,\quad \forall\ x\in X,$

$$\beta^{*^T}g(x^*)=0.$$

Theorem 3.2 (Strong Duality) *Assume the point $(\overline{x},\overline{y},\overline{\lambda},\overline{p})$is a weak minimum for the (MONLP) problem with a fix $\lambda=\overline{\lambda},\ r=\overline{r}$and that:*
(i) $[\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})+\nabla_y\{\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})\overline{p}\}]$*is negative definite,*
(ii) $\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})$*is nonsingular, and*
(iii) The assumptions of *Theorem* 3.1 are correct.

Then, with equal values of objective functions for the (MONLP) and (MONLD) problems, it point $(\overline{x},\overline{y},\overline{\lambda},\overline{r})$ is feasible to solve the (MONLD) problem.

Proof Because the point $(\overline{x},\overline{y},\overline{\lambda},\overline{p})$ is a weak minimum for the (MONLP) problem, Lemma 3.1, states that there exists $\alpha\in K^*,\ \ \beta\in(C_2^*)=C_2,\ \ (\alpha,\beta)\neq 0,(x,y)\in C_1\times C_2,$ $\lambda,\ \ p\in K^*$ such that.

$$\begin{aligned}&[\alpha\nabla_x f(\overline{x},\overline{y})+(\beta-(\alpha^T e)\overline{y})^T\nabla_{yx}(\overline{\lambda}^T f)(\overline{x},\overline{y})\\&\quad+(\beta-(\alpha^T e)\overline{y})^T\nabla_x\{\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})\overline{p}\}](x-\overline{x})\\&\quad+[(\alpha-(\alpha^T e)\overline{\lambda})^T\nabla_y f(\overline{x},\overline{y})+(\beta-(\alpha^T e)\overline{y}-(\alpha^T e)\overline{p})\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})\\&\quad+(\beta-(\alpha^T e)\overline{y})^T\nabla_y\{\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})\overline{p}\}](y-\overline{y})\\&\quad+[(\beta-(\alpha^T e)\overline{\lambda})^T\nabla_y f(\overline{x},\overline{y})+(\beta-(\alpha^T e)\overline{y})^T\nabla_{yy}f(\overline{x},\overline{y})\overline{p}](\lambda-\overline{\lambda})\\&\quad+[(\beta-(\alpha^T e)\overline{y})^T\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})](p-\overline{p})\geq 0\end{aligned}\tag{6}$$

$$\beta[\nabla_y(\overline{\lambda}^T f)(\overline{x},\overline{y})+\overline{p}^T\nabla_{yy}(\overline{\lambda}^t f)(\overline{x},\overline{y})]=0\tag{7}$$

We claim that $\alpha\neq 0$, if a substitute is used $y\in C_2, x=\overline{x}\in C_1,\ \ \lambda=\overline{\lambda}\in K^*$ *and* $p=\overline{p}\in K^*$ the inequality (6) becomes

$$\begin{aligned}&[(\alpha-(\alpha^T e)\overline{\lambda})^T\nabla_y f(\overline{x},\overline{y})+(\beta-(\alpha^T e)\overline{y}-(\alpha^T e)\overline{p})\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})\\&\quad+(\beta-(\alpha^T e)\overline{y})^T\nabla_y\{\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})\overline{p}\}](y-\overline{y})\geq 0\end{aligned}\tag{8}$$

If $\alpha=0$ and belongs to K* and when $y=\beta+\overline{y}\in C_2$ we have.
$\beta\ [\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})+\nabla_y\{\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})\overline{p}\}]\beta\geq 0.$
We can conclude from assumption (i) that we obtained that $\beta=0$ this is not possible since,$(\alpha,\beta)\neq 0$ as a result,$\alpha\neq 0$.
When we substitute $x=\overline{x},\ \ y=\overline{y}$ *and* $\lambda=\overline{\lambda}$ in (6), we get

$$[(\beta-(\alpha^T e)\overline{y})^T\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})](p-\overline{p})\geq 0\ \ \forall\ p\in K^*\ \ \Rightarrow$$

$$(\beta-(\alpha^T e)\overline{y})^T\nabla_{yy}(\overline{\lambda}^T f)(\overline{x},\overline{y})=0$$

