Journal of Global Optimization (2023) 85:595–613
https://doi.org/10.1007/s10898-022-01224-1

New bounds for nonconvex quadratically constrained quadratic programming

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Received: 2 May 2021 / Accepted: 11 August 2022 / Published online: 26 August 2022

Abstract
In this paper, we study some bounds for nonconvex quadratically constrained quadratic programs (QCQPs). We propose two types of bounds for QCQPs, quadratic and cubic bounds. We use affine functions as Lagrange multipliers for quadratic bounds. We demonstrate that most semidefinite relaxations can be obtained as the dual of a quadratic bound. In addition, we study bounds obtained by changing the ground set. For cubic bounds, in addition to affine multipliers we employ quadratic functions. We provide a comparison between the proposed cubic bound and typical bounds for standard quadratic programs. Moreover, we report comparison results of some quadratic and cubic bounds.

Keywords Quadratically constrained quadratic programming · Semidefinite relaxation · Reformulation-linearization technique

1 Introduction

We consider the following quadratically constrained quadratic program, QCQP,

$$\begin{aligned} \min\ & x^T Q_0 x + 2c_0^T x \\ \text{s.t.}\ & x^T Q_i x + 2c_i^T x \le b_i, \quad i = 1, \dots, m \\ & Ax = d, \\ & l \le x \le u, \end{aligned} \tag{QCQP}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, Q_i $(i = 0, 1, \dots, m)$ are $n \times n$ real symmetric matrices, A is a $p \times n$ real matrix, c_i $(i = 0, 1, \dots, m)$ and d are vectors in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, and b_i $(i = 1, \dots, m)$ are real scalars. We assume that $-\infty < l_i \le u_i < \infty$ for $i = 1, \dots, n$. Without loss of generality, we may assume that $l = 0$ and $u = e$, where e represents the vector of ones in $\mathbb{R}^n$. We remark that general QCQPs with bounded feasible set can be formulated as (QCQP).

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Quadratically constrained quadratic programming is a fundamental problem in optimization theory and practice. QCQPs arise in many applications including economic equilibria, facility location and circle packing problems [3, 26, 29]. Furthermore, most combinatorial optimization problems including max-cut problem and clique problem can be cast as QCQPs [4, 10]. In addition to the aforementioned problems, Madani et al. [16] showed that any polynomial optimization problem can be cast as a QCQP. When the matrices Q_i $(i = 0, 1, ..., m)$ are positive semidefinite, (QCQP) will be a convex optimization problem, and consequently it is polynomially solvable. Nevertheless, as QCQPs include a wide range of NP-hard optimization problems, QCQP is NP-hard in general [28].

A typical class of optimization methods for handling QCQPs is branch-and-bound method. In this approach, the general problem is divided into some subproblems, which are called nodes. At each node, a lower bound is computed by a relaxation or a bound. In general, the generated lower determines that a node will be fathomed or branched. The effectiveness of a branch-and-bound method rests mainly on the tightness of generated lower bounds and their computational time. Most relaxations and bounds for QCQPs are mainly based on the reformulation-linearization technique (RLT), convex relaxations and semidefinite relaxations [13, 24, 27]. The most effective relaxation methods are based on semidefinite relaxation (SDR) [3, 11]. Due to the efficiency of this approach, many SDRs have been proposed; see [3, 31] for review and comparisons.

Recently, the author proposed a new dual for linearly constrained quadratic programming [30], in which affine functions are regarded as Lagrange multipliers. In this paper, similar to this method, we present two types of bounds for QCQPs, quadratic and cubic bounds. For quadratic bounds, we employ affine functions as Lagrange multipliers. We illustrate that most SDRs can be interpreted as the dual of a quadratic bound. In addition, we introduce some bounds which are obtained by changing the ground set.

For cubic bounds, we apply quadratic functions as Lagrange multipliers. We give some conditions under which the proposed bound is exact. We demonstrate that the cubic bound is equivalent to the bound obtained by Parrilo hierarchy for standard quadratic programs. The interested reader can see Chapter 5 in [20] for more details on Parrilo hierarchy.

The paper is organized as follows. After reviewing our notations, in Sect. 2 we introduce quadratic bounds. Section 3 is devoted to cubic bounds. In Sect. 4, we illustrate the effectiveness of some quadratic and cubic bounds by presenting its numerical performance on some QCQPs.

1.1 Notation

The following notation is used throughout the paper. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. Let A_i stand for the ith row of matrix A. Vectors are considered to be column vectors and T denotes transposition operation. We employ e_i to represent the ith unit vector, and vector e stands for the vector of ones. We denote the identity matrix by I. For symmetric matrices A and H, we use notation $A \succeq H$ to denote $A - H$ is positive semidefinite. The inner product of A and H is defined and denoted as $A \bullet H = \text{trace}(AH)$. A symmetric $n \times n$ matrix Q is called copositive if the bilinear form $x^T Qx$ is non-negative on non-negative orthant. For $x \in \mathbb{R}^n$, $\text{diag}(x)$ stands for the diagonal matrix whose entries on the diagonal are the components of x. Moreover, for $n \times n$ matrix Q, $\text{Diag}(Q)$ denotes a column vector with $\text{Diag}(Q)_i = Q_{ii}$.

For a set $\mathcal{Z} \subseteq \mathbb{R}^n$, we use the notations $\text{int}(\mathcal{Z})$ and $\text{conv}(\mathcal{Z})$ for the interior and the convex hull of $\mathcal{Z}$, respectively. $\mathbb{R}^n_+$ denotes non-negative orthant. We use B to represent box $[0, 1]^n$. The dual cone of K is denoted and defined as $K^* := \{y : y^T x \geq 0,\ \forall x \in K\}$.

We use $\mathcal{A}(\mathbb{R}^n)$ and $\mathcal{Q}(\mathbb{R}^n)$ to represent affine and quadratic functions on $\mathbb{R}^n$. We denote non-negative affine and quadratic functions on $\mathcal{Z} \subseteq \mathbb{R}^n$ by $\mathcal{A}_+(\mathcal{Z})$ and $\mathcal{Q}_+(\mathcal{Z})$, respectively, i.e., $\mathcal{A}_+(\mathcal{Z}) = \{\alpha \in \mathcal{A}(\mathbb{R}^n) : \alpha(x) \geq 0\, \forall x \in \mathcal{Z}\}$ and $\mathcal{Q}_+(\mathcal{Z}) = \{q \in \mathcal{Q}(\mathbb{R}^n) : q(x) \geq 0\, \forall x \in \mathcal{Z}\}$. We denote the matrix representation of a quadratic function $q(x) = x^T Qx + 2c^T x + b$ by $\mathcal{M}(q) = \begin{pmatrix} Q & c \\ c^T & b \end{pmatrix}$.

2 Quadratic bounds

In this section, we propose some quadratic bounds for QCQPs. Let $\mathcal{Z} = \{x \in \mathbb{R}^n : x \in B, Ax = d\}$ and $F = \{x \in \mathbb{R}^n : x \in \mathcal{Z}, x^T Q_i x + 2c_i^T x \leq b_i, i = 1, ..., m\}$. Because $\mathcal{A}_+(\mathcal{Z})$ is a polyhedral set, it follows that the representation of $\mathcal{A}_+(\mathcal{Z})$ in $\mathbb{R}^{n+1}$ is a polyhedral cone [17]. We propose the following problem as a new quadratic bound for (QCQP),

$$\begin{aligned}
&\max \ell \\
&\text{s. t. } x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^{m} \lambda_i (x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^{p} \alpha_i(x)(A_i x - d_i) - \\
&\qquad \sum_{i=1}^{n} \beta_i(x) x_i + \sum_{i=1}^{n} \gamma_i(x)(x_i - 1) \in \mathcal{Q}_+(\mathbb{R}^n), \\
&\quad \alpha_i \in \mathcal{A}(\mathbb{R}^n), \qquad\qquad\quad i = 1, ..., p \\
&\quad \lambda_i \geq 0,\ \beta_i, \gamma_i \in \mathcal{A}_+(\mathcal{Z}), \quad i = 1, ..., n.
\end{aligned} \tag{1}$$

Problem (1) can be regarded as a Lagrangian dual for (QCQP), for which the dual variables corresponding to linear constraints are replaced with affine functions. We remark that, due to the non-homogeneous Farkas' Lemma, $\alpha(x) = f^T x + g$ belongs to $\mathcal{A}_+(\mathcal{Z})$ if and only if there exist $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^n_+$ with $f \geq A^T \lambda - \mu$ and $g \geq -d^T \lambda + e^T \mu$. Note that the quadratic function $q(x) = x^T Qx + 2c^T x + b$ is non-negative on $\mathbb{R}^n$ if and only if matrix $\mathcal{M}(q)$ is positive semidefinite, and accordingly problem (1) can be formulated as a semidefinite program, which has $O(n^2)$ variables.

