Journal of Scientific Computing (2021) 89:34
https://doi.org/10.1007/s10915-021-01636-3

Error Constants for the Semi-Discrete Galerkin Approximation of the Linear Heat Equation

Makoto Mizuguchi[1] · Mitsuhiro T. Nakao[2] · Kouta Sekine[3] · Shin'ichi Oishi[4,5]

Received: 30 September 2020 / Revised: 2 September 2021 / Accepted: 7 September 2021 /
Published online: 29 September 2021

Abstract
In this paper, we propose $L^2(J;\, H_0^1(\Omega))$ and $L^2(J;\, L^2(\Omega))$ norm error estimates that provide the explicit values of the error constants for the semi-discrete Galerkin approximation of the linear heat equation. The derivation of these error estimates shows the convergence of the approximation to the weak solution of the linear heat equation. Furthermore, explicit values of the error constants for these estimates play an important role in the computer-assisted existential proofs of solutions to semi-linear parabolic partial differential equations. In particular, the constants provided in this paper are better than the existing constants and, in a sense, the best possible.

Keywords semi-discrete Galerkin approximation · Error constant · A priori error estimate · Best possible

Mathematics Subject Classification 65N15 · 65N30 · 35K05

1 Introduction

In this paper, we propose norm error estimates that provide explicit values of error constants for the semi-discrete Galerkin approximation of the linear heat equation.

Let $\Omega \subset \mathbb{R}^N (N \in \mathbb{N})$ be a bounded Lipschitz domain. $L^2(\Omega)$ denotes the real Hilbert space endowed with inner product $(u, v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)dx$ and norm $\|u\|_{L^2(\Omega)} := \sqrt{(u, u)_{L^2(\Omega)}}$ for $u, v \in L^2(\Omega)$. The real Hilbert space $H_0^1(\Omega)$ is endowed with inner product

✉ Makoto Mizuguchi
mmizuguchi168@g.chuo-u.ac.jp

1 Department of Information and System Engineering, Chuo University, Tokyo, Japan

2 Faculty of Science and Engineering, Waseda University, Tokyo, Japan

3 Department of Information and Communication Systems Engineering, Chiba Institute of Technology, Chiba, Japan

4 Department of Applied Mathematics, Faculty of Science and Engineering, Waseda University, Tokyo, Japan

5 CREST, JST, Tokyo, Japan

$a(u,v) := (\nabla u, \nabla v)_{L^2(\Omega)}$ and norm $\|u\|_{H_0^1(\Omega)} := \sqrt{a(u,u)}$ for $u, v \in H_0^1(\Omega)$, where any function u in $H_0^1(\Omega)$ vanishes on the boundary of Ω. Let $H^{-1}(\Omega)$ be the dual space of $H_0^1(\Omega)$ and $\langle \cdot, \cdot \rangle$ be the real dual product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We identify $u \in H_0^1(\Omega)$ with $u \in L^2(\Omega)$ and with $u \in H^{-1}(\Omega)$ based on the Gelfand triple $H_0^1(\Omega) \subset L^2(\Omega) = L^2(\Omega)^* \subset H^{-1}(\Omega)$ (all inclusions are dense with continuous injections), where $L^2(\Omega)^*$ denotes the dual space of $L^2(\Omega)$. Let $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined by

$$\langle \mathcal{A}u, v \rangle = a(u, v) \ \forall v \in H_0^1(\Omega).$$

We also define as $\mathcal{W} = \{u \in H_0^1(\Omega) \mid \mathcal{A}u \in L^2(\Omega)\}$, where the regularities of the functions in $\mathcal{W}$ are dependent on shapes of the domain Ω; (see e.g., [4]).

For parameter $h > 0$, the function space V_h denotes a finite-dimensional subspace of $H_0^1(\Omega)$. We define the Ritz projection $R_h : H_0^1(\Omega) \to V_h$ as

$$a(u - R_h u, v_h) = 0 \ \ \forall v_h \in V_h. \tag{1}$$

Assume that the constant C_h satisfies

$$\|u - R_h u\|_{H_0^1(\Omega)} \le C_h \|\mathcal{A}u\|_{L^2(\Omega)} \ \ \forall u \in \mathcal{W}, \tag{2}$$

where $C_h \to 0$ as $h \to 0$. Then, Aubin-Nitsche's trick implies

$$\|u - R_h u\|_{L^2(\Omega)} \le C_h \|u - R_h u\|_{H_0^1(\Omega)} \ \ \forall u \in H_0^1(\Omega). \tag{3}$$

The estimates (2) and (3) derive very meaningful inequalities for the numerical analysis of elliptic partial differential equations (PDEs); (see e.g., [1]). In particular, explicit values of C_h play an important role in computer-assisted existential proofs of solutions to elliptic PDEs; (see e.g., [12]). Therefore, many estimates for obtaining the values have been proposed and applied to computer-assisted existential proofs of solutions to semi-linear elliptic PDEs; (see e.g., [6–8,10,11,15,17] and references therein).