Furthermore, we know from assumption (ii) that.

$$\beta = (\alpha^T e)\overline{y} \tag{9}$$

If we put $y = \overline{y},\ \ \lambda = \overline{\lambda}\ \ and\ \ \ p = \overline{p}$ in (6), we get

$$\alpha\ \nabla_x f(\overline{x}, \overline{y})(x - \overline{x}) \geq 0\ \Rightarrow\ \nabla_x(\overline{\lambda}^T f)(\overline{x}, \overline{y})(x - \overline{x}) = 0\ \ \forall\ x \in C_1. \tag{10}$$

That implies

$$x,\ \overline{x} \in C_1,\ \ x + \overline{x} \in C_1\ \Rightarrow\ x\nabla_x(\overline{\lambda}^T f)(\overline{x}, \overline{y}) = 0 \tag{11}$$

The following is obtained by differentiating the relationship (11) w. r. t.x

$$x\nabla_{xx}(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + \nabla_x(\overline{\lambda}^T f)(\overline{x}, \overline{y}) = 0\ \Rightarrow\ \nabla_x(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + \overline{r}^T\nabla_{xx}(\overline{\lambda}^T f)(\overline{x}, \overline{y}) \in C_1^*$$

This means it point $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{r})$ is feasible to solve the (MONLD) problem.
If we put $x = 0,\ \ x = \overline{x}$ in (10), we get $\overline{x}^T\nabla_x(\overline{\lambda}^T f)(\overline{x}, \overline{y}) = 0$.
From the differential w. r. t.x, we get

$$\overline{x}^T\nabla_x(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + x^T\nabla_{xx}(\overline{\lambda}^T f)(\overline{x}, \overline{y})\overline{r} = 0 \tag{I}$$

Substituting Eq. (9) into Eq. (8) yields

$$[(\alpha - (\alpha^T e)\overline{\lambda})^T\nabla_y f(\overline{x}, \overline{y}) - (\alpha^T e)\overline{p}^T\nabla_{yy}(\overline{\lambda}^T f)(\overline{x}, \overline{y})](y - \overline{y}) \geq 0\ \Rightarrow$$
$$\{\nabla_y f(\overline{x}, \overline{y}) - [\nabla_y(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + \overline{p}^T\nabla_{yy}(\overline{\lambda}^T f)(\overline{x}, \overline{y})]\}(y - \overline{y}) \geq 0 \tag{12}$$

As a result of (7), and because $\beta \neq 0$ we obtained

$$\nabla_y f(\overline{x}, \overline{y})(y - \overline{y}) \geq 0\ \ \forall\ \ y \in C_2 \Rightarrow \tag{13}$$

$$\forall\ y,\ \ \overline{y} \in C_1,\ \ y + \overline{y} \in C_1\ \Rightarrow\ y\nabla_y(\overline{\lambda}^T f)(\overline{x}, \overline{y}) \geq 0 \tag{14}$$

We get the following result by differentiating the above inequality w. r. t.y

$$y\nabla_{yy}(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + \nabla_y(\overline{\lambda}^T f)(\overline{x}, \overline{y}) \geq 0\ \Rightarrow\ \nabla_y(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + \overline{p}^T\nabla_{yy}(\overline{\lambda}^T f)(\overline{x}, \overline{y}) \in C_2^*$$

When we put $y = 0\ \ and\ \ y = \overline{y}$ in (13) and (14) respectively, we get

$$\overline{y}^T\nabla_y f(\overline{x}, \overline{y}) = 0.$$

From the differential equation w. r. t.y, we get the following when put $y = p$.

$$y^T\nabla_y(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + y^T\nabla_{yy}(\overline{\lambda}^T f)(\overline{x}, \overline{y})\overline{p} = 0 \tag{II}$$

Thus, from (I) and (II), the (MONLP) and (MONLD) problems are equal in the values of objective functions.