One crucial question regarding this bound is well-definedness. In the next proposition, we prove that problem (1) is feasible and generates a finite lower bound for (QCQP).

Proposition 1 *Let (QCQP) have a feasible point. Then problem (1) gives a finite lower bound.*

Proof Similar to the proof of Proposition 2 in [30], it is shown that there exist $\gamma_i \in \mathcal{A}_+(\mathcal{Z})$ for $i = 1, ..., n$ such that $x^T Q_0 x + 2c_0^T x + \sum_{i=1}^n \gamma_i(x)(x_i - 1)$ is strictly convex. So for suitable choice of ℓ, we have $x^T Q_0 x + 2c_0^T x + \sum_{i=1}^k \gamma_i(x)(x_i - 1) - \ell \in \mathcal{Q}_+(\mathbb{R}^n)$, which shows the feasibility of (1). Due to the feasibility of (QCQP), the first constraint of (1) implies that the optimal value of (1) is finite and it is a lower bound for (QCQP). □

The proof of Proposition 1 reveals that problem (1) is feasible for each quadratic function as an objective function of (QCQP). Indeed, the problem is strongly feasible. A conic optimization problem is called strongly feasible if it is feasible and remains feasible for all sufficiently small perturbations of right side of linear constraints [22]. As problem (1) is convex, it is natural to ask about the dual thereof. The dual of problem (1) may be written as

$$
\begin{aligned}
\min\ & Q_0 \bullet X + 2c_0^T x \\
\text{s.t.}\ & Q_i \bullet X + 2c_i^T x \le b_i, && i = 1, \dots, m \\
& X A_i^T = d_i x, && i = 1, \dots, p \\
& Ax = d, \\
& X \ge 0, \\
& e x^T - X \ge 0, \\
& X - e x^T - x e^T + e e^T \ge 0, \\
& X - x x^T \succeq 0.
\end{aligned} \tag{2}
$$

We refer the reader to [30] for the details of computations. In problem (2), we did not write redundant constraint $x \in B$ [25]. Problem (2) is a well-known relaxation, called Shor relaxation with partial first-level RLT [1, 3]. Anstreicher [1] proposed SDR (2) as a combination of RLT and Shor relaxation. He showed that SDR (2) can generate lower bounds tighter than either technique. Bao et al. [3] established that SDR (2) and doubly non-negative relaxation provide the same lower bound. The doubly non-negative relaxation is analogous to problem (2), but the constraint $X A_i^T = d_i x$ is replaced with $X \bullet A_i^T A_i = d_i^2$, $i = 1, \dots, m$. Note that since problem (1) is strongly feasible, strong duality holds [22], and consequently problems (1) and (2) generate the same lower bound.

Since the ground set of (QCQP) is not $\mathbb{R}^n$, the bound may be improved if one replaces $\mathcal{Q}_+(\mathbb{R}^n)$ with other sets. Bomze [5] took advantage of this idea and proposed some results about global optimality conditions for QCQPs. As the feasible set of (QCQP) is subset of positive orthant, one replacement for $\mathcal{Q}_+(\mathbb{R}^n)$ can be quadratic functions with non-negative coefficients. In this case, we get the following bound

$$
\begin{aligned}
\max\ & \ell \\
\text{s.t.}\ & x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^{m} \lambda_i (x^T Q_i x + 2c_i^T x - b_i) + \\
& \quad \sum_{i=1}^{p} \alpha_i(x)(A_i x - d_i) - \sum_{i=1}^{n} \beta_i(x) x_i + \sum_{i=1}^{n} \gamma_i(x)(x_i - 1) \in \mathcal{Q}^N(\mathbb{R}^n), \\
& \lambda_i \ge 0,\ \alpha_i \in \mathcal{A}(\mathbb{R}^n), \quad i = 1, \dots, m \\
& \beta_i, \gamma_i \in \mathcal{A}_+(\mathcal{Z}), \quad i = 1, \dots, p.
\end{aligned} \tag{3}
$$

where $\mathcal{Q}^N(\mathbb{R}^n)$ denotes quadratic functions with non-negative coefficients. The above problem can be formulated as a linear program. Indeed, by the non-homogeneous Farkas' Lemma, constraints $\beta_i, \gamma_i \in \mathcal{A}_+(\mathcal{Z})$ may be written as some linear inequalities and the rest of constraints are linear. It can be shown that problem (3) is the dual of a linear RLT [25].

Another interesting substitute for $\mathcal{Q}_+(\mathbb{R}^n)$ may be non-negative quadratic functions on B. Recall that $B = [0, 1]^n$. In this case, the following program provides a bound

$$
\begin{aligned}
\max\ & \ell \\
\text{s.t.}\ & x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^{m} \lambda_i (x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^{p} \alpha_i(x)(A_i x - d_i) \in \mathcal{Q}_+(B), \\
& \lambda \ge 0,\ \alpha_i \in \mathcal{A}(\mathbb{R}^n), \quad i = 1, \dots, m.
\end{aligned} \tag{4}
$$

Since for each $q \in \mathcal{Q}(\mathbb{R}^n)$, there exists ℓ with $q - \ell \in \mathcal{Q}_+(B)$, problem (4) is always feasible. Needless to say, bound (4) dominates all the above-mentioned bounds. Nevertheless, this bound is not necessarily exact for general QCQPs. Note that a bound or a relaxation is said to be exact if it provides a lower bound equal to the optimal value of the problem under question. Next theorem gives some sufficient conditions for exactness.

Theorem 1 *Bound (4) is exact if*

$$x^T Q_i x + 2c_i^T x - b_i \leq (or \geq) 0, \quad \forall x \in \mathcal{Z}, i = 1, ..., m.$$

Proof First we prove the case that there does not exist any quadratic constraint. By Lemma 4 in [8],

$$\mathcal{Q}_+(B) = \left\{ \begin{pmatrix} x^T & 1 \end{pmatrix} Q \begin{pmatrix} x \\ 1 \end{pmatrix} : Q \in K_B^* \right\},$$
$$\mathcal{Q}_+(\mathcal{Z}) = \left\{ \begin{pmatrix} x^T & 1 \end{pmatrix} Q \begin{pmatrix} x \\ 1 \end{pmatrix} : Q \in K_{\mathcal{Z}}^* \right\},$$

where

$$K_B = \operatorname{conv} \left\{ zz^T : z \in \mathbb{R}_+^{n+1}, z_i \leq z_{n+1} \; i = 1, ..., n \right\},$$
$$K_{\mathcal{Z}} = \operatorname{conv} \left\{ zz^T : z \in \mathbb{R}_+^{n+1}, \begin{pmatrix} A & -d \end{pmatrix} z = 0, \; z_i \leq z_{n+1} \; i = 1, ..., n \right\}.$$

