Journal of Global Optimization (2019) 75:131–141
https://doi.org/10.1007/s10898-019-00810-0

Weak minimal elements and weak minimal solutions of a nonconvex set-valued optimization problem

M. Chinaie[1] · F. Fakhar[1] · M. Fakhar[1,3] · H. R. Hajisharifi[2,3]

Received: 1 January 2019 / Accepted: 11 July 2019 / Published online: 18 July 2019

Abstract

In this paper, we characterize the nonemptiness of the set of weak minimal elements for a nonempty subset of a linear space. Moreover, we obtain some existence results for a nonconvex set-valued optimization problem under weaker topological conditions.

Keywords Algebraic interior · Linear space · Set-valued optimization · Vector closure

1 Introduction and preliminaries

Let Y be a real linear space ordered by a convex cone $C \subseteq Y$ which is assumed to be proper; i.e., $\{0\} \neq C \neq Y$. Let K be a nonempty set and $F : K \rightrightarrows Y$ be a set-valued mapping with nonempty values. A general form of set-valued optimization problem is usually defined as follows:

$$\text{(SOP)} \quad \min F(x) \quad \text{subject to} \quad x \in K.$$

There are two approaches to defining the solutions of this problem: the vector approach [3,14,17] and the set approach [15,16]. In the vector approach, $\bar{x} \in K$ is a solution of the problem (SOP), whenever $F(\bar{x})$ contains a weak minimal element or minimal element of $F(K) = \cup_{x \in K} F(x)$. In the set approach, it is necessary to introduce an ordering for sets and find a minimal element of subset $\{F(x) : x \in K\}$ of $P(Y)$, where $P(Y)$ is the set of

✉ M. Fakhar
fakhar@sci.ui.ac.ir

M. Chinaie
m.chinaei@yahoo.com

F. Fakhar
f.fakhar66@gmail.com

H. R. Hajisharifi
h.hajisharifi@yahoo.com

1 Department of Mathematics, University of Isfahan, Isfahan 81745-163, Iran

2 Department of Mathematics, University of Khansar, P. O. Box 87916854163, Khansar, Iran

3 School of Mathematics, Institute for Research in Fundamental Sciences, P. O. Box 19395-5746, Tehran, Iran

all nonempty subsets of Y. Moreover, in most articles, Y endowed with a vector topology in both approaches. In this case, the closedness condition and the existence of interior points of C are assumed. In recent years, many authors studied vector optimization and set-valued optimization problems in the case where the image space has no topology; see [1,2,9,10,14, 21]. Most of such results are established by convexity assumption on the image of set-valued objective mappings. Therefore, some algebraic counterparts of the usual topological tools was introduced such as vectorial closedness, algebraic interior and relative algebraic interior; see [1,2,9,10,13,15,21] and the references therein. However, these notions are weaker than their topological counterparts, whenever the linear space is equipped with a vector topology.

A useful technique for investigating a set-valued optimization problem is to convert it into appropriate scalar problem by scalarization functionals. The Gerstewitz scalarization functional plays an important role in nonconvex vector optimization problems. In the literature, this functional has been generalized in order to study of set-valued optimization problems when the image space is a topological vector space; see [5,7,8,12].

The aim of this paper is to investigate the existence of weak minimal solutions of nonconvex set-valued optimization problems under weaker topological conditions. For this purpose, we first study some connections between vector closure, q-vector closure and their topological counterparts. Then, some characterizations of the nonemptiness of the set of weak minimal elements for a nonempty subset of Y are presented. In the final section, by applying these results we obtain some existence results for weak minimal solutions of set-valued mappings in the vector approach.

In what follows, we give some notions, notation and results that will be used in this paper. We denote the set of real numbers (resp. nonnegative real numbers and nonnegative integers) by $\mathbb{R}$ (resp. $\mathbb{R}_+$ and $\mathbb{N}$). A subset C of Y is said to be a cone, whenever $tC \subseteq C$ for all $t \in \mathbb{R}_+$. If the cone C is convex, then it is called a convex cone. Clearly, the cone C is convex if and only if $C + C \subseteq C$. If Y endowed with a vector topology τ, it is denoted by (Y, τ). If A is a subset of (Y, τ), then $\text{int}(A)$ and $\text{cl}(A)$ stand for the topological interior and the topological closure of A, respectively. Recall that every linear space Y can be endowed with the strongest locally convex topology $\tau_c := \tau_{\mathcal{P}}$, where $\mathcal{P}$ is the family of all semi-norms defined on Y, $\tau_{\mathcal{P}}$ is called the core convex topology; see [15]. The interior and the closure of A in the core convex topology are denoted by $\text{int}_c(\text{A})$ and $\text{cl}_c(\text{A})$, respectively.