To show that the point $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ is the weak maximum for the (MONLD) problem; otherwise, there exists a feasible point $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{r})$ such that

$$f(\overline{u}, \overline{v}) - \overline{u}^T\nabla_u(\overline{\lambda}^T f)(\overline{u}, \overline{v})e - (\overline{u}^T\nabla_{uu}(\overline{\lambda}^T f)(\overline{u}, \overline{v})\overline{r})e - f(\overline{x}, \overline{y}) + \overline{x}^T\nabla_x(\overline{\lambda}^T f)(\overline{x}, \overline{y})e$$
$$+ (\overline{x}^T\nabla_{xx}(\lambda^T f)(\overline{x}, \overline{y})\overline{p})e \in \text{ int } K.$$

Since

$$\nabla_x(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + \overline{x}^T\nabla_{xx}(\overline{\lambda}^T f)(\overline{x}, \overline{y}) = \nabla_y(\overline{\lambda}^T f)(\overline{x}, \overline{y}) + \overline{y}^T\nabla_{yy}(\overline{\lambda}^T f)(\overline{x}, \overline{y})$$

We got

$$f(\overline{u},\overline{v})-\overline{u}^T\nabla_u(\overline{\lambda}^T f)(\overline{u},\overline{v})e-(\overline{u}^T\nabla_{uu}(\overline{\lambda}^T f)(\overline{u},\overline{v})\overline{r})e-f(\overline{x},\overline{y})+\overline{y}^T\nabla_y(\overline{\lambda}^T f)(\overline{x},\overline{y})e$$
$$+(\overline{y}^T\nabla_{yy}(\lambda^T f)(\overline{x},\overline{y})\overline{p})e\in \text{int } K \quad \text{(III)}$$

This contradicts the weak duality theorem.

The following example illustrates the results of the strong duality theorem.

Example 3.2.1 Let the point $(\overline{x},\overline{y},\overline{\lambda},\overline{p})$ be a weak minimum for the (MONLP)1 problem. Then, based on Lemma 3.1, there exists.

$\alpha\in K^*,\ \ \beta\in(C_2^*)=C_2,\ \ (\alpha,\beta)\neq 0,\ (x,y)\in C_1\times C_2,\ \lambda,\ \ p\in K^*$ such a thing as

$$[\alpha(3x^2,3x^2)](x-\overline{x})+[(\alpha-(\alpha^Te)\overline{\lambda})^T(-3y^2,0)+(\beta-(\alpha^Te)\overline{y})^T-(\alpha^Te)\overline{p})(-6\lambda_1 y,0)$$
$$+(\beta-(\alpha^Te)\overline{y})^T(-6\lambda_1,0)\overline{p}](y-\overline{y})+[(\beta-(\alpha^Te)\overline{\lambda})^T(-3y^2,0)$$
$$+(\beta-(\alpha^Te)\overline{y})^T(-6\lambda_1 y,0)\overline{p}](\lambda-\overline{\lambda})+[(\beta-(\alpha^Te)\overline{y})^T)-(6\lambda_1 y,0)](p-\overline{p})\geq 0 \quad (15)$$

Equation (7) takes the form of

$$\beta\ [(-3\lambda_1 y^2,0)+(-6\lambda_1 y,0)\overline{p}]=0 \quad (16)$$

If we put it $\alpha\neq 0,\ x=\overline{x}\in C_1,\ \lambda=\overline{\lambda}\in K^* and\ p=\overline{p}\in K^*$ this way $\forall y\in C_2$, the inequality (15) becomes

$$[(\alpha-(\alpha^Te)\overline{\lambda})^T(-3y^2,0)+(\beta-(\alpha^Te)\overline{y}-(\alpha^Te)\overline{p})(-6\lambda_1 y,0)$$
$$+(\beta-(\alpha^Te)\overline{y})^T(-6\lambda_1,0)\overline{p}](y-\overline{y})\geq 0 \quad (17)$$

If $\alpha=0\in K^*\ and\ \ put\ \ y=\beta+\overline{y}\in C_2\ \ \Rightarrow.$
We get from (i) that

$$\beta=0,\ \ (\alpha,\beta)\neq 0\Rightarrow\ \ \alpha\neq 0$$

We get the following by substituting $x=\vec{x},\ \ y=\overline{y},\ \ \lambda=\overline{\lambda}$ in relation (13).