In this paper, we propose two norm error estimates, which provide the best possible error constants using only C_h in (2) for the semi-discrete Galerkin approximation of the linear heat equation. Let $J = (t_0, t_1)$ $(0 \le t_0 < t_1 < \infty)$. For any function $v : J \times \Omega \to \mathbb{R}$, we introduce the shortened form $v(t) := v(t, \cdot)$ and $\partial_t v(t) := (\partial_t v)(t, \cdot)$, where ∂_t denotes the weak derivative for $t \in J$. For any real Hilbert space Y, $L^2(J; Y)$ is defined by the function space of Lebesgue integrable functions $J \ni t \mapsto v(t) \in Y$ endowed with the norm $\|v\|_{L^2(J;Y)} := \sqrt{\int_J \|v(s)\|_Y^2 ds}$ for $v \in L^2(J; Y)$. Let $H^1(J; Y)$ denote the set of weak differentiable functions for J endowed with the norm $\|v\|_{H^1(J;Y)} = \sqrt{\int_J \left(\|v(s)\|_Y^2 + \|\partial_s v(s)\|_Y^2\right) ds}$ for $v \in H^1(J; Y)$. The function space $C^0([t_0, t_1]; L^2(\Omega))$ is defined by the set of continuous functions as $[t_0, t_1] \ni t \mapsto v(t) \in L^2(\Omega)$. Let $Z := H^1(J; H^{-1}(\Omega)) \cap L^2(J; H_0^1(\Omega))$ be endowed with the norm $\|v\|_Z = \sqrt{\|v\|^2_{H^1(J;H^{-1}(\Omega))} + \|v\|^2_{L^2(J;H_0^1(\Omega))}}$. Let $w_0 \in L^2(\Omega)$ and $f \in L^2(J; H^{-1}(\Omega))$. We define the weak solution as the function $w \in Z$ that satisfies the linear heat equation:

$$\begin{cases} \langle \partial_t w(t), v \rangle + a(w(t), v) = \langle f(t), v \rangle \ \ \forall v \in H_0^1(\Omega), \ \text{a.e. } t \in J \\ w(t_0) = w_0. \end{cases} \tag{4}$$

Let $V_{J,h} := H^1(J; V_h)$. We define the semi-discrete Galerkin approximation of (4) as the function $w_h \in V_{J,h}$ that satisfies

$$\begin{cases} \langle \partial_t w_h(t), v_h \rangle + a(w_h(t), v_h) = \langle f(t), v_h \rangle \ \ \forall v_h \in V_h, \ \text{a.e. } t \in J \\ w_h(t_0) = \hat{w}_0, \end{cases} \tag{5}$$

where $\hat{w}_0 \in V_h$ is any approximation of w_0 in (4). The error estimates for the semi-discrete Galerkin approximation have been proposed in, for example, $L^2(\Omega)$, $H^1(\Omega)$, $L^\infty(\Omega)$, $L^2(J; H_0^1(\Omega))$, and $L^2(J; L^2(\Omega))$ norms; (see e.g., [16]). The regularities of w_0 and f required for deriving the convergence of the semi-discrete Galerkin approximation w_h to the weak solution w have been studied. For instance, for $w_0 \in L^2(\Omega)$ and $f \in L^2(J; H^{-1}(\Omega))$, $\|w - w_h\|_Z \to 0$ as $h \to 0$ holds under some assumptions [2, Theorem 3.2 and 3.3]. In these studies, there is a case in which an $L^2(J; L^2(\Omega))$ norm error estimate of the form $\|w - w_h\|_{L^2(J;L^2(\Omega))} \le E_h \|w - w_h\|_{L^2(J;H_0^1(\Omega))}$ is derived. The estimate of such a form is called the parabolic Aubin-Nitsche's trick; (see e.g., [2, Theorem 3.5]).

