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ERRATUM

Erratum to: Multivariate McCormick relaxations

Jaromił Najman[1] · Dominik Bongartz[1] · Angelos Tsoukalas[2] · Alexander Mitsos[1]

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Abstract We provide a correction of the closed-form solutions for the multivariate McCormick relaxations of the binary product provided by Tsoukalas and Mitsos (JOGO, 59:633–662, 2014). The original closed-form solution may provide a function that is a non-convex relaxation or a convex function that is not a relaxation or a function that is neither convex nor a valid relaxation in some special cases. We prove the validity of the new closed-form solution.

In [1] Tsoukalas and Mitsos introduced the multivariate McCormick relaxations and in particular the multivariate McCormick relaxation of the binary product of functions. To provide a better overview, in the following we only consider the convex relaxation in detail. All results are analogously applicable to the concave relaxation for which we directly provide the closed-form solution. We adopt all assumptions made in [1]. The convex relaxation of $g(z) = mult(f_1(z), f_2(z)) \equiv f_1(z) f_2(z)$ with $f_i : Z \subset \mathbb{R}^n \to \mathbb{R}$ is given by:

$$\begin{aligned} g^{cv}(z) = \min_{x_i \in [f_i^L, f_i^U]} \; & \max\{H_1(\boldsymbol{x}), H_2(\boldsymbol{x})\} \\ \text{s.t. } & f_1^{cv}(z) \le x_1 \le f_1^{cc}(z) \\ & f_2^{cv}(z) \le x_2 \le f_2^{cc}(z) \end{aligned} \tag{1}$$

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✉ Alexander Mitsos
amitsos@alum.mit.edu

1 RWTH Aachen University, AVT - Aachener Verfahrenstechnik, Process Systems Engineering, Turmstrasse 46, 52056 Aachen, Germany

2 Olayan School of Business, American University of Beirut, Riad El-Solh Beirut 1107 2020, Lebanon

with

$$H_1(\mathbf{x}) = f_2^L x_1 + f_1^L x_2 - f_1^L f_2^L,\ H_2(\mathbf{x}) = f_2^U x_1 + f_1^U x_2 - f_1^U f_2^U.$$

The concave relaxation is given by:

$$\begin{aligned} g^{cc}(z) = \max_{x_i \in [f_i^L, f_i^U]} & \min\left\{ f_2^L x_1 + f_1^U x_2 - f_1^U f_2^L,\ f_2^U x_1 + f_1^L x_2 - f_1^L f_2^U \right\} \\ \text{s.t. } & f_1^{cv}(z) \le x_1 \le f_1^{cc}(z) \\ & f_2^{cv}(z) \le x_2 \le f_2^{cc}(z) \end{aligned} \tag{2}$$

where f_i^L, f_i^U denote bounds for f_i, i.e., $f_i^L \le f_i(z) \le f_i^U$ for all $z \in Z$ and f_i^{cv}, f_i^{cc} are convex and concave relaxations of f_i.

In [1] the authors also provide a closed-form solution for relaxation (1). The closed form they provide for the convex relaxation is given by

$$g_{old}^{cv}(z) = \min \left\{ \begin{array}{l} \max \left\{ \begin{array}{l} f_2^U f_1^{cv}(z) + f_1^U mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cv}(z) + \zeta\right) - f_1^U f_2^U, \\ f_2^L f_1^{cv}(z) + f_1^L mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cv}(z) + \zeta\right) - f_1^L f_2^L \end{array} \right\}, \\ \max \left\{ \begin{array}{l} f_2^U f_1^{cc}(z) + f_1^U mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cc}(z) + \zeta\right) - f_1^U f_2^U, \\ f_2^L f_1^{cc}(z) + f_1^L mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cc}(z) + \zeta\right) - f_1^L f_2^L \end{array} \right\}, \\ \max \left\{ \begin{array}{l} f_2^U mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z) - \zeta}{\kappa}\right) + f_1^U f_2^{cv}(z) - f_1^U f_2^U, \\ f_2^L mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z) - \zeta}{\kappa}\right) + f_1^L f_2^{cv}(z) - f_1^L f_2^L \end{array} \right\}, \\ \max \left\{ \begin{array}{l} f_2^U mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cc}(z) - \zeta}{\kappa}\right) + f_1^U f_2^{cc}(z) - f_1^U f_2^U, \\ f_2^L mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cc}(z) - \zeta}{\kappa}\right) + f_1^L f_2^{cc}(z) - f_1^L f_2^L \end{array} \right\} \end{array} \right\} \tag{3}$$

with

$$\kappa = \frac{f_2^L - f_2^U}{f_1^U - f_1^L},\ \zeta = \frac{f_1^U f_2^U - f_1^L f_2^L}{f_1^U - f_1^L}.$$