$$[(\beta-(\alpha^Te)\overline{y})^T(-6\lambda_1 y,0)](p-\overline{p})\geq 0\ \forall\ p\in K^*\ \ \Rightarrow$$

$$(\beta-(\alpha^Te)\overline{y})^T(-6\lambda_1 y,0)=0$$

In addition, from (ii), we obtain

$$\beta=(\alpha^Te)\overline{y} \quad (18)$$

In addition, when we substitute $y=\overline{y},\ \ \lambda=\overline{\lambda}\ and\ \ p=\overline{p}$ in relation (13), we get an

$$\alpha(3x^2,3x^2)(x-\overline{x})\geq 0\ \ \Rightarrow\ \ (3x^2,3x^2)(x-\overline{x})\geq 0\ \forall\ x\in C_1$$

For each $x,\ \ \overline{x}\in C_1,\ \ x+\overline{x}\in C_1\ \ \Rightarrow\ \ x(3x^2,3x^2)\geq 0.$
Then, by differentiating this inequality w. r. t. x, we obtain

$$x(6\lambda_1 x,6\lambda_2 x)+(3\lambda_1 x^2,2\lambda_2 x^2)\geq 0\ \ \Rightarrow\ \ (3\lambda_1 x^2,3\lambda_2 x^2)+r(6\lambda_1 x,6\lambda_2 x)\in C_1$$

That shows the point $(\overline{x},\overline{y},\overline{\lambda},\overline{r})$ is feasible for the (MONLD) problem.

Furthermore, if we put $x = 0,\ \ x = \overline{x}$, we get $\overline{x}^T(3\lambda_1 x^3, 3\lambda_2 x^3) + x(6\lambda_1 x, 6\lambda_2 x)\overline{r} = 0$. Substituting (16) into (15), we get the following:

$$[(\alpha - (\alpha^T e)\overline{\lambda})^T(-3y^2, 0) - (\alpha^T e)\overline{p}^T(-6\lambda_1 y, 0)](y - \overline{y}) \geq 0 \ \Rightarrow$$
$$\{(-3y^2, 0) - [(-3\lambda_1 y^2, 0) + \overline{p}^T(-6\lambda_1 y, 0)]\}(y - \overline{y}) \geq 0 \tag{19}$$

Because $\beta \neq 0$ (16) implies

$$(-3y^2, 0)(y - \overline{y}) \geq 0 \ \ \forall \ \ y \in C_2 \tag{20}$$

For each $y,\ \overline{y} \in C_1,\ \ y + \overline{y} \in C_1 \ \ \Rightarrow$

$$y(-3\lambda_1 y^2, 0) \geq 0 \tag{21}$$

By differentiating the above inequality w. r. t. y, we obtain the following:

$$y(-6\lambda_1 y, 0) + (-3\lambda_1 y^2, 0) \geq 0 \ \ \Rightarrow$$

$$(-3\lambda_1 y^2, 0) + \overline{p}^T(-6\lambda_1 y, 0) \in C_2$$

When we put $y = 0,\ \ y = \overline{y}$ in (17) and (19), respectively, we obtain.
$\overline{y}^T(-3y^2, 0) = 0$.
Then comes the differential w. r. t. y we get for $y = p$

$$y^T(-3\lambda_1 y^2, 0) + y^T(-6\lambda_1 y, 0)\overline{p} = 0 \tag{22}$$

Therefore, the (MONLP) and (MONLD) problems have equal objective function values.

To demonstrate that point is the weak maximum for the (MONLD) problem; otherwise, there exists a feasible point $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{r})$ such that

$$(u^3 - v^3, u^3) - \overline{u}^T(3\lambda_1 u^2, 3\lambda_2 u^2)e - \overline{u}^T(6\lambda_1 u, 6\lambda_2 u)re - (x^3 - y^3, x^3) + x(3\lambda_1 x^2, 3\lambda_2 x^2)e$$
$$+ \overline{x}^t(6\lambda_1 x, 6\lambda_2 x)\overline{p})e \in \text{ int. } K$$

If $p = r = 1$ we have

$$(u^3 - v^3 - 3\lambda_1 u^3 - 6\lambda_1 u^2 - x^3 + y^3 - 3\lambda_1 y^3 - 6\lambda_1 y^2,\ u^3 - 3\lambda_2 u^3 - 6\lambda_2 u^2 - x^2) \in Int. K$$

That contradicts the theorem.

