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C^∞-Regularization by Noise of Singular ODE's

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Abstract

In this paper we construct a new type of noise of fractional nature that has a strong regularizing effect on differential equations. We consider an equation driven by a highly irregular vector field and study the effect of this noise on such dynamical systems. We employ a new method to prove existence and uniqueness of global strong solutions, where classical methods fail because of the "roughness" and non-Markovianity of the driving process. In addition, we prove the rather remarkable property that such solutions are infinitely many times classically differentiable with respect to the initial condition in spite of the vector field being discontinuous. The technique used in this article corresponds, in a certain sense, to the Nash–Moser iterative scheme in combination with a new concept of "higher order averaging operators along highly fractal stochastic curves". This approach may provide a general principle for the study of regularization by noise effects in connection with important classes of partial differential equations.

Keywords Regularization by noise · Singular SDE's · Stochastic flows · Malliavin calculus · Compactness criterion

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1 Introduction

1.1 Background and Main Result

Consider the ordinary differential equation (ODE)

$$\frac{d}{dt}X_t^x = b(t, X_t^x), \quad X_0 = x, \quad 0 \le t \le T \tag{1}$$

for a vector field $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$.

It is well-known that the ODE (1) admits the existence of a unique solution X_t, $0 \le t \le T$, if b is a Lipschitz function of linear growth, uniformly in time. Further, if in addition $b \in C^k([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$, $k \ge 1$, then the flow associated with the ODE (1) inherits the regularity from the vector field, that is

$$(x \longmapsto X_t^x) \in C^k(\mathbb{R}^d; \mathbb{R}^d).$$

However, well-posedness of the ODE (1) in the sense of existence, uniqueness and the regularity of solutions or flow may fail, if the driving vector field b lacks regularity, that is if b e.g. is not Lipschitz or discontinuous.

In this article we aim at studying the restoration of well-posedness of the ODE (1) in the above sense by perturbing the equation via a specific noise process $\mathbb{B}_t$, $0 \le t \le T$, that is we are interested to analyze strong solutions to the following stochastic differential equation (SDE)

$$X_t^x = x + \int_0^t b(t, X_s^x)ds + \mathbb{B}_t, \quad 0 \le t \le T, \tag{2}$$

where the driving process $\mathbb{B}_t$, $0 \le t \le T$ is a stationary Gaussian process with non-Hölder continuous paths given by

$$\mathbb{B}_t = \sum_{n \ge 1} \lambda_n B_t^{H_n,n}. \tag{3}$$

Here $B_\cdot^{H_n,n}$, $n \ge 1$ are independent fractional Brownian motions in $\mathbb{R}^d$ with Hurst parameters $H_n \in (0, \frac{1}{2})$, $n \ge 1$ such that

$$H_n \searrow 0$$

for $n \longrightarrow \infty$. Further, $\sum_{n\ge1} |\lambda_n| < \infty$ for $\lambda_n \in \mathbb{R}$, $n \ge 1$.

We recall (for $d = 1$) that a fractional Brownian motion $B_\cdot^H$ with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process on some probability space with a covariance structure $R_H(t, s)$ given by

$$R_H(t, s) = E\left[B_t^H B_s^H\right] = \frac{1}{2}(s^{2H} + t^{2H} + |t - s|^{2H}), \quad t, s \ge 0.$$

In addition, $B_\cdot^H$ has a version with Hölder continuous paths with an exponent strictly smaller than H. The fractional Brownian motion coincides with the Brownian motion for $H = \frac{1}{2}$, but is neither a semimartingale nor a Markov process, if $H \ne \frac{1}{2}$. We also recall here that a fractional Brownian motion $B_\cdot^H$ has a representation in terms of a stochastic integral as

$$B_t^H = \int_0^t K_H(t, u)dW_u, \tag{4}$$

where $W_\cdot$ is a Wiener process and where $K_H(t,\cdot)$ is an integrable kernel. See Sect. 2 and e.g. [55] and the references therein for more information about fractional Brownian motion.

In fact, on the other hand, the SDE (2) can be also naturally recast for $Y_t^x := X_t^x - \mathbb{B}_t$ in terms of the ODE

$$Y_t^x = x + \int_0^t b^*(s, Y_s^x) ds, \tag{5}$$

where $b^*(t, y) := b(t, y + \mathbb{B}_t)$ is a "randomization" of the input vector field b.

Using Malliavin calculus combined with integration-by-parts techniques based on Fourier analysis, we want to show in this paper the existence of a unique global strong solution $X_\cdot^x$ to (2) with a stochastic flow which is *smooth*, that is

$$(x \longmapsto X_t^x) \in C^\infty(\mathbb{R}^d; \mathbb{R}^d) \quad \text{a.e. for all} \quad t, \tag{6}$$

when the driving vector field b is *singular*, that is more precisely, when

$$b \in \mathcal{L}_{2,p}^q := L^q([0,T]; L^p(\mathbb{R}^d; \mathbb{R}^d)) \cap L^1(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R}^d))$$

for $p, q \in (2, \infty]$. More precisely, the main result of our paper, whose proof will be given in Sect. 5, is the following:

Theorem 1.1 *Assume that the conditions for* $\lambda = \{\lambda_i\}_{i=1}^\infty$ *with respect to* $\mathbb{B}_\cdot = \mathbb{B}_\cdot^H$ *in Theorem 4.16 hold. Suppose that* $b \in \mathcal{L}_{2,p}^q$, $p, q \in (2, \infty]$. *Let* $U \subset \mathbb{R}^d$ *be an open and bounded set and* X_t, $0 \le t \le T$ *the solution of* (2). *Then for all* $t \in [0, T]$ *we have that*

$$X_t^\cdot \in \bigcap_{k \ge 1} \bigcap_{\alpha > 2} L^2(\Omega, W^{k,\alpha}(U)).$$

We think that the latter result is rather surprising since it seems to contradict the paradigm in the theory of (stochastic) dynamical systems that solutions to ODE's or SDE's inherit their regularity from the driving vector fields.

We also mention that Theorem 1.1 is the first result[1] in the literature on the C^∞-regularization by noise of singular ODEs in the sense of (6).

1.2 Possible Applications to PDEs and the Theory of Dynamical Systems

We expect that the regularizing effect of the noise in (2) will also pay off dividends in PDE theory and in the study of dynamical systems with respect to singular SDE's:

For example, if $X_\cdot^x$ is a solution to the ODE (1) on $[0, \infty)$, then $X : [0, \infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ may have the interpretation of a flow of a fluid with respect to the velocity field $u = b$ of an incompressible inviscid fluid, which is described by a solution to an incompressible Euler equation

$$u_t + (Du)u + \nabla P = 0, \;\; \nabla \cdot u = 0, \tag{7}$$

where $P : [0, \infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is the pressure field.

[1] In this context, we would like to point out that related results to ours appeared in the literature about 2 years after the completion of our paper. See Harang and Perkowski [34], which is partially based on our article, or Galeati and Gubinelli [27, 28], in which the authors also analyze path-by-path solutions for (distributional) vector fields in Besov–Hölder spaces and other types of regularizing noise. See also Galeati [26] for an overview.

Since solutions to (7) may be singular, a deeper analysis of the regularity of such solutions also necessitates the study of ODE's (1) with irregular vector fields. See the seminal works of Di Perna and Lions [22] and Ambrosio [3] in connection with the construction of (generalized) flows associated with singular ODE's.

In the context of stochastic regularization of the ODE (1) in the sense of (2), however, the obtained results in this article naturally give rise to the question, whether the constructed smooth stochastic flow in (6) may be used for the study of regular solutions of a stochastic version of the Euler equation (7).

Regarding applications to the theory of stochastic dynamical systems one may study the behaviour of orbits with respect to solutions to SDE's (2) with singular vector fields at sections on a 2-dimensional sphere (Theorem of Poincaré-Bendixson).

Another potential application pertains to a modified (stochastic) version of the Theorem of Kupka–Smale [62], which may lead to a corresponding result on the residuality of hyperbolic critical elements, for which pairs of critical elements have transversal invariant manifolds, in the case of singular SDE's on smooth compact manifolds. The proof of the classical Kupka–Smale theorem (see e.g. [56]) is based on the study of the properties of the differential of the smooth (time-homogeneous) vector fields. However, in the case of a singular SDE, the proof may rely on the investigation of the smooth stochastic flow (6) in Theorem 1.1.

Finally, we point out that our method for proving Theorem 1.1 allows for the to study of dynamical systems associated with (2) in a *deterministic* sense, that is in the path-by-path sense of Davie [17]. The latter means that one can find a measurable subset Ω^* (depending on the initial value x) in the space of continuous functions on $[0, T]$ with probability mass (i.e. Wiener measure mass) 1 such that for all $\omega \in \Omega^*$ the deterministic ODE

$$f(t, x) = x + \int_0^t b(t, f(s, x))ds + \mathbb{B}_t(\omega),\ 0 \leq t \leq T$$

has a unique solution $f(\cdot, x)$ in the space of continuous functions on $[0, T]$.

Moreover, when the vector field b in (2) is essentially bounded and integrable, one can also show that Ω^* can be chosen to be independent of the initial value x and that $f(t, \cdot) \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ for all t. See [6] in connection with Theorem 1.1.

1.3 Discussion of Previous Results and Our Approach

We mention that well-posedness in the sense of existence and uniqueness of strong solutions to (1) via regularization of noise was first found by Zvonkin [69] in the early 1970s in the one-dimensional case for a driving process given by the Brownian motion, when the vector field b is merely bounded and measurable. Subsequently the latter result, which can be considered a milestone in SDE theory, was extended to the multidimensional case by Veretennikov [65].

Other more recent results on this topic in the case of Brownian motion were e.g. obtained by Krylov and Röckner [37], where the authors established existence and uniqueness of strong solutions under some integrability conditions on b. See also the works of Gyöngy and Krylov [31] and Gyöngy and Martinez [32]. As for a generalization of the result of Zvonkin [69] to the case of stochastic evolution equations on a Hilbert space, we also mention the striking paper of Da Prato et al. [18], who constructed strong solutions for bounded and measurable drift coefficients by employing solutions of infinite-dimensional Kolmogorov equations in connection with a technique known as the "Itô–Tanaka–Zvonkin trick".

The common general approach used by the above mentioned authors for the construction of strong solutions is based on the so-called Yamada–Watanabe principle [67]: The authors prove the existence of a weak solution (by means of e.g. Skorokhod's or Girsanov's theorem) and combine it with the property of pathwise uniqueness of solutions, which is shown by using solutions to (parabolic) PDE's, to eventually obtain strong uniqueness. As for this approach in the case of certain classes of Lévy processes the reader may consult Priola [59] or Zhang [68] and the references therein.

Let us comment on here that the methods of the above authors, which are essentially limited to equations with Markovian noise, cannot be directly used in connection with our SDE (2). The reason for this is that the initial noise in (2) is not a Markov process. Furthermore, it is even not a semimartingale due to the properties of fractional Brownian motion.