For more details on the following concepts and for some examples of these notions we can refer to [1,13–15,19].

Let A be a nonempty subset of Y.

- The set
$$\text{cor}(A) := \{\bar{x} \in A : \forall x \in Y, \exists \lambda > 0 \; s.t. \; \bar{x} + [0, \lambda]x \subseteq A\},$$
is called the algebraic interior of A. The set A is said to be algebraically open (resp. algebraically solid) if and only if $A = \text{cor}(A)$ (resp. $\text{cor}(A) \neq \emptyset$).
- If $q \in Y$, the set
$$\text{vcl}_q(A) := \{x \in Y : \forall \lambda > 0, \; \exists \lambda' \in [0, \lambda] \; s.t. \; x + \lambda' q \in A\},$$
is called q-vector closure of A. If $A = \text{vcl}_q(A)$, then we say that A is q-vectorially closed.
- The set
$$\text{vcl}(A) := \bigcup_{q \in Y} \text{vcl}_q(A),$$

is called the vector closure of A. The set A is called vectorially closed, whenever $A = \text{vcl(A)}$.

It is clear that $\text{vcl}_q(A) = \bigcap_{\lambda>0}(A - [0, \lambda]q)$, and so

$$\text{vcl}(A) = \bigcup_{q \in Y} \bigcap_{\lambda>0} (A - [0, \lambda]q).$$

Notice that if A is a nonempty subset of a topological vector space (Y, τ) and $q \in Y\backslash\{0\}$, then from proposition 1 of [1] we have that

$$A \subseteq \text{vcl}_q(A) \subseteq \text{vcl}(A) \subseteq \text{cl(A)}.$$

Moreover, $A = \text{vcl}_q(A)$, for $q = 0$.

We define a binary relation $\preceq$ on Y as follows:

$$x \preceq y \Leftrightarrow y - x \in C.$$

An element $\bar{y} \in A$ is called a minimal element of A, if

$$(\{\bar{y}\} - C) \cap A \subseteq \{\bar{y}\} + C.$$

Let C be algebraically solid. An element $\bar{y} \in A$ is said to be a weak minimal element of A, if

$$(\{\bar{y}\} - \text{cor}(C)) \cap A = \emptyset.$$

The set of minimal elements and weak minimal elements of the set A are denoted by $\text{Min}(A, C)$ and $\text{WMin}(A, C)$, respectively.

An element $\bar{x} \in K$ is a weak minimal solution of the problem (SOP), whenever there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y}$ is a weak minimal element of the set $F(K)$; see [14,15,17,22].

The following proposition will be used in the sequel. This result was proved in [1,2,14].

Proposition 1 *Let A be a nonempty subset of Y, and let C be algebraically solid. Then,*

$$\text{cor}(A + C) = A + \text{cor}(C).$$

2 Some properties of vectorially closed sets

In this section, we give some relationships among vector closure and q-vector closure of the Minkowski sum of a nonempty subset A of Y and C. We also show that some algebraic notions and their topological counterparts coincide under suitable conditions.

Proposition 2 *Let C be algebraically solid, and let A be a nonempty subset of Y. Then, $\text{vcl}_q(A + C) = \text{vcl}(A + C)$ for every $q \in \text{cor}(C)$.*

Proof Let $q \in \text{cor(C)}$. It is enough to show that $\text{vcl}(A+C) \subseteq \text{vcl}_q(A+C)$. If $x \in \text{vcl}(A+C)$, then there exist $z \in Y$ and $t_n \downarrow 0$ such that $x + t_n z \in A + C$. Since $q \in \text{cor}(C)$, there exists $\lambda > 0$ such that $q + [-\lambda, \lambda]z \subseteq C$. Since there exists $m \in \mathbb{N}$ such that $\frac{1}{m} \leq \lambda$, we conclude that

$$x + mt_n q = x + t_n z + mt_n q - t_n z = x + t_n z + mt_n \left(q - \frac{1}{m} z\right) \in A + C + C = A + C.$$

Hence, $x \in \text{vcl}_q(A + C)$ and so $\text{vcl}(A + C) \subseteq \text{vcl}_q(A + C)$. □

Proposition 3 *Let A be a nonempty subset of Y, and let $q \in -C$. Then,* $\mathrm{vcl}_q(A+C) = A+C$.