Example 3.2.2 Assume the point $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ is a weak minimum for the (MONLP)2 problem; then, according to Lemma 3.1, there exists.
$\alpha \in K^*,\ \ \beta \in (C_2^*),\ \ (\alpha, \beta) \neq 0,\ (x, y) \in C_1 \times C_2,\ \ \lambda, p \in K^*$ such that

$$\beta^T[(2x, -2x)](x - \overline{x}) + [(\alpha^T - (\alpha^T e)\overline{\lambda})^T(0, 2y) + (\beta^T - \alpha^T e)\overline{y} - (\alpha^T e)\overline{p})(0, 2\lambda_2)](y - \overline{y})$$
$$+ [(\beta - (\alpha^T e)\overline{\lambda})^T(0, 2y) + (\beta - (\alpha^T e)\overline{y})^T(0, 2)\overline{p}](\lambda - \overline{\lambda})$$
$$+ [(\beta - (\alpha^T e)\overline{y})^T(0, 2\lambda_2)](p - \overline{p}) \geq 0 \tag{6"}$$

Also, Eq. (7) takes the form:

$$\beta[(0, 2\lambda_2 y) + \overline{p}^T(0, 2\lambda_2)] = 0 \tag{7"}$$

When we put $x = \overline{x} \in C_1,\ \ \lambda = \overline{\lambda} \in K^*,\ \ p = \overline{p}\ \forall\, y \in C_2$, then inequality $(6)^{"}$ becomes.

$$[(\alpha - (\alpha^T e)\overline{\lambda})^T(0, 2y) + (\beta - (\alpha^T e)\overline{y} - (\alpha^T e)\overline{p})(0, 2\lambda_2)] \geq 0 \tag{8"}$$

If $\alpha = 0 \in K^*, \;\; y = \beta + \overline{y} \in C_2$ we get

$$\beta\,(0, 2\lambda_2) \geq 0$$

Using assumption (i), we obtain $\beta = 0$.
This has not been possible since $(\alpha, \beta) \neq 0$ then $\alpha \neq 0$.
And if we put $x = \overline{x}, \;\; y = \overline{y}, \;\; \lambda = \overline{\lambda}$ it into Eq. (6"), we get

$$[(\beta - (\alpha^T e)\overline{y})^T (0, 2\lambda_2)](p - \overline{p}) \geq 0 \;\; \forall \;\; p \in K^*$$

Therefore, we obtained the following from assumption (ii),

$$\beta = (\alpha^T e)\overline{y} \tag{9"}$$

When we put $y = \overline{y}, \;\; \lambda = \overline{\lambda}, \;\; p = \overline{p}$ in inequality (6"), we get

$$\begin{aligned} &\alpha(2x, -2x)(x - \overline{x}) \geq 0 \;\; \Rightarrow \\ &(2x, -2x)(x - \overline{x}) = 0 \;\; \forall \;\; x \in C_1 \end{aligned} \tag{10"}$$

Furthermore, for each

$$\begin{aligned} &x, \;\; \overline{x} \in C_1, \;\; x + \overline{x} \in C_1 \;\;\; \Rightarrow \\ &x(2\lambda_1 x, -2\lambda_2 x) = 0 \end{aligned} \tag{11"}$$

When we differentiate the above equation w. r. t. x, we get

$$(2\lambda_1, -2\lambda_2) + (2\lambda_1 x, -2\lambda_2 x) = 0$$

This implies its point a feasible for the (MONLD)2 problem.
If we put $x = 0, \;\; x = \overline{x}$ in Eq. (10"), we get

$$\overline{x}(2\lambda_1 x, -2\lambda_2 x) = 0$$

Differentiating this equation w. r. t. x, we obtain.

$$\overline{x}(2\lambda_1 x, -2\lambda_2 x) + x(2\lambda_1, -2\lambda_2)\,r = 0 \tag{I"}$$

Substituting from (9") into (8") yields

$$\begin{aligned} &[(\alpha - (\alpha^T e)\overline{\lambda})^T (0, 2\overline{y}) - (\alpha^T e)\overline{p}^T (0, 2\lambda_2)](y - \overline{y}) \geq 0 \;\; \Rightarrow \\ &\{(0, 2\overline{y}) - [(0, 2\lambda_2 \overline{y}) + \overline{p}^T (0, 2\lambda_2)]\}(y - \overline{y}) \geq 0 \end{aligned} \tag{12"}$$