In addition, we point out that our approach is diametrically opposed to the Yamada–Watanabe principle: We first construct a strong solution to (2) by using Mallliavin calculus. Then we verify uniqueness in law of solutions, which enables us to establish strong uniqueness, that is we use the following principle:

$$\boxed{\text{Strong existence}} + \boxed{\text{Uniqueness in law}} \Rightarrow \boxed{\text{Strong uniqueness}}.$$

Moreover, our approach for the construction of strong solutions of singular SDE's (2) in connection with smooth stochastic flows is not based on techniques from Markov or semimartingale theory as commonly used in the literature. In fact, our construction method has its roots in a series of papers [9, 46–48]. See also [33] in the case of SDE's driven by Lévy processes, [25, 49] regarding the study of singular stochastic partial differential equations or [8, 10] in the case of functional SDE's.

Finally, let us also mention some results in the literature on the existence and uniqueness of strong solutions of singular SDE's driven by a non-Markovian noise in the case of fractional Brownian motion:

The first results in this direction were obtained by Nualart and Ouknine [53, 54] for one-dimensional SDE's with additive noise. For example, using the comparison theorem, the authors in [53] are able to derive unique strong solutions to such equations for locally unbounded drift coefficients and Hurst parameters $H < \frac{1}{2}$.

More recently, Catellier and Gubinelli [15] developed a construction method for solutions of multidimensional singular SDE's with additive fractional noise and $H \in (0, 1)$ for vector fields b in the Besov–Hölder space $B^{\alpha+1}_{\infty,\infty}$, $\alpha \in \mathbb{R}$. Here the solutions obtained are even *path-by-path* in the sense of Davie [17], which is a stronger property than that of strong uniqueness, and the construction technique of the authors relies on the Leray–Schauder–Tychonoff fixed point theorem and a comparison principle based on an average translation operator. Further, the drift part of the SDE is given by a non-linear Young type of integral. See also Amine et al. [6], where the authors use Malliavin calculus to establish path-by-path solutions in the sense of Davie, when the drift part of the SDE is a classical Lebesgue integral and the vector field is essentially bounded and integrable. We remark that the approach in Catellier and Gubinelli [15] fails to work in the latter case, since non-linear Young type of integrals don't necessarily coincide with classical Lebesgue integrals, in general.

In this context let us also mention the recent works by Hu et al. [36], which deals with the study of the Brox diffusion, and Butkovski and Mytnik [14], where the authors obtain results on the regularization by (space time white) noise of solutions to a non-Lipschitz stochastic heat equation and the associated flow. Here path-by-path unique solutions in the sense of Davie [17] are proved.

Another recent result which is based on Malliavin techniques very similar to our paper can be found in Baños et al. [9]. Here the authors proved the existence of unique strong solutions for coefficients

$$b \in L^{1,\infty}_{\infty,\infty} := L^1(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d; L^\infty([0,T]; \mathbb{R}^d))$$

for sufficiently small $H \in (0, \frac{1}{2})$.

The approach in [9] is different from the above mentioned ones and the results for vector fields $b \in L^{1,\infty}_{\infty,\infty}$ are not in the scope of the techniques in [15]. See also [10] in the case of fractional noise driven SDE's with a distributional drift.

Let us now turn to results in the literature on the well-posedness of singular SDE's under the aspect of the regularity of stochastic flows:

If we assume that the vector field b in the ODE (1) is not smooth, but merely require that $b \in W^{1,p}$ and $\nabla \cdot b \in L^\infty$, then it was shown in [22] the existence of a unique generalized flow X associated with the ODE (1). See also [3] for a generalization of the latter result to the case of vector fields of bounded variation.

On the other hand, if b in ODE (1) is less regular than required in [3, 22], then a flow may even not exist in a generalized sense.

However, the situation changes, if we regularize the ODE (1) by an (additive) noise:

For example, if the driving noise in the SDE (2) is chosen to be a Brownian noise, or more precisely if we consider the SDE

$$dX_t = u(t, X_t)dt + dB_t, \quad s, t \geq 0, \quad X_s = x \in \mathbb{R}^d$$

with the associated stochastic flow $\varphi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$, the authors in [49] could prove for merely bounded and measurable vector fields b a regularizing effect of the Brownian motion on the ODE (1), that is they could show that $\varphi_{s,t}$ is a stochastic flow of Sobolev diffeomorphisms with

$$\varphi_{s,t}, \varphi_{s,t}^{-1} \in L^2(\Omega; W^{1,p}(\mathbb{R}^d; w))$$

for all s, t and $p \in (1, \infty)$, where $W^{1,p}(\mathbb{R}^d; w)$ is a weighted Sobolev space with weight function $w : \mathbb{R}^d \to [0, \infty)$. Further, as an application of the latter result, which rests on techniques similar to those used in this paper, the authors also study solutions of a singular stochastic transport equation with multiplicative noise of Stratonovich type.

Another work in this direction with applications to Navier–Stokes equations, which invokes similar techniques as introduced in [49], deals with globally integrable $u \in L^{r,q}$ for $r/d + 2/q < 1$ (r stands here for the spatial variable and q for the temporal variable). In this context, we also mention the paper [24], where the authors present an alternative method to the above mentioned ones based on solutions to backward Kolmogorov equations. See also [23]. We also refer to [59, 68] in the case of α-stable processes.

On the other hand if we consider a noise in the SDE (2), which is rougher than a Brownian motion with respect to the path properties and given by a fractional Brownian motion for small Hurst parameters, one can even observe a stronger regularization by noise effect on the ODE (1): For example, using Malliavin techniques very similar to those in our paper, the authors in [9] are able to show for vector fields $b \in L^{1,\infty}_{\infty,\infty}$ the existence of higher order Fréchet differentiable stochastic flows

$$(x \mapsto X_t^x) \in C^k(\mathbb{R}^d) \quad \text{a.e. for all} \quad t,$$

provided $H = H(k)$ is sufficiently small.

Another result related to the regularity of flows in connection with fractional Brownian motion can be found in Catellier and Gubinelli [15], where the authors under certain conditions obtain Lipschitz continuity of the associated stochastic flow for drift coefficients b in the Besov–Hölder space $B^{\alpha+1}_{\infty,\infty}$, $\alpha \in \mathbb{R}$.

The method we aim at employing in this paper for the construction of strong solutions rests on a compactness criterion for square integrable functionals of a cylindrical Brownian motion from Malliavin calculus, which is a generalization of that in [19], applied to solutions $X^{x,n}_{\cdot}$

$$dX_t^{x,n} = b_n(t, X_t^{x,n})dt + d\mathbb{B}_t, \quad X_0^{x,n} = x, \quad n \geq 1,$$

where $b_n, n \geq 0$ are smooth vector fields converging to $b \in \mathcal{L}^q_{2,p}$. Then using variational techniques based on Fourier analysis, we prove that X_t^x as a solution to (2) is the strong L^2-limit of $X_t^{x,n}$ for all t.

To be more specific (in the case of time-homogeneous vector fields), we "linearize" the problem of finding strong solutions by applying Malliavin derivatives D^i in the direction of Wiener processes W^i with respect to the corresponding representations of $B^{H_i,i}_{\cdot}$ in (4) in connection with (3) and get the linear equation

$$D_t^i X_u^{x,n} = \int_t^u b_n^{\prime}(X_s^{x,n}) D_t^i X_s^{x,n} ds + K_H(u,t) I_d, 0 \leq t < u, n \geq 1, \tag{8}$$

where $b_n^{\prime}$ denotes the spatial derivative of b_n, K_H the kernel in (4) and $I_d \in \mathbb{R}^{d\times d}$ the unit matrix. Picard iteration then yields

$$D_t^i X_u^{x,n} = K_H(u,t) I_d + \sum_{m\geq 1} \int_{t<s_1<\cdots<s_m<u} b_n^{\prime}(X_{s_m}^{x,n}) \dots b_n^{\prime}(X_{s_1}^{x,n}) K_H(s_1,t) I_d ds_1 \dots ds_m. \tag{9}$$

In a next step, in order to "get rid of" the derivatives of b_n in (9), we use Girsanov's change of measure in connection with the following "local time variational calculus" argument:

$$\begin{aligned} \int_{0<s_1<\cdots<s_n<t} \kappa(s) D^\alpha f(\mathbb{B}_s) ds &= \int_{\mathbb{R}^{dn}} D^\alpha f(z) L_\kappa^n(t,z) dz \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^{dn}} f(z) D^\alpha L_\kappa^n(t,z) dz, \end{aligned} \tag{10}$$

for $\mathbb{B}_s := (\mathbb{B}_{s_1}, \dots, \mathbb{B}_{s_n})$ and smooth functions $f : \mathbb{R}^{dn} \longrightarrow \mathbb{R}$ with compact support, where D^α stands for a partial derivative of order $|\alpha|$ for a multi-index α). Here, $L_\kappa^n(t,z)$ is a spatially differentiable local time of $\mathbb{B}_{\cdot}$ on a simplex scaled by a non-negative integrable function $\kappa(s) = \kappa_1(s) \dots \kappa_n(s)$.

Using the latter enables us to derive upper bounds based on Malliavin derivatives D^i of the solutions in terms of continuous functions of $\|b_n\|_{\mathcal{L}^q_{2,p}}$, which we can use in connection with a compactness criterion for square integrable functionals of a cylindrical Brownian motion to obtain the strong solution as a L^2-limit of approximating solutions.

Based on similar previous arguments we also verify that the flow associated with (2) for $b \in \mathcal{L}^q_{2,p}$ is smooth by using an estimate of the form

$$\sup_t \sup_{x\in U} E\left[\left\| \frac{\partial^k}{\partial x^k} X_t^{x,n} \right\|^\alpha\right] \leq C_{p,q,d,H,k,\alpha,T}\left(\|b_n\|_{\mathcal{L}^q_{2,p}}\right), n \geq 1$$

for arbitrary $k \geq 1$, where $C_{p,q,d,H,k,\alpha,T} : [0,\infty) \to [0,\infty)$ is a continuous function, depending on $p, q, d, H = \{H_n\}_{n\geq 1}, k, \alpha, T$ for $\alpha \geq 1$ and $U \subset \mathbb{R}^d$ a fixed bounded domain. See Theorem 5.1.

We also mention that the method used in this article significantly differs from that in [9] and related works, since the underlying noise of $\mathbb{B}_\cdot$ in (2) is of infinite-dimensional nature, that is a cylindrical Brownian motion. The latter however, requires in this paper the application of an infinite-dimensional version of the compactness criterion in [19] tailored to the driving noise $\mathbb{B}_\cdot$.

1.4 Discussion of Some General Principles Underlying Our Methodology

It is crucial to note here that the above technique explained in the case of perturbed ODE's of the form (2) reveals or strongly hints at a general principle, which could be used to study important classes of PDE's in connection with conservation laws or fluid dynamics. In fact, we believe that the following underlying principles may play a major role in the analysis of solutions to PDE's.

1.4.1 Nash–Moser Principle

The idea of this principle, which goes back to Nash [52] and Moser [50], can be (roughly) explained as follows:

Assume a function Φ of class C^k. Then the Nash–Moser technique pertains to the study of solutions u to the equation

$$\Phi(u) = \Phi(u_0) + f, \tag{11}$$

where $u_0 \in C^\infty$ is given and where f is a "small" perturbation.