By contrast, there are few results of studies for the explicit values of the error constants. Nakao et al. started pioneering studies with the constants and they have shown that for w in (4) and w_h in (5),

$$\|w - w_h\|_{L^2(J;H_0^1(\Omega))} \le 2C_h \|f\|_{L^2(J;L^2(\Omega))} \tag{6}$$

$$\|w - w_h\|_{L^2(J;L^2(\Omega))} \le 4C_h \|w - w_h\|_{L^2(J;H_0^1(\Omega))}, \tag{7}$$

where they assume that $t_0 = 0$, $w_0 = \hat{w}_0 = 0$, $f \in L^2(J; L^2(\Omega))$, and Ω is a bounded convex polygonal or polyhedral domain [14, Theorem 4, 5]. Furthermore, these estimates (6) and (7) have been applied to verified numerical computations for semi-linear parabolic PDEs [14]. Currently, following the estimates in (6) and (7), methods, which are related to verified numerical computations to semi-linear parabolic PDEs, have been proposed; (see e.g., [5,9,13] and references therein).

In this paper, we provide sharp $L^2(J; H_0^1(\Omega))$ and $L^2(J; L^2(\Omega))$ norm error estimates, which contribute to improving methods for computer-assisted proofs for semi-linear parabolic PDEs, assuming $w_0 \in L^2(\Omega)$, $\hat{w}_0 \in V_h$, and a bounded Lipschitz domain Ω. First, we derive an $L^2(J; H_0^1(\Omega))$ norm error estimate.

Theorem 1 *For w and w_h defined by (4) and (5) with $f \in L^2(J; L^2(\Omega))$, we have*

$$\|w - w_h\|_{L^2(J;H_0^1(\Omega))} \le \sqrt{\|w_0 - \hat{w}_0\|_{L^2(\Omega)}^2 + C_h^2 \left(\|f\|_{L^2(J;L^2(\Omega))}^2 + \|\hat{w}_0\|_{H_0^1(\Omega)}^2 \right)}.$$

Corollary 1 follows immediately from Theorem 1 with $w_0 = \hat{w}_0 = 0$.

Corollary 1 *We use the same notation and assumptions as in Theorem 1 and assume that $w_0 = \hat{w}_0 = 0$ in (4) and (5). Then, we obtain*

$$\|w - w_h\|_{L^2(J;H_0^1(\Omega))} \le C_h \|f\|_{L^2(J;L^2(\Omega))}.$$

Next, we provide the parabolic Aubin-Nitsche's trick as the following theorem:

Theorem 2 *For w and w_h defined by (4) and (5), we have*

$$\|w - w_h\|_{L^2(J;L^2(\Omega))} \le \sqrt{\|R_h \mathcal{A}^{-1}(w_0 - \hat{w}_0)\|_{H_0^1(\Omega)}^2 + C_h^2 \|w - w_h\|_{L^2(J;H_0^1(\Omega))}^2}.$$

We define $P_h : L^2(\Omega) \to V_h$ as

$$(u - P_h u, v_h)_{L^2(\Omega)} = 0 \ \ \forall v_h \in V_h. \tag{8}$$

Because $R_h \mathcal{A}^{-1}(w_0 - \hat{w}_0) = 0$ when $\hat{w}_0 = P_h w_0$, Theorem 2 with $\hat{w}_0 = P_h w_0$ leads to Corollary 2.

Corollary 2 *We use the same notation and assumptions as in Theorem 2 and assume that* $\hat{w}_0 = P_h w_0$ *in (5). Then, we obtain*

$$\|w - w_h\|_{L^2(J;L^2(\Omega))} \leq C_h \|w - w_h\|_{L^2(J;H_0^1(\Omega))}.$$

Assuming that $t_0 = 0$ and $w_0 = \hat{w}_0 = 0$, Corollaries 1 and 2 immediately yield sharper estimates than (6) and (7). Each of the constants derived by Corollaries 1 and 2 should be the best possible in the sense that we only use the error constant C_h for the Ritz projection in (2).

In this paper, we prove Theorem 1 in Sect. 2 and Theorem 2 in Sect. 3.

2 Proof of Theorem 1

We provide the proof of Theorem 1.