In the following, we show that the closed-form solution (3) is not always correct. More specifically, in some special cases it is neither convex nor a valid relaxation, see Fig. 1, and sometimes it is convex but not a relaxation, see Fig. 2. We give a simple counter example for each case and discuss the issue. Subsequently, we provide a corrected closed-form solution for the multivariate McCormick binary product of functions.

Example 1 Consider $g(z) = mult(f_1(z), f_2(z))$ with $f_1(z) = (z+1)^2$ and $f_2(z) = (z-1)^6 + 1$ on $Z = [0, 1]$. We use exact bounds for f_1, f_2 given by $f_1^L = 1,\ f_1^U = 4,\ f_2^L = 1,\ f_2^U = 2$. We use envelopes for the relaxations of f_1, f_2 given by:

$$\begin{aligned} f_1^{cv}(z) &= (z+1)^2, & f_1^{cc}(z) &= 1 + 3z, \\ f_2^{cv}(z) &= (z-1)^6 + 1, & f_2^{cc}(z) &= 2 - z. \end{aligned}$$

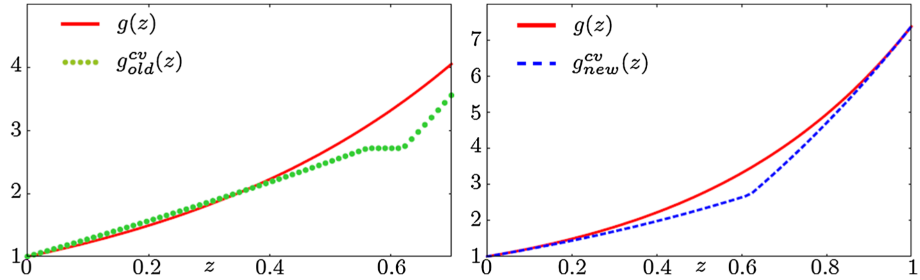

Fig. 1 The old wrong closed form g^{cv}_{old} (3) gives a function that is neither convex nor a valid relaxation. The new correct formula g^{cv}_{new} (5) provides the desired convex relaxation for $g(z) = \exp(z) \cdot \exp(z)$ on $Z = [0, 1]$. Note that g^{cv}_{old} is plotted only over $Z = [0, 0.7]$ to make the issues more visible

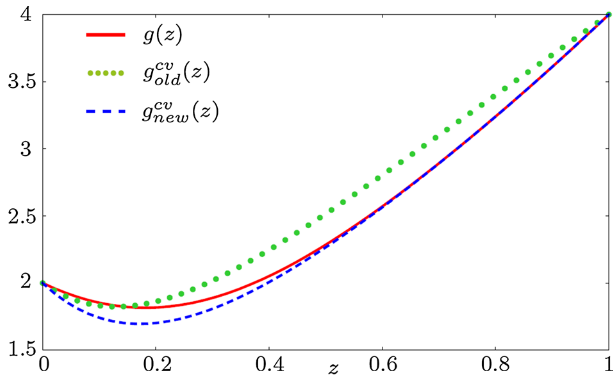

Fig. 2 The old wrong closed form g^{cv}_{old} (3) gives a convex but not valid relaxation, while the new correct formula g^{cv}_{new} (5) provides the convex relaxation for $g(z) = (z+1)^2\left((z-1)^6+1\right)$ on $Z = [0, 1]$