In the setting of our paper, the latter equation corresponds to the SDE (2) with a (non-deterministic) perturbation given by $f = \mathbb{B}_\cdot$ (or $\varepsilon\mathbb{B}_\cdot$ for small $\varepsilon > 0$). Then, using this principle, the problem of studying solutions to (11) is "linearized" by analyzing solutions to the linear equation

$$\Phi'(u)v = g, \tag{12}$$

where Φ' stands for the Fréchet derivative of Φ. The study of the latter problem, however, usually comes along with a "loss of derivatives", which can be measured by "tame" estimates based on a (decreasing) family of Banach spaces E_s, $0 \le s < \infty$ with norms $|\cdot|_s$ such that $\cap_{s\ge 0} E_s = C^\infty$. Typically, $E_s = C^s$ (Hölder spaces) or $E_s = H^s$ (Sobolev spaces).

In our situation, Eq. (12) has its analogon in (8) with respect to the (stochastic Sobolev) derivative D^i (or the Fréchet derivative D in connection with flows).

Roughly speaking, in the case of Hölder spaces, assume that

$$\Phi'(u)\psi(u) = Id$$

for a linear mapping $\psi(u)$, which satisfies the "tame" estimate:

$$|\psi(u)g|_\alpha \le C(|g|_{\alpha+\lambda} + |g|_\lambda (1 + |u|_{\alpha+r}))$$

for numbers $\lambda, r \ge 0$ and $\alpha \ge 0$. In addition, require a similar estimate with respect to $\Phi''(u)$. Then, there exists in a certain neighbourhood W of the origin such that for $f \in W$ Eq. (11) has a solution $u(f) \in C^\alpha$. Solution here means that there exists a sequence u_j, $j \ge 1$ in C^∞ such that for all $\varepsilon > 0$, $u_j \longrightarrow u$ in $C^{\alpha-\varepsilon}$ and $\Phi(u_j) \longrightarrow \Phi(u_0) + f$ in $C^{\alpha+\lambda-\varepsilon}$ for $j \longrightarrow \infty$. The proof of the latter result rests on a Newton approximation scheme and results from Littlewood-Paley theory. See also [1] and the references therein.

1.4.2 Signature of Higher Order Averaging Operators Along a Highly Fractal Stochastic Curve

In fact another, but to the best of our knowledge new principle, which comes into play in connection with our technique for the study of perturbed ODE's, is the "extraction" of information from "signatures" of *higher order averaging operators* along a highly irregular or fractal stochastic curve $\gamma_t = \mathbb{B}_t$ of the form

$$\begin{aligned}
&(T_t^{0,\gamma,l_1,\dots,l_k}(b)(x), T_t^{1,\gamma,l_1,\dots,l_k}(b)(x), T_t^{2,\gamma,l_1,\dots,l_k}(b)(x), \dots) \\
&\quad = (I_d, \int_{\mathbb{R}^d} b(x^{(1)} + z_1)\Gamma_\kappa^{1,l_1,\dots,l_k}(z_1)dz_1, \\
&\qquad \int_{\mathbb{R}^{2d}} b^{\otimes 2}(x^{(2)} + z_2)\Gamma_\kappa^{2,l_1,\dots,l_k}(z_2)dz_2, \int_{\mathbb{R}^{3d}} b^{\otimes 3}(x^{(3)} + z_3)\Gamma_\kappa^{3,l_1,\dots,l_k}(z_3)dz_3, \dots) \\
&\qquad \in \mathbb{R}^{d\times d} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \cdots
\end{aligned} \tag{13}$$

where $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a "rough", that is a merely (locally integrable) Borel measurable vector field and

$$\Gamma_\kappa^{n,l_1,\dots,l_k}(z_n) = (D^{\alpha^{j_1,\dots,j_{n-1},j,l_1,\dots,l_k}} L_\kappa^n(t, z_n))_{1\le j_1,\dots,j_{n-1},j\le d}$$

for multi-indices $\alpha^{j_1,\dots,j_{n-1},j,l_1,\dots,l_k} \in \mathbb{N}_0^{nd}$ of order $\left|\alpha^{j_1,\dots,j_{n-1},j,l_1,\dots,l_k}\right| = n + k - 1$ for all (fixed) $l_1, \dots, l_k \in \{1, \dots, d\}$, $k \ge 0$ and $x^{(n)} := (x, \dots, x) \in \mathbb{R}^{nd}$. Here L_κ^n is the local time from (10) and the multiplication of $b^{\otimes n}(z_n)$ and $\Gamma_\kappa^{n,l_1,\dots,l_k}(z_n)$ in the above signature is defined via tensor contraction as

$$(b^{\otimes n}(z_n)\Gamma_\kappa^{n,l_1,\dots,l_k}(z_n))_{ij} = \sum_{j_1,\dots,j_{n-1}=1}^{d} (b^{\otimes n}(z_n))_{ij_1,\dots,j_{n-1}} (\Gamma_\kappa^{n,l_1,\dots,l_k}(z_n))_{j_1,\dots,j_{n-1}j}, n \ge 2.$$

If $k = 0$, we simply set

$$T_t^{n,\gamma,l_1,\dots,l_k}(b)(x) = T_t^{n,\gamma}(b)(x) = \int_{\mathbb{R}^d} b(z)L_\kappa^1(t, z)dz$$

for all $n \ge 1$.

The motivation for the concept (13) for rough vector fields b comes from the integration by parts formula (10) applied to each summand of (9) (under a change of measure), which can be written in terms of $T_u^{n,\gamma,l_1,\dots,l_k}(b)(x)$ for $k = 1$.

Higher order derivatives $(D^i)^k$ (or alternatively Fréchet derivatives D^k of order k) in connection with (9) give rise to the definition of operators $T_u^{n,\gamma,l_1,\dots,l_k}(b)(x)$ for general $k \ge 1$ (see Sect. 5).

For example, if $n = 1$, $k = 2$, $\kappa \equiv 1$, then we have for (smooth) b that

$$\begin{aligned}
\int_0^t b''(x + \gamma_s)ds &= \int_0^t b''(x + \mathbb{B}_s)ds \\
&= \left(\int_{\mathbb{R}^d} b(x^{(1)} + z_1)(D^2 L_\kappa^1(t, z_1))_{l_1,l_2}dz_1\right)_{1\le l_1,l_2\le d} \\
&= \left(\int_{\mathbb{R}^d} b(x^{(1)} + z_1)\Gamma_\kappa^{1,l_1,l_2}(z_1)dz_1\right)_{1\le l_1,l_2\le d} \\
&= \left(T_t^{1,\gamma,l_1,l_2}(b)(x)\right)_{1\le l_1,l_2\le d} \in \mathbb{R}^d \otimes \mathbb{R}^d.
\end{aligned} \tag{14}$$

In the case, when $n = 1$, $k = 0$, $\kappa \equiv 1$ and $\gamma_t = B_t^H$ a fractional Brownian motion for $H < \frac{1}{2}$, the first order averaging operator $T_t^{1,\gamma}$ along the curve γ_t in (13) coincides with that in Catellier and Gubinelli [15] given by

$$T_t^{\gamma}(b)(x) = \int_0^t b\left(x + B_s^H\right) ds,$$

which was used by the authors- as mentioned before- to study the regularization effect of γ_t on ODE's perturbed by such curves. For example, if $b \in B_{\infty,\infty}^{\alpha+1}$ (Besov–Hölder space) with $\alpha > 2 - \frac{1}{2H}$, then the corresponding SDE (2) driven by $B_\cdot^H$ admits a unique Lipschitz flow. The reason why the latter authors "only" obtain Lipschitz flows and not higher regularity may be that they do not take into account in their analysis information coming from higher order averaging operators $T_t^{n,\gamma,l_1,\ldots,l_k}$ for $n > 1$, $k \geq 1$. Here in this article, we rely in fact on the information based on such higher order averaging operators to be able to study C^∞-regularization effects with respect to flows.

Let us also mention here that Tao and Wright [64] actually studied averaging operators of the type T_t^{γ} along (smooth) *deterministic* curves γ_t and the improvement in bounds of such operators on L^p along such curves. See also [35, 51] and the recent work of [30] and the references therein.

On the other hand, in view of the possibility of a geometric study of the regularity of solutions to ODE's or PDE's, it would be (motivated by (14) natural to replace the signatures in (13) by the following family of signatures for rough vector fields b:

$$\begin{aligned} S_t^n(b)(x) &:= \left(1, T_t^{n,\gamma}(b)(x), \left(T_t^{n,\gamma,l_1}(b)(x)\right)_{1\leq l_1\leq d}, \left(T_t^{n,\gamma,l_1,l_2}(b)(x)\right)_{1\leq l_1,l_2\leq d}, \ldots\right) \\ &\in T(\mathbb{R}^d) := \prod_{k\geq 0}(\otimes_{i=1}^k \mathbb{R}^d), n \geq 1, \end{aligned}$$

where we use the convention $\otimes_{i=1}^0 \mathbb{R}^d = \mathbb{R}$. The space $T(\mathbb{R}^d)$ becomes an associative algebra under tensor multiplication. Then the regularity of solutions to ODE's or PDE's can be analyzed by means of such signatures in connection with Lie groups $\mathfrak{G} \subset T_1(\mathbb{R}^d) := \{(g_0, g_1, \ldots) \in T(\mathbb{R}^d) : g_0 = 1\}$.

In this context, it would be conceivable to derive a Chen-Strichartz type of formula [63] by means of $S_t^n(b)$ in connection with a sub-Riemannian geometry for the study of flows. See [11] and the references therein.

1.4.3 Removal of a "thin" Set of "worst case" Input Data via Noisy Perturbation

As explained before well-posedness of the ODE (1) can be restored by "randomization" or perturbation of the input vector field b in (5). The latter suggests that this procedure leads to a removal of a "thin" set of "worst case" input data, which do not allow for regularization or the restoration of well-posedness. It would be interesting here to develop methods for the measurement of the size of such "thin" sets

The organization of our article is as follows: In Sect. 2 we discuss the mathematical framework of this paper. Further, in Sect. 3 we derive important estimates via variational techniques based on Fourier analysis, which are needed later on for the proofs of the main results of this paper. Section 4 is devoted to the construction of unique strong solutions to the SDE (2). Finally, in Sect. 5 we show C^∞-regularization by noise $\mathbb{B}_\cdot$ of the singular ODE (1).

1.5 Notation

Throughout the article, we will usually denote by C a generic constant. If π is a collection of parameters then C_π will denote a collection of constants depending only on the collection π. Given differential structures M and N, we denote by $C_c^\infty(M; N)$ the space of infinitely many times continuously differentiable function from M to N with compact support. For a complex number $z \in \mathbb{C}$, $\overline{z}$ denotes the conjugate of z and $\boldsymbol{i}$ the imaginary unit. Let E be a vector space, we denote by $|x|$, $x \in E$ the Euclidean norm. For a matrix A, we denote $|A|$ its determinant and $\|A\|_\infty$ its maximum norm.