Remark that $\mathcal{M}(\mathcal{Q}_+(B))^* = K_B$ and $\mathcal{M}(\mathcal{Q}_+(\mathcal{Z}))^* = K_{\mathcal{Z}}$. We establish that $\mathcal{Q}_+(\mathcal{Z}) = \mathcal{Q}_+(B) + \{\sum_{i=1}^p \alpha_i(x)(A_i x - d_i) : \alpha_i \in \mathcal{A}(\mathbb{R}^n)\}$. The inclusion $\supseteq$ is trivial. We prove the inclusion $\subseteq$ by contradiction. Let $q(x) = x^T Q x + 2c^T x + c_0 \in \mathcal{Q}_+(\mathcal{Z})$ while $q \notin \mathcal{Q}_+(B) + \{\sum_{i=1}^p \alpha_i(x)(A_i x - d_i) : \alpha_i \in \mathcal{A}(\mathbb{R}^n)\}$. By separation theorem, there exists

$$O \in \mathcal{M}\left(\mathcal{Q}_+(B)\right)^* \cap \mathcal{M}\left(\left\{\sum_{i=1}^p \alpha_i(x)(A_i x - d_i) : \alpha_i \in \mathcal{A}(\mathbb{R}^n)\right\}\right)^*,$$

with $\mathcal{M}(q) \bullet O = -1$. Because $\mathcal{M}(\mathcal{Q}_+(B))^* = K_B$, we have $O = \sum_{k=1}^l z^k (z^k)^T$, where $z^k \in \{z \in \mathbb{R}_+^{n+1} : z_i \leq z_{n+1} \; i = 1, ..., n\}$ $(k = 1, ..., l)$. As $(a_i x - d_i)(a_i x - d_i)$ and $-(a_i x - d_i)(a_i x - d_i)$ are members of $\{\sum_{i=1}^p \alpha_i(x)(A_i x - d_i) : \alpha_i \in \mathcal{A}(\mathbb{R}^n)\}$ for $i = 1, ..., p$, we have $O \in K_{\mathcal{Z}}$. This implies that $O \in \mathcal{M}(\mathcal{Q}_+(\mathcal{Z}))^*$. Thus, we have $\mathcal{M}(q) \bullet O \geq 0$ which contradicts the assumption $\mathcal{M}(q) \bullet O = -1$.

Now we consider the case that quadratic constraints exist. If $x^T Q_i x + 2c_i^T x \leq b_i$ for each $x \in \mathcal{Z}$ and $i = 1, ..., m$, the quadratic constraints are redundant and theorem follows from the first part.

For the case that $x^T Q_i x + 2c_i^T x \geq b_i$ for each $x \in \mathcal{Z}$ and $i = 1, ..., m$, we establish that the dual cones corresponding to $\mathcal{M}(\mathcal{Q}_+(F))$ and $\mathcal{M}(\mathcal{Q}_+(\mathcal{Z}) + \{\sum_{i=1}^m \lambda_i (x^T Q_i x + 2c_i^T x - b_i) : \lambda \leq 0\})$ are the same. The following inclusion is immediate,

$$\mathcal{M}\left(\mathcal{Q}_+(F)\right)^* \subseteq \mathcal{M}\left(\mathcal{Q}_+(\mathcal{Z}) + \left\{\sum_{i=1}^m \lambda_i (x^T Q_i x + 2c_i^T x - b_i) : \lambda \leq 0\right\}\right)^*.$$

We establish the reverse inclusion. Suppose that

$$O \in \mathcal{M}\left(\mathcal{Q}_+(\mathcal{Z}) + \left\{\sum_{i=1}^m \lambda_i (x^T Q_i x + 2c_i^T x - b_i) : \lambda \leq 0\right\}\right)^*.$$

By the representation of $\mathcal{Q}_+(\mathcal{Z})$, we have $O = \sum_{k=1}^{l} z^k (z^k)^T$, where

$$z^k \in \left\{ z \in \mathbb{R}_+^{n+1} : \begin{pmatrix} A & -d \end{pmatrix} z = 0, z_i \leq z_{n+1} \; i = 1, ..., n \right\}, \quad k = 1, ..., l.$$

By the assumption $\begin{pmatrix} Q_i & c_i \\ c_i^T & b_i \end{pmatrix} \bullet z_k z_k^T \geq 0$ ($i = 1, ..., m, k = 1, ..., l$), we have $z_i^T O z_i = 0$. Suppose that

$$K_F = \operatorname{conv}\{zz^T : z \in \mathbb{R}_+^{n+1}, \begin{pmatrix} A & -d \end{pmatrix} z = 0, z_i \leq z_{n+1} \; i = 1, ..., n, z^T \mathcal{M}(q_j) z \leq 0,$$
$$j = 1, ..., m\}.$$

As $\mathcal{Q}_+(F) = \left\{ \begin{pmatrix} x^T & 1 \end{pmatrix} Q \begin{pmatrix} x \\ 1 \end{pmatrix} : Q \in K_F^* \right\}$, we have $O \in \mathcal{M}(\mathcal{Q}_+(F))^*$, which completes the proof. □

It is worth mentioning that Theorem 1 can be proved by using strong duality for conic programs and Proposition 6 in [3], but here we present a new proof. The next proposition states that bound (4) is exact for linearly constrained quadratic programs with binary variables.

Proposition 2 *Bound (4) is exact for linearly constrained quadratic programs with binary variables.*

Proof Consider the problem

$$\begin{aligned} \min \; & x^T Q_0 x + 2c_0^T x \\ \text{s.t.} \; & A_i^T x = d_i, \qquad i = 1, ..., p \\ & x_i \in \{0, 1\} \qquad i \in \mathcal{I} \\ & 0 \leq x \leq e, \end{aligned}$$

where index set $\mathcal{I} \subseteq \{1, ..., n\}$ denotes binary variables. This problem can be formulated as

$$\begin{aligned} \min \; & x^T Q_0 x + 2c_0^T x \\ \text{s.t.} \; & A_i^T x = d_i, \qquad i = 1, ..., p \\ & x_i(1 - x_i) \leq 0, \qquad i \in \mathcal{I} \\ & 0 \leq x \leq e. \end{aligned}$$

As all conditions of Theorem 1 holds for the above problem, bound (4) is exact for linearly constrained quadratic programs with binary variables. □

By Theorem 2.6 in [7] and strong duality for conic programs $q \in \mathcal{A}_+(B)$ if and only if the following system has a solution

$$q(x) + \sum_{i=1}^{n} \alpha_i(x, s)(x_i + s_i - 1) \in \mathcal{Q}_+(\mathbb{R}_+^{2n}),$$
$$\alpha_i(x, s) \in \mathcal{A}(\mathbb{R}_+^{2n}), \quad i = 1, ..., n,$$

where variables $s_1, ..., s_n$ are slack variables. A quadratic function $q(x) = x^T Q x + 2c^T x + b \in \mathcal{Q}_+(\mathbb{R}_+^n)$ if and only if matrix $\mathcal{M}(q)$ is copositive [5]. Therefore, bound (4) can be formulated as a copostive program. Copostive programs are intractable in general. In fact, they are NP-hard. Nonetheles, there exist efficient methods which approximate copositive cone [6, 20].

It is well-known that Shor relaxation is the dual of (QCQP) when affine multipliers are constant functions. In addition, we showed that the dual of (1) is Shor relaxation with partial first-level RLT. It is may be of interest to know whether other SDRs can be also obtained in this manner. In the sequel, we will demonstrate that some SDRs can be obtained as the dual of bounds in the form of (1) by a suitable choice of affine multipliers or adding some valid cuts.

Let $m_c = \{i : Q_i \succeq 0, Q_i \neq 0\}$. In the rest of the section, we make the assumption that for each $i \in m_c$ there exists $\bar{x}^i \in \mathbb{R}^n$ such that $(\bar{x}^i)^T Q_i \bar{x}^i + 2c^T \bar{x}^i < b_i$. Due to the semipositiveness of Q_i $(i \in m_c)$, there exists a matrix R_i with $Q_i = R_i^T R_i$. By the Schur Complement Lemma, $x^T Q_i x + 2c_i^T x \leq b_i$ is equivalent to

$$\begin{pmatrix} I & -R_i x \\ -x^T R_i^T & -2c_i^T x + b_i, \end{pmatrix} \succeq 0.$$

So affine function $\alpha(x) = f^T x + g$ is non-negative on $L_i = \{x \in \mathbb{R}^n : x^T Q_i x + 2c_i^T x \leq b_i\}$ if and only if the optimal value of the following semidefinite program is greater than or equal to $-g$.

$$\begin{aligned} &\min\ f^T x \\ &\text{s.t.}\ \begin{pmatrix} I & -R_i x \\ -x^T R_i^T & -2c_i^T x + b_i, \end{pmatrix} \succeq 0. \end{aligned} \tag{5}$$