When we evaluate (3), we get $\kappa = -\frac{1}{3}$, $\zeta = \frac{7}{3}$ and see that $g^{cv}_{old}(z)$ is given by

$$\begin{aligned}
g^{cv}_{old}(z) &= \max\left\{\begin{array}{l} f_2^U mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z)-\zeta}{\kappa}\right) + f_1^U f_2^{cv}(z) - f_1^U f_2^U, \\ f_2^L mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z)-\zeta}{\kappa}\right) + f_1^L f_2^{cv}(z) - f_1^L f_2^L \end{array}\right\} \\
&= \max\left\{\begin{array}{l} 2mid\left(f_1^{cv}(z), f_1^{cc}(z), 7-3f_2^{cv}(z)\right) + 4f_2^{cv}(z) - 8, \\ mid\left(f_1^{cv}(z), f_1^{cc}(z), 7-3f_2^{cv}(z)\right) + f_2^{cv}(z) - 1 \end{array}\right\} \\
&= mid\left(f_1^{cv}(z), f_1^{cc}(z), 7-3f_2^{cv}(z)\right) + f_2^{cv}(z) - 1 \\
&= f_1^{cc}(z) + f_2^{cv}(z) - 1 \\
&= (z-1)^6 + 3z + 1
\end{aligned} \tag{4}$$

As can be seen in Fig. 2, the resulting function g^{cv}_{old} is convex but not a relaxation of g.

We now discuss what causes the mistake and give a correct closed-form solution for (1). The envelope of the binary product in Example 1 is constructed over the exact bounds of f_1, f_2 given by $X = \left[f_1^L, f_1^U\right] \times \left[f_2^L, f_2^U\right]$ and is strictly monotonically increasing over X with its minimum in the corner point $\left(f_1^L, f_1^U\right)$. The minimum of the envelope over the box given by $\left[f_1^{cv}(z), f_1^{cc}(z)\right] \times \left[f_2^{cv}(z), f_2^{cc}(z)\right]$ is then attained in the corner point $\left(f_1^{cv}(z), f_2^{cv}(z)\right)$.

Formula (3) falsely gives the corner point $\left(f_1^{cc}(z), f_2^{cv}(z)\right)$ as the solution of formulation (1) because it holds that

$$mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z)-\zeta}{\kappa}\right) = f_1^{cc}(z) \text{ over } Z=[0,1].$$

Note that similar examples can be constructed where (3) excludes the optimal corner $\left(f_1^{cc}(z), f_2^{cc}(z)\right)$ for the convex relaxation and the corners $\left(f_1^{cv}(z), f_2^{cc}(z)\right)$, $\left(f_1^{cc}(z), f_2^{cv}(z)\right)$ for the concave relaxation. In the proof of Lemma 1, it becomes clear why the corners $\left(f_1^{cv}(z), f_2^{cc}(z)\right)$, $\left(f_1^{cc}(z), f_2^{cv}(z)\right)$ cannot be excluded by the $mid(\ldots)$ term when computing the convex relaxation. The same argumentation applies to the corners $\left(f_1^{cv}(z), f_2^{cv}(z)\right)$, $\left(f_1^{cc}(z), f_2^{cc}(z)\right)$ when regarding the concave relaxation.

To avoid the exclusions described above, we correct the current closed-form solutions by explicitly adding the two corners that can be excluded by the $mid(\ldots)$ terms in the case of a monotonic envelope of the binary product. We add $\left(f_1^{cv}(z), f_2^{cv}(z)\right)$, $\left(f_1^{cc}(z), f_2^{cc}(z)\right)$ to the closed form of the multivariate convex relaxation and we add the corners $\left(f_1^{cv}(z), f_2^{cc}(z)\right)$, $\left(f_1^{cc}(z), f_2^{cv}(z)\right)$ to the closed form of the multivariate concave relaxation. The new closed-form solutions for the multivariate McCormick relaxations of the binary product of functions are then given by

