Journal of Mathematical Chemistry (2020) 58:1420–1436
https://doi.org/10.1007/s10910-020-01134-1

ORIGINAL PAPER

Existence and properties of solutions for boundary value problems based on the nonlinear reactor dynamics

Aleksandra Orpel[1]

Received: 9 February 2020 / Accepted: 28 April 2020 / Published online: 23 May 2020

Abstract
We deal with the existence of positive solutions for the following class of nonlinear equation $u''(t) + Au'(t) + g(t, u(t), v(t)) = 0$ a.e. in (0, 1), with boundary conditions $u'(0) = 0$, $u'(1) + Au(1) = 0$, where v is a functional parameter. The form of the problem is associated with the classical model described by Markus and Amundson. We show the existence of at least one positive solution of this problem and discuss its properties. Moreover we describe conditions that guarantee the continuous dependence of solution on parameter v also in the case of the lack of the uniqueness of a solution. The results are based on the clasical fixed point methods. Our approach allows us to consider both sub and superlinear nonlinearities which may be singular with respect to the first variable.

Keywords Positive solutions · Reactor model · Fix point methods · Dependence on functional parametres

1 Introduction

The research on multipoint boundary value problems containing nonlinear ODEs has been enjoying of increasing interests for many years (see [12, 28, 29, 31, 34, 35, 41, 42, 44, 49] and references therein). Many real-life problems modeled by such BVPs arise in various areas of applied mathematics: in chemical or physical phenomena, in the electrohydrodynamics and astrophysics (see, among others, [1, 4, 5, 30]). We would like to join the discussion and investigate the following equation

$$u''(t) + Au'(t) + g(t, u(t), v(t)) = 0, \text{ for a.a. } t \in (0, 1) \tag{1}$$

with boundary conditions

✉ Aleksandra Orpel
aleksandra.orpel@wmii.uni.lodz.pl

[1] Faculty of Mathematics and Computer Science, University of Lodz, Lodz, Poland

$$u'(0) = 0 \text{ and } u'(1) + Au(1) = 0, \tag{2}$$

where $v \in V \subset L^2(0,1)$ and $A > 0$. We consider (1)–(2) in the case when some mild assumptions are satisfied. Precisely, in the paper we assume the below conditions.

(H1) There exist numbers $b > 0$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, such that $g : (0,1) \times [0,b] \times \mathbb{R} \to [0,+\infty)$ is a Caratheodory function and for all $t \in (\alpha,\beta) \subset [0,1]$, $v \in V \subset L^2(0,1)$, $g(t,u,v(t)) > 0$ for $u \in [0,b]$.

(H2) There exists $\varphi \in L^2(0,1)$ such that for all $t \in (\alpha,\beta) \subset [0,1]$, $v \in V$, the following inequalities hold $\max_{u\in[0,b]} g(t,u,v(t)) \le \varphi(t)$ and

$$\int_0^1 \varphi(t)dt \le \frac{A}{A+1}b. \tag{3}$$

Remark 1 Let us note that (**H1**)–(**H2**) are satisfied by many functions $g(t, u, v)$ which are polynomials and exponential, logarithmic or fractional functions with respect to the second variable. Moreover g is not necessarily smooth in u and v in whole plane $\mathbb{R}^2$ and it can be singular with respect to t. We discuss an example of such nonlinearities at the end of the paper.

2 Motivations

The form of this equation is associated with the classical model arising in the theory of chemical reactors described by Markus and Amundson in [1] which is our main motivation. Precisely, their model concerns a single exothermic homogeneous reaction involving a few chemical species which occurs in a tubular reactor of length 1. Let u denote the dimensionless temperature in this reactor. Then u is modeled as follows

$$\begin{aligned} u''(t) + Au'(t) + Bf(u(t)) &= 0, \text{ for all } t \in (0,1) \\ u'(0) &= 0 \\ u'(1) + Au(1) &= 0, \end{aligned}$$

where A and B are given constants and f describes the rates of chemical production of the species. Finally, the solution u and the stoichiometric coefficients of the species involved in the reaction allow us to describe their concentration (see also e.g. [7, 8]). The classical paper [8] presents results concerning the existence and uniqueness of positive solutions also in the case when f depends on t. However, the authors consider the nonlinearity f which has to be sufficiently smooth and satisfy some conditions concerning monotonicity with respect to u. Moreover, the monotonicity of the function $u \mapsto f(t,u)/u$ in a certain interval is also assumed there.

Let us note that according to assumptions (**H1**) and (**H2**) the nonlinearity f can be a polynomial thus our approach works also for Lane-Emden type equations which have attracted the interests of many authors lately. It is associated with the fact that such equations model a reaction-diffusion process (see e.g. [13, 23, 37] and references therein). It appears that analytical solitons of such problems are useful in the optimization of this process (see e.g. [48]). Therefore, it is very important to find exact or at least approximate solutions when nonlinearities are given in an explicit form. To this effect the approach based on Taylor series is often applied. In a very recent paper [23], such method allows the authors to find the exact solution for the following Lane-Emden system of nonlinear equations with singularity

$$\begin{cases} u''(x) + \frac{1}{x}u'(x) - v^3(x)(u^2(x) + 1) = 0, \; x \in (0, 1) \\ v''(x) + \frac{3}{x}v'(x) + v^5(x)(u^2(x) + 3) = 0, \; x \in (0, 1) \\ \quad u(0) = 1, u'(0) = 0 \text{ and } v(0) = 1, v'(0) = 0. \end{cases}$$

Similar approach is applied also in [27] devoted to the Bratu's equation with Dirichlet boundary conditions, namely

$$u''(x) + \lambda e^{-u(x)} = 0, \; x \in (0, 1)$$

which models the instability of the moving jet in the electrospining (see e.g. [9, 26, 33]). Let us note that this is the special case of our equation for $g(x, u, v) = e^{-u}$, $A = 0$ and $B = \lambda$. In the recent paper [20], this simple and effective method based on Taylor series is also applied for (1) with the coefficient A replaced by x^2 and $g \equiv 0$. In this case we obtain the following convection-diffusion equation

$$u''(x) + x^2u'(x) = 0 \text{ in } (0, l). \tag{4}$$

Precisely, (4), together with boundary conditions $u(0) = 0$ and $u'(l) = 0$, arise when we consider a rotating disc electrode which is rotated to make the diffusion layer as small as possible (see e.g. [3, 10, 20, 40]).