Note that problem (5) can be reformulated as a second-order cone program. Recently, Zheng et al. [31] proposed some SDRs for QCQPs. In fact, they introduced a unified framework for generating convex relaxations for QCQPs. They propose the following SDR for (QCQP),

$$\begin{aligned} &\min Q_0 \bullet X + 2c_0^T x \\ &\text{s.t.}\ Q_i \bullet X + 2c_i^T x \leq b_i, \qquad i = 1, \ldots, m \\ &\quad XA_i^T = d_i x, \qquad i = 1, \ldots, p \\ &\quad Ax = d, \\ &\quad X \geq 0,\ X - xx^T \succeq 0, \\ &\quad ex^T - X \geq 0, \\ &\quad X - ex^T - xe^T + ee^T \geq 0, \\ &\quad \begin{pmatrix} x_k I & R_i X e_k \\ (R_i X e_k)^T & -2c_i^T X e_k + b_i e_k^T x \end{pmatrix} \succeq 0, \qquad i = 1, \ldots, m_c, k = 1, \ldots, n \\ &\quad \begin{pmatrix} (1 - x_k) I & -R_i X e_k + R_i x \\ (-R_i X e_k + R_i x)^T & 2c_i^T X e_k - (2c_i^T + b_i e_k^T) x + b_k \end{pmatrix} \succeq 0, i = 1, \ldots, m_c, k = 1, \ldots, n, \end{aligned} \tag{6}$$

and they call it SDP relaxation with rank-2 second-order cone valid inequalities. Note that the above SDR is obtained by adding the last two constraints of problem (6) to Shor relaxation with partial first-level RLT. We demonstrate that SDR (6) is the dual of the following bound,

$$\begin{aligned} &\max \ell \\ &\text{s.t.}\ x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^{m} \lambda_i (x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^{p} \alpha_i(x)(A_i x - d_i) - \\ &\qquad \sum_{i=1}^{n} \big(\beta_i(x) + \sum_{j \in m_c} \beta_{ij}(x)\big) x_i + \sum_{i=1}^{n} \big(\gamma_i(x) + \sum_{j \in m_c} \gamma_{ij}(x)\big)(x_i - 1) \in \mathcal{Q}_+(\mathbb{R}^n), \end{aligned}$$

$$\begin{aligned}
&\alpha_i \in \mathcal{A}(\mathbb{R}^n), && i = 1, ..., p \\
&\lambda \geq 0,\ \beta_i, \gamma_i \in \mathcal{A}_+(\mathcal{Z}), && i = 1, ..., n \\
&\beta_{ij}, \gamma_{ij} \in \mathcal{A}_+(L_j), && i = 1, ..., n,\ j \in m_c.
\end{aligned} \tag{7}$$

It is easily seen (7) is a bound for (QCQP). For convenience, to show bound (7) is the dual of (6) we consider the QCQP

$$\begin{aligned}
&\min x^T Q_0 x + 2c_0^T x \\
&\text{s. t. } x^T Q_1 x + 2c_1^T x \leq b_1, \\
&\ a_1^T x \leq d_1,
\end{aligned} \tag{8}$$

which has a convex quadratic constraint and a linear inequality constraint. Bound (7) for problem (8) is formulated as

$$\begin{aligned}
&\max\ \ell \\
&\text{s. t. } x^T Q_0 x + 2c_0^T x - \ell + \alpha(x^T Q_1 x + 2c_1^T x - b_1) + (f^T x + g)(a_1^T x - d_1) \in \mathcal{Q}_+(\mathbb{R}^n), \\
&\qquad \alpha \geq 0,\ f^T x + g \in \mathcal{A}_+(L_1).
\end{aligned} \tag{9}$$

As $\mathrm{int}(L_1) \neq \emptyset$, strong duality holds for problem (5). Accordingly, $f^T x + g \in \mathcal{A}_+(L_1)$ is equivalent that the optimal value of the following semidefinite program is greater than or equal to $-g$,

$$\begin{aligned}
&\max -I \bullet Y - b_1 y_0 \\
&\text{s. t.} -R_1 y - c_1 y_0 = \frac{1}{2} f, \\
&\begin{pmatrix} Y & y \\ y^T & y_0 \end{pmatrix} \succeq 0.
\end{aligned}$$

Hence, problem (9) is reformulated as follows,

$$\begin{aligned}
&\max \ell \\
&\text{s. t. } x^T Q_0 x + 2c_0^T x - \ell + \alpha(x^T Q_1 x + 2c_1^T x - b_1) + \\
&\qquad\qquad (2(-R_1 y - c_1 y_0)^T x + g)(a_1^T x - d_1) \in \mathcal{Q}_+(\mathbb{R}^n), \\
&\quad - I \bullet Y - b_1 y_0 \geq -g, \\
&\quad \alpha \geq 0,\ \begin{pmatrix} Y & y \\ y^T & y_0 \end{pmatrix} \succeq 0.
\end{aligned}$$

By a little algebra, the dual of the above problem may be written as follows,

$$\begin{aligned}
&\min\ Q_0 \bullet X + 2c_0^T x \\
&\text{s. t. } Q_1 \bullet X + 2c_1^T x \leq b_1, \\
&\quad a_1^T x \leq d_1, \\
&\quad X - xx^T \succeq 0, \\
&\quad \begin{pmatrix} (d_1 - a_1^T x) I & -R_1 X a_1 + d_1 R_1 x \\ (-R_1 X a_1 + d_1 R_1 x)^T & 2c_1^T X a_1 - (2d_1 c_1 + b_1 a_1)^T x + b_1 d_1 \end{pmatrix} \succeq 0,
\end{aligned}$$

which clarifies the point that bound (7) is the dual of problem (6). Since problem (7) is strongly feasible, strong duality also holds.

Adding valid cuts is a typical method to tighten the relaxation gap. Zheng et al. [31] introduced a class of quadratic valid cuts for QCQP and they proposed a new SDR by using these valid cuts. Their method generates a quadratic valid cut as follows. Let $F \subseteq \Omega$. Suppose that $u \in \operatorname{int}(\mathbb{R}^n_+)$ and $0 < u_\Omega = \{\max\ u^T x\ :\ x \in \Omega\}$. They showed that for $S \succeq 0$, the convex quadratic inequality $x^T Sx - u_\Omega \operatorname{Diag}(S)^T \operatorname{diag}(u)^{-1} x \leq 0$ is valid for (QCQP); see [31, Proposition 3]. We remark that the set of generated cuts by this method forms a convex cone in $\mathcal{Q}(\mathbb{R}^n)$.

By the above discussion, one can extend bound (7) as follows,

$$
\begin{aligned}
&\max \ell \\
&\text{s.t.}\, x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^m \lambda_i (x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^p \alpha_i(x)(A_i x - d_i) - \\
&\qquad \sum_{i=1}^n (\beta_i(x) + \sum_{j\in m_c} \beta_{ij}(x)) x_i + \sum_{i=1}^n (\gamma_i(x) + \sum_{j\in m_c} \gamma_{ij}(x))(x_i - 1) + \\
&\qquad \sum_{i\in\mathbb{R}} \mu_i (x^T S_i x - u_\Omega \operatorname{Diag}(S_i)^T \operatorname{diag}(u)^{-1} x) \in \mathcal{Q}_+(\mathbb{R}^n), \\
&\quad \alpha_i \in \mathcal{A}(\mathbb{R}^n), && i = 1, \dots, p \\
&\quad \lambda \geq 0,\ \beta_i, \gamma_i \in \mathcal{A}_+(\mathcal{Z}), && i = 1, \dots, n \\
&\quad \beta_{ij}, \gamma_{ij} \in \mathcal{A}_+(L_j), && i = 1, \dots, n,\ j \in m_c \\
&\quad \mu_i \geq 0,\ S_i \succeq 0, && i \in \mathbb{R},
\end{aligned}
$$

which is a non-convex optimization problem with infinite number of constraints and variables. As mentioned above, the set of valid cuts is a convex cone, so the above bound may be formulated as the following semidefinite program,

$$
\begin{aligned}
&\max \ell \\
&\text{s.t.}\ x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^m \lambda_i (x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^p \alpha_i(x)(A_i x - d_i) - \\
&\qquad \sum_{i=1}^n (\beta_i(x) + \sum_{j\in m_c} \beta_{ij}(x)) x_i + \sum_{i=1}^n (\gamma_i(x) + \sum_{j\in m_c} \gamma_{ij}(x))(x_i - 1) + \\
&\qquad x^T S x - u_\Omega \operatorname{Diag}(S)^T \operatorname{diag}(u)^{-1} x \in \mathcal{Q}_+(\mathbb{R}^n), \\
&\quad \alpha_i \in \mathcal{A}(\mathbb{R}^n),\ i = 1, \dots, p \\
&\quad \lambda \geq 0,\ S \succeq 0, \beta_i, \gamma_i \in \mathcal{A}_+(\mathcal{Z}),\ i = 1, \dots, n \\
&\quad \beta_{ij}, \gamma_{ij} \in \mathcal{A}_+(L_j), \qquad i = 1, \dots, n,\ j \in m_c.
\end{aligned} \tag{10}
$$