$$g_{new}^{cv}(z) = \min\left\{\begin{array}{l}
\max\left\{\begin{array}{l} f_2^U f_1^{cv}(z) + f_1^U mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cv}(z)+\zeta\right) - f_1^U f_2^U, \\ f_2^L f_1^{cv}(z) + f_1^L mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cv}(z)+\zeta\right) - f_1^L f_2^L \end{array}\right\}, \\
\max\left\{\begin{array}{l} f_2^U f_1^{cc}(z) + f_1^U mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cc}(z)+\zeta\right) - f_1^U f_2^U, \\ f_2^L f_1^{cc}(z) + f_1^L mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cc}(z)+\zeta\right) - f_1^L f_2^L \end{array}\right\}, \\
\max\left\{\begin{array}{l} f_2^U mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z)-\zeta}{\kappa}\right) + f_1^U f_2^{cv}(z) - f_1^U f_2^U, \\ f_2^L mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z)-\zeta}{\kappa}\right) + f_1^L f_2^{cv}(z) - f_1^L f_2^L \end{array}\right\}, \\
\max\left\{\begin{array}{l} f_2^U mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cc}(z)-\zeta}{\kappa}\right) + f_1^U f_2^{cc}(z) - f_1^U f_2^U, \\ f_2^L mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cc}(z)-\zeta}{\kappa}\right) + f_1^L f_2^{cc}(z) - f_1^L f_2^L \end{array}\right\}, \\
\max\left\{\begin{array}{l} f_2^U f_1^{cv}(z) + f_1^U f_2^{cv}(z) - f_1^U f_2^U, \\ f_2^L f_1^{cv}(z) + f_1^L f_2^{cv}(z) - f_1^L f_2^L \end{array}\right\}, \\
\max\left\{\begin{array}{l} f_2^U f_1^{cc}(z) + f_1^U f_2^{cc}(z) - f_1^U f_2^U, \\ f_2^L f_1^{cc}(z) + f_1^L f_2^{cc}(z) - f_1^L f_2^L \end{array}\right\}
\end{array}\right\} \quad (5)$$

with

$$\kappa = \frac{f_2^L - f_2^U}{f_1^U - f_1^L}, \ \zeta = \frac{f_1^U f_2^U - f_1^L f_2^L}{f_1^U - f_1^L},$$

and

$$g_{new}^{cc}(z) = \max \left\{ \begin{array}{l} \min \left\{ \begin{array}{l} f_2^L f_1^{cv}(z) + f_1^U mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cv}(z) + \zeta\right) - f_1^U f_2^L, \\ f_2^U f_1^{cv}(z) + f_1^L mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cv}(z) + \zeta\right) - f_1^L f_2^U \end{array} \right\}, \\ \min \left\{ \begin{array}{l} f_2^L f_1^{cc}(z) + f_1^U mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cc}(z) + \zeta\right) - f_1^U f_2^L, \\ f_2^U f_1^{cc}(z) + f_1^L mid\left(f_2^{cv}(z), f_2^{cc}(z), \kappa f_1^{cc}(z) + \zeta\right) - f_1^L f_2^U \end{array} \right\}, \\ \min \left\{ \begin{array}{l} f_2^L mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z) - \zeta}{\kappa}\right) + f_1^U f_2^{cv}(z) - f_1^U f_2^L, \\ f_2^U mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cv}(z) - \zeta}{\kappa}\right) + f_1^L f_2^{cv}(z) - f_1^L f_2^U \end{array} \right\}, \\ \min \left\{ \begin{array}{l} f_2^L mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cc}(z) - \zeta}{\kappa}\right) + f_1^U f_2^{cc}(z) - f_1^U f_2^L, \\ f_2^U mid\left(f_1^{cv}(z), f_1^{cc}(z), \frac{f_2^{cc}(z) - \zeta}{\kappa}\right) + f_1^L f_2^{cc}(z) - f_1^L f_2^U \end{array} \right\}, \\ \min \left\{ \begin{array}{l} f_2^L f_1^{cv}(z) + f_1^U f_2^{cc}(z) - f_1^U f_2^L, \\ f_2^U f_1^{cv}(z) + f_1^L f_2^{cc}(z) - f_1^L f_2^U \end{array} \right\}, \\ \min \left\{ \begin{array}{l} f_2^L f_1^{cc}(z) + f_1^U f_2^{cv}(z) - f_1^U f_2^L, \\ f_2^U f_1^{cc}(z) + f_1^L f_2^{cv}(z) - f_1^L f_2^U \end{array} \right\} \end{array} \right\} \tag{6}$$

with

$$\kappa = \frac{f_2^U - f_2^L}{f_1^U - f_1^L}, \ \zeta = \frac{f_1^U f_2^L - f_1^L f_2^U}{f_1^U - f_1^L}.$$

Lemma 1 shows that formulas (5) and (6) are correct.