It is worth emphasizing that the special case of our equation with $A = 0$ gives also the cubic-quintic Duffing oscillator equation

$$u''(x) + u(x) + u^3(x) + u^5(x) = 0$$

with certain initial conditions, widely discussed by Chowdhury et al. [6, 21]. In the latter paper the estimation of the frequency of a nonlinear conservative oscillator is given. In [6], the following generalization of the above equation

$$u''(x) + \omega^2u(x) = \varepsilon f(u(x)),$$

is discussed in the case when the nonlinearity f is odd (i.e. $f(-x) = -f(x)$), with ε being a certain constant and ω denoting the angular frequency. The authors obtain higher-order approximate solutions applying the approach based on Harmonic Balance Method (HBM). Similar initial problem for the general non-linear oscillator equation is investigated in [2], where the Laplace transform together with the variational iteration method allow the authors to obtain the first order approximate solution.

As far as the mathematical point of view is concerned, existence results for two-point BVPs are widely discussed, for example, in [29], where the following equation is investigated

$$u'' + sign(1-\alpha)q(t)f(u,u')u' = 0$$

with the boundary conditions $u(0)=0$ and $u'(1)=\alpha u'(0)$ or $u(1)=0$ and $u'(1)=\alpha u'(0)$. Papers [31] and [41] are devoted to the three-point problem with the following conditions

$$u'(0) = 0 \text{ and } u'(1) = \delta u(\eta), \tag{5}$$

where $\delta > 0$ and $\eta \in (0,1)$, in the case when the nonlinearity is independent of the derivative u' of solution u. These papers are based mostly on the monotone iterative methods. The problem with the general nonlinearity g and the boundary conditions (5) is investigated, among others, in [42]. The authors combine the upper and lower solution methods with the monotone iterative technique. They consider both well ordered and reverse ordered upper and lower solutions. Their approach works in the case when there exist lower and upper solutions α and $\beta \in C^2[0,1]$ of the problem

$$u'' + f(t,u,u') = 0,$$

where $f : D \to \mathbb{R}$ is continuous on $D := \{(t,u,v) \in [0,1]\times\mathbb{R}^2 : \beta(t) \le u \le \alpha(t)\}$. Moreover there exist $M \ge 0$ and $N \ge 0$ such that for all $(t,u_1,v),(t,u_2,v) \in D$,

$$u_1 \le u_2 \Rightarrow f(t,u_2,v) - f(t,u_1,v) \le M(u_2 - u_1) \tag{6}$$

and

$$|f(t,u,v_2) - f(t,u,v_1)| \le N|v_2 - v_1|.$$

The authors also assume the estimate for the nonlinearity f of the form $|f(t,u,v)| \le \phi(|v|)$ for all $(t,u,v) \in D$, where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is continuous and satisfies $\max_{t\in[0,1]} \alpha(t) - \min_{t\in[0,1]} \beta(t) \le \int_{l_0}^{\infty} \frac{s}{\phi(s)} ds$.

In [35], R. Ma presents the extension of the Erbe and Wang's results ([12]) for two-point and his own results for three-point BVPs ([34]). Precisely, that paper is devoted to the existence of positive solutions for the m-point boundary value problem and the results are based on the Krasnosielskii fixed point theorem in cones. In [36] the Leray-Schauder continuation theorem is applied to obtain the existence of solutions for m-point BVPs for the second order differential equation when nonlinearity f is either superlinear or sublinear.

Although the nonlinearities, discussed in the papers mentioned above, have more general form we cannot apply these results to our problem because we do not assume conditions like (6) and consider nonlinearity which can be singular with respect to the first variable. We focus on the BVP (1)–(2) with nonlinearities satisfying quite general conditions. Here we want to emphasize that our approach allows us to consider nonlinearities which are not necessarily smooth in whole domain and do not satisfy any conditions concerning the monotonicity. We also omit growth conditions on $g(x,\cdot,\cdot)$ either at zero or at plus infinity. Therefore our approach covers both sublinear and superlinear cases. Moreover, g can be also singular with respect to the first variable. The goal of this paper is to prove the existence of bounded nonincreasing and positive solutions with the bounded derivative. We also try to describe how the solution depends on the functional parameter v. Precisely, we show that the sequence of positive solutions $\{u_n\}_{n\in\mathbb{N}}$, corresponding to the sequence of parameters $\{v_n\}_{n\in\mathbb{N}} \subset V$, tends uniformly (up to a subsequence) in [0, 1] to u_0, provided that the sequence of parameters is convergent almost everywhere in $(0, T)$ to $v_0 \in V$. Moreover, we prove that u_0 is a solutions of our problem (1)–(2) with $v = v_0$. Usually such results are based on the uniqueness of a solution (see [11, 45] and references therein). We consider the problem in the case of the lack of such condition which is more challenging than in the previous case when the solution is unique (see also [38] and [39]).

3 Existence results

In the first step we consider our problem with parameter $v \in V$ fixed. Then (1) takes the following form

$$\begin{aligned} u''(t) + Au'(t) + f(t, u(t)) =&0, \text{ for a.a. } t \in (0, 1)\\ u'(0) =&0 \\ u'(1) + Au(1) =&0 \end{aligned} \tag{7}$$

with $f(t, u) := g(t, u, v(t))$, $t \in (0, 1), u \in [0, b]$.