2 Framework and Setting

In this section we recollect some specifics on Fourier analysis, shuffle products, fractional calculus and fractional Brownian motion which will be extensively used throughout the article. The reader might consult [43, 44] or [21] for a general theory on Malliavin calculus for Brownian motion and [55, Chapter 5] for fractional Brownian motion. For a more detailed theory on harmonic analysis and Fourier transform the reader is referred to [29].

2.1 Fourier Transform

In the course of the paper we will make use of the Fourier transform. There are several definitions in the literature. In the present article we have taken the following: let $f \in L^1(\mathbb{R}^d)$ then we define its *Fourier tranform*, denoted it by $\widehat{f}$, by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi \boldsymbol{i} \langle x, \xi \rangle_{\mathbb{R}^d}} dx, \quad \xi \in \mathbb{R}^d. \tag{15}$$

The above definition can be actually extended to functions in $L^2(\mathbb{R}^d)$ and it makes the operator $L^2(\mathbb{R}^d) \ni f \mapsto \widehat{f} \in L^2(\mathbb{R}^d)$ a linear isometry which, by polarization, implies

$$\langle \widehat{f}, \widehat{g} \rangle_{L^2(\mathbb{R}^d)} = \langle f, g \rangle_{L^2(\mathbb{R}^d)}, \quad f, g \in L^2(\mathbb{R}^d),$$

where

$$\langle f, g \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(z) \overline{g(z)} dz, \quad f, g \in L^2(\mathbb{R}^d).$$

2.2 Shuffles

Let $k \in \mathbb{N}$. For given $m_1, \ldots, m_k \in \mathbb{N}$, denote

$$m_{1:j} := \sum_{i=1}^{j} m_i,$$

e.g. $m_{1:k} = m_1 + \cdots + m_k$ and set $m_0 := 0$. Denote by $S_m = \{\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}\}$ the set of permutations of length $m \in \mathbb{N}$. Define the set of *shuffle permutations* of length $m_{1:k} = m_1 + \cdots m_k$ as

$$S(m_1, \ldots, m_k) := \{\sigma \in S_{m_{1:k}} : \sigma(m_{1:i} + 1) < \cdots < \sigma(m_{1:i+1}),\ i = 0, \ldots, k-1\},$$

and the m-dimensional simplex in $[0, T]^m$ as

$$\Delta^m_{t_0,t} := \{(s_1, \ldots, s_m) \in [0, T]^m : t_0 < s_1 < \cdots < s_m < t\}, \quad t_0, t \in [0, T], \quad t_0 < t.$$

Let $f_i : [0, T] \to [0, \infty)$, $i = 1, \ldots, m_{1:k}$ be integrable functions. Then, we have

$$\begin{aligned}&\prod_{i=0}^{k-1} \int_{\Delta^{m_i}_{t_0,t}} f_{m_{1:i}+1}(s_{m_{1:i}+1}) \cdots f_{m_{1:i+1}}(s_{m_{1:i+1}}) ds_{m_{1:i}+1} \cdots ds_{m_{1:i+1}} \\ &\quad = \sum_{\sigma^{-1} \in S(m_1,\ldots,m_k)} \int_{\Delta^{m_{1:k}}_{t_0,t}} \prod_{i=1}^{m_{1:k}} f_{\sigma(i)}(w_i) dw_1 \cdots dw_{m_{1:k}}.\end{aligned} \tag{16}$$

The above is a trivial generalisation of the case $k = 2$ where

$$\begin{aligned}&\int_{\substack{t_0 < s_1 \cdots < s_{m_1} < t \\ t_0 < s_{m_1+1} < \cdots < s_{m_1+m_2} < t}} \prod_{i=1}^{m_1+m_2} f_i(s_i)\, ds_1 \cdots ds_{m_1+m_2} \\ &= \sum_{\sigma^{-1} \in S(m_1,m_2)} \int_{t_0 < w_1 < \cdots < w_{m_1+m_2} < t} \prod_{i=1}^{m_1+m_2} f_{\sigma(i)}(w_i) dw_1 \cdots dw_{m_1+m_2},\end{aligned} \tag{17}$$

which can be for instance found in [42].

We will also need the following formula. Given indices $j_0, j_1, \ldots, j_{k-1} \in \mathbb{N}$ such that $1 \le j_i \le m_{i+1}$, $i = 1, \ldots, k-1$ and we set $j_0 := m_1 + 1$. Introduce the subset $S_{j_1,\ldots,j_{k-1}}(m_1, \ldots, m_k)$ of $S(m_1, \ldots, m_k)$ defined as

$$\begin{aligned}S_{j_1,\ldots,j_{k-1}}(m_1, \ldots, m_k) := \Big\{&\sigma \in S(m_1, \ldots, m_k) : \sigma(m_{1:i} + 1) < \cdots < \sigma(m_{1:i} + j_i - 1), \\ &\sigma(l) = l,\ m_{1:i} + j_i \le l \le m_{1:i+1},\ i = 0, \ldots, k-1\Big\}.\end{aligned}$$

We have

$$\begin{aligned}&\int_{\Delta^{m_k}_{t_0,t} \times \Delta^{m_{k-1}}_{t_0,s_{m_{1:k-1}+j_{k-1}}} \times \cdots \times \Delta^{m_1}_{t_0,s_{m_1+j_1}}} \prod_{i=1}^{m_{1:k}} f_i(s_i)\, ds_1 \cdots ds_{m_{1:k}} \\ &= \int_{\substack{t_0 < s_1 < \cdots < s_{m_1} < s_{m_1+j_1} \\ t_0 < s_{m_1+m_2+1} < \cdots < s_{m_1+m_2} < s_{m_1+m_2+j_2} \\ \vdots \\ t_0 < s_{m_1+\cdots m_{k-1}+1} < \cdots < s_{m_1+\cdots+m_k} < t}} \prod_{i=1}^{m_{1:k}} f_i(s_i)\, ds_1 \cdots ds_{m_{1:k}} \\ &= \sum_{\sigma^{-1} \in S_{j_1,\ldots,j_{k-1}}(m_1,\ldots,m_k)} \int_{t_0 < w_1 < \cdots < w_{m_{1:k}} < t} \prod_{i=1}^{m_{1:k}} f_{\sigma(i)}(w_i)\, dw_1 \cdots dw_{m_{1:k}}.\end{aligned} \tag{18}$$

$$\#S(m_1, \ldots, m_k) = \frac{(m_1 + \cdots + m_k)!}{m_1! \cdots m_k!},$$

where # denotes the number of elements in the given set. Then by using Stirling's approximation, one can show that

$$\#S(m_1, \ldots, m_k) \le C^{m_1+\cdots+m_k}$$

for a large enough constant $C > 0$. Moreover,

$$\#S_{j_1,\ldots,j_{k-1}}(m_1, \ldots, m_k) \le \#S(m_1, \ldots, m_k).$$

2.3 Fractional Calculus

We pass in review here some basic definitions and properties on fractional calculus. The reader may consult [41, 61] for more information about this subject.

Suppose $a, b \in \mathbb{R}$ with $a < b$. Further, let $f \in L^p([a, b])$ with $p \geq 1$ and $\alpha > 0$. Introduce the *left-* and *right-sided Riemann–Liouville fractional integrals* by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy$$

for almost all $x \in [a, b]$, where Γ stands for the Gamma function.

Furthermore, for an integer $p \geq 1$, denote by $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) the image of $L^p([a, b])$ of the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then we define the *left-* and *right-sided Riemann–Liouville fractional derivatives* by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} dy$$

and

$$D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b \frac{f(y)}{(y-x)^{\alpha}} dy.$$

The above left- and right-sided derivatives of f can be represented as follows:

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right),$$
$$D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right).$$

By construction one also finds the relations

$$I_{a+}^{\alpha}(D_{a+}^{\alpha} f) = f$$

for all $f \in I_{a+}^{\alpha}(L^p)$ and

$$D_{a+}^{\alpha}(I_{a+}^{\alpha} f) = f$$

for all $f \in L^p([a, b])$ and similarly for I_{b-}^{α} and D_{b-}^{α}.

2.4 Fractional Brownian Motion

Consider a d-dimensional *fractional Brownian motion* $B_t^H = (B_t^{H,(1)}, \ldots, B_t^{H,(d)})$, $0 \leq t \leq T$ with Hurst parameter $H \in (0, 1/2)$. So $B_\cdot^H$ is a centered Gaussian process with covariance structure

$$(R_H(t,s))_{i,j} := E[B_t^{H,(i)} B_s^{H,(j)}] = \delta_{i,j} \frac{1}{2} \left(t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad i, j = 1, \ldots, d,$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.

One finds that $E[|B_t^H - B_s^H|^2] = d|t-s|^{2H}$. The latter implies that $B_\cdot^H$ has stationary increments and Hölder continuous trajectories of index $H-\varepsilon$ for all $\varepsilon \in (0, H)$. In addition, one also checks that the increments of $B_\cdot^H$, $H \in (0, 1/2)$ are not independent. This fact however, complicates the study of e.g. SDE's driven by the such processes compared to the Wiener setting. Another difficulty one is faced with in connection with such processes is that they are not semimartingales, see e.g. [55, Proposition 5.1.1].

In what follows let us briefly discuss the construction of fractional Brownian motion via an isometry. In fact, this construction can be done componentwise. Therefore, for convenience we confine ourselves to the one-dimensional case. We refer to [55] for further details.

Let us denote by $\mathcal{E}$ the set of step functions on $[0, T]$ and by $\mathcal{H}$ the Hilbert space, which is obtained by the closure of $\mathcal{E}$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping $1_{[0,t]} \mapsto B_t^H$ has an extension to an isometry between $\mathcal{H}$ and the Gaussian subspace of $L^2(\Omega)$ associated with B^H. We denote the isometry by $\varphi \mapsto B^H(\varphi)$.

The following result, which can be found in (see [55, Proposition 5.1.3]), provides an integral representation of $R_H(t, s)$, when $H < 1/2$:

Proposition 2.1 *Let $H < 1/2$. The kernel*

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + \left(\frac{1}{2} - H\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right],$$

where $c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+1/2)}}$ being β the Beta function, satisfies

$$R_H(t,s) = \int_0^{t \wedge s} K_H(t,u) K_H(s,u) du. \tag{19}$$

The kernel K_H also has a representation in terms of a fractional derivative as follows

$$K_H(t,s) = c_H \Gamma\left(H + \frac{1}{2}\right) s^{\frac{1}{2}-H} \left(D_{t^-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}}\right)(s).$$

Let us now introduce a linear operator $K_H^* : \mathcal{E} \to L^2([0, T])$ by

$$(K_H^* \varphi)(s) = K_H(T, s)\varphi(s) + \int_s^T (\varphi(t) - \varphi(s)) \frac{\partial K_H}{\partial t}(t, s) dt$$

for every $\varphi \in \mathcal{E}$. We see that $(K_H^* 1_{[0,t]})(s) = K_H(t, s) 1_{[0,t]}(s)$. From this and (19) we obtain that K_H^* is an isometry between $\mathcal{E}$ and $L^2([0, T])$ which has an extension to the Hilbert space $\mathcal{H}$.