Lemma 1 *The closed-form solution for $g^{cv}(z)$ given by (1) is given by (5) and the closed-form solution for $g^{cc}(z)$ given by (2) is given by (6).*

Proof We prove validity of the convex formula (5). The proof for the validity of (6) is analogous.

Problem (1) minimizes $\max\{H_1(\boldsymbol{x}), H_2(\boldsymbol{x})\}$ over the two-dimensional box $\mathbb{B} = \left[f_1^{cv}(z), f_1^{cc}(z)\right] \times \left[f_2^{cv}(z), f_2^{cc}(z)\right]$. Let $\boldsymbol{x}^*$ denote an optimal solution point of problem (1), which by compactness exists. If it holds that $H_1(\boldsymbol{x}^*) \neq H_2(\boldsymbol{x}^*)$, then (1) is equivalent to minimizing the appropriate $H_j(\boldsymbol{x})$ over $\mathbb{B}$, where j is determined by

$$j \in \arg\max_{i \in \{1,2\}} H_i(\boldsymbol{x}^*). \tag{7}$$

With j determined by (7), problem (1) reduces to a linear program. Indeed, there exists a neighborhood of $\boldsymbol{x}^*$, $\mathcal{N}(\boldsymbol{x}^*, \varepsilon)$, such that $H_j(\boldsymbol{x}) = \max\{H_1(\boldsymbol{x}), H_2(\boldsymbol{x})\}$ for all $\boldsymbol{x} \in \mathcal{N}(\boldsymbol{x}^*, \varepsilon)$, yielding

$$H_j\left(\boldsymbol{x}^*\right) = \min_{\boldsymbol{x} \in \mathbb{B} \cap \mathcal{N}(\boldsymbol{x}^*, \varepsilon)} H_j(\boldsymbol{x}) = \min_{\boldsymbol{x} \in \mathbb{B}} H_j(\boldsymbol{x}),$$

where the second equality follows from convexity of H_j.

Therefore, a solution of (1) has to lie at one of the corners of $\mathbb{B}$, given by the set $\mathbb{B}^c = \{DL, UL, DR, UR\}$, with

$$DL = \left(f_1^{cv}(z), f_2^{cv}(z)\right), \ UL = \left(f_1^{cv}(z), f_2^{cc}(z)\right),$$
$$DR = \left(f_1^{cc}(z), f_2^{cv}(z)\right), UR = \left(f_1^{cc}(z), f_2^{cc}(z)\right).$$

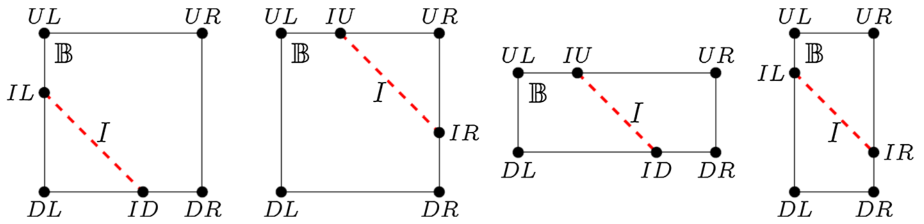

Fig. 3 Four possible cases of $\cap I \neq \emptyset$. The box $\mathbb{B}$ is given by the four corners $\{DL, UL, DR, UR\}$ and the two additional points emerge from the intersection with I. In each case the six corners are elements of $\hat{P} = \mathbb{B} \cap P$. I is represented by the *dashed line*

If, on the contrary, $H_1(\boldsymbol{x}^*) = H_2(\boldsymbol{x}^*)$, it follows that the intersection of $\mathbb{B}$ with the line $I = \{\boldsymbol{x} | x_2 = \kappa x_1 + \zeta\}$, with κ and ζ defined in (5), is non-empty and that (1) is equivalent to minimizing $H_1(\boldsymbol{x})$ over $\mathbb{B} \cap I$. Indeed,

$$\min_{\boldsymbol{x} \in \mathbb{B} \cap I} H_1(\boldsymbol{x}) \geq \min_{\boldsymbol{x} \in \mathbb{B}} H_1(\boldsymbol{x}) \geq H_1\left(\boldsymbol{x}^*\right)$$