Proof Suppose that $y \in \mathrm{vcl}_q(A+C)$. Then for every $\lambda > 0$ there exists $0 \leq \lambda' \leq \lambda$ such that $y + \lambda' q \in A + C$. Therefore, $y \in A + C - \lambda' q \subseteq A + C$, and so the proof is completed. □

Notice that, if E is a convex subset of a topological vector space (Y, τ) and $\mathrm{int}(E) \neq \emptyset$, then $\mathrm{int}(E) = \mathrm{cor}(E)$; see [13,20]. From this fact and Proposition 2, we have the following result.

Theorem 1 *Let A be a nonempty subset of a topological vector space (Y, τ), and let $q_0 \in$* $\mathrm{int}(C)$. *Then, we have*

$$\mathrm{vcl}_{q_0}(A+C) = \mathrm{vcl}(A+C) = \mathrm{cl}(A+C).$$

Proof From Proposition 2 and the remainder before this theorem, we have $\mathrm{vcl}_{q_0}(A+C) = \mathrm{vcl}(A+C)$. Now, let $\mathcal{N}$ be a neighborhood base of 0 in Y. Since for $\delta > 0$, $C - [0, \delta]q_0 = C - \delta q_0$, then

$$\begin{aligned} \mathrm{vcl}(A+C) \subseteq \mathrm{cl}(A+C) &= \bigcap_{V \in \mathcal{N}} (A + C + V) \\ &\subseteq \bigcap_{q \in \mathrm{int}(C)} (A + C + C - q) \subseteq \bigcap_{\delta > 0} (A + C - \delta q_0) \\ &= \bigcap_{\delta > 0} (A + C - [0, \delta] q_0) = \mathrm{vcl}_{q_0}(A+C), \end{aligned}$$

and the conclusion follows. □

As a consequence of the above theorem we obtain the following result due to Qui and He [19, Proposition 2.3].

Corollary 1 *Suppose that D is a nonempty convex cone of a topological vector space (Y, τ). Then, D is closed if and only if D is q-vectorially closed for all $q \in \mathrm{int}(D)$.*

Proof In Theorem 1, we set $A := D$ and $C := D$. Since $A + C = D$, for each $q \in \mathrm{int}(D)$, we get $\mathrm{vcl}_q(D) = \mathrm{cl}(D)$, hence the conclusion holds. □

In [15] it was proved that $\mathrm{int}_c(E) = \mathrm{cor}(E)$ for all convex subsets E of Y. From this fact by considering $\tau := \tau_c$ in Theorem 1, we establish the following corollary.

Corollary 2 *Let A be a nonempty subset of Y, and let C be algebraically solid. Then,* $\mathrm{vcl}(A+C) = \mathrm{cl}_c(A+C)$.

The following examples show that in Theorem 1, condition $\mathrm{int}(C) \neq \emptyset$ is necessary.

Example 1 Let f be any discontinuous linear functional on a topological vector space (Y, τ). Let $A = \{0\}$, and let $C = \{x \in Y : f(x) = 0\}$. Then $\mathrm{int}(C) = \emptyset$ and $\mathrm{vcl}(A+C) = \mathrm{vcl}(C) = C$. Since f is discontinuous, C is dense in Y and therefore, $\mathrm{cl}(A+C) = \mathrm{cl}(C) = Y$.

Note that in Example 1, $\mathrm{int}(C) = \mathrm{cor}(C) = \emptyset$. But in the following example we show that Theorem 1 is not valid even if $\mathrm{int}(C) = \emptyset$ and $\mathrm{cor}(C) \neq \emptyset$. Therefore, in Theorem 1 we can not replace $q_0 \in \mathrm{int}(C)$ with $q_0 \in \mathrm{cor}(C)$.

Example 2 Let $Y = \ell_\infty$ be equipped with the weak* topology $\sigma(\ell_\infty, \ell_1)$, and let S_{ℓ_∞} be the unit sphere of ℓ_∞. Assume that

$$C = \ell_\infty^+ := \{x = (x_n) \in \ell_\infty : x_n \geq 0, \forall n \in \mathbb{N}\}, \quad \text{and} \quad A = S_{\ell_\infty} \cap \ell_\infty^+.$$

We show that

(i) $\text{int}_{w^*}(C) = \emptyset$, where $\text{int}_{w^*}(C)$ denotes the interior of C with respect to the weak* topology.
(ii) $\text{cor}(C) \neq \emptyset$.
(iii) $\text{vcl}(A + C) = A + C$.
(iv) $0 \notin A + C$, $0 \in \text{cl}_{w^*}(A + C)$, and so $\text{cl}_{w^*}(A + C) \neq A + C$, where $\text{cl}_{w^*}(A + C)$ denotes the closure of $A + C$ with respect to the weak* topology.

Now, we prove the above assertions.