Since $\beta \neq 0$ Eq. (7") takes the form.

$$(0, 2\overline{y})(y - \overline{y}) \geq 0 \;\; \forall \;\; y \in C_2 \tag{13"}$$

For each $y, \;\; \overline{y} \in C_1, \;\;\; y + \overline{y} \in C_1$, we have.

$$\overline{y}(0, 2\lambda_2 \overline{y}) \geq 0 \tag{14"}$$

By differentiable w. r. t. y, we get

$$\begin{aligned} &\overline{y}(0, 2\overline{\lambda}_2) + (0, 2\lambda_2 \overline{y}) \geq 0 \;\;\; \Rightarrow \\ &(0, 2\overline{\lambda}_2 \overline{y}) + \overline{p}(0, 2\lambda_2) \in C_2^* \end{aligned}$$

When we put $y = 0, \;\; y = \overline{y}$ in the inequalities (13") and (14"), respectively, we get

$$\overline{y}(0, 2\overline{\lambda}_2 \overline{y}) = 0$$

By differentiable w. r. t. y, we obtain for $y = p$.

$$\overline{y}^T(0, 2\overline{\lambda}_2\overline{y}) + \overline{y}^T(0, 2\overline{\lambda}_2)\overline{p} = 0 \tag{II"}$$

Therefore, from (I)" and (II)", the (MONLP)2 and (MONLD)2 problems are equal-value objective functions.

To demonstrate that this point is the weak maximum for the (MONLD)2 problem, otherwise, there exists a feasible point $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{r})$ such that

$$(u^2, v^2 - u^2) - \overline{u}^T(2\lambda_1 u, -2\lambda_2 u)e - \overline{u}^T(2\lambda_1, -2\lambda_2)\, r\, e - (x^2, y^2 - x^2) + \overline{x}^T(2\lambda_1 x, -2\lambda_2 x)\, e + \overline{x}^T(2\lambda_1, -2\lambda_2)\ \overline{p}\, e\ \in \text{int.}\, K$$

This contradicts the theorem, so the proof is complete.

Theorem 3.3 (Strict Converse Duality). *Assume that there is $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{r})$a weak maximum for the (MONLD) problem with a fix $\lambda = \overline{\lambda}$, $p = \overline{p}$and that:*

(i) $[\nabla_{uu}(\overline{\lambda}^T f)(\overline{u}, \overline{v}) + \nabla_u\{\nabla_{uu}(\overline{\lambda}^T f)(\overline{u}, \overline{v})\overline{r}\}]$*is positive definite,*

(ii)$\nabla_{uu}(\overline{\lambda}^T f)(\overline{u}, \overline{v})$*is nonsingular, in addition to.*

(iii) Theorem 3. 1, assumptions are held.

Then, for the (MONLP) problem, the point $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{r})$is feasible, and the value objective functions for the (MONLP) and (MONLD) problems have the same value.

The proof is similar to Theorem 3. 2.

4 Self-duality

Mathematical programming is called self-dual if it is formally identical to its dual (see Mishra [15, 16] and Kassem [12]), i.e., the dual can be recast in the form of a primal.

Definition 4.1 The function $f(x, y)$ is skew-symmetrical if $\forall x, y$ we have $f(x, y) = -f(y, x)$.

Assume the function f is skew-symmetric, $m = n,\ p = r\ and\ C_1 = C_2$.

We rewrite the (MONLD) problem as the following minimization problem:

$$(\text{MONLD}') : \quad K - \min\ -f(u, v) + [u^T\nabla_u(\lambda^T f)(u, v)]e + [u^T\nabla_{uu}(\lambda^T f)(u, v)r]e$$

$subject\ to\ (u, v) \in C_1 \times C_2$,
$\nabla_u(\lambda^T f)(u, v) + r^T\nabla_{uu}(\lambda^T f)(u, v) \in C_1^*$,
$\lambda,\ r \in K^*,\quad e \in \text{int}\ K,\quad \lambda^T e = 1$.