For a $\varphi \in \mathcal{H}$ one proves the following representations for K_H^*:

$$(K_H^* \varphi)(s) = c_H \Gamma\left(H + \frac{1}{2}\right) s^{\frac{1}{2}-H} \left(D_{T^-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \varphi(u)\right)(s),$$

$$(K_H^* \varphi)(s) = c_H \Gamma\left(H + \frac{1}{2}\right) \left(D_{T^-}^{\frac{1}{2}-H} \varphi(s)\right)(s)$$
$$+ c_H \left(\frac{1}{2} - H\right) \int_s^T \varphi(t)(t-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{t}{s}\right)^{H-\frac{1}{2}}\right) dt.$$

On the other hand one also gets the relation $\mathcal{H} = I_{T^-}^{\frac{1}{2}-H}(L^2)$ (see [20] and [2, Proposition 6]).

Using the fact that K_H^* is an isometry from $\mathcal{H}$ into $L^2([0,T])$, the d-dimensional process $W = \{W_t, t \in [0,T]\}$ given by

$$W_t := B^H((K_H^*)^{-1}(1_{[0,t]}))$$

is a Wiener process and the process B^H can be represented as

$$B_t^H = \int_0^t K_H(t,s)dW_s. \tag{20}$$

See [2].

In the sequel, we denote by $W_\cdot$ a standard Wiener process on a given probability space endowed with the natural filtration generated by W augmented by all P-null sets. Further, $B_\cdot := B_\cdot^H$ stands for the fractional Brownian motion with Hurst parameter $H \in (0,1/2)$ given by the representation (20).

In the following, we need a version of Girsanov's theorem for fractional Brownian motion which goes back to [20, Theorem 4.9]. Here we state the version given in [53, Theorem 3.1]. In preparation of this, we introduce an isomorphism K_H from $L^2([0,T])$ onto $I_{0+}^{H+\frac{1}{2}}(L^2)$ associated with the kernel $K_H(t,s)$ in terms of the fractional integrals as follows, see [20, Theorem 2.1]

$$(K_H\varphi)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}}\varphi, \quad \varphi \in L^2([0,T]).$$

Using the latter and the properties of the Riemann–Liouville fractional integrals and derivatives, one finds that the inverse of K_H is given by

$$(K_H^{-1}\varphi)(s) = s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H}\varphi(s), \quad \varphi \in I_{0+}^{H+\frac{1}{2}}(L^2). \tag{21}$$

Hence, if φ is absolutely continuous, see [53], one can prove that

$$(K_H^{-1}\varphi)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\varphi'(s), \quad a.e. \tag{22}$$

Theorem 2.2 (Girsanov's theorem for fBm) *Let $u = \{u_t, t \in [0,T]\}$ be an $\mathcal{F}$-adapted process with integrable trajectories and set $\widetilde{B}_t^H = B_t^H + \int_0^t u_s ds, \quad t \in [0,T]$. Assume that*

(i) $\int_0^\cdot u_s ds \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$, *$P$-a.s.*
(ii) $E[\xi_T] = 1$ *where*

$$\xi_T := \exp\left\{-\int_0^T K_H^{-1}\left(\int_0^\cdot u_r dr\right)(s)dW_s - \frac{1}{2}\int_0^T K_H^{-1}\left(\int_0^\cdot u_r dr\right)^2(s)ds\right\}.$$

Then the shifted process $\widetilde{B}^H$ is an $\mathcal{F}$-fractional Brownian motion with Hurst parameter H under the new probability $\widetilde{P}$ defined by $\frac{d\widetilde{P}}{dP} = \xi_T$.

Remark 2.3 For the multidimensional case, define

$$(K_H\varphi)(s) := ((K_H\varphi^{(1)})(s), \ldots, (K_H\varphi^{(d)})(s))^*, \quad \varphi \in L^2([0,T];\mathbb{R}^d),$$

where $*$ denotes transposition. Similarly for K_H^{-1} and K_H^*.

Finally, we mention a crucial property of the fractional Brownian motion which was proven by [57] for general Gaussian vector fields.

Let $m \in \mathbb{N}$ and $0 =: t_0 < t_1 < \cdots < t_m < T$. Then for every $\xi_1, \ldots, \xi_m \in \mathbb{R}^d$ there exists a positive finite constant $C > 0$ (depending on m) such that

$$Var\left[\sum_{j=1}^{m} \left\langle \xi_j, B_{t_j}^H - B_{t_{j-1}}^H \right\rangle_{\mathbb{R}^d}\right] \geq C \sum_{j=1}^{m} |\xi_j|^2 E\left[\left|B_{t_j}^H - B_{t_{j-1}}^H\right|^2\right]. \tag{23}$$

The above property is known as the *local non-determinism* property of the fractional Brownian motion. A stronger version of the local non-determinism, which we want to make use of in this paper and which is referred to as *two sided strong local non-determinism* in the literature, is also satisfied by the fractional Brownian motion: There exists a constant $K > 0$, depending only on H and T, such that for any $t \in [0, T]$, $0 < r < t$,

$$Var\left[B_t^H | \left\{B_s^H : |t - s| \geq r\right\}\right] \geq K r^{2H}. \tag{24}$$

The reader may e.g. consult [57] or [66] for more information on this property.

3 A New Regularizing Process

Throughout this article we operate on a probability space $(\Omega, \mathfrak{A}, P)$ equipped with a filtration $\mathcal{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ where $T > 0$ is fixed, generated by a process $\mathbb{B}_\cdot = \mathbb{B}_\cdot^H = \{\mathbb{B}_t^H, t \in [0, T]\}$ to be defined later and here $\mathfrak{A} := \mathcal{F}_T$.

Let $H = \{H_n\}_{n\geq 1} \subset (0, 1/2)$ be a sequence of numbers such that $H_n \searrow 0$ for $n \longrightarrow \infty$. Also, consider $\lambda = \{\lambda_n\}_{n\geq 1} \subset \mathbb{R}$ a sequence of real numbers such that there exists a bijection

$$\{n : \lambda_n \neq 0\} \to \mathbb{N} \tag{25}$$

and

$$\sum_{n=1}^{\infty} |\lambda_n| \in (0, \infty). \tag{26}$$

Let $\{W_\cdot^n\}_{n\geq 1}$ be a sequence of independent d -dimensional standard Brownian motions taking values in $\mathbb{R}^d$ and define for every $n \geq 1$,

$$B_t^{H_n,n} = \int_0^t K_{H_n}(t, s) dW_s^n = \left(\int_0^t K_{H_n}(t, s) dW_s^{n,1}, \ldots, \int_0^t K_{H_n}(t, s) dW_s^{n,d}\right)^*. \tag{27}$$

By construction, $B_\cdot^{H_n,n}$, $n \geq 1$ are pairwise independent d-dimensional fractional Brownian motions with Hurst parameters H_n. Observe that $W_\cdot^n$ and $B_\cdot^{H_n,n}$ generate the same filtrations, see [55, Chapter 5, p. 280]. We will be interested in the following stochastic process

$$\mathbb{B}_t^H = \sum_{n=1}^{\infty} \lambda_n B_t^{H_n,n}, \quad t \in [0, T]. \tag{28}$$

Finally, we need another technical condition on the sequence $\lambda = \{\lambda_n\}_{n\geq 1}$, which is used to ensure continuity of the sample paths of $\mathbb{B}_\cdot^H$:

$$\sum_{n=1}^{\infty} |\lambda_n| E\left[\sup_{0\leq s\leq 1} |B_s^{H_n,n}|\right] < \infty, \tag{29}$$

where $\sup_{0\leq s\leq 1}\left|B_s^{H_n,n}\right| \in L^1(\Omega)$ indeed, see e.g. [12].

The following theorem gives a precise definition of the process $\mathbb{B}_\cdot^H$ and some of its relevant properties.

Theorem 3.1 *Let $H = \{H_n\}_{n\geq 1} \subset (0, 1/2)$ be a sequence of real numbers such that $H_n \searrow 0$ for $n \longrightarrow \infty$ and $\lambda = \{\lambda_n\}_{n\geq 1} \subset \mathbb{R}$ satisfying (25), (26) and (29). Let $\{B_\cdot^{H_n,n}\}_{n=1}^{\infty}$ be a sequence of d-dimensional independent fractional Brownian motions with Hurst parameters H_n, $n \geq 1$, defined as in (27). Define the process*

$$\mathbb{B}_t^H := \sum_{n=1}^{\infty} \lambda_n B_t^{H_n,n}, \quad t \in [0, T],$$

where the convergence is P-a.s. and $\mathbb{B}_t^H$ is a well defined object in $L^2(\Omega)$ for every $t \in [0, T]$.

Moreover, $\mathbb{B}_t^H$ is normally distributed with zero mean and covariance given by

$$E\left[\mathbb{B}_t^H (\mathbb{B}_s^H)^*\right] = \sum_{n=1}^{\infty} \lambda_n^2 R_{H_n}(t, s) I_d,$$

where $*$ denotes transposition, I_d is the d-dimensional identity matrix and $R_{H^n}(t, s) := \frac{1}{2}\left(s^{2H_n} + t^{2H_n} - |t - s|^{2H_n}\right)$ denotes the covariance function of the components of the fractional Brownian motions $B_t^{H_n,n}$.

The process $\mathbb{B}_\cdot^H$ has stationary increments. It does not admit any version with Hölder continuous paths of any order. $\mathbb{B}_\cdot^H$ has no finite p-variation for any order $p > 0$, hence $\mathbb{B}_\cdot^H$ is not a semimartingale. It is not a Markov process and hence it does not possess independent increments.

Finally, under condition (29), $\mathbb{B}_\cdot^H$ has P -a.s. continuous sample paths.

Proof One can verify, employing Kolmogorov's three series theorem, that the series converges P-a.s. and we easily see that

$$E[|\mathbb{B}_t^H|^2] = d\sum_{n=1}^{\infty} \lambda_n^2 t^{2H_n} \leq d(1+t)\sum_{n=1}^{\infty} \lambda_n^2 < \infty,$$

where we used that $x^\alpha \leq 1 + x$ for all $x \geq 0$ and any $\alpha \in [0, 1]$.

The Gaussianity of $\mathbb{B}_t^H$ follows simply by observing that for every $\theta \in \mathbb{R}^d$,

$$E\left[\exp\left\{\boldsymbol{i}\langle\theta, \mathbb{B}_t^H\rangle_{\mathbb{R}^d}\right\}\right] = e^{-\frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{d} \lambda_n^2 t^{2H_n}\theta_j^2},$$

where we used the independence of $B_t^{H_n,n}$ for every $n \geq 1$. The covariance formula follows easily again by independence of $B_t^{H_n,n}$.

The stationarity follows by the fact that $B^{H_n,n}$ are independent and stationary for all $n \geq 1$.