Suppose that $\text{int}_{w^*}(C) \neq \emptyset$ and $a = (a_n) \in \text{int}_{w^*}(C)$. Therefore, there exists a w^*-neighborhood U of 0 such that $a + U \subseteq C$. We can assume that $U = \{y \in \ell_\infty : |< x_i, y >| < \epsilon, i = 1, \ldots, n\}$, for some $x_1, \ldots, x_n \in \ell_1$ and $\epsilon > 0$. Hence, by considering each x_i as a functional on ℓ_∞ we have

$$a + \bigcap_{i=1}^{n} \ker(x_i) \subseteq a + U \subset C.$$

Since ℓ_∞ is an infinite dimensional space, then $\bigcap_{i=1}^{n} \ker(x_i)$ is a nontrivial subspace of ℓ_∞. Let $y = (y_j) \in \bigcap_{i=1}^{n} \ker(x_i)$ and $y \neq 0$, then there exists $k \in \mathbb{N}$ such that $y_k \neq 0$. Therefore, there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that

$$a_k + \lambda y_k < 0. \tag{1}$$

On the other hand, we have $\lambda y \in \bigcap_{i=1}^{n} \ker(x_i)$. Hence,

$$a + \lambda y \in a + U \subseteq C,$$

which contradicts (1). Therefore, (i) holds.

Taking τ the norm topology on ℓ_∞, we have

$$\text{int}_\tau C = \left\{x \in \ell_\infty : \inf_{n \in \mathbb{N}} x_n > 0\right\}.$$

Hence, $\text{cor}(C) = \text{int}_\tau(C) \neq \emptyset$ and so (ii) holds. Since $\text{cl}_\tau(A + C) = A + C$ and $A + C \subseteq \text{vcl}(A + C) \subseteq \text{cl}_\tau(A + C)$, then (iii) holds.

It is clear that $0 \notin A + C$. Moreover, if e_n is the sequence with a 1 in the nth coordinate and 0 in the other coordinates, then $e_n \in A + C$ and this sequence $\sigma(\ell_\infty, \ell_1)$-convergence to 0. Consequently, $0 \in \text{cl}_{w^*}(A + C)$, and therefore (iv) holds.

3 Weak minimal elements

Assume that C is algebraically solid and A is a nonempty subset of Y. In this section, we first give some equivalent conditions for the nonemptyness of the set $\text{WMin}(A, C)$. Then, by using these results some sufficient conditions for the existence of the weak minimal elements of A are given. Moreover, by using the nonconvex scalarization functional, we show that the set of weak minimal elements of A is nonempty under suitable conditions.

Theorem 2 *The following statements are equivalent:*

(i) *$A+C$ is not algebraically open;*
(ii) $\mathrm{WMin}(A, C)$ *is nonempty;*
(iii) $\mathrm{WMin}(A+C, C)$ *is nonempty.*

Proof We have $\mathrm{WMin}(A, C) = A\backslash(A+\mathrm{cor}(C))$ (see [15], Eq. (7.14) for $\varsigma = 0$). So,

$$\mathrm{WMin}(A+C, C) = (A+C)\backslash(A+\mathrm{cor}(C)) \supseteq \mathrm{WMin}(A, C).$$

Hence, $(ii) \Rightarrow (iii)$. On the other hand, if $\mathrm{WMin}(A, C) = \emptyset$, then $A \subseteq A+\mathrm{cor}(C)$. Hence,

$$A+C \subseteq A+\mathrm{cor}(C)+C = A+\mathrm{cor}(C),$$

and so $\mathrm{WMin}(A+C, C) = \emptyset$. Therefore, $(iii) \Rightarrow (ii)$.

Since by Proposition 1 $\mathrm{cor}(A+C) = A+\mathrm{cor}(C)$ and $A+\mathrm{cor}(C) \subseteq A+C$, we deduce that conditions (i) and (iii) are equivalent. □

Recall that the set A is called C-proper, if $A+C \neq Y$. In the following we give a characterization of C-properness.

Proposition 4 *The following statements are equivalent:*

(i) $A+\mathrm{cor}(C) = Y$*;*
(ii) *A is not C-proper;*
(iii) $\mathrm{vcl}_q(A+C) = Y$.