Proof For $t \in J$, it follows from (5) with $v_h = R_h(w - w_h)(t) \in V_h$ that

$$\begin{aligned}
&\langle f(t) - \mathcal{A}w_h(t) - \partial_t w_h(t), R_h(w - w_h)(t)\rangle \\
&= \langle f(t), R_h(w - w_h)(t)\rangle - a(w_h(t), R_h(w - w_h)(t)) - \langle \partial_t w_h(t), R_h(w - w_h)(t)\rangle \\
&= \langle f(t), R_h(w - w_h)(t)\rangle - \langle f(t), R_h(w - w_h)(t)\rangle \\
&= 0.
\end{aligned} \tag{9}$$

From (4) with $v = (w - w_h)(t)$,

$$\begin{aligned}
&\langle \partial_t(w - w_h)(t), (w - w_h)(t)\rangle + a((w - w_h)(t), (w - w_h)(t)) \\
&= \langle f(t) - \mathcal{A}w_h(t) - \partial_t w_h(t), (w - w_h)(t)\rangle \\
&= \langle f(t) - \mathcal{A}w_h(t) - \partial_t w_h(t), (I - R_h)(w - w_h)(t)\rangle \\
&\quad + \langle f(t) - \mathcal{A}w_h(t) - \partial_t w_h(t), R_h(w - w_h)(t)\rangle.
\end{aligned}$$

The equality (9) yields

$$\begin{aligned}
&\langle \partial_t(w - w_h)(t), (w - w_h)(t)\rangle + a((w - w_h)(t), (w - w_h)(t)) \\
&= \langle f(t) - \mathcal{A}w_h(t) - \partial_t w_h(t), (I - R_h)(w - w_h)(t)\rangle \\
&= a(\mathcal{A}^{-1}(f(t) - \mathcal{A}w_h(t) - \partial_t w_h(t)), (I - R_h)(w - w_h)(t)) \\
&= a((I - R_h)\mathcal{A}^{-1}(f(t) - \mathcal{A}w_h(t) - \partial_t w_h(t)), (w - w_h)(t)) \\
&= a((I - R_h)\mathcal{A}^{-1}(f(t) - \partial_t w_h(t)), (w - w_h)(t)),
\end{aligned}$$

where the last equality holds because $(I - R_h)w_h(t) = 0$ for $w_h(t) \in V_h$. Because $f(t) - \partial_t w_h(t) \in L^2(\Omega)$, it follows from (2) that

$$\begin{aligned}
&\langle \partial_t(w - w_h)(t), (w - w_h)(t)\rangle + a((w - w_h)(t), (w - w_h)(t)) \\
&\leq \|(I - R_h)\mathcal{A}^{-1}(f(t) - \partial_t w_h(t))\|_{H_0^1(\Omega)} \|(w - w_h)(t)\|_{H_0^1(\Omega)} \\
&\leq C_h \|f(t) - \partial_t w_h(t)\|_{L^2(\Omega)} \|(w - w_h)(t)\|_{H_0^1(\Omega)}.
\end{aligned} \tag{10}$$

Note that $w - w_h \in Z \subset C^0([t_0, t_1]; L^2(\Omega))$ and

$$\left(\frac{dk}{dt}\right)(t) = 2\langle \partial_t (w - w_h)(t), (w - w_h)(t)\rangle \quad t \in J$$

are satisfied, where $k(t) := \|(w - w_h)(t)\|^2_{L^2(\Omega)}$; (see e.g., [3, Theorem 3 in Sect. 5.9]). Integrating both sides of (10) on J yields,

$$\begin{aligned} &\frac{1}{2}\|(w - w_h)(t_1)\|^2_{L^2(\Omega)} - \frac{1}{2}\|(w - w_h)(t_0)\|^2_{L^2(\Omega)} + \|w - w_h\|^2_{L^2(J;H^1_0(\Omega))} \\ &= \int_J \langle \partial_s (w - w_h)(s), (w - w_h)(s)\rangle ds + \int_J a((w - w_h)(s), (w - w_h)(s)) ds \\ &\leq C_h \|f - \partial_t w_h\|_{L^2(J;L^2(\Omega))} \|w - w_h\|_{L^2(J;H^1_0(\Omega))}. \end{aligned} \tag{11}$$

We consider an estimate of $\|f - \partial_t w_h\|_{L^2(J;L^2(\Omega))}$. Equation (5) with $v_h = \partial_t w_h(t) \in V_h$ provides that

$$\|\partial_t w_h(t)\|^2_{L^2(\Omega)} + a(w_h(t), \partial_t w_h(t)) = (f(t), \partial_t w_h(t))_{L^2(\Omega)}$$

holds. Integrating on J yields

$$\|\partial_t w_h\|^2_{L^2(J;L^2(\Omega))} + \int_J a(w_h(s), \partial_s w_h(s)) ds = \int_J (f(s), \partial_s w_h(s))_{L^2(\Omega)} ds. \tag{12}$$