The dual of bound (10) may be written as follows,

$$
\begin{aligned}
&\min Q_0 \bullet X + 2c_0^T x \\
&\text{s.t.}\ Q_i \bullet X + 2c_i^T x \leq b_i, && i = 1, \dots, m \\
&\quad X A_i^T = d_i x, && i = 1, \dots, p \\
&\quad Ax = d,
\end{aligned}
$$

$$
\begin{aligned}
& X \geq 0,\ X - xx^T \succeq 0, \\
& ex^T - X \geq 0, \\
& X - ex^T - xe^T + ee^T \geq 0, \\
& \begin{pmatrix} x_k I & R_i X e_k \\ (R_i X e_k)^T & -2c_i^T X e_k + b_i e_k^T x \end{pmatrix} \succeq 0, \qquad\qquad i = 1, \ldots, m_c, k = 1, \ldots, n \\
& \begin{pmatrix} (1 - x_k) I & -R_i X e_k + R_i x \\ (-R_i X e_k + R_i x)^T & 2c_i^T X e_k - (2c_i^T + b_i e_k^T) x + b_k \end{pmatrix} \succeq 0, i = 1, \ldots, m_c, k = 1, \ldots, n \\
& u_\Omega \operatorname{diag}(u)^{-1} \operatorname{diag}(x) - X \succeq 0, \qquad (11)
\end{aligned}
$$

which is the SDR proposed in [31]; see problem (SDP_{α_u}). Since problem (10) is also strongly feasible, we have strong duality. Here we just investigate some well-known SDRs and show that they can be interpreted as the dual of a bound in the form of (1). However, by similar arguments one can show that most SDRs can be obtained as the dual of a bound in the form of (1).

We conclude the section by mentioning some points. As the dual of the proposed bounds are well-known SDRs, we have just reinvented the wheel. Of course, this statement is correct, but viewing SDRs from this aspect can supply us with more tools for analyzing a SDR method. Furthermore, it paves the road for introducing and analyzing new relaxations or bounds. For instance, one can extend bound (7) as follows,

$$
\begin{aligned}
& \max \ell \\
& \text{s.t. } x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^k \lambda_i (x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^k \alpha_i(x)(A_i x - d_i) - \\
& \qquad \sum_{i=1}^k \beta_i(x) x_i + \sum_{i=1}^k \gamma_i(x)(x_i - 1) \in \mathcal{Q}_+(\mathbb{R}^n), \\
& \quad \alpha_i \in \mathcal{A}(\mathbb{R}^n), \qquad\qquad i = 1, \ldots, p \\
& \quad \lambda_i \geq 0,\ \beta_i, \gamma_i \in \mathcal{A}_+(V), \quad i = 1, \ldots, n, \qquad (12)
\end{aligned}
$$

where $V = \{x \in \mathcal{Z} : x^T Q_i x + 2c_i^T x \leq b_i,\ i \in m_c\}$. As $\mathcal{A}_+(\mathcal{Z}) \cup_{i \in m_c} \mathcal{A}_+(L_i) \subseteq \mathcal{A}_+(V)$, bound (12) dominates (7). Therefore, the dual of (12) leads to a SDR which dominates (6). Here, it is assumed that $\operatorname{int}(\cap_{i \in m_c} L_i) \cap \mathcal{Z} \neq \emptyset$. It is worth mentioning that bound (12) is exact for QCQP in the following form,

$$
\begin{aligned}
& \min\ x^T Q_0 x + 2c_0^T x \\
& \text{s.t. } x^T Q_1 x + 2c_1^T x \leq b_1, \\
& \quad a_1^T x \leq d_1,
\end{aligned}
$$

in which $Q_1 \succeq 0$, $a_1 \in \mathbb{R}^n$, $d_1 \in \mathbb{R}$ and the feasible set has non-empty interior; see [30, Proposition 1]. The aforementioned problem may be interpreted as an extension of S-Lemma. We refer the interested reader to [21] for more information about S-Lemma.

In the same line, one can formulate the following bound which dominates (10),

$$
\begin{aligned}
& \max \ell \\
& \text{s.t. } x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^k \lambda_i (x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^k \alpha_i(x)(A_i x - d_i) -
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{i=1}^{k} \beta_i(x)x_i + \sum_{i=1}^{k} \gamma_i(x)(x_i - 1) + x^T Sx - u_\Omega \operatorname{Diag}(S)^T \operatorname{diag}(u)^{-1} x \in \mathcal{Q}_+(\mathbb{R}^n), \\
& \alpha_i \in \mathcal{A}(\mathbb{R}^n), \qquad\qquad\qquad i = 1, ..., p \\
& \lambda \geq 0, \;\; S \succeq 0, \beta_i, \gamma_i \in \mathcal{A}_+(W), \qquad i = 1, ..., n,
\end{aligned} \tag{13}
$$

where $W = \{x \in V : x^T Sx - u_\Omega \operatorname{Diag}(S)^T \operatorname{diag}(u)^{-1}x \leq 0, \;\forall S \succeq 0\}$. Note that if there exists $\bar{x} \in \mathcal{Z}$ with $\bar{x}^T S\bar{x} - u_\Omega \operatorname{Diag}(S)^T \operatorname{diag}(u)^{-1}\bar{x} < 0$ for $S \succeq 0$ and $\bar{x}^T Q_i \bar{x} + 2c_i^T \bar{x} < b_i$ $(i \in m_c)$, then $f^T x + g \in \mathcal{A}_+(W)$ is equivalent to the consistency of the system

$$
\begin{aligned}
& f^T x + g + \lambda^T (Ax - d) + \mu^T (x - e) - \nu^T x + x^T Sx - u_\Omega \operatorname{Diag}(S)^T \operatorname{diag}(u)^{-1}x \in \mathcal{Q}_+(\mathbb{R}^n), \\
& \mu, \nu \geq 0, \;\; S \succeq 0.
\end{aligned}
$$

Thus, bound (13) is reformulated as a semidefinite program, and consequently its dual gives a SDR which dominates (11).

To the best knowledge of author, bounds (12) and (13) or their dual have not been proposed in the literature. Another point about the proposed bounds is that they not only provide a lower bound, but also give a convex underestimator. The given convex underestimator can be employed in optimization methods for generating a solution.

It is well-known when an optimal solution of a SDR has rank one the SDR is exact [15]. The next proposition gives necessary and sufficient conditions for exactness. For convenience to state the proposition, we consider bound (1). Let F_{opt} denote the optimal solution set of (QCQP).