Proof $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. Now, assume that $\mathrm{vcl}_q(A+C) = Y$. Then, $A+C-[0,\delta]q = Y$ for all $\delta > 0$. Fix $v_0 \in \mathrm{cor}(C)$ and take $y \in Y$. Hence, there exists $\delta > 0$ such that $v_0+[-\delta,\delta]q \subseteq \mathrm{cor}(C)$. It follows that $y - v_0 \in A+C-[0,\delta]q$, and so $y = a+c+v_0-\lambda q$ for some $a \in A$, $c \in C$ and $\lambda \in [0,\delta]$. Thus, $y \in A+C+\mathrm{cor}(C) = A+\mathrm{cor}(C)$. Therefore, $Y \subseteq A+\mathrm{cor}(C)$, and so $(iii) \Rightarrow (i)$. □

As a consequence of Theorems 1, 2 and Proposition 4 we obtain the following corollary.

Corollary 3 *Let $A+C$ be vectorially closed. Then,* $\mathrm{WMin}(A, C)$ *is nonempty if and only if A is C-proper.*

Proof Let A be C-proper. By vectorially closedness of $A+C$ and Theorem 1, we have that $\mathrm{cl}_c(A+C) = \mathrm{vcl}(A+C) = A+C$. From Proposition 1, we have $\mathrm{cor}(A+C) = A+\mathrm{cor}(C)$, and so $\mathrm{int}_c(A+C) = A+\mathrm{cor}(C)$. Since A is C-proper and (Y, τ_c) is connected, we deduce that $A+C \neq A+\mathrm{cor}(C)$. Therefore, Theorem 2 follows that $\mathrm{WMin}(A, C)$ is nonempty.

Conversely, if $\mathrm{WMin}(A, C)$ is nonempty, then from Theorem 2, $A+C \neq A+\mathrm{cor}(C)$. Hence, by Proposition 4, $A+C \neq Y$. □

Remark 1 In the case where (Y, τ) is a topological vector space, in view of Example 2, the vectorially closedness of $A+C$ is weaker assumption than the closedness of $A+C$.

For a nonempty convex subset E of Y, an element a of E is an extreme point of E if there is no proper open line segment which contains a and lies entirely in E; see [4].

Corollary 4 *If $A+C$ is convex and the set of extreme points of $A+C$ is nonempty, then* $\mathrm{Min}(A, C)$ *is nonempty, and furthermore* $\mathrm{WMin}(A, C) \neq \emptyset$.

Proof We show that every extreme point of $A + C$ is a minimal element of A. Suppose on the contrary that there is an extreme point $\bar{y}$ of $A + C$ and it is not a minimal element of A. Therefore,

$$(\{\bar{y}\} - (C\backslash - C)) \cap A \neq \emptyset.$$

Hence, there exists $x \in A$ such that $x - \bar{y} \in -(C\backslash - C)$, and so $\bar{y} \in x + (C\backslash - C)$. Therefore, it can be written as $\bar{y} = x + c$, where $x \in A$ and $c \in C\backslash\{0\}$. Thus, $\bar{y} = \frac{1}{2}(x + 2c) + \frac{1}{2}x$, where $x, x + 2c \in A + C$. It follows that $\bar{y}$ is not an extreme point of $A + C$ which is a contradiction. □

In the following example we show that conditions in Corollaries 3 and 4 are not superfluous.

Example 3 Let $Y = \mathbb{R}^2$, let $C = \mathbb{R}^2_+$, and let $A = \{(x, y) \in \mathbb{R}^2 : x > 0, y \leq \frac{1}{x}\}$. Then, C is an algebraically solid convex cone and A is C-proper. However, $A + C$ is not vectorially closed and it is convex with no extreme point. It is clear that $\mathrm{WMin}(A, C) = \emptyset$. Hence, vectorially closedness of $A + C$ in Corollary 3 or the existence of extreme point for $A + C$ in Corollary 4 are not superfluous conditions.

Let $q \in Y\backslash\{0\}$, and let $\emptyset \neq E \subseteq Y$. The nonconvex separation functional (or the Gerstewitz function) $\phi^q_E : Y \to \mathbb{R} \cup \{\pm\infty\}$ which was introduced in [6,17,18] defined by E and q is

$$\phi^q_E(y) = \begin{cases} +\infty, & y \notin \mathbb{R}q - E, \\ \inf\{t \in \mathbb{R} : \quad y \in tq - E\}, & otherwise. \end{cases}$$

For the main properties of this functional one can refer to [3,9–11,15,17] and the references therein.

According to Theorem 4 of [9], Lemma 2.3 and Remark 2.4 of [10], and Theorem 2.1 of [21], we collect some properties of the nonconvex separation functional which will be used in what follows.