Remark 2 Let us note that for g satisfying (**H1**) and (**H2**), we can state that f satisfies the below conditions.

(**H1'**) There exist numbers $b > 0$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, such that $f : (0, 1) \times [0, b] \to [0, +\infty)$ is a Caratheodory function and for all $t \in (\alpha, \beta) \subset [0, 1]$, $f(t, u) > 0$ for $u \in [0, b]$.

(**H2'**) There exists $\varphi \in L^2(0, 1)$ such that for all $t \in (\alpha, \beta) \subset [0, 1]$, the following conditions hold $\max_{u\in[0,b]} f(t, u) \le \varphi(t)$ and

$$\int_0^1 \varphi(t)dt \le \frac{A}{A+1}b.$$

Theorem 3 *If* (**H1'**) *and* (**H2'**) *hold then there exists a solution* $\overline{u} \in C^1([0, 1]) \cap W^{2,2}(0, 1)$ *of* (7) *such that for all* $t \in [0, 1]$, $0 \le \overline{u}(t) \le b$ *and* $-b \le \overline{u}'(t) \le 0$. *Moreover for all* $\alpha \le t \le 1$, $\overline{u}'(t) < 0$, *which means that* $\overline{u}$ *is decreasing in* $(\alpha, 1)$.

Proof Let us consider the set

$$X := \{u \in C^1([0, 1]), 0 \le u(t) \le b, -b \le u' \le 0, u'(0) = 0, u'(1) + Au(1) = 0\}.$$

We can treat our problem as a fixed point problem for the following integral operator

$$Tu(t) := -\int_0^t e^{-Ar}\int_0^r e^{As}\overline{f}(s,u(s))dsdr + \frac{e^{-A}}{A}\int_0^1 e^{As}\overline{f}(s,u(s))ds$$
$$+\int_0^1 e^{-Ar}\int_0^r e^{As}\overline{f}(s,u(s))dsdr,$$

where

$$\overline{f}(s,z) = \begin{cases} f(s,0) & \text{for} \quad z<0,\ s\in(0,1) \\ f(s,z) & \text{for} \quad 0\le z\le b,\ s\in(0,1) \\ f(s,b) & \text{for} \quad z>b,\ s\in(0,1). \end{cases}$$

Taking into account (**H1'**) and (**H2'**) we infer that operator T is well defined and maps $C([0,1])$ into itself. We show that $TX \subset X$ and T is completely continuous. Let us take an arbitrary $u\in X$. Then for $\overline{u} := Tu$ we have $\overline{u}'(t) = -e^{-At}\int_0^t e^{As}\overline{f}(s,u(s))ds$ and further we see immediately that $\overline{u}'(0)=0$, $\overline{u}'(1)+A\overline{u}(1)=0$, $\overline{u}' \le 0$ in $[0,1]$ and $\overline{u}' < 0$ in $(\alpha,1]$. Moreover for all $t\in(0,1)$, we have

$$\overline{u}'(t) = -e^{-At}\int_0^t e^{As}\overline{f}(s,u(s))ds \ge -\int_0^1 \max_{u\in[0,b]}\overline{f}(s,u)ds \ge -b.$$

We can also obtain the following estimates

$$\overline{u}(1) = \frac{e^{-A}}{A}\int_0^1 e^{As}\overline{f}(s,u(s))ds > 0$$

and

$$\overline{u}(0) \le \frac{A+1}{A}\int_0^1 \max_{u\in[0,b]}\overline{f}(s,u)ds \le b.$$

Both assertions and the fact that $\overline{u}$ is nonincreasing in $[0,1]$ lead to the chain of inequalities for all $t\in[0,1]$,

$$0 < \overline{u}(1) \le \overline{u}(t) \le \overline{u}(0) \le b.$$

Finally, we have derived that $TX \subset X$. Our task is now to show that operator T is completely continuous in $C([0,1])$. We prove this fact using the standard reasoning (see e.g. [43]) and start with the proof of the continuity of T. To this effect we take arbitrary $u_0 \in C([0,1])$ and a sequence $(u_n)_{n\in\mathbb{N}} \subset C([0,1])$, such that $u_n \to u_0$ in $C([0,1])$ with the sup-norm $||u||_C = \sup_{t\in[0,1]}|u(t)|$. Then we obtain

$$\begin{aligned}
&||Tu_0 - Tu_n||_C \\
&\quad \le \int_0^1 |\bar{f}(s,u_n(s)) - \bar{f}(s,u_0(s))|ds + \frac{1}{A}\int_0^1 |\bar{f}(s,u_0(s)) - \bar{f}(s,u_n(s))|ds \\
&\qquad + \int_0^1 |\bar{f}(s,u_0(s)) - \bar{f}(s,u_n(s))|ds \\
&\quad = \left(2 + \frac{1}{A}\right)\int_0^1 |\bar{f}(s,u_n(s)) - \bar{f}(s,u_0(s))|ds.
\end{aligned}$$