Since $\nabla_u f(u, v) = -\nabla_v f(v, u)\ and\ \nabla_{uu} f(u, v) = -\nabla_{vv} f(v, u)$ then, the above (MONLD') problem has taken the form:

$$(\text{MONLD}')_1 : \quad K - \min\ f(v, u) - [u^T\nabla_v(\lambda^T f)(v, u)]e - [u^T\nabla_{vv}(\lambda^T f)(v, u)r]e$$

$subject\ to\ (v, u) \in C_2 \times C_1$,
$-\nabla_v(\lambda^T f)(v, u) - r^T\nabla_{vv}(\lambda^T f)(v, u) \in C_2^*$,
$\lambda,\ r \in K^*,\quad e \in \text{int}\ K,\quad \lambda^T e = 1$.

This means the $(MONLD')$ problem is formally identical to the (MONLP) problem, that is, the objective and constraint functions are identical. Therefore, the problem is self-dual.

Then, the point $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ of feasibility for the (MONLP) problem implies the point $(\overline{y}, \overline{x}, \overline{\lambda}, \overline{r})$ of feasibility for the (MONLD) problem and vice versa.

Theorem 4.1 (Self-Duality) *If the point* $(x_0, y_0, \lambda_0, r_0)$*is jointly optimal for the self-dual problem, then there exists a point* $(y_0, x_0, \lambda_0, r_0)$*such that.*

$$
\begin{aligned}
& f(y_0, x_0) - (y_0^T \nabla_y (\lambda_0^T f)(y_0, x_0))e - (y_0^T \nabla_{yy} (\lambda_0^T f)(y_0, x_0) r_0)e \\
&= f(x_0, y_0) - (x_0^T \nabla_x (\lambda_0^T f)(x_0, y_0))e - (x_0^T \nabla_{xx} (\lambda_0^T f)(x_0, y_0) r_0)e \\
&= 0
\end{aligned}
$$

Proof As shown above, the problem $(MONLD')$ is formally identical to the (MONLP) problem. Then the point $(x_0, y_0, \lambda_0, r_0)$ is optimal for the $(MONLD')$ problem, which implies the point $(y_0, x_0, \lambda_0, r_0)$ is optimal for the (MONLP) problem. From the symmetric duality and the $(MONLD')$ problem, we derive the following:

$$
\begin{aligned}
& f(y_0, x_0) - (y_0^T \nabla_y (\lambda_0^T f)(y_0, x_0))e - (y_0^T \nabla_{yy} (\lambda_0^T f)(y_0, x_0) r_0)e \\
&= f(x_0, y_0) - (x_0^T \nabla_x (\lambda_0^T f)(x_0, y_0))e - (x_0^T \nabla_{xx} (\lambda_0^T f)(x_0, y_0) r_0)e \\
&= -f(y_0, x_0) + (y_0^T \nabla_y (\lambda_0^T f)(y_0, x_0))e + (y_0^T \nabla_{yy} (\lambda_0^T f)(y_0, x_0) r_0)e \\
&= 0
\end{aligned}
$$

5 Special Cases

1. Using $\eta(x, u) = x - u,\ \eta(v, y) = v - y$ *and* $p = r = 0$ the (MONLP) and (MONLD) problems are related to the problems studied in [13].
2. If we put $p = r = 0$ and the objective functions do not contain the terms $y^T \nabla_y (\lambda^T f)(x, y)$, $u^T \nabla_u (\lambda^T f)(u, v)$, then the (MONLP) and (MONLD) problems are reduced to the problems studied in [12].

6 Conclusions

In this work, we presented generalizations of the cone-preinvexity functions and studied a pair of second-order symmetric dualities for the (MONLP) problems under these generalizations of the cone-preinvexity functions. In addition, we established and proved the weak duality, strong duality, strict converse duality, and self-duality theorems under these generalizations of the cone-preinvexity functions. Finally, four nontrivial numerical examples are presented to show that the results of the weak and strong duality theorems are true.

In the future, I will study this idea for higher-order fractional vector optimization problems.

Funding Open access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB). The Egyptian Knowledge Bank (EKB) and The Science, Technology and innovation Funding Authority (STDF) have partnered to provide open access funding.

Declarations

Conflict of interest The author states that he has no financial interest in the subject matter of this research. There are no competing interests to declare.

Ethical Standards Acceptance of the journal's ethical standards and agreement to follow them: The author accepts the journal's ethical standards.

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