We set $\varphi^q_E(y) := \phi^{-q}_{-E}(y)$, for each nonempty subset E of Y, $q \in Y\backslash\{0\}$ and $y \in Y$. If $\Re \in \{\leq, <, =, \geq, >\}$ and $r \in \mathbb{R}$, then we set

$$L\left(\varphi^q_E, r, \Re\right) := \left\{z \in Y : \varphi^q_E(z)\Re r\right\}.$$

Note that $L(\varphi^q_E, r, \Re) = L(\varphi^q_E, 0, \Re) - rq$ for every $\Re \in \{\leq, <, =, \geq, >\}$, and for each $r \in \mathbb{R}$. Recall that the set E is called free disposal with respect to C, if $E + C = E$.

Proposition 5 [10] *Let E be a nonempty subset of Y, and let $q \in cor(C)$. If E is free disposal with respect to C, then*

(i) $L(\varphi^q_E, 0, \leq) = \mathrm{vcl}_q(E)$;
(ii) $L(\varphi^q_E, 0, <) = \mathrm{cor}(E)$;
(iii) $L(\varphi^q_E, 0, =) = \mathrm{vcl}_q(E)\backslash\mathrm{cor}(E)$.

Recession cone of A is denoted by $O^+A := \{u \in Y : a + tu \in A, \forall a \in A, \forall t \geq 0\}$. In the following, we express some properties.

Remark 2 It is easy to check that:

(i) if C is a convex cone, then $O^+(C) = C$;
(ii) if A is a free disposal set with respect to C, then $C \subseteq O^+(A)$.

Lemma 1 *Let A be C-proper, and let $q \in cor(C)$. Then, φ^q_{A+C} has finite values.*

Proof Since $A + C$ is free disposal, by Remark 2(ii) we have

$$q \in \mathrm{cor}(C) \subseteq \mathrm{cor}(O^+(A + C)).$$

On the other hand, A is C-proper, therefore by Theorem 2.2 of [21], φ^q_{A+C} has finite values. □

Remark 3 If A is C-proper and $q \in \mathrm{cor}(C)$, then by Lemma 1 φ^q_{A+C} has finite values. Hence, by Theorem 5.2.3(b)–(c) of [15] we establish that

$$y + \mathbb{R}q \nsubseteq A + C, \quad \forall y \in Y.$$

Proposition 6 *Let $A+C$ be vectorially closed, and let there exist $q \in \mathrm{cor}(C)$ and $\bar{y} \in Y$ such that $\varphi^q_{A+C}(\bar{y}) = 0$. Then, $\mathrm{WMin}(A, C) \neq \emptyset$. Furthermore, there exists $y_1 \in \mathrm{WMin}(A, C)$ such that $y_1 \preceq \bar{y}$.*

Proof From Proposition 1, we have $\mathrm{cor}(A + C) = A + \mathrm{cor}(C)$. Consequently, Proposition 5(iii) follows that

$$\{y : \varphi^q_{A+C}(y) = 0\} = \mathrm{vcl}_q(A + C)\backslash(A + \mathrm{cor}(C)).$$

Since $A + C$ is vectorially closed, by Proposition 2, it is q-vectorially closed. So,

$$\bar{y} \in (A + C)\backslash(A + \mathrm{cor}(C)). \tag{2}$$

Hence, $A + C \neq A + \mathrm{cor}(C)$ and so by Theorem 2, $\mathrm{WMin}(A, C) \neq \emptyset$. Furthermore, by (2) there exists $y_1 \in A$ such that $\bar{y} \in y_1 + C$, and $y_1 \notin A + \mathrm{cor}(C)$, because if $y_1 \in A + \mathrm{cor}(C)$, then

$$\bar{y} \in y_1 + C \subseteq A + \mathrm{cor}(C),$$

which contradicts (2). Therefore, $\bar{y} - y_1 \in C$ and $(y_1 - \mathrm{cor}(C)) \cap A = \emptyset$. Hence, $y_1 \preceq \bar{y}$ and $y_1 \in \mathrm{WMin}(A, C)$. □

Proposition 7 *Let A be C-proper. Then, for each $q \in \mathrm{cor}(C)$ there exists $\bar{z} \in Y$ such that $\varphi^q_{A+C}(\bar{z}) = 0$.*

Proof By Lemma 1, $\varphi^q_{A+C}(y)$ is finite for every $y \in Y$. If we set $\bar{z} = y + \varphi^q_{A+C}(y)q$, then $\varphi^q_{A+C}(\bar{z}) = 0$. □

Now, by Corollary 3, Propositions 6 and 7, we present some conditions equivalent to the existence of weak minimal elements.