Because $w_h \in H^1(J; V_h)$, we have

$$\begin{aligned} &\int_J a(w_h(s), \partial_s w_h(s)) ds \\ &= \int_J \int_\Omega \nabla w_h(s, x) \cdot \nabla \partial_s w_h(s, x) dx ds \\ &= \int_J \frac{d}{ds}\left(\int_\Omega |\nabla w_h(s, x)|^2 dx\right) ds - \int_J \int_\Omega \nabla \partial_s w_h(s, x) \cdot \nabla w_h(s, x) dx ds \\ &= \int_J \left(\frac{dg}{ds}\right)(s) ds - \int_J a(w_h(s), \partial_s w_h(s)) ds, \end{aligned}$$

where $g(t) := a(w_h(t), w_h(t)) = \|w_h(t)\|^2_{H^1_0(\Omega)}$. Since $w_h \in H^1(J; H^1_0(\Omega)) \subset C^0([t_0, t_1]; H^1_0(\Omega))$; (see e.g., [3, Theorem 2 in Sect. 5.9]), we obtain

$$\begin{aligned} \int_J a(w_h(s), \partial_s w_h(s)) ds &= \frac{1}{2}\int_J \left(\frac{dg}{ds}\right)(s) ds \\ &= \frac{1}{2}\left(\|w_h(t_1)\|^2_{H^1_0(\Omega)} - \|w_h(t_0)\|^2_{H^1_0(\Omega)}\right). \end{aligned} \tag{13}$$

From (12) and (13),

$$\begin{aligned} &\|f - \partial_t w_h\|^2_{L^2(J;L^2(\Omega))} \\ &= \|f\|^2_{L^2(J;L^2(\Omega))} - 2\int_J (f(s), \partial_s w_h(s))_{L^2(\Omega)} ds + \|\partial_t w_h\|^2_{L^2(J;L^2(\Omega))} \\ &= \|f\|^2_{L^2(J;L^2(\Omega))} - 2\|\partial_t w_h\|^2_{L^2(J;L^2(\Omega))} - 2\int_J a(w_h(s), \partial_s w_h(s)) ds + \|\partial_t w_h\|^2_{L^2(J;L^2(\Omega))} \\ &= \|f\|^2_{L^2(J;L^2(\Omega))} - \|w_h(t_1)\|^2_{H^1_0(\Omega)} + \|w_h(t_0)\|^2_{H^1_0(\Omega)} - \|\partial_t w_h\|^2_{L^2(J;L^2(\Omega))} \\ &\leq \|f\|^2_{L^2(J;L^2(\Omega))} + \|w_h(t_0)\|^2_{H^1_0(\Omega)}. \end{aligned} \tag{14}$$

It follows from (11), (14), and the additive geometric mean that

$$
\begin{aligned}
&\frac{1}{2}\|(w-w_h)(t_1)\|^2_{L^2(\Omega)} + \|w-w_h\|^2_{L^2(J;H^1_0(\Omega))} \\
&\le \frac{1}{2}\|(w-w_h)(t_0)\|^2_{L^2(\Omega)} \\
&\quad + C_h\sqrt{\|f\|^2_{L^2(J;L^2(\Omega))} + \|w_h(t_0)\|^2_{H^1_0(\Omega)}}\,\|w-w_h\|_{L^2(J;H^1_0(\Omega))} \\
&\le \frac{1}{2}\|(w-w_h)(t_0)\|^2_{L^2(\Omega)} + \frac{C_h^2}{2}\left(\|f\|^2_{L^2(J;L^2(\Omega))} + \|w_h(t_0)\|^2_{H^1_0(\Omega)}\right) \\
&\quad + \frac{1}{2}\|w-w_h\|^2_{L^2(J;H^1_0(\Omega))}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\|(w-w_h)(t_1)\|^2_{L^2(\Omega)} + \|w-w_h\|^2_{L^2(J;H^1_0(\Omega))} \\
&\le \|(w-w_h)(t_0)\|^2_{L^2(\Omega)} + C_h^2\left(\|f\|^2_{L^2(J;L^2(\Omega))} + \|w_h(t_0)\|^2_{H^1_0(\Omega)}\right).
\end{aligned}
$$

Because $w(t_0) = w_0$ and $w_h(t_0) = \hat{w}_0$, this proof is complete. □

3 Proof of Theorem 2

We provide notation and lemmas, that are used for proving Theorem 2. Because $w - w_h \in Z \subset C^0([t_0, t_1]; L^2(\Omega))$, for $t \in [t_0, t_1]$, we may define

$$
z(t) := \mathcal{A}^{-1}(w-w_h)(t) \in H^1_0(\Omega), \quad z_h(t) := R_h\mathcal{A}^{-1}(w-w_h)(t) \in V_h. \tag{15}
$$

We show Lemma 1, which is to be used to prove Theorem 2.