Proposition 3 *Bound (1) is exact if and only if there exists feasible point $\bar{\lambda}$, $\bar{\alpha}_i$ $(i = 1, ..., p)$, $\bar{\beta}_i, \bar{\gamma}_i$ $(i = 1, ..., n)$ and $\bar{\ell}$ with*

$$
\begin{aligned}
& \bar{x} \in \operatorname*{argmin}_{x \in \mathbb{R}^n} x^T Q_0 x + 2c_0^T x - \bar{\ell} + \sum_{i=1}^{m} \bar{\lambda}_i (x^T Q_i x + 2c_i^T x - b_i) + \\
& \qquad\qquad \sum_{i=1}^{p} \bar{\alpha}_i(x)(A_i x - d_i) - \sum_{i=1}^{n} \bar{\beta}_i(x)x_i + \sum_{i=1}^{n} \bar{\gamma}_i(x)(x_i - 1), \\
& \bar{x}^T Q_0 \bar{x} + 2c_0^T \bar{x} = \bar{\ell}, \qquad \forall \bar{x} \in F_{opt}.
\end{aligned}
$$

Proof Let bound (1) be exact and suppose that $\bar{\lambda}$, $\bar{\alpha}_i$ $(i = 1, ..., p)$, $\bar{\beta}_i, \bar{\gamma}_i$ $(i = 1, ..., n)$ and $\bar{\ell}$ is an optimal solution. As the bound is exact, we have $\bar{x}^T Q_0 \bar{x} + 2c_0^T \bar{x} = \bar{\ell}$ for $\bar{x} \in F_{opt}$. In the light of $q(x) = x^T Q_0 x + 2c_0^T x - \bar{\ell} + \sum_{i=1}^{k} \bar{\lambda}_i (x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^{k} \bar{\alpha}_i(x)(A_i x - d_i) - \sum_{i=1}^{k} \bar{\beta}_i(x)x_i + \sum_{i=1}^{k} \bar{\gamma}_i(x)(x_i - 1) \in \mathcal{Q}_+(\mathbb{R}^n)$ and $q(\bar{x}) = 0$, we have

$$
\bar{x} \in \operatorname{argmin}\{q(x) : x \in \mathbb{R}^n\},
$$

which completes the if part. The only-if part is immediate. □

It is worth mentioning that as strong duality holds for all proposed bounds, exactness of SDRs and bounds are equivalent. Moreover, bound (1) or SDR (2) are exact for general QCQP if and only if $n = 2$; see [2] for more details.

3 Cubic bounds

In this section, we propose cubic bounds for QCQP. Until now, we have used affine functions as dual variables. The most important point for applying other functions is that the obtained problem should be tractable.

Due to the structure of (QCQP), one may consider the following convex cones for dual variables,

1. $\mathcal{Q}_+^c(\mathcal{Z})$: non-negative convex quadratic functions on $\mathcal{Z}$,
2. $\mathcal{Q}^N(\mathbb{R}^n)$: quadratic functions with non-negative coefficients.

Both the above-mentioned cones have non-empty interior and verifying the membership of a given quadratic function is tractable. Verifying $q \in \mathcal{Q}^N(\mathbb{R}^n)$ is straightforward. By alternative theorem [17], $q(x) = x^T \hat{Q} x + 2\hat{c}^T c + \hat{c}_0$ belongs to $\mathcal{Q}_+^c(\mathcal{Z})$ if and only if there exist $\lambda \in \mathbb{R}^p$ and $\mu, \nu \in \mathbb{R}_+^n$ with

$$\mathcal{M}\left(x^T \hat{Q} x + 2\hat{c}^T c + \hat{c}_0 + \lambda^T(Ax - d) + \mu^T(x - e) - \nu^T x\right) \succeq 0.$$

By employing quadratic functions as dual variables, we are faced with checking non-negativity of a cubic function. Of course, a cubic function may not be non-negative on $\mathbb{R}^n$, unless it is quadratic. So it appears our effort by substituting some classes of quadratic functions for affine functions was in vain. Nevertheless, checking non-negativity of some classes of cubic functions might be tractable on non-negative orthant. For instance, one may consider the following sets of cubic functions,

1. $\mathcal{C}_+^c(\mathbb{R}_+^n)$: non-negative convex cubic functions on $\mathbb{R}_+^n$,
2. $\mathcal{C}^N(\mathbb{R}^n)$: cubic functions with non-negative coefficients.

Both sets are convex cones with non-empty interior. In addition, to check a cubic function belongs to these cones is tractable. Let $\kappa(x) = Tx^3 + xQx + cx + c_0$ be a cubic function, where T is a symmetric tensor of order 3. Note that $Tx^3 = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n T_{i,j,k} x_i x_j x_k$. Verifying $\kappa \in \mathcal{C}^N(\mathbb{R}^n)$ is straightforward. To check $\kappa \in \mathcal{C}_+^c(\mathbb{R}_+^n)$, we need first to impose the following conditions

$$\begin{aligned} &Te_i \succeq 0, \ i = 1, ..., n, \\ &Q \succeq 0, \end{aligned}$$

which guarantees convexity of κ on $\mathbb{R}_+^n$. As κ is convex, its optimal value can be obtained by primal interior point methods. As a result, membership verification is tractable in this case, but cannot be checked explicitly by some linear (matrix) inequalities.

Another replacement for $\mathcal{C}_+^c(\mathbb{R}_+^n)$ or $\mathcal{C}^N(\mathbb{R}^n)$ may be the set of quadratically Sum-of-Squares. We call a cubic function $\kappa(x) = Tx^3 + x^T Qx + c^T x + c_0$ quadratically Sum-of-Squares if $T(x^{(2)})^3 + (x^{(2)})^T Q(x^{(2)}) + c(x^{(2)}) + c_0$ is Sum-of-Squares, where $x^{(2)} = (x_1^2, ..., x_n^2)$. Note that one can check whether a polynomial is Sum-of-Squares by solving a semidefinite program [12].

By the above discussion, we propose the following bound for (QCQP),

$$\begin{aligned} &\max \ \ell \\ &\text{s.t. } x^T Q_0 x + 2c_0^T x - \ell + \sum_{i=1}^m \lambda_i(x)(x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^p \alpha_i(x)(A_i x - d_i) - \\ &\qquad \sum_{i=1}^n \beta_i(x) x_i + \sum_{i=1}^n \gamma_i(x)(x_i - 1) - \kappa(x) \in \mathcal{Q}_+(\mathbb{R}^n), \end{aligned}$$

$$\begin{aligned}
&\lambda_i \in \mathcal{A}_+(X),\ \alpha_j \in \mathcal{Q}(\mathbb{R}^n), \quad i = 1, ..., m,\ j = 1, ..., p\\
&\beta_i, \gamma_i \in \mathcal{Q}_+^c(\mathcal{Z}), \qquad\qquad\quad i = 1, ..., n\\
&\kappa \in \mathcal{C}^N(\mathbb{R}^n).
\end{aligned} \tag{14}$$

Problem (14) may be formulated as a semidefinite program and it has $O(n^3)$ variables. Similar to Proposition 1, one can show that bound (14) is always finite and generates a lower bound greater than or equal to that of (1).

One may wonder how the generated bound given by problem (14) can be improved. The straightforward method for tightening can be enlargement of feasible set. In problem (14), we have linear and quadratic function variables. One can adopt methods in Sect. 2 to tighten bound (14). The following proposition gives necessary and sufficient conditions under which bound (14) is exact.

Proposition 4 *Bound (14) is exact if and only if there exists feasible point $\bar{\lambda}_i$ $(i = 1, ..., m)$, $\bar{\alpha}_i$ $(i = 1, ..., p)$, $\bar{\beta}_i, \bar{\gamma}_i$ $(i = 1, ..., n)$, $\bar{\ell}$ and $\bar{\kappa}$ with*

$$\begin{aligned}
\bar{x} \in \operatorname*{argmin}_{x \in \mathbb{R}^n} x^T Q_0 x + 2c_0^T x - \bar{\ell} + \sum_{i=1}^{m} \bar{\lambda}_i(x)(x^T Q_i x + 2c_i^T x - b_i) +& \\
\sum_{i=1}^{p} \bar{\alpha}_i(x)(A_i x - d_i) - \sum_{i=1}^{n} \bar{\beta}_i(x) x_i + \sum_{i=1}^{n} \bar{\gamma}_i(x)(x_i - 1) - \bar{\kappa}(x)& \\
\bar{x}^T Q_0 \bar{x} + 2c_0^T \bar{x} = \bar{\ell},\ \forall \bar{x} \in F_{opt}.&
\end{aligned}$$