The process $\mathbb{B}_\cdot^H$ could a priori be very irregular. Since $\mathbb{B}_\cdot^H$ is a stochastically continuous separable process with stationary increments, we know by [45, Theorem 5.3.10] that either $\mathbb{B}^H$ has P-a.s. continuous sample paths on all open subsets of $[0, T]$ or $\mathbb{B}^H$ is P-a.s.

unbounded on all open subsets on $[0, T]$. Under condition (29) and using the self-similarity of the fractional Brownian motions we see that

$$E\left[\sup_{s\in[0,T]} |\mathbb{B}_s^H|\right] \leq \sum_{n=1}^{\infty} |\lambda_n| T^{H_n} E\left[\sup_{s\in[0,1]} |B_s^{H_n,n}|\right] \\ \leq (1+T)\sum_{n=1}^{\infty} |\lambda_n| E\left[\sup_{s\in[0,1]} |B_s^{H_n,n}|\right] < \infty$$

and hence by Belyaev's dichotomy for separable stochastically continuous processes with stationary increments (see e.g. [45, Theorem 5.3.10]) there exists a version of $\mathbb{B}_\cdot^H$ with continuous sample paths.

Trivially, $\mathbb{B}_\cdot^H$ is never Hölder continuous since for arbitrary small $\alpha > 0$ there is always $n_0 \geq 1$ such that $H_n < \alpha$ for all $n \geq n_0$ and since the sequence λ satisfies (25) cancellations are not possible. Further, one also argues that $\mathbb{B}_\cdot^H$ is neither Markov nor has finite variation of any order $p > 0$ which then implies that $\mathbb{B}_\cdot^H$ is not a semimartingale. □

We will refer to (28) as a *regularizing* cylindrical fractional Brownian motion with associated Hurst sequence H or simply a regularizing fBm.

Next, we state a version of Girsanov's theorem which actually shows that Eq. (31) admits a weak solution. Its proof is mainly based on the classical Girsanov theorem for a standard Brownian motion in Theorem 2.2.

Theorem 3.2 (Girsanov) *Let $u : [0,T] \times \Omega \to \mathbb{R}^d$ be a (jointly measurable) $\mathcal{F}$-adapted process with integrable trajectories such that $t \mapsto \int_0^t u_s ds$ belongs to the domain of the operator $K_{H_{n_0}}^{-1}$ from (21) for some $n_0 \geq 1$.*

Define the $\mathbb{R}^d$-valued process

$$\widetilde{\mathbb{B}}_t^H := \mathbb{B}_t^H + \int_0^t u_s ds.$$

Define the probability $\widetilde{P}_{n_0}$ in terms of the Radon–Nikodym derivative

$$\frac{d\widetilde{P}_{n_0}}{dP_{n_0}} := \xi_T,$$

where

$$\xi_T^{n_0} := \exp\left\{-\int_0^T K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot u_s ds\right)(s) dW_s^{n_0} - \frac{1}{2}\int_0^T \left|K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot u_s ds\right)(s)\right|^2 ds\right\}.$$

If $E[\xi_T^{n_0}] = 1$, then $\widetilde{\mathbb{B}}_\cdot^H$ is a regularizing $\mathbb{R}^d$-valued cylindrical fractional Brownian motion with respect to $\mathcal{F}$ under the new measure $\widetilde{P}_{n_0}$ with Hurst sequence H.

Proof Indeed, write

$$
\begin{aligned}
\widetilde{\mathbb{B}}_t^H &= \int_0^t u_s ds + \lambda_{n_0} B_t^{H_{n_0}, n_0} + \sum_{n \neq n_0}^{\infty} \lambda_n B_t^{H_n, n} \\
&= \lambda_{n_0} \left(\frac{1}{\lambda_{n_0}} \int_0^t u_s ds + B_t^{H_{n_0}, n_0} \right) + \sum_{n \neq n_0}^{\infty} \lambda_n B_t^{H_n, n} \\
&= \lambda_{n_0} \left(\frac{1}{\lambda_{n_0}} \int_0^t u_s ds + \int_0^t K_{H_{n_0}}(t,s) dW_s^{n_0} \right) + \sum_{n \neq n_0}^{\infty} \lambda_n B_t^{H_n, n} \\
&= \lambda_{n_0} \left(\int_0^t K_{H_{n_0}}(t,s) d\widetilde{W}_s^{n_0} \right) + \sum_{n \neq n_0}^{\infty} \lambda_n B_t^{H_n, n},
\end{aligned}
$$

where

$$
\widetilde{W}_t^{n_0} := W_t^{n_0} + \int_0^t K_{H_{n_0}}^{-1} \left(\frac{1}{\lambda_{n_0}} \int_0^{\cdot} u_r dr \right)(s) ds.
$$

Then it follows from Theorem 2.2 or [54, Theorem 3.1] that

$$
\widetilde{B}_t^{H_{n_0}, n_0} := \int_0^t K_{H_{n_0}}(t,s) d\widetilde{W}_s^{n_0}
$$

is a fractional Brownian motion with Hurst parameter H_{n_0} under the measure

$$
\begin{aligned}
\frac{d\widetilde{P}_{n_0}}{dP_{n_0}} = \exp\Bigg\{ &- \int_0^T K_{H_{n_0}}^{-1} \left(\frac{1}{\lambda_{n_0}} \int_0^{\cdot} u_s ds \right)(s) dW_s^{n_0} \\
&- \frac{1}{2} \int_0^T \left| K_{H_{n_0}}^{-1} \left(\frac{1}{\lambda_{n_0}} \int_0^{\cdot} u_s ds \right)(s) \right|^2 ds \Bigg\}.
\end{aligned}
$$

Hence,

$$
\widetilde{\mathbb{B}}_t^H = \sum_{n=1}^{\infty} \lambda_n \widetilde{B}_t^{H_n, n},
$$

where

$$
\widetilde{B}_t^{H_n, n} = \begin{cases} B_t^{H_n, n} & \text{if} \quad n \neq n_0, \\ \widetilde{B}_t^{H_{n_0}, n_0} & \text{if} \quad n = n_0 \end{cases},
$$

defines a regularizing $\mathbb{R}^d$-valued cylindrical fractional Brownian motion under $\widetilde{P}_{n_0}$. □

Remark 3.3 In the above Girsanov theorem we just modify the law of the drift plus one selected fractional Brownian motion with Hurst parameter H_{n_0}. In our proof later, we show that actually $t \mapsto \int_0^t b(s, \mathbb{B}_s^H) ds$ belongs to the domain of the operators $K_{H_n}^{-1}$ for any $n \geq 1$ but only large $n \geq 1$ satisfy Novikov's condition for arbitrary selected values of $p, q \in (2, \infty]$.

Consider now the following stochastic differential equation with the driving noise $\mathbb{B}_{\cdot}^H$, introduced earlier:

$$
X_t = x + \int_0^t b(s, X_s) ds + \mathbb{B}_t^H, \quad t \in [0, T], \tag{30}
$$

where $x \in \mathbb{R}^d$ and b is regular.

The following result summarises the classical existence and uniqueness theorem and some of the properties of the solution. Existence and uniqueness can be conducted using the classical arguments of $L^2([0, T] \times \Omega)$-completeness in connection with a Picard iteration argument.

Theorem 3.4 *Let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable in $\mathbb{R}^d$ with bounded derivative uniformly in $t \in [0, T]$ and such that there exists a finite constant $C > 0$ independent of t such that $|b(t, x)| \leq C(1 + |x|)$ for every $(t, x) \in [0, T] \times \mathbb{R}^d$. Then Eq. (30) admits a unique global strong solution which is P -a.s. continuously differentiable in x and Malliavin differentiable in each direction W^i, $i \geq 1$ of $\mathbb{B}^H_\cdot$. Moreover, the space derivative and Malliavin derivatives of X satisfy the following linear equations*

$$\frac{\partial}{\partial x} X_t = I_d + \int_0^t b'(s, X_s) \frac{\partial}{\partial x} X_s ds, \quad t \in [0, T]$$

and

$$D^i_{t_0} X_t = \lambda_i K_{H_i}(t, t_0) I_d + \int_{t_0}^t b'(s, X_s) D^i_{t_0} X_s ds, \quad i \geq 1, \quad t_0, t \in [0, T], \quad t_0 < t,$$

where b' denotes the space Jacobian matrix of b, I_d the d -dimensional identity matrix and $D^i_{t_0}$ the Malliavin derivative along W^i, $i \geq 1$. Here, the last identity is meant in the L^p -sense $[0, T]$.

4 Construction of the Solution

We aim at constructing a Malliavin differentiable unique global $\mathcal{F}$ -strong solution to the following equation

$$dX_t = b(t, X_t)dt + d\mathbb{B}^H_t, \quad X_0 = x \in \mathbb{R}^d, \quad t \in [0, T], \tag{31}$$

where the differential is interpreted formally in such a way that if (31) admits a solution $X_\cdot$, then

$$X_t = x + \int_0^t b(s, X_s)ds + \mathbb{B}^H_t, t \in [0, T],$$

whenever it makes sense. Denote by $L^q_p := L^q([0, T]; L^p(\mathbb{R}^d; \mathbb{R}^d))$, $p, q \in [1, \infty]$ the Banach space of integrable functions such that

$$\|f\|_{L^q_p} := \left(\int_0^T \left(\int_{\mathbb{R}^d} |f(t, z)|^p dz \right)^{q/p} dt \right)^{1/q} < \infty,$$

where we take the essential supremum's norm in the cases $p = \infty$ and $q = \infty$.

In this paper, we want to reach the class of discontinuous coefficients $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ in the Banach space

$$\mathcal{L}^q_{2,p} := L^q([0, T]; L^p(\mathbb{R}^d; \mathbb{R}^d)) \cap L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)), \quad p, q \in (2, \infty],$$

of functions $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ with the norm

$$\|f\|_{\mathcal{L}^q_{2,p}} = \|f\|_{L^q_p} + \|f\|_{L^1_\infty}$$

for chosen $p, q \in (2, \infty]$, where

$$L^1_\infty := L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)).$$

We will show existence and uniqueness of strong solutions of Eq. (31) driven by a d-dimensional regularizing fractional Brownian motion with Hurst sequence H with coefficients b belonging to the class $\mathcal{L}^q_{2,p}$. Moreover, we will prove that such solutions are Malliavin differentiable and infinitely many times differentiable with respect to the initial value x, where $d \geq 1$, $p, q \in (2, \infty]$ are arbitrary.

Remark 4.1 We would like to remark that with the method employed in the present article, the existence of weak solutions and the uniqueness in law, holds for drift coefficients in the space L^q_p. In fact, as we will see later on, we need the additional space L^1_∞ to obtain unique strong solutions.

This solution is neither a semimartingale, nor a Markov process, and it has very irregular paths. We show in this paper that the process $\mathbb{B}^H_\cdot$ is a right noise to use in order to produce infinitely classically differentiable flows of (31) for highly irregular coefficients.

To construct a solution the main key is to approximate b by a sequence of smooth functions b_n a.e. and denoting by $X^n = \{X^n_t, t \in [0, T]\}$ the approximating solutions, we aim at using an *ad hoc* compactness argument to conclude that the set $\{X^n_t\}_{n\geq 1} \subset L^2(\Omega)$ for fixed $t \in [0, T]$ is relatively compact.