Lemma 1 *The function z_h defined by (15) is in $H^1(J; H^1_0(\Omega))$ and we have*

$$
\partial_t z_h = R_h\mathcal{A}^{-1}\partial_t(w-w_h).
$$

Proof We first verify that $R_h\mathcal{A}^{-1}\partial_t(w-w_h) \in L^2(J; H^1_0(\Omega))$. Since $R_h\mathcal{A}^{-1} : H^{-1}(\Omega) \to V_h$ is a bounded operator, we only have to show that $\partial_t(w-w_h) \in L^2(J; H^{-1}(\Omega))$. We have $\partial_t w_h \in L^2(J; V_h) \subset L^2(J; H^1_0(\Omega))$ because of $w_h \in H^1(J; V_h)$. We can consider $\partial_t w_h$ as $\partial_t w_h \in L^2(J; H^{-1}(\Omega))$ and conclude that $\partial_t(w-w_h) \in L^2(J; H^{-1}(\Omega))$. Therefore, we have $R_h\mathcal{A}^{-1}\partial_t(w-w_h) \in L^2(J; V_h) \subset L^2(J; H^1_0(\Omega))$.

Next, we show that $z_h \in H^1(J; H^1_0(\Omega))$ and $\partial_t z_h = R_h\mathcal{A}^{-1}\partial_t(w-w_h)$. Let the function space $C^\infty_0(J)$ be the set of infinitely differentiable functions with compact support on J. For any $\phi \in C^\infty_0(J)$, it follows that

$$
\begin{aligned}
\int_J \partial_s z_h(s)\phi(s)ds &= -\int_J z_h(s)\frac{d\phi}{ds}(s)ds \\
&= -\int_J R_h\mathcal{A}^{-1}(w-w_h)(s)\frac{d\phi}{ds}(s)ds \\
&= -R_h\mathcal{A}^{-1}\int_J (w-w_h)(s)\frac{d\phi}{ds}(s)ds,
\end{aligned}
$$

where the last equation is led by the boundedness of $R_h\mathcal{A}^{-1} : H^{-1}(\Omega) \to V_h$; (see e.g., [18, Corollary 2 on Sect. 5 in Chapter V]). It follows from $\partial_t(w - w_h) \in L^2(J; H^{-1}(\Omega))$ and the boundedness of $R_h\mathcal{A}^{-1} : H^{-1}(\Omega) \to V_h$ that

$$\begin{aligned}\int_J \partial_s z_h(s)\phi(s)ds &= R_h\mathcal{A}^{-1}\int_J \partial_s(w - w_h)(s)\phi(s)ds\\ &= \int_J R_h\mathcal{A}^{-1}\partial_s(w - w_h)(s)\phi(s)ds.\end{aligned}$$

Since $R_h\mathcal{A}^{-1}\partial_t(w - w_h) \in L^2(J; H_0^1(\Omega))$, we have $z_h \in H^1(J; H_0^1(\Omega))$ and $\partial_t z_h = R_h\mathcal{A}^{-1}\partial_t(w - w_h)$. □

Now, we prove Theorem 2.

Proof For $t > t_0$, substituting $v = z_h(t)$ into (4) and $v_h = z_h(t)$ in (5) yields

$$\begin{aligned}&\langle \partial_t(w - w_h)(t), z_h(t)\rangle + a((w - w_h)(t), z_h(t))\\ &= \langle \partial_t w(t), z_h(t)\rangle + a(w(t), z_h(t)) - (\langle \partial_t w_h(t), z_h(t)\rangle + a(w_h(t), z_h(t)))\\ &= \langle f(t), z_h(t)\rangle - \langle f(t), z_h(t)\rangle\\ &= 0. \end{aligned} \tag{16}$$

Because the bilinear form a is symmetric, it follows from (16) that for $t > t_0$,

$$\begin{aligned}\|w(t) - w_h(t)\|^2_{L^2(\Omega)} &= ((w - w_h)(t), (w - w_h)(t))_{L^2(\Omega)}\\ &= a(z(t), (w - w_h)(t))\\ &= a((z - z_h)(t), (w - w_h)(t)) + a(z_h(t), (w - w_h)(t))\\ &= a((z - z_h)(t), (w - w_h)(t)) + a((w - w_h)(t), z_h(t))\\ &= a((z - z_h)(t), (w - w_h)(t)) - \langle \partial_t(w - w_h)(t), z_h(t)\rangle\\ &= a((z - z_h)(t), (w - w_h)(t)) - a(\mathcal{A}^{-1}\partial_t(w - w_h)(t), z_h(t)).\end{aligned}$$