Bearing in mind (**H1’**), we note that for all $s \in [0,1]$,

$$\psi_n(s) := |\bar{f}(s,z_0(s)) - \bar{f}(s,z_n(s))| \to 0 \text{ when } n \to \infty,$$

and, according to the definition of $\bar{f}$ and (**H2’**), we get

$$\psi_n(s) \le 2\max_{u\in[0,b]} f(s,u).$$

Moreover, we have

$$\int_0^1 \psi_n(s)ds \le 2\int_0^1 \varphi(s)ds < +\infty.$$

Therefore, the Lebesgue dominated convergence theorem leads to the conclusion that

$$\int_0^1 \psi_n(s)ds \to 0 \text{ for } n \to \infty$$

and consequently

$$||Tu_0 - Tu_n||_C \to 0 \text{ for } n \to \infty.$$

Finally, we can state that T is continuous at arbitrary $z_0 \in C([0,1])$, namely T is continuous as an operator from $C([0, 1])$ into itself. Now we prove that T is compact applying the Ascoli-Arzelá Lemma. To this end we prove that T maps bounded subsets of $C([0, 1])$ into relatively compact subsets of $C([0, 1])$. Therefore, we take any $R > 0$ and consider the closed ball $B := \{u \in C([0,1]),\ ||u||_C \le R\}$. Our task is now to prove that the image of B, i.e. $T(B) := \{Tu \in C([0,1]),\ ||u||_C \le R\}$ is relatively compact in $C([0, 1])$. We start with the proof of equicontinuity of functions from $T(B)$. To this effect we take any t_0 and $t_n \to t_0^+$. Now, by the definition of $\bar{f}$, we have for all $u_0 \in B$,

$$
\begin{aligned}
&|Tu_0(t_n) - Tu_0(t_0)| \\
&\quad \le \int_{t_0}^{t_n} \int_0^r \bar{f}(s, u_0(s)) ds dr \le \int_{t_0}^{t_n} \int_0^1 \bar{f}(s, u_0(s)) ds dr \\
&\quad \le \int_{t_0}^{t_n} \int_0^1 \max_{u \in [0,b]} f(s,u) ds dr \le \frac{A}{A+1} b\big(t_n - t_0\big),
\end{aligned}
$$

which implies

$$\lim_{n\to\infty} |Tu_0(t_n) - Tu_0(t_0)| = 0,$$

uniformly with respect to $u_0 \in B$. We obtain the same conclusion for $t_n \to t_0^-$ similarly. Finally, we can derive that the family of functions from $T(B)$ is equicontinuous. To check the equiboundedness of this family it suffices to note the following estimate

$$\sup_{t\in[0,1]} |Tu(t)| \le \frac{2(A+1)}{A} \int_0^1 \max_{u\in[0,b]} f(s,u) ds < +\infty,$$

for all $u \in T(B)$. Owing to the Ascoli-Arzela Lemma, we conclude that $T(B)$ is relatively compact in $C([0, 1])$.

To sum up, we have proved that T is a completely continuous operator which maps the nonempty, closed and convex subset X of $C([0, 1])$ into X. With the Schauder's fixed point theorem in mind, we derive that there exists at least one fixed point $u_0 \in X$ of operator T. By the definition of T and condition (**H1'**) and (**H2'**) we state that $u_0 \in C^1([0, 1]) \cap W^{2,2}(0, 1)$ and u_0 is positive. Finally, u_0 is a positive solution (7). □

4 Dependence on functional parameters

Now we consider problem (1)–(2) and formulate our main result.

Theorem 4 *Assume hypotheses* (**H1**) *and* (**H2**). *Let* $\{v_n\}_{n\in\mathbb{N}} \subset V$ *be a sequence of parameters convergent pointwisely to a certain* $v_0 \in V$. *Additionally, let us suppose that for each* $n \in \mathbb{N}$, $u_n \in X$ *and denotes a solution of* (1)–(2) *with* $v = v_n$. *Then the sequence* $\{u_n\}_{n\in\mathbb{N}} \subset X$ *possesses a subsequence which converges uniformly to a certain element* $u_0 \subset X$ *which is a solution of* (1)–(2) *corresponding to parameter* $v = v_0$.

Proof We start with the observation that, by Theorem 3, for each parameter v_n there exists at least one positive solution $u_n \in X$ of (1)–(2). Since $u_n \in X$, we have that $\{u_n\}_{n\in\mathbb{N}}$ and $\{u_n'\}_{n\in\mathbb{N}}$ are bounded in [0, 1] and, consequently, in $W^{1,2}(0, 1)$. Thus, going if necessary to a subsequence, we state that $\{u_n\}_{n\in\mathbb{N}}$ tends weakly to a certain

$u_0 \in W^{1,2}(0,1)$. Applying the Rellich-Kondrashov theorem we get the uniform convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u_0 in $[0, 1]$. Now we consider an auxiliary sequence

$$p_n(t) = u_n'(t)e^{At},\ t \in [0,1]$$

which is also bounded, since for all $t \in [0,1]$ the following estimate holds

$$|p_n(t)| \le |u_n'(t)|e^{At} \le be^A.$$

Moreover, by the construction of u_n in the proof of Theorem 3 , we get for all $t \in (0,1)$,

$$\begin{aligned} |p_n'(t)| =& \left|\left(u_n'(t)e^{At}\right)'\right| = \left|\left(\int_0^t e^{As}g(s,u_n(s),v_n(s))ds\right)'\right| \\ =& |e^{At}g(t,u_n(t),v_n(t))| \le e^A\varphi(t). \end{aligned}$$

Taking into account the fact that $\varphi \in L^2(0,1)$, we obtain the boundedness of $\{p_n'\}_{n\in\mathbb{N}}$ in $L^2(0,1)$ and finally, we get the boundedness in the norm of $W^{1,2}(0,1)$. This fact yields that $\{p_n\}_{n\in\mathbb{N}}$, up to a subsequence, is weakly convergent to a certain $p_0 \in W^{1,2}(0,1)$. We use again the Rellich-Kondrashov theorem which leads to the uniform convergence (up to a subsequence) of $\{p_n\}_{n\in\mathbb{N}}$ to p_0 in $[0, 1]$. Since for all $t \in [0,1]$, $u_n'(t) = e^{-At}p_n(t)$, we derive that $\{u_n'\}_{n\in\mathbb{N}}$ tends to u_0' pointwisely in $[0, 1]$, $u_0'(t) = e^{-At}p_0(t)$ and

$$\begin{aligned} u_0'(0) =& p_0(0) = \lim_{n\to\infty} p_n(0) = 0, \\ u_0'(1) + Au_0(1) =& \lim_{n\to\infty} e^{-A}p_n(1) + A\lim_{n\to\infty} u_n(1) \\ =& \lim_{n\to\infty}\left(u_n'(1) + Au_n(1)\right) = 0. \end{aligned}$$