Proof Analogous to Proposition 3 is proved. □

In the same line, one could consider quadratics or linear functions as Lagrange multipliers for which the optimal value of problem (14) are non-negative. Indeed, one may consider $q(x) = x^T \hat{Q} x + 2\hat{c}^T x + \hat{c}_0$ eligible if the following system has a solution

$$\begin{aligned}
&x^T \hat{Q} x + 2\hat{c}^T x + \hat{c}_0 + \sum_{i=1}^{m} \lambda_i(x)(x^T Q_i x + 2c_i^T x - b_i) + \sum_{i=1}^{p} \alpha_i(x)(A_i x - d_i) - \\
&\quad \sum_{i=1}^{n} \beta_i(x) x_i + \sum_{i=1}^{n} \gamma_i(x)(x_i - 1) - \kappa(x) \in \mathcal{Q}_+(\mathbb{R}^n),\\
&\lambda_i \in \mathcal{A}_+(X),\ \alpha_j \in \mathcal{Q}(\mathbb{R}^n),\ i = 1, ..., m,\ j = 1, ..., p\\
&\beta_i, \gamma_i \in \mathcal{Q}_+^c(\mathcal{Z}), \qquad\qquad i = 1, ..., n\\
&\kappa \in \mathcal{C}^N(\mathbb{R}^n).
\end{aligned}$$

Thus, we obtain a hierarchy for tackling (QCQP). This hierarchy is increasing and each problem is formulated as a semidefinite program. If we consider problem (14) as a first problem of hierarchy, the number of variables of kth problem is of $O(n^{k+2})$. The most important inquiry concerning this method is its convergence in finite steps. In addition, if it is convergent in finite steps, what the order of k will be. As the subject of the paper is quadratic and cubic bounds, we leave these questions for further research.

The following example demonstrates that problem (14) could generate a bound tighter than the proposed bounds in Sect. 2.

Example 1 Consider the nonconvex QCQP,

$$\min -8x_1^2 - x_2^2 + x_3^2 - 5x_4^2 + 14x_1x_2 + 10x_1x_4 + 4x_2x_4 - 20x_2$$

$$\begin{aligned}
\text{s. t. } & 2x_1^2 + 2x_2^2 + 4x_1x_2 + 8x_1 + 6x_2 + x_4 \le 8 \\
& -8x_1^2 - 5x_2^2 + 2x_1x_4 - 8x_1x_2 - 4x_1 + 4x_2 + 2x_4 \le -4, \\
& 2x_1^2 + x_2^2 + 4x_4^2 + 2x_1 + x_4 \le 4, \\
& x_1 + 2x_2 + 2x_3 + x_4 = 3, \\
& x \in B.
\end{aligned}$$

The problem has two convex quadratic constraints, one nonconvex quadratic constraint and nine linear constraints, with the optimal value of -8.0008 and the optimal solution $\big(0.4203,\ 0.4942,\ 0.7956,\ 0\big)$. We set $u = \big(1,\ 2,\ 2,\ 1\big)^T$ and $u_\Omega = max\{u^T x :\ x \in F\} = 3.9145$. The performance of the bounds are listed in Table 1, which lb denotes the generated lower bound.

Bound	Shor relaxation	Bound (1)	Bound (7)	Bound (10)	Bound (12)	Bound (13)	Bound (14)
lb	−44.0945	−15.2676	−13.3647	−13.2294	−13.2518	−11.8969	−8.0008

As seen bound (14) is exact.

In the rest of the section, we investigate the relationship between bound (14) and the conventional bounds for QCQPs. Due to the computational burdensome, cubic bounds are not commonplace and they have been applied just for some classes of QCQPs such as standard quadratic programs. Of course, we can provide a comparison between bound (14) and general polynomial optimization methods, including the Lasserre hierarchy [12], with $O(n^3)$ variables, but we prefer bounds tailored for QCQPs.

Consider the standard quadratic program,

$$\begin{aligned}
\min\ & x^T Qx \\
\text{s. t. } & \textstyle\sum_{i=1}^n x_i = 1, \\
& x \ge 0.
\end{aligned} \tag{StQP}$$

It is well-known that (StQP) is solvable in polynomial time provided Q is either positive semidefinite or negative semidefinite on standard simplex. In general, however, (StQP) is NP-hard [6]. Suppose that Δ denotes the standard simplex.

Let ℓ_Q denote the optimal value of (StQP). We remark that optimizing a quadratic function on standard simplex can be formulated as (StQP). This is resulted from the fact that for each $x \in \Delta$, we have $x^T Qx + 2c^T x = x^T (Q + ec^T + ce^T)x$.

One effective method for handling (StQP) is Parrilo hierarchy [6]. In this method, for $r = 0, 1, \ldots$ the following problem gives a lower bound

$$p_Q^r = max\{\ell : Q - \ell ee^T \in \mathcal{P}^r\}, \tag{15}$$

where $\mathcal{P}^r = \{A : (\sum_{i=1}^n x_i^2)^r (\sum_{i=1}^n \sum_{j=1}^n x_i^2 A_{ij} x_j^2) \in \Sigma[x]\}$ and $\Sigma[x]$ denotes the set of all sum of square polynomials. It is well-known for sufficiently large r, p_Q^r is equal to the optimal value of (StQP) [6]. In addition, the number of variables of (15) is of $O(n^{r+2})$ [20].

Bound (14) is formulated as follows for (StQP),

$$
\begin{aligned}
&\max \ell \\
&\text{s.t. } x^T Qx - \ell - \sum_{i=1}^{n} \alpha_i(x)x_i + \alpha_{n+1}(x)(e^T x - 1) - \kappa(x) \in \mathcal{Q}_+(\mathbb{R}^n), \\
&\quad \alpha \in \mathcal{Q}_+^c(\Delta),\ i = 1, ..., n \\
&\quad \kappa \in \mathcal{C}^N(\mathbb{R}^n).
\end{aligned} \tag{16}
$$

As mentioned earlier, the number of variables of (16) is of $O(n^3)$. So, one may wonder what the relationship between p_Q^1 and the optimal value of problem (16) is. The following theorem says these bounds are equivalent. Before we get to the proof, let us mention some points. It is shown in [6, 20], the symmetric matrix $B \in \mathcal{P}^1$ if and only if there exist symmetric matrices $K^{(1)}, ..., K^{(n)}$ with

$$B - K^{(i)} \succeq 0, \qquad i = 1, ..., n \tag{17}$$

$$K_{ii}^{(i)} = 0, \qquad i = 1, ..., n \tag{18}$$

$$K_{ii}^{(j)} + 2K_{ij}^{(i)} = 0, \qquad i \neq j \tag{19}$$

$$K_{jk}^{(i)} + K_{ik}^{(j)} + K_{ij}^{(k)} \geq 0, \qquad i > j > k. \tag{20}$$

Let convex quadratic function $q(x) = x^T Sx + 2c^T x + c_0$ be nonnegative on Δ. It is easily seen that $\left(\sum_{i=1}^n x_i\right)^2 q\left((\sum_{i=1}^n x_i)^{-1} x\right)$ is homogeneous polynomial of degree two. So for some symmetric matrix Q, we have

$$\left(\sum_{i=1}^{n} x_i\right)^2 q\left((\sum_{i=1}^{n} x_i)^{-1} x\right) = x^T Qx.$$

As $q \in \mathcal{Q}_+^c(\Delta)$, there exist nonnegative multipliers λ_i $(i = 1, ..., n)$ and λ_{n+1} with

$$q(x) - \sum_{i=1}^{n} \lambda_i x_i + \lambda_{n+1}(e^T x - 1) \in \mathcal{Q}_+(\mathbb{R}^n).$$

By the replacement of x with $(\sum_{i=1}^n x_i)^{-1} x$ and multiplication of $(\sum_{i=1}^n x_i)^2$, it is readily seen that Q can be represented as a summation of a positive semidefinite matrix and a nonnegative matrix.