Because $R_h\mathcal{A}^{-1}\partial_t(w - w_h) = \partial_t z_h$ holds from Lemma 1, we obtain

$$\begin{aligned}&\|w(t) - w_h(t)\|^2_{L^2(\Omega)}\\ &= a((z - z_h)(t), (w - w_h)(t)) - a(R_h\mathcal{A}^{-1}\partial_t(w - w_h)(t), z_h(t))\\ &= a((z - z_h)(t), (w - w_h)(t)) - a(\partial_t z_h(t), z_h(t)).\end{aligned} \tag{17}$$

Integrating both sides of (17) for $t \in J$, we obtain

$$\begin{aligned}&\|w - w_h\|^2_{L^2(J;L^2(\Omega))}\\ &= \int_J a((z - z_h)(s), (w - w_h)(s))ds - \int_J a(\partial_s z_h(s), z_h(s))ds\\ &\le \left|\int_J a((z - z_h)(s), (w - w_h)(s))ds\right| - \int_J a(\partial_s z_h(s), z_h(s))ds\\ &\le \int_J |a((z - z_h)(s), (w - w_h)(s))|ds - \int_J a(\partial_s z_h(s), z_h(s))ds\\ &\le \sqrt{\int_J \|(z - z_h)(s)\|^2_{H_0^1(\Omega)}ds}\,\|w - w_h\|_{L^2(J;H_0^1(\Omega))} - \int_J a(\partial_s z_h(s), z_h(s))ds\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\int_J \|(I-R_h)\mathcal{A}^{-1}(w-w_h)(s)\|^2_{H^1_0(\Omega)} ds} \|w-w_h\|_{L^2(J;H^1_0(\Omega))} \\
&\quad - \int_J a(\partial_s z_h(s), z_h(s)) ds \\
&\leq C_h \|w-w_h\|_{L^2(J;L^2(\Omega))} \|w-w_h\|_{L^2(J;H^1_0(\Omega))} - \int_J a(\partial_s z_h(s), z_h(s)) ds,
\end{aligned}
$$

where because $(w-w_h)(t) \in L^2(\Omega)$ $(t \in [t_0, t_1])$, the last inequality follows from (2). It follows from (13), where w_h is replaced by z_h, that

$$
\begin{aligned}
&\|w-w_h\|^2_{L^2(J;L^2(\Omega))} \\
&\leq C_h \|w-w_h\|_{L^2(J;L^2(\Omega))} \|w-w_h\|_{L^2(J;H^1_0(\Omega))} - \frac{1}{2}\left(\|z_h(t_1)\|^2_{H^1_0(\Omega)} - \|z_h(t_0)\|^2_{H^1_0(\Omega)}\right) \\
&\leq C_h \|w-w_h\|_{L^2(J;L^2(\Omega))} \|w-w_h\|_{L^2(J;H^1_0(\Omega))} + \frac{1}{2}\|z_h(t_0)\|^2_{H^1_0(\Omega)} \\
&= \sqrt{C_h^2 \|w-w_h\|^2_{L^2(J;H^1_0(\Omega))} \|w-w_h\|^2_{L^2(J;L^2(\Omega))}} + \frac{1}{2}\|z_h(t_0)\|^2_{H^1_0(\Omega)} \\
&\leq \frac{C_h^2}{2}\|w-w_h\|^2_{L^2(J;H^1_0(\Omega))} + \frac{1}{2}\|w-w_h\|^2_{L^2(J;L^2(\Omega))} + \frac{1}{2}\|z_h(t_0)\|^2_{H^1_0(\Omega)},
\end{aligned}
$$

where the last inequality follows from the additive geometric mean. Therefore, we have

$$
\|w-w_h\|^2_{L^2(J;L^2(\Omega))} \leq \|z_h(t_0)\|^2_{H^1_0(\Omega)} + C_h^2 \|w-w_h\|^2_{L^2(J;H^1_0(\Omega))}.
$$

□

4 Conclusion

We proposed $L^2(J; H^1_0(\Omega))$ and $L^2(J; L^2(\Omega))$ norm error estimates that provide explicit values of the error constants for the semi-discrete Galerkin approximation of the linear heat equation (4) in Theorems 1 and 2, respectively. Furthermore, we derived Corollaries 1 and 2 as special cases of Theorems 1 and 2, respectively. The estimates in Corollaries 1 and 2 are sharper than those given by Nakao et al. [14]. Moreover, we showed that these constants coincide with C_h in (2). From this fact we believe that our error estimates should be, in a sense, the best possible. Therefore, our results contribute to the theoretical and numerical basis for computer-assisted existential proofs of solutions to semi-linear parabolic PDEs.