Therefore $u_0 \in X$. Moreover we have for all $h \in W_0^{1,2}(0,1)$,

$$\begin{aligned} &\int_0^1 p_0(t)h'(t)dt \\ &= \lim_{n\to\infty}\int_0^1 p_n(t)h'(t)dt = \lim_{n\to\infty}\int_0^1 (-p_n'(t))h(t)dt \\ &= \lim_{n\to\infty}\int_0^1 e^{At}g(t,u_n(y),v_n(t))h(t)dt \\ &= \lim_{n\to\infty}\int_0^1 e^{At}g(t,u_0(y),v_0(t))h(t)dt, \end{aligned}$$

where the last equality follows from the continuity of $g(t,\cdot,\cdot)$, the estimates given in (**H2**) and the Lebesgue dominated theorem. By the du Bois-Reymond lemma we infer that

$$-\left(p_0(t)\right)' = e^{At}g(t,u_0(y),v_0(t))$$

what can be rewritten as follows

$$-\left(u_0'(t)e^{At}\right)' = e^{At}g(t, u_0(y), v_0(t)).$$

Since p_0 is absolutely continuous and $u_0'(t) = e^{-At}p_0(t)$, we derive that u_0' possesses the derivative almost everywhere in (0, 1) which belongs to $L^2(0,1)$. Thus we get a.e. in (0, 1)

$$u_0''(t) + Au_0'(t) + g(t, u_0(y), v_0(t)) = 0.$$

This means that $u_0 \in X$ and u_0 is a solution of (1)–(2) with the limit parameter $v = v_0$. □

Now we apply our results to the explicit problem with g such that $g(t, \cdot, v)$ has an exponential growth. Moreover $g(\cdot, u, v)$ is singular at 0 and $g(x, \cdot, v)$ does not satisfy any conditions associated with the monotonicity in [0, 2] (see e.g. assumptions in [8]).

Example 1 Let us consider problem (1)–(2) with g given as follows

$$\begin{aligned} & g(y, u, v) \\ & = \left[\overline{\alpha}(t)e^{u-1}e^v + \overline{\beta}(t)\frac{u^2}{(u+4)(5-u)}v + \frac{\overline{\gamma}(t)}{\sqrt[5]{t}}(u-1)^2(2-u)v^2\right] \end{aligned} \tag{8}$$

where $\overline{\alpha}, \overline{\beta}, \overline{\gamma} \in L^\infty(0,1)$ are nonnegative, $\overline{\alpha}(t) \neq 0$ a.e. in (0, 1) and

$$2\left(ess\sup\overline{\alpha}\right)e^2 + \left(ess\sup\overline{\beta}\right) + 5\left(ess\sup\overline{\gamma}\right) \le \frac{4A}{A+1}.$$

Consider the set $V := \{w \in L^2(0,1); ess\sup|w| \le 1\}$ and the sequence of parameters given as follows $v_n(t) = t^n$, $t \in [0,1]$. It is clear that v_n tends pointwisely to $v_0(t) = 0$ for $t \in [0,1)$ and $v_0(t) = 1$ for $t = 1$. Then, for $b = 2$, we have the following estimate for a.a. $t \in (0,1)$, $u \in [0,b]$ and $v \in V$,

$$\begin{aligned} & g(t, u, v(t)) \\ & \quad = \left[\overline{\alpha}(t)e^{u-1}e^{v(t)} + \overline{\beta}(t)\frac{u^2}{(u+4)(5-u)}v(t) + \frac{\overline{\gamma}(t)}{\sqrt[5]{t}}(u-1)^2(2-u)v^2(t)\right] \\ & \quad \le \left[\overline{\alpha}(t)e^2 + \frac{1}{2}\overline{\beta}(t) + \frac{2}{\sqrt[5]{t}}\overline{\gamma}(t)\right] \end{aligned}$$

and

$$\int_0^1 \max_{u\in[0,b]} g(t,u,v(t))dt$$
$$\leq \left(ess\sup\overline{\alpha}\right)e^2 + \frac{1}{2}\left(ess\sup\overline{\beta}\right) + 2\left(ess\sup\overline{\gamma}\right)\int_0^1 \frac{dt}{\sqrt[5]{t}}$$
$$\leq 2\frac{A}{A+1},$$

where the last inequality follows from the estimates made on $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$. Moreover, $g(t,u,0) > 0$ for all $t \in (0,1)$, $u \in [0,2]$ and $\max_{u\in[0,b]} g(t,u,v(t))$ belongs to $L^2(0,1)$. Finally, we can state that g satisfies (**H1**) and (**H2**). Applying Theorem 3 we derive that for each parameter v_n, where $n \in \mathbb{N}$, there exists at least one positive, nonincreasing solution u_n such that $u_n(t) \in (0,2]$ and $u_n'(t) \in [-2,0)$ for all $t \in (0,1)$. Moreover, by Theorem 4, we obtain (up to a subsequence) the uniform convergence of $\{u_n\}_{n\in\mathbb{N}}$ to a certain u_0 in [0, 1], where u_0 is a solution of (1)–(2) with g given by (8) and $v = v_0$.