and

$$\min_{\boldsymbol{x} \in \mathbb{B} \cap I} H_1(\boldsymbol{x}) \leq \min_{\boldsymbol{x} \in \mathbb{B} \cap I} \max\{H_1(\boldsymbol{x}), H_2(\boldsymbol{x})\} = H_1\left(\boldsymbol{x}^*\right),$$

yielding $\min_{\boldsymbol{x} \in \mathbb{B} \cap I} H_1(\boldsymbol{x}) = H_1(\boldsymbol{x}^*)$. $\mathbb{B} \cap I$ is the non-empty intersection of a line with a box and is either a point or a line segment. Therefore, also in this case, (1) is equivalent to a linear program with an optimal solution at the edge of the (potentially degenerate) intersection $\mathbb{B} \cap I$. The intersection of I with the lines defining the box $\mathbb{B}$, see Fig. 3, give the set $I^c = \{IL, IR, ID, IU\}$ of candidate points for an optimal solution, with

$$IL = \left(f_1^{cv}(z), \kappa f_1^{cv}(z) + \zeta\right),\ IR = \left(f_1^{cc}(z), \kappa f_1^{cc}(z) + \zeta\right),$$
$$ID = \left(\frac{f_2^{cv}(z) - \zeta}{\kappa}, f_2^{cv}(z)\right),\ IU = \left(\frac{f_2^{cc}(z) - \zeta}{\kappa}, f_2^{cc}(z)\right).$$

It follows that the union $P = \mathbb{B}^c \cup I^c$, always includes an optimal solution to problem (1), which can be reformulated as

$$\min_{\boldsymbol{x} \in \mathbb{B} \cap P} \max\{H_1(\boldsymbol{x}), H_2(\boldsymbol{x})\}.$$

By definition we have $\mathbb{B}^c \subset \mathbb{B}$. Furthermore, let $M(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \in \mathbb{R}^2$ be a mapping that maps three collinear points to the middle one, and let

$$\widehat{IL} = M(DL, UL, IL),\ \widehat{IR} = M(DR, UR, IR),$$
$$\widehat{ID} = M(DL, DR, ID),\ \widehat{IU} = M(UL, UR, IU).$$

Note that, although the domain of $M(\alpha_1, \alpha_2, \alpha_3)$ is $\mathbb{R}^2$, it can be expressed by the one dimensional $mid(\ldots)$ term, but we still introduce $M(\ldots)$ to avoid confusion. With $\hat{I}^c = \left\{\widehat{IL}, \widehat{IR}, \widehat{ID}, \widehat{IU}\right\}$ and $\hat{P} = \mathbb{B}^c \cup \hat{I}^c$, it is easy to see that $\hat{P} = \mathbb{B} \cap P$. Observe, for example, that $\widehat{IL} = IL$ if and only if $IL \in \mathbb{B}$; otherwise, it evaluates to DL or UL. Therefore (1) is further equivalent to

$$\min_{x \in \hat{P}} \max\{H_1(\boldsymbol{x}), H_2(\boldsymbol{x})\}.$$

This is a closed-form solution. It remains to show that we can drop UL and DR from $\hat{P}$, obtaining the proposed corrected formula (5), without affecting the result.

We show it for the corner UL, the proof for DR is analogous. We argue that UL is given either by M (*UL, UR, IU*) or M (*DL, UL, IL*). Assume to the contrary that IU is to the right of *UL* and *IL* is below *UL*. That is, assume that $\frac{f_2^{cc}(z)-\zeta}{\kappa} > f_1^{cv}(z)$ and $\kappa f_1^{cv}(z)+\zeta < f_2^{cc}(z)$. This would imply that IU is to the right and above IL and that the line I, passing from IU to IL has positive slope, contradicting $\kappa < 0$. □

Note that formula (3) in [1], in addition to UL and DR, also incorrectly dropped DL and UR.

Consider Example 1 again, with the correct closed form (5). We obtain the correct relaxation shown in Fig. 2.

Reference

1. Tsoukalas, A., Mitsos, A.: Multivariate McCormick relaxations. J. Global Optim. **59**, 633–662 (2014)