Acknowledgements We appreciate editors in this journal and reviewers' useful comments for improving quality of this paper.

Funding This work was supported by CREST, JST Grant No. JPMJCR14D4, JSPS KAKENHI No.18K13462, and JSPS KAKENHI No.18K03434.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Brenner, S., Scott, R.: The Mathematical Theory of Finite Element Methods, 2nd edn. Springer, New York (2007)
2. Chrysafinos, K., Hou, L.S.: Error estimates for semidiscrete finite element approximations of linear and semilinear parabolic equation under minimal regularity assumptions. SIAM J. Numer. Anal. **40**, 282–306 (2002). https://doi.org/10.1137/S0036142900377991
3. Evans, L.: Partial Differential Equations, 2nd edn. American Mathematical Society, USA (2010)
4. Grisvard, P.: Elliptic problems in nonsmooth domains. Soc. Ind. Appl. Math. (2011). https://doi.org/10.1137/1.9781611972030
5. Hashimoto, K., Kimura, T., Minamoto, T., Nakao, M.T.: Constructive error analysis of a full-discrete finite element method for the heat equation. Jpn. J. Ind. Appl. Math. (2019). https://doi.org/10.1007/s13160-019-00362-6
6. Kikuchi, F., Xuefeng, L.: Determination of the Babuška-Aziz constant for the linear triangular finite element. Jpn. J. Ind. Appl. Math. **23**, 75–82 (2006). https://doi.org/10.1007/BF03167499
7. Kimura, S., Yamamoto, N.: On explicit bounds in the error for the H_0^1-projection into piecewise polynomial spaces. Bull. Inform. Cybernet. **31**, 109–115 (1999)
8. Kinoshita, T., Hashimoto, K., Nakao, M.T.: On the L^2 a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator. Numer. Funct. Anal. Optim. **30**, 289–305 (2009). https://doi.org/10.1080/01630560802679364
9. Kinoshita, T., Kimura, T., Nakao, M.: On the a posteriori estimates for inverse operators of linear parabolic equations with applications to the numerical enclosure of solutions for nonlinear problems. Numer. Math. **126**, 679–701 (2014). https://doi.org/10.1007/s00211-013-0575-z
10. Kobayashi, K.: A constructive a priori error estimation for finite element discretizations in a non-convex domain using singular functions. Jpn. J. Ind. Appl. Math. **26**, 493–516 (2009). https://doi.org/10.1007/BF03186546
11. Liu, X., Oishi, S.: Verified eigenvalue evaluation for the laplacian over polygonal domains of arbitrary shape. SIAM J. Numer. Anal. **51**, 1634–1654 (2013). https://doi.org/10.1137/120878446
12. Nakao, M.: A numerical approach to the proof of existence of solutions for elliptic problems. Jpn. J. Ind. Appl. Math. **5**(2), 313–332 (1988). https://doi.org/10.1007/BF03167877
13. Nakao, M., Kimura, T., Kinoshita, T.: Constructive a priori error estimates for a full discrete approximation of the heat equation. SIAM J. Numer. Anal. **51**, 1525–1541 (2013). https://doi.org/10.1137/120875661
14. Nakao, M., Kinoshita, T., Kimura, T.: On a posteriori estimates of inverse operators for linear parabolic initial-boundary value problems. Computing **94**, 151–162 (2012). https://doi.org/10.1007/s00607-011-0180-x
15. Nakao, M., Yamamoto, N.: A guaranteed bound of the optimal constant in the error estimates for linear triangular element. Top. Numer. Anal. **15**, 165–173 (2001)
16. Thomee, V.: Galerkin Finite Element Methods for Parabolic Problems, 2nd edn. Springer, Berlin, Heidelberg (2007)
17. Yamamoto, N., Nakao, M.: Numerical verifications of solutions for elliptic equations in nonconvex polygonal domains. Numer. Math. **65**, 503–521 (1993). https://doi.org/10.1007/BF01385765
18. Yosida, K.: Functional Analysis, 6th edn. Springer, Berlin, Heidelberg (1995)