5 Additional remarks

5.1 Variational principle

Let us note that for each parameter v fixed, (1) can be treated as the Euler-Lagrange equation for the following functional

$$J(u) = \int_0^1 e^{At}\left(\frac{1}{2}\left|u'(t)\right|^2 - G(y,u(t),v(t))\right)dt + \frac{A}{2}e^A u^2(1) \tag{9}$$

where $G(t,u,v(t)) = \int_0^u g(t,z,v(t))dz$. Taking into account the boundary conditions (2), we have to consider J in the space consisted of all $C^1([0,1])$ satisfying the conditions $u'(0) = 0$ and $u(1) + Au'(1) = 0$. The necessary condition of optimality, namely $\delta J = 0$, where δJ denotes the first variation of J, gives the Euler-Lagrange equation

$$\frac{d}{dt}(e^{At}u'(t)) = e^{At}g(y,u(t),v(t))dt$$

and the transversality conditions

$$\begin{cases} u'(0) = 0 \\ e^A u'(1) = -Ae^A u(1). \end{cases}$$

Finally, we obtain our problem (1)–(2).

5.2 Variational iteration methods (VIM)

Variational iteration method has been attracting attention of many researchers for years. It was developed by He in his papers (see, among others, [16–22, 24, 25, 26]). It is worth emphasizing that this approach is successfully applied to many linear and nonlinear problems without any restrictive assumptions. The VIM gives the solution as a limit of sequence which is obtained during rapidly convergent successive approximation procedure that can lead to the exact solution if such a solution exists. If exact solution cannot be obtained (e.g. for the Airy equation), the procedure allows us to find a few number of approximations, see [47] and references therein. In Sect. 3 we showed the existence of positive and nonincreasing solution for our problem. Now, we consider VIM as a tool which can help us to obtain the approximate solution. We start with a correction functional for equation (1) with a given parameter v, in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\big[u_n''(s) + g(s, \widetilde{u}_n(s), v(s))\big]\,ds,$$

where λ is a general Lagrange multiplier, which can be calculated optimally with help of variation theory, and $\widetilde{u}_n$ is identified as a restricted variation, namely $\delta g(s, \widetilde{u}_n(s), v(s)) = 0$.

In our case the multiplier can be described in the general form $\lambda = \lambda(s-t)$ (see [2] and also [46] or [47]). This method can be divided into two main steps. The first one is to find the multiplier applying the restricted variation and the integration by parts. In the other step, having u_n, we consider the iteration schema, without restricted variation, which allows us to calculate successive element $u_{n+1.}$ As a starting approximation u_0 we can take any selective function. Finally, the possible solution is obtained as the following limit $\lim_{n\to\infty} u_n(t) = u(t)$. Thus, the iteration schema is given as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s-t)\big[u_n''(s) + g(s, u_n(s), v(s))\big]\,ds.$$

The number of iterations, which is necessary to obtain the reasonable level of accuracy, depends on the form of nonlinearity g. As an example we can consider the problem discussed in [47], where the author investigates, among others, the problem (1)–(2) without derivative term $Au\prime$ for nonlinearity $g(t,u,v) = -\frac{nu}{u+k}$, where $n, k > 0$ and n and k are positive constants associated with the reaction rate and the Michaelis constant, namely $n = 0.76129$ and $k = 0.03119$ (see e.g. [32]). There the multiplier is $(s-t)$ and the correction functional takes the form

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left[u_n''(s) - \frac{0.76129u_n(s)}{u_n(s) + 0.03119}\right]ds.$$

In that paper, the author choses the constant function as a starting point u_0 and presents the the second approximation of the form

$$u_2(t) = 0,4952605157 + 0,3580933694 \cdot x^2 + 0,002556615536 \cdot x^4 + 0,0006956073111 \cdot x^6 + O(x^7).$$

Here, we emphasize that in [47] the general problem

$$\begin{cases} u''(t) + \frac{\alpha}{x} u'(t) = f(t, u(t)), \ t \in (0, 1) \\ u'(0) = 0, \ \beta u(1) + \gamma u'(1) = \delta \end{cases}$$

is considered for constants α, β, γ, δ, where $\alpha \geq 0$, $\gamma > 0$ and the nonlinearity f is continuos and continuously differentiable with respect to the second variable. Let us note that, in general, our assumptions do not guarantee such conditions.

Applying the same multiplier and the starting approximation $u_0 = const$ as in [47], we can describe the iteration schema for the nonlinearity presented in the Example 1 with $v \equiv 1$, $A = 1$, $\overline{\alpha} \equiv \frac{e^{-2}}{16}$, $\overline{\beta} \equiv \frac{1}{8}$, $\overline{\gamma} \equiv \frac{1}{5}$. Then we have

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\Big[u_n''(s) + \frac{e^{-1}}{16} e^{u_n(s)-1} + \frac{u_n^2(s)}{8(u_n(s)+4)(5-u_n(s))} + \frac{(u_n(s)-1)^2(2-u_n(s))}{5\sqrt[5]{s}}\Big] ds.$$

Let us recall that our approach guarantees the existence of solution with values in $[0, b]$ (see Example 1, where $b = 2$). Thus we consider the starting point also in this interval, so we get

$$\begin{aligned} u_0(t) &= a \in (0, 2), \\ u_1(t) &= a + \int_0^t (s-t)\left[\frac{e^{a-2}}{16} + \frac{a^2}{8(a+4)(5-a)} + \frac{(a-1)^2(2-a)}{5\sqrt[5]{s}}\right] ds \\ &= a + \left(\frac{e^{a-2}}{16} + \frac{a^2}{8(a+4)(5-a)}\right)\left(-\frac{1}{2}t^2\right) - (a-1)^2(2-a)\frac{5}{36}t^{\frac{9}{5}} \end{aligned}$$

It is clear that $u_1'(0) = 0$. Now, taking into account the boundary conditions, we get $u_1'(1) + u_1(1) = 0$ which gives

$$\frac{540e^{a-2} - 16\,736a + 27ae^{a-2} - 27a^2e^{a-2} + 7942a^2 - 944a^3 - 560a^4 + 112a^5 + 4480}{288a^2 - 288a - 5760} = 0$$

and further $a = 0.320\,26$. Finally, for all $t \in [0, 1]$,

$$u_1(t) = 0.320\,26 - 0.107\,79t^{\frac{9}{5}} - 6.\,142\,8 \times 10^{-3}t^2$$

is a positive and decreasing function with values less than 2. It means that u_1 has the same properties as the solution that existence is guaranteed in Sect. 3.