As for the regularity of the mapping $x \mapsto X^x_t$, we are interested in proving that it is infinitely many times differentiable. It is known that the SDE $dX_t = b(t, X_t)dt + dB^H_t$, $X_0 = x \in \mathbb{R}^d$ admits a unique strong solution for irregular vector fields $b \in L^{1,\infty}_{\infty,\infty}$ and that the mapping $x \mapsto X^x_t$ belongs, P-a.s., to C^k if $H = H(k, d) < 1/2$ is small enough. Hence, by adding the noise $\mathbb{B}^H_\cdot$, we should expect the solution of (31) to have a smooth flow.

Hereunder, we establish the following main result, which will be stated later on in this Section in a more precise form (see Theorem 4.16):

Let $b \in \mathcal{L}^q_{2,p}$, $p, q \in (2, \infty]$ and assume that $\lambda = \{\lambda_i\}_{i\geq 1}$ in (28) satisfies certain growth conditions to be specified later on. Then there exists a unique (global) strong solution $X = \{X_t, t \in [0, T]\}$ of equation (31). Moreover, for every $t \in [0, T]$, X_t is Malliavin differentiable in each direction of the Brownian motions W^n, $n \geq 1$ in (27).

The proof of Theorem 4.16 consists of the following steps:

(1) First, we give the construction of a weak solution $X_\cdot$ to (31) by means of Girsanov's theorem for the process $\mathbb{B}^H_\cdot$, that is we introduce a probability space $(\Omega, \mathfrak{A}, P)$, on which a regularizing fractional Brownian motion $\mathbb{B}^H_\cdot$ and a process $X_\cdot$ are defined, satisfying the SDE (31). However, a priori $X_\cdot$ is not adapted to the natural filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ with respect to $\mathbb{B}^H_\cdot$.

(2) In the next step, consider an approximation of the drift coefficient b by a sequence of compactly supported and infinitely continuously differentiable functions (which always exists by standard approximation results) $b_n : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $n \geq 0$ such that $b_n(t, x) \to b(t, x)$ for a.e. $(t, x) \in [0, T] \times \mathbb{R}^d$ and such that $\sup_{n\geq 0} \|b_n\|_{\mathcal{L}^q_{2,p}} \leq M$ for some finite constant $M > 0$. Then by the previous Section we know that for each smooth coefficient b_n, $n \geq 0$, there exists unique strong solution $X^n = \{X^n_t, t \in [0, T]\}$ to the SDE

$$dX^n_t = b_n(t, X^n_t)du + d\mathbb{B}^H_t, \quad 0 \leq t \leq T, \quad X^n_0 = x \in \mathbb{R}^d. \tag{32}$$

Then we prove that for each $t \in [0, T]$ the sequence X^n_t converges weakly to the conditional expectation $E[X_t|\mathcal{F}_t]$ in the space $L^2(\Omega)$ of square integrable random variables.

(3) By the previous Section we have that for each $t \in [0, T]$ the strong solution X_t^n, $n \geq 0$, is Malliavin differentiable, and that the Malliavin derivatives $D_s^i X_t^n$, $i \geq 1$, $0 \leq s \leq t$, with respect to W^i in (27) satisfy

$$D_s^i X_t^n = \lambda_i K_{H_i}(t,s) I_d + \int_s^t b_n'(u, X_u^n) D_s^i X_u^n du,$$

for every $i \geq 1$ where b_n' is the Jacobian of b_n and I_d the identity matrix in $\mathbb{R}^{d\times d}$. Then, we apply an infinite-dimensional compactness criterion for square integrable functionals of a cylindrical Wiener process based on Malliavin calculus to show that for every $t \in [0, T]$ the set of random variables $\{X_t^n\}_{n\geq 0}$ is relatively compact in $L^2(\Omega)$. The latter, however, enables us to prove that X_t^n converges strongly in $L^2(\Omega)$ to $E[X_t|\mathcal{F}_t]$. Further we find that $E[X_t|\mathcal{F}_t]$ is Malliavin differentiable as a consequence of the compactness criterion.

(4) We verify that $E[X_t|\mathcal{F}_t] = X_t$. So it follows that X_t is $\mathcal{F}_t$-measurable and thus a strong solution on our specific probability space.

(5) Uniqueness in law is enough to guarantee pathwise uniqueness.

In view of the above scheme, we go ahead with step (1) by first providing some preparatory lemmas in order to verify Novikov's condition for $\mathbb{B}_\cdot^H$. Consequently, a weak solution can be constructed via a change of measure.

Lemma 4.2 *Let $\mathbb{B}_\cdot^H$ be a d-dimensional regularizing fBm and $p, q \in [1, \infty]$. Then for every Borel measurable function $h : [0, T] \times \mathbb{R}^d \to [0, \infty)$ we have*

$$E\left[\int_0^T h\left(t, \mathbb{B}_t^H\right) dt\right] \leq C\|h\|_{L_p^q}, \tag{33}$$

where $C > 0$ is a constant depending on p, q, d and H. Also,

$$E\left[\exp\left\{\int_0^T h\left(t, \mathbb{B}_t^H\right) dt\right\}\right] \leq A(\|h\|_{L_p^q}), \tag{34}$$

where A is an analytic function depending on p, q, d and H.

Proof Let $\mathbb{B}_\cdot^H$ be a d-dimensional regularizing fBm. By a conditioning argument and by the independence of increments of the Brownian motion it is easy to see that for every Borel measurable function h we have

$$E\left[\int_{t_0}^T h(t_1, \mathbb{B}_{t_1}^H) dt_1 \middle| \mathcal{F}_{t_0}\right]$$
$$\leq \int_{t_0}^T \int_{\mathbb{R}^d} h(t_1, Y+z)(2\pi)^{-d/2}\sigma_{t_0,t_1}^{-d} \exp\left(-\frac{|z|^2}{2\sigma_{t_0,t_1}^2}\right) dz\Big|_{Y=\sum_{n=1}^\infty \lambda_n \int_0^{t_0} K_{H_n}(t_1,s)dW_s^n} dt_1.$$

Here

$$\sigma_{t_0,t_1}^2 := Var\left[\sum_{n\geq 1} \lambda_n \int_{t_0}^{t_1} K_{H_n}(t_1,s) dW_s^n\right]$$
$$= \sum_{n\geq 1} \lambda_n^2 \int_{t_0}^{t_1} (K_{H_n}(t_1,s))^2 ds.$$

On the other hand, one finds that

$$\int_{t_0}^{t_1} (K_{H_n}(t_1,s))^2 ds \geq C_{H_n} |t_1 - t_0|^{2H_n}$$

for a constant $C_{H_n} > 0$ which depends on H_n.

So

$$\sigma^2_{t_0,t_1} \geq \sum_{n\geq 1} \lambda_n^2 C_{H_n} \left|t_1 - t_0\right|^{2H_n}.$$

Applying Hölder's inequality, first w.r.t. z and then w.r.t. t_1 we arrive at

$$E\left[\int_{t_0}^T h(t_1, \mathbb{B}^H_{t_1})dt_1 \middle| \mathcal{F}_{t_0}\right] \leq C\left(\int_{t_0}^T \left(\int_{\mathbb{R}^d} h(t_1, x_1)^p dx_1\right)^{q/p} dt_1\right)^{1/q} \left(\int_{t_0}^T \left(\sigma^2_{t_0,t_1}\right)^{-dq'(p'-1)/2p'} dt_1\right)^{1/q'},$$

for some finite constant $C > 0$. The time integral is finite for arbitrary values of d, q' and p'. To see this, use the bound $\sum_n a_n \geq a_{n_0}$ for $a_n \geq 0$ and for all $n_0 \geq 1$. Hence,

$$\int_{t_0}^T \left(\sum_{n=1}^\infty \lambda_n^2 C_n (t_1 - t_0)^{2H_n}\right)^{-dq'(p'-1)/2p'} dt_1 \leq \left(\lambda_{n_0}^2 C_{n_0}\right)^{-dq'(p'-1)/2p'} \int_{t_0}^T (t_1 - t_0)^{-H_{n_0} dq'(p'-1)/p'} dt_1,$$

then for fixed d, q' and p' choose n_0 so that $H_{n_0} dq'(p'-1)/p' < 1$. Actually, the above estimate already implies that all exponential moments are finite by [58, Lemma 1.1]. Here, though we need to derive the explicit dependence on the norm of h.

Altogether,

$$E\left[\int_{t_0}^T h(t_1, \mathbb{B}^H_{t_1})dt_1 \middle| \mathcal{F}_{t_0}\right] \leq C\left(\int_{t_0}^T \left(\int_{\mathbb{R}^d} h(t_1, x_1)^p dx_1\right)^{q/p} dt_1\right)^{1/q}, \qquad (35)$$

and setting $t_0 = 0$ this proves (33).

In order to prove (34), Taylor's expansion yields

$$E\left[\exp\left\{\int_0^T h(t, \mathbb{B}^H_t)dt\right\}\right] = 1 + \sum_{m=1}^\infty E\left[\int_0^T \int_{t_1}^T \cdots \int_{t_{m-1}}^T \prod_{j=1}^m h(t_j, \mathbb{B}^H_{t_j}) dt_m \cdots dt_1\right].$$

Using (35) iteratively we have

$$E\left[\exp\left\{\int_0^T h(t, \mathbb{B}^H_t)dt\right\}\right] \leq \frac{C^m}{(m!)^{1/q}} \left(\int_0^T \left(\int_{\mathbb{R}^d} h(t,x)^p dx\right)^{q/p} dt\right)^{m/q} = \frac{C^m \|h\|^m_{L^q_p}}{(m!)^{1/q}},$$

and the result follows with $A(x) := \sum_{m=1}^\infty \frac{C^m}{(m!)^{1/q}} x^m$. □

Lemma 4.3 *Let $\mathbb{B}^H_\cdot$ be a d-dimensional regularizing fBm and assume $b \in L^q_p$, $p, q \in [2, \infty]$. Then for every $n \geq 1$,*

$$t \mapsto \int_0^t b(s, \mathbb{B}^H_s) ds \in I_{0+}^{H_n + \frac{1}{2}}(L^2([0,T])), \quad P-a.s.,$$

i.e. the process $t \mapsto \int_0^t b(s, \mathbb{B}^H_s)ds$ belongs to the domain of the operator $K^{-1}_{H_n}$ for every $n \geq 1$, P-a.s.