Corollary 5 *If $A + C$ is vectorially closed, then the following conditions are equivalent.*

(i) *A is C-proper.*
(ii) *For each $q \in \mathrm{cor}(C)$, there exists $\bar{z} \in Y$ such that $\varphi^q_{A+C}(\bar{z}) = 0$.*
(iii) *$\mathrm{WMin}(A, C)$ is nonempty.*

4 Weak minimal solutions of set-valued mappings

In this section, by using the results of Sect. 3, we obtain some existence results relative to weak minimal solutions of set-valued mappings. In what follows, let (X, τ_1) and (Y, τ_2) be two topological vector spaces, and let C be an algebraically solid convex cone in Y.

A set-valued mapping $F : X \rightrightarrows Y$ is said to be upper semicontinuous, whenever $\{x \in X : F(x) \subseteq U\}$ is open for any open subset U of Y; see [15]. It is well known that F is upper semicontinuous if and only if the set $\{x : F(x) \cap W \neq \emptyset\}$ is closed for every closed subset W of Y; see Proposition 3.1.5 of [15].

As a consequence of Corollary 3 we have the following theorem.

Theorem 3 *Let C be closed, let K be a nonempty compact subset of X, and let $F : K \rightrightarrows Y$ be upper semicontinuous. If $F(x) + C$ is vectorially closed for every $x \in K$, and $F(K)$ is C-proper, then the problem* (SOP) *has a weak minimal solution.*

Proof From Corollary 3, it is enough to show that $F(K) + C$ is vectorially closed. If $q \in \text{cor}(C)$ and $y \in \text{vcl}(F(K)+C)$, then by Proposition 2, $y \in \text{vcl}_q(F(K)+C)$. Thus, $y+\lambda q \in F(K) + C$, for every $\lambda > 0$. Therefore, for each $\lambda > 0$ there exists $x_\lambda \in K$ such that $y + \lambda q \in F(x_\lambda) + C$. If $\lambda_0 > 0$ and $\lambda < \lambda_0$, then

$$y + \lambda_0 q = y + \lambda q + (\lambda_0 - \lambda)q \in F(x_\lambda) + C.$$

Therefore, for $\lambda < \lambda_0$ we have

$$F(x_\lambda) \cap (y + \lambda_0 q - C) \neq \emptyset. \tag{3}$$

Since K is compact, without loss of generality we can assume that there exists $x_0 \in K$ such that $x_\lambda \to x_0$, as $\lambda \to 0$. Since F is upper semicontinuous, and $(y + \lambda_0 q - C)$ is closed, by (3) we conclude that

$$F(x_0) \cap (y + \lambda_0 q - C) \neq \emptyset, \quad \forall \lambda_0 > 0.$$

Therefore,

$$y + \lambda_0 q \in F(x_0) + C, \quad \forall \lambda_0 > 0.$$

Thus, $y \in \text{vcl}_q(F(x_0) + C)$. Since $F(x_0) + C$ is vectorially closed, $y \in F(x_0) + C$, and so $y \in F(K) + C$. Hence, $F(K) + C$ is vectorially closed. □

The following result shows that the image of a set-valued mapping can be C-proper under suitable conditions.

Proposition 8 *Let* $\text{int}(C) \neq \emptyset$*, and let K be a nonempty compact subset of X. If $F : K \rightrightarrows Y$ is upper semicontinuous and $F(x)$ is C-proper, for any $x \in K$, then $F(K)$ is C-proper.*

Proof If $q \in -\text{int}(C)$, then $\text{int}(C) + q$ is an open neighborhood of 0. Therefore, for every $x \in K$ there is an open neighborhood U_x of x in X such that $F(U_x \cap K) \subseteq F(x)+\text{int}(C)+q$ (note that F is upper semicontinuous). Since K is compact and $K = \bigcup_{x\in K}(U_x \cap K)$, there exists a finite subset $\{x_1, \ldots, x_n\}$ of K such that $K = \bigcup_{i=1}^{n}(U_{x_i} \cap K)$. Hence,

$$F(K) = \bigcup_{i=1}^{n} F(U_{x_i} \cap K) \subseteq \bigcup_{i=1}^{n} F(x_i) + C + q.$$

If $y \in Y$, then by Remark 3, $y + \mathbb{R}q \nsubseteq F(x_i) + C$, for each $i \in \{1, \ldots, n\}$. Thus, for each $i \in \{1, \ldots, n\}$ there exists $t_i \in \mathbb{R}$ such that $y+t_i q \notin F(x_i)+C$. Now, if $t_0 > \max\{t_1, \ldots, t_n\}$, then we show that $y + t_0 q \notin F(x_i) + C$ for each $i \in \{1, \ldots, n\}$. Assume on the contrary that there exists some $i \in \{1, \ldots, n\}$ such that $y + t_0 q \in F(x_i) + C$. So,

$$y + t_i q = y + t_0 q + (t_i - t_0)q \in F(x_i) + C + \text{int}(C) \subseteq F(x_i) + C,$$

which is a contradiction. Hence, $y + (t_0 + 1)q \notin F(x_i) + C + q$ for each $i \in \{1, \ldots, n\}$, and therefore $y + (t_0 + 1)q \notin \bigcup_{i=1}^{n} F(x_i) + C + q$. It follows that $y + (t_0 + 1)q \notin F(K) + C$, and so $F(K) + C \neq Y$. □

As a consequence of Theorem 3 and Proposition 8, we obtain the following corollary.