5.3 Other methods

Let us note that our nonlinearity is not, in general, differentiable. Therefore, we cannot apply the useful and powerful approach based on the Taylor series like in e.g. [20, 23, 27], since we do not know if we can present solution in this form. We cannot apply the method like in ([6]) either. It is associated with the fact that our nonlinearity is not, in general odd.

Another approach often applied in such problems is the homotopy perturbation method which is described in the classical paper [15], where the nonlinerity is an analytic function with respect to the first variable. In our case we are able to apply such technique if we assume that g is sufficiently smooth in t. Then, we could consider the approach presented in e.g. [15] or [24], with the homotopy given, for example, as follows

$$H(v,p) = (1-p)\left[v''(t) + g_1(v(t)) - u_0''(t) - g_1(u_0(t))\right] + p\left[g(v(t)) - g(u_0(t))\right],$$

for $p \in [0,1]$, $t \in (0,1)$, where g_1 is a linear part of g (if exists), to obtain an approximate solution.

6 Conclusion

In this paper, the boundary value nonlinear problem is discussed in the case, when the singularity is associated with the nonlinearity g. We propose the methods based on two classical tools. The first one is the Schauder fixed point theorem which allows us to obtain the existence of at least one positive nonincreasing solution. The other one is the du Bois-Reymods lemma, which together with the Rellich-Kondrashov theorem, gives the continuous dependence of solutions on functional parameters. This result can be treated as a starting step in the methods which give the approximate solutions, because it guarantees the existence of such solution.

The natural question arises wether it is possible to consider the fractal modification of our equation as its done in [27] for Bratu's equation. It is important, especially in the light of the paper [22] where the authors emphasize that physical laws depends on the scale. Generally, the approaches to the phenomenon associated, for example, with thermodynamics are based on conventional continuum mechanics. When the continuum assumptions are neglected, we obtain quite different theory in which fractal calculus can give new information ([22] and references therein). The approach enriched with the fractal derivatives is also necessary in the model of E reaction when electron transfer is to be improved. In this case the porous electrodes are applied and (4) cannot be applied (see [20] and references therein). For the time being, the question is open in the general case of nonlinearity.

Acknowledgements The author is grateful to anonymous referees for their careful reading of the first version of this manuscript and their constructive comments. Their suggestions substantially influenced the final version of the manuscript.

References

1. N.R. Amundson, L. Markus, Nonlinear boundary value problems arising in chemical reactor theory. J. Differ. Equ. **4**, 102-l 13 (1968)
2. N. Anjum, J.H. He, Laplace transform: making the variational iteration method easier. Appl. Math. Lett. **92**, 134–8 (2019)
3. P. Beran, F. Opekar, Rotating disk electrodes. J. Electroanal. Chem. Interfacial Electrochem. **69**, 1–105 (1976)
4. P.L. Chamber, On the solution of the Poisson–Boltzmann equation with the application to the theory of thermal explosions. J. Chem. Phys. **20**, 1795–1797 (1952)
5. S. Chandrasekhar, *Introduction to the Study of Stellar Structure* (Dover, New York, 1967)
6. M.S.H. Chowdhury, M.A. Hosen, K. Ahmad, M.Y. Ali, A.F. Ismail, High-order approximate solutions of strongly nonlinear cubic-quintic Duffing oscillator based on the harmonic balance method. Results Phys. **7**, 3962–3967 (2017)
7. D.S. Cohen, Positive solutions of nonlinear eigenvalue problems: applications to nonlinear reactor dynamics. Arch. Ration. Mech. Anal. **26**, 305–315 (1967)
8. D.S. Cohen, T.W. Laetsch, Nonlinear boundary value problems suggested by chemical reactor theory. J. Differ. Equ. **7**, 217–226 (1970)
9. A. Colantoni, K. Boubaker, Electro-spun organic nanofibers elaboration process investigations using comparative analytical solutions. Carbohydr. Polym. **101**, 307–312 (2014)
10. J. Dolinska, M. Holdynski, M. Opallo, Electrochemical behaviour of suspended redox-tagged carbon nanotubes at a rotating disc electrode. Electrochem. Commun. **99**, 32–35 (2019)
11. P. Eloe, J. Henderson, Uniqueness implies existence and uniqueness conditions for a class of (k + j)-point boundary value problems for nth order differential equations. Math. Nachr. **284**, 229–239 (2011)
12. L.H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations. Proc. AMS **120**, 743–748 (1994)
13. T.C. Hao, F.Z. Cong, Y.F. Shang, An efficient method for solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions and error estimate. J. Math. Chem. **56**, 2691–2706 (2018)
14. J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput. Meth. Appl. Mech. Eng. **167**, 57–68 (1998)
15. J.H. He, Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng. **178**, 257–262 (1999)
16. J.H. He, Variational iteration method for autonomous ordinary differential systems. Appl. Math. Comput. **114**(2/3), 115–123 (2000)
17. J.H. He, Some asymptotic methods for strongly nonlinear equations. Int. J. Modern Phys. B **20**(10), 1141–1199 (2006)
18. J.H. He, Variational iteration method - some recent results and new interpretations. J. Comput. Appl. Math. **207**(1), 3–17 (2007)
19. J.H. He, An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering. Int. J. Modern Phys. B **22**(21), 3487–3578 (2008)
20. J.H. He, A simple approach to one-dimensional convection–diffusion equation and its fractional modification for E reaction arising in rotating disk electrodes. J. Electroanal. Chem. (2019). https://doi.org/10.1016/j.jelechem.2019.113565