Theorem 2 *Bounds p_Q^1 and (16) are equivalent.*

Proof First, we show that the optimal value of (16) is less than or equal to p_Q^1. Let $\bar{\alpha}_i(x) = x^T S_i x + 2c_i^T x + g_i$, $i = 1, ..., n+1$, $\bar{\kappa}$ and $\bar{\ell}$ be an optimal solution of (16). (Without loss of generality, it is assumed (16) attains its optimal solution.) As $\bar{\kappa}$ is nonnegative on standard simplex, $(e^T x)^3 \bar{\kappa}((e^T x)^{-1} x)$ is homogeneous polynomial of degree three with nonnegative coefficients. Thus, for nonnegative symmetric matrix K_i, $i = 1, ..., n$, we have $(e^T x)^3 \bar{\kappa}((e^T x)^{-1} x) = \sum_{i=1}^n x_i (x^T K_i x)$. Furthermore, $(e^T x)^2 \bar{\alpha}_i((e^T x)^{-1} x) = x^T (L_i + M_i)x$, $i = 1, ..., n$, where L_i and M_i are non-negative and positive semidefinite, respectively. Therefore, we have

$$\sum_{i=1}^{n} x_i x^T (Q - \bar{\ell} ee^T - K_i - L_i - M_i - L_0 - M_0)x = 0,$$

where $L_0 \geq 0$ and $M_0 \succeq 0$. Due to the positive semidefiniteness of diagonal matrices with nonnegative elements and the convexity of nonnegative and positive semidefinite matrices, with a little algebra, we get symmetric matrices $\bar{K}_i$, $i = 1, ..., n$, which satisfy (17)–(20). Hence, $\bar{\ell} \leq p_Q^1$.

Now, we prove that p_Q^1 is less than or equal to the optimal value of (16). Similar to the former case, we assume that optimal solution is attained. Hence, there exist symmetric matrices K_i, $i = 1, ..., n$, which satisfy (17)–(20) for $B = Q - p_Q^1 ee^T$. Let $M_i = Q - p_Q^1 ee^T - K_i$, $i = 1, ..., n$, we have

$$(e^T x)\left(x^T(Q - p_Q^1 ee^T)x\right) - \sum_{i=1}^{n} x_i x^T M_i x \in \mathcal{C}^N(\mathbb{R}^n).$$

Hence,

$$x^T Qx - p_Q^1 - \sum_{i=1}^{n} x_i x^T M_i x + (e^T x - 1)\left(x^T(Q - p_Q^1 ee^T)x\right) \in \mathcal{C}^N(\mathbb{R}^n),$$

which completes the proof. □

The proof of Theorem 2 reveals that one can replace $\mathcal{Q}_+^c(\Delta)$ with the set of homogeneous convex quadratics in problem (16). Hence, problem (16) is equivalent to the following problem

$$\begin{aligned}
&\max \ \ell \\
&\text{s. t. } x^T Qx - \ell - \sum_{i=1}^{n}(x^T S_i x)x_i + \alpha_{n+1}(x)(e^T x - 1) - \kappa(x) \in \mathcal{Q}_+(\mathbb{R}^n), \\
&\quad S_i \succeq 0, \qquad i = 1, ..., n \\
&\quad \kappa \in \mathcal{C}^N(\mathbb{R}^n).
\end{aligned}$$

We conclude the section by noting that bound (14) dominates semidefinite relaxations obtained in the Lasserre Hierarchy with the same order of variables for QCQPs. Strictly speaking, as bound (14) applies $\mathcal{Q}_+^c(\mathcal{Z})$ instead of $\mathcal{Q}_+^c(\mathbb{R}^n)$, it can generate tighter bounds in comparison with the Lasserre hierarchy with the same order of variables. Moreover, bound (14) and RLT-level 2 are not necessarily relevant.

4 Computational results

As mentioned above, two important factors which determine the efficiency of a given bound are the tightness of the generated bound and its computational time. In the section, we compare quadratic bounds (1) and (12) and cubic bound (14). To this end, we illustrate numerical performance of the above-mentioned bounds on three groups of test problems. The codes and the test problems are publicly available at https://github.com/molsemzamani/QCQP.

We implemented the bounds using MATLAB 2020b, and the computations were run on a laptop computer with Intel Core i7 CPU, 1.8 GHz, and 16 GB of RAM. To solve semidefinite programs, we employed MOSEK in Matlab environment [18]. We applied YALMIP to pass bounds (1), (12) and (14) to MOSEK [14]. To obtain the optimal value of (QCQP), we modeled the problem via AMPL [9] and we employed BARON 21.1.13 [23] as a solver.

Table 1 First group of instances with 20 decision variables

Bound (1)			Bound (12)			Bound (14)		
μ	σ	M	μ	σ	M	μ	σ	M
0.0332	0.0401	0.1509	0.0327	0.0398	0.1501	0.0002	0.0007	0.0036

Table 2 Second group of instances with 25 decision variables

Bound (1)			Bound (12)			Bound (14)		
μ	σ	M	μ	σ	M	μ	σ	M
0.0260	0.0316	0.1321	0.0256	0.0311	0.1318	0.0001	0.0006	0.0042

Table 3 Second group of instances with 30 decision variables

Bound (1)			Bound (12)			Bound (14)		
μ	σ	M	μ	σ	M	μ	σ	M
0.0159	0.0211	0.1083	0.0157	0.0207	0.1081	0.0005	0.0023	0.0149

The test problems used in the section were randomly generated. Indeed, matrices A, Q_i's and vector c_i's were generated by MATLAB's function randn. The randn function generates a sample of a Gaussian random variable with mean 0 and standard deviation 1. To ensure feasibility, we set $d = A(\frac{1}{2}e)$ and $b_i = \frac{1}{4}e^T Q_i e + c_i^T e + 0.1$, $i = 1..., m$. In addition, to generate a convex constraint, we replaced non-positive definite matrix Q_i with $Q_i - (\lambda_{\min}(Q_i) - 0.1)I$, where $\lambda_{\min}(Q_i)$ denotes the smallest eigenvalue of Q_i.

We generated three groups of instances. Each group included fifty problems with the same number of variables in the form of (QCQP). To measure the accuracy of a lower bound, similar to [19], we use the relaxation gap

$$R_G = \frac{f_{opt} - \ell_b}{\max(|\ell_b|, 10^{-3})} \times 100,$$

where ℓ_b is the generated lower bound and f_{opt} is the best upper bound provided by BARON with the relaxation gap 0.02 and the maximum running time of 1000 seconds.

For the first group of the instances, we generated QCQPs with 20 decision variables, three non-convex quadratic constraints, one convex quadratic constraint and two linear equality constraints. The performance of the bounds are summarized in Table 1. In Tables 1, 2 and 3 for 50 instances in terms of the relaxation gap, μ and σ denote mean and standard deviation, respectively, and M stands for the maximum relaxation gap.

The second group of the instances included 50 QCQPs with 25 decision variables, three non-convex quadratic constraints, two convex quadratic constraints and two linear equality constraints. Table 2 reports computational performances.

We considered 50 QCQPs with 30 decision variables, four non-convex quadratic constraints, two convex quadratic constraints and two linear equality constraints for the last group of instances. The performance of the bounds are presented in Table 3.

Table 4 reports computational time and exactness. In this table, T denotes the average running time for fifty instances and E denotes the number of instances out of fifty for which

Table 4 Computational time and exactness

Test problems	Bound (1)		Bound (12)		Bound (14)		T_{BARON}
	T	E	T	E	T	E	
First group	0.12	10	1.51	10	4.23	24	218.1
Second group	0.30	8	4.12	8	18.22	22	843.9
Third group	0.54	8	11.83	9	72.09	22	1000

the relaxation gap is less than 10^{-4}. The last column, T_{BARON}, denotes the average running time of BARON with the relaxation gap 0.02 and maximum running time 1000 seconds.

As seen from the tables, all bounds in question outperform lower bounds generated by BARON with respect to both time and accuracy. In addition, the improvement of bound (12) in comparison with (1) is not considerable. Indeed, both method generated almost near lower bounds while the computation time for bound (1) is a small fraction of that of bound (12).

The running time of bound (14) is considerably larger than that of bound (1) and (12). Nevertheless, bound (14) could generate a tight lower bound for all instances. This point may be of interest as solvers spend considerable amount of time for improving the lower bound. Hence, incorporating the following quadratic constraint

$$x^T Q_0 x + 2c_0^T x \geq \ell_c,$$

may reduce running time (ℓ_c denotes the generated bound by problem (14)).

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