Corollary 6 *Let C be closed, and let $\text{int}(C) \neq \emptyset$. Assume that K is a nonempty compact subset of X, $F : K \rightrightarrows Y$ is upper semicontinuous, $F(x)$ is C-proper and $F(x) + C$ is vectorially closed for every $x \in K$. Then, the problem* (SOP) *has a weak minimal solution.*

Acknowledgements The authors would like to thank the associate editor and reviewers for their constructive comments, which helped us to improve the paper. We also thank Professor Nicolas Hadjisavvas and Professor Constantin Zalinescu who read our manuscript and provided us with valuable comments. The third and fourth authors were partially supported by a Grant from IPM (Nos. 96550414, 98460038).

References

1. Adán, M., Novo, V.: Weak efficiency in vector optimization using a closure of algebraic type under some cone-convexlikeness. Eur. J. Oper. Res. **149**, 641–653 (2003)
2. Adán, M., Novo, V.: Proper efficiency in vector optimization on real linear spaces. J. Optim. Theory Appl. **121**, 515–540 (2004)
3. Chen, G., Huang, X., Yang, X.: Vector Optimization. Springer, Berlin (2005)
4. Conway, J.: A Course in Functional Analysis. Springer, New York (1985)
5. Dhingra, M., Lalitha, C.S.: Approximate solutions and scalarization in set-valued optimization. Optimization **66**, 1793–1805 (2017)
6. Gerstewitz, Ch., Iwanow, E.: Dualität für nichtkonvexe Vektoroptimierungsprobleme. Wiss. Z. Tech. Hochsch. Ilmenau. **31**, 61–81 (1985)
7. Gutiérrez, C., Jiménez, B., Miglierina, E., Molho, E.: Scalarization in set optimization with solid and nonsolid ordering cones. J. Glob. Optim. **61**, 525–552 (2015)
8. Gutiérrez, C., Miglierina, E., Molho, E., Novo, V.: Pointwise well-posedness in set optimization with cone proper sets. Nonlinear Anal. **75**, 1822–1833 (2012)
9. Gutiérrez, C., Novo, V., Ródenas-Pedregosa, J.L., Tanaka, T.: Nonconvex separation functional in linear spaces with applications to vector equilibria. SIAM J. Optim. **26**, 2677–2695 (2016)
10. Gutiérrez, C., Kassay, G., Novo, V., Ródenas-Pedregosa, J.L.: Ekeland variational principles in vector equilibrium problems. SIAM J. Optim. **27**, 2405–2425 (2017)
11. Göpfert, A., Riahi, H., Tammer, Ch., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
12. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**, 1–18 (2007)
13. Holmes, R.B.: Geometric Functional Analysis and Its Applications. Springer, New York-Heidelberg (1975)
14. Jahn, J.: Vector Optimization: Theory, Applications and Extensions. Springer, Berlin (2011)
15. Khan, A.A., Tammer, C., Zălinescu, C.: Set-Valued Optimization: An Introduction with Applications. Vector Optimization. Springer, Heidelberg (2015)
16. Kuroiwa, D.: The natural criteria in set-valued optimization. Surikaisekikenkyusho Kokyuroku **1031**, 85–90 (1998)
17. Luc, D.T.: Theory of Vector Optimization. Springer, Berlin (1989)
18. Pascoletti, A., Serafini, P.: Scalarizing vector optimization problems. J. Optim. Theory Appl. **42**, 499–524 (1984)
19. Qiu, J.H., He, F.: A general vectorial Ekeland's variational principle with a p-distance. Acta Math. Sin. **29**, 1655–1678 (2013)
20. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970)

21. Weidner, P.: Gerstewitz functionals on linear spaces and functionals with uniform sublevel sets. J. Optim. Theory Appl. **173**, 812–827 (2017)
22. Zhang, W.Y., Li, S.J., Teo, K.L.: Well-posedness for set optimization problems. Nonlinear Anal. **71**, 3769–3778 (2009)