21. J.H. He, The simplest approach to nonlinear oscillators. Results Phys. **15**, 102546 (2019). https://doi.org/10.1016/j.rinp.2019.102546
22. J.H. He, Q.T. Ain, New promises and future challenges of fractal calculus: from two-scale thermodynamics to fractal variational principle. Therm. Sci. (2020). https://doi.org/10.2298/TSCI200127065H
23. J.H. He, F.Y. Ji, Taylor series solution for Lane–Emden equation. J. Math. Chem. **57**(8), 1932–1934 (2019)
24. J.H. He, Y. Wu, Homotopy perturbation method for nonlinear oscillators with coordinate-dependent mass. Results Phys. **10**, 270–271 (2018)
25. J.H. He, W. Xh, Variational iteration method: new development and applications. Comput. Math. Appl. **54**(7/8), 881–894 (2007)
26. J.H. He, H.Y. Kong, R.X. Chen, Variational iteration method for Bratu-like equation arising in electrospinning. Carbohydr. Polym. **105**, 229–230 (2014)
27. C.H. He, Y. Shen, F.Y. Ji, J.H. He, Taylor series solution for fractal Bratu-type equation arising in electrospinning process. Fractals **28**(1), 2050011 (2020). https://doi.org/10.1142/S0218348X20500115
28. J. Henderson, B. Karna, C.C. Tisdell, Existence of solutions for three-point boundary value problems for second order equations. Proc. Am. Math. Soc. **133**, 1365–1369 (2004)
29. G.L. Karakostas, P.C. Tsamatos, Positive solutions of a boundary-value problem for second order ordinary differential equations. Electron. J. Differ. Equ. **49**, 1–9 (2000)
30. J.B. Keller, Electrohydrodynamics I. The equilibrium of a charged gas in a container. J. Ration. Mech. Anal. **5**, 715–724 (1956)
31. F. Li, M. Jia, X. Liu, C. Li, G. Li, Existence and uniqueness of solutions of second-order three-point boundary value problems with upper and lower solutions in the reversed order. Nonlinear Anal. **68**, 2381–2388 (2008)
32. P.M. Lima, L. Morgado, Numerical modeling of oxygen diffusion in cells with Michaelis–Menten uptake kinetics. J. Math. Chem. **48**, 145–58 (2010)
33. H.Y. Liu, P. Wang, A short remark on WAN model for electrospinning and bubble electrospinning and its development. Int. J. Nonlinear Sci. Numer. Simul. **16**(1), 1–2 (2015)
34. R. Ma, Positive solutions for a nonlinear three-point boundary-value problem. Electron. J. Differ. Equ. **34**, 1–8 (1998)
35. R. Ma, Existence of positive solutions for second order m-point boundary value problems. Ann. Pol. Math. LXXIX **3**, 256–276 (2002)
36. R.Y. Ma, N. Castaneda, Existence of solutions of nonlinear m-point boundary value problems. J. Math. Anal. Appl. **256**, 556–567 (2001)
37. H. Madduri, P. Roul, A fast-converging iterative scheme for solving a system of Lane–Emden equationsarising in catalytic diffusion reactions. J. Math. Chem. **57**, 570–582 (2019)
38. D. O'Regan, A. Orpel, Eigenvalue problem for ODEs with a perturbed q-Laplace operator. Dyn. Syst. Appl. **24**(1), 97–112 (2015)
39. A. Orpel, Nonlinear BVPS with functional parameters. J. Differ. Equ. **246**, 1500–1522 (2009)
40. P. Pirabaharan, R. Saravanakumar, L. Rajendran, The theory of steady state current for chronoamperometric and cyclic voltammetry on rotating disk electrodes for EC' and ECE reactions. Electrochim. Acta **313**, 441–456 (2019)
41. M. Singh, A.K. Verma, Picard type iterative scheme with initial iterates in reverse order for a class of nonlinear three point BVPs. Int. J.Differ. Equ. (2013). https://doi.org/10.1155/2013/728149
42. M. Singh, A.K. Verma, On amonotone iterative method for a class of three point nonlinear nonsingular BVPs with upper and lower solutions in reverse order. J. Appl. Math. Comput. **43**, 99–114 (2013)
43. R. Stańczy, Positive solutions for superlinear elliptic equations. J. Math. Anal. Appl. **283**, 159–166 (2003)
44. A.K. Verma, M. Singh, Existence of solutions for three-point BVPS arising in bridge design. Electron. J. Differ. Equ. **173**, 1–11 (2014)
45. G. Vidossich, On the continuous dependence of solutions of boundary value problems for ordinary differential equations. J. Differ. Equ. **82**, 1–14 (1989)
46. A.M. Wazwaz, The variational iteration method for analytic treatment for linear and nonlinear ODEs. Appl. Math. Comput. **212**, 120–134 (2009)

47. A.M. Wazwaz, The variational iteration method for solving nonlinear singular boundary value problems arising in various physical models. Commun. Nonlinear Sci. Numer. Simul. **16**, 3881–3886 (2011)
48. A.M. Wazwaz, Solving the non-isothermal reaction–diffusion model equations in a spherical catalyst by the variational iteration method. Chem. Phys. Lett. **679**, 132–136 (2017)
49. J.R.L. Weeb, Positive solutions of some three-point boundary value problems via fixed point theory. Nonlinear Anal. **47**, 4319–4332 (2001)