Proof Using the property that $D_{0^+}^{H+\frac{1}{2}} I_{0^+}^{H+\frac{1}{2}}(f) = f$ for $f \in L^2([0,T])$ we need to show that for every $n \geq 1$,

$$D_{0^+}^{H_n+\frac{1}{2}} \int_0^{\cdot} |b(s, \mathbb{B}_s^H)| ds \in L^2([0,T]), \quad P-a.s.$$

Indeed,

$$\begin{aligned}\left| D_{0^+}^{H_n+\frac{1}{2}} \left(\int_0^{\cdot} |b(s, \mathbb{B}_s^H)| ds \right)(t) \right| &= \frac{1}{\Gamma\left(\frac{1}{2}-H_n\right)} \left(\frac{1}{t^{H_n+\frac{1}{2}}} \int_0^t |b(u, \mathbb{B}_u^H)| du \right. \\ &\quad \left. + \left(H+\frac{1}{2}\right) \int_0^t (t-s)^{-H_n-\frac{3}{2}} \int_s^t |b(u, \mathbb{B}_u^H)| du ds \right) \\ &\leq \frac{1}{\Gamma\left(\frac{1}{2}-H_n\right)} \left(\frac{1}{t^{H_n+\frac{1}{2}}} + \left(H+\frac{1}{2}\right) \int_0^t (t-s)^{-H_n-\frac{3}{2}} ds \right) \\ &\quad \int_0^t |b(u, \mathbb{B}_u^H)| du.\end{aligned}$$

Hence, for some finite constant $C_{H,T} > 0$ we have

$$\left| D_{0^+}^{H+\frac{1}{2}} \left(\int_0^{\cdot} |b(s, \tilde{\mathbb{B}}_s^H)| ds \right)(t) \right|^2 \leq C_{H,T} \int_0^T |b(u, \mathbb{B}_u^H)|^2 du$$

and taking expectation the result follows by Lemma 4.2 applied to $|b|^2$. □

We are now in a position to show that Novikov's condition is met if n is large enough.

Proposition 4.4 *Let $\mathbb{B}_t^H$ be a d-dimensional regularizing fractional Brownian motion with Hurst sequence H. Assume $b \in L_p^q$, $p, q \in (2, \infty]$. Then for every $\mu \in \mathbb{R}$, there exists n_0 with $H_n < \frac{1}{2} - \frac{1}{p}$ for every $n \geq n_0$ and such that for every $n \geq n_0$ we have*

$$E\left[\exp\left\{\mu \int_0^T \left| K_{H_n}^{-1}\left(\frac{1}{\lambda_n} \int_0^{\cdot} b(r, \mathbb{B}_r^H) dr\right)(s)\right|^2 ds\right\}\right] \leq C_{\lambda_n, H_n, d, \mu, T}(\|b\|_{L_p^q})$$

for some real analytic function $C_{\lambda_n, H_n, d, \mu, T}$ depending only on λ_n, H_n, d, T and μ.

In particular, there is also some real analytic function $\widetilde{C}_{\lambda_n, H_n, d, \mu, T}$ depending only on λ_n, H_n, d, T and μ such that

$$E\left[\mathcal{E}\left(\int_0^T K_{H_n}^{-1}\left(\frac{1}{\lambda_n} \int_0^{\cdot} b(r, \mathbb{B}_r^H) dr\right)^*(s) dW_s^n\right)^\mu\right] \leq \widetilde{C}_{H,d,\mu,T}(\|b\|_{L_p^q}),$$

for every $\mu \in \mathbb{R}$.

Proof By Lemma 4.3 both random variables appearing in the statement are well defined. Then, fix $n \geq n_0$ and denote $\theta_s^n := K_{H_n}^{-1}\left(\frac{1}{\lambda_n} \int_0^{\cdot} |b(r, \mathbb{B}_r^H)| dr\right)(s)$. Then using relation (22) we have

$$\begin{aligned}|\theta_s^n| &= \left| \frac{1}{\lambda_n} s^{H_n - \frac{1}{2}} I_{0^+}^{\frac{1}{2}-H_n} s^{\frac{1}{2}-H_n} |b(s, \mathbb{B}_s^H)| \right| \\ &= \frac{1/|\lambda_n|}{\Gamma\left(\frac{1}{2}-H_n\right)} s^{H_n-\frac{1}{2}} \int_0^s (s-r)^{-\frac{1}{2}-H_n} r^{\frac{1}{2}-H_n} |b(r, \mathbb{B}_r^H)| dr. \qquad (36)\end{aligned}$$

Observe that since $H_n < \frac{1}{2} - \frac{1}{p}$, $p \in (2, \infty]$ we may take $\varepsilon \in [0, 1)$ such that $H_n < \frac{1}{1+\varepsilon} - \frac{1}{2}$ and apply Hölder's inequality with exponents $1+\varepsilon$ and $\frac{1+\varepsilon}{\varepsilon}$, where the case $\varepsilon = 0$ corresponds to the case where b is bounded. Then we get

$$|\theta_s^n| \leq C_{\varepsilon,\lambda_n,H_n} s^{\frac{1}{1+\varepsilon}-H_n-\frac{1}{2}} \left(\int_0^s |b(r, \mathbb{B}_r^H)|^{\frac{1+\varepsilon}{\varepsilon}} dr \right)^{\frac{\varepsilon}{1+\varepsilon}}, \tag{37}$$

where

$$C_{\varepsilon,\lambda_n,H_n} := \frac{\Gamma\left(1-(1+\varepsilon)(H_n+1/2)\right)^{\frac{1}{1+\varepsilon}} \Gamma\left(1+(1+\varepsilon)(1/2-H_n)\right)^{\frac{1}{1+\varepsilon}}}{\lambda_n \Gamma\left(\frac{1}{2}-H_n\right) \Gamma\left(2(1-(1+\varepsilon)H_n)\right)^{\frac{1}{1+\varepsilon}}}.$$

Squaring both sides and using the fact that $|b| \geq 0$ we have the following estimate

$$|\theta_s^n|^2 \leq C_{\varepsilon,\lambda_n,H_n}^2 s^{\frac{2}{1+\varepsilon}-2H_n-1} \left(\int_0^T |b(r, \mathbb{B}_r^H)|^{\frac{1+\varepsilon}{\varepsilon}} dr \right)^{\frac{2\varepsilon}{1+\varepsilon}}, \quad P-a.s.$$

Since $0 < \frac{2\varepsilon}{1+\varepsilon} < 1$ and $|x|^\alpha \leq \max\{\alpha, 1-\alpha\}(1+|x|)$ for any $x \in \mathbb{R}$ and $\alpha \in (0, 1)$ we have

$$\int_0^T |\theta_s^n|^2 ds \leq C_{\varepsilon,\lambda_n,H_n,T} \left(1 + \int_0^T |b(r, \mathbb{B}_r^H)|^{\frac{1+\varepsilon}{\varepsilon}} dr \right), \quad P-a.s. \tag{38}$$

for some constant $C_{\varepsilon,\lambda_n,H_n,T} > 0$. Then estimate (34) from Lemma 4.2 with $h = C_{\varepsilon,\lambda_n,H_n,T}\, \mu\, b^{\frac{1+\varepsilon}{\varepsilon}}$ with $\varepsilon \in [0, 1)$ arbitrarily close to one yields the result for $p, q \in (2, \infty]$. □

Let $(\Omega, \mathfrak{A}, \widetilde{P})$ be some given probability space which carries a regularizing fractional Brownian motion $\widetilde{\mathbb{B}}_\cdot^H$ with Hurst sequence $H = \{H_n\}_{n\geq 1}$ and set $X_t := x + \widetilde{\mathbb{B}}_t^H$, $t \in [0, T]$, $x \in \mathbb{R}^d$. Set $\theta_t^{n_0} := \left(K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}} \int_0^\cdot b(r, X_r) dr\right)\right)(t)$ for some fixed $n_0 \geq 1$ such that Proposition 4.4 can be applied and consider the new measure defined by

$$\frac{dP_{n_0}}{d\widetilde{P}_{n_0}} = Z_T^{n_0},$$

where

$$Z_t^{n_0} := \prod_{n=1}^\infty \mathcal{E}\left(\theta_\cdot^{n_0}\right)_t := \exp\left\{ \int_0^t \left(\theta_s^{n_0}\right)^* dW_s^{n_0} - \frac{1}{2}\int_0^t |\theta_s^{n_0}|^2 ds \right\}, \quad t \in [0, T].$$

In view of Proposition 4.4 the above random variable defines a new probability measure and by Girsanov's theorem, see Theorem 3.2, the process

$$\mathbb{B}_t^H := X_t - x - \int_0^t b(s, X_s) ds, \quad t \in [0, T] \tag{39}$$

is a regularizing fractional Brownian motion on $(\Omega, \mathfrak{A}, P_{n_0})$ with Hurst sequence H. Hence, because of (39), the couple $(X, \mathbb{B}_\cdot^H)$ is a weak solution of (31) on $(\Omega, \mathfrak{A}, P_{n_0})$. Since $n_0 \geq 1$ is fixed we will omit the notation P_{n_0} and simply write P.

Henceforth, we confine ourselves to the filtered probability space $(\Omega, \mathfrak{A}, P)$, $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ which carries the weak solution $(X, \mathbb{B}_\cdot^H)$ of (31).

Remark 4.5 In order to establish existence of a strong solution, the main difficulty here is to show that $X_\cdot$ is $\mathcal{F}$ -adapted. In fact, in this case $X_t = F_t(\mathbb{B}^H_\cdot)$ for some progressively measurable functional F_t, $t \in [0,T]$ on $C([0,T];\mathbb{R}^d)$ and for any other stochastic basis $(\hat{\Omega}, \hat{\mathfrak{A}}, \hat{P}, \hat{\mathbb{B}})$ one gets that $X_t := F_t(\hat{\mathbb{B}}_\cdot)$, $t \in [0,T]$, is a solution to SDE (31), which is adapted with respect to the natural filtration of $\hat{\mathbb{B}}_\cdot$. But this exactly gives the existence of a strong solution to SDE (31).

We take a weak solution $X_\cdot$ of (31) and consider $E[X_t|\mathcal{F}_t]$. The next result corresponds to step (2) of our program.

Lemma 4.6 *Let $b_n : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $n \geq 1$, be a sequence of compactly supported smooth functions converging a.e. to b such that $\sup_{n\geq 1}\|b_n\|_{L_p^q} < \infty$. Let $t \in [0,T]$ and X_t^n denote the solution of (31) when we replace b by b_n. Then for every $t \in [0,T]$ and continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$ of at most linear growth we have that*

$$\varphi(X_t^n) \xrightarrow{n\to\infty} E\left[\varphi(X_t)|\mathcal{F}_t\right],$$

weakly in $L^2(\Omega)$.

Proof Let us assume, without loss of generality, that $x = 0$. In the course of the proof we always assume that for fixed $p, q \in (2,\infty]$ then $n_0 \geq 1$ is such that $H_{n_0} < \frac{1}{2} - \frac{1}{p}$ and hence Proposition 4.4 can be applied.

First we show that

$$\begin{aligned}
&\mathcal{E}\left(\frac{1}{\lambda_{n_0}}\int_0^t K_{H_{n_0}}^{-1}\left(\int_0^\cdot b_n(r,\mathbb{B}_r^H)dr\right)^*(s)dW_s^{n_0}\right)\\
&\quad\to \mathcal{E}\left(\int_0^t K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b(r,\mathbb{B}_r^H)dr\right)^*(s)dW_s^{n_0}\right)
\end{aligned} \tag{40}$$

in $L^p(\Omega)$ for all $p \geq 1$. To see this, note that

$$K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b_n(r,\mathbb{B}_r^H)dr\right)(s) \to K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b(r,\mathbb{B}_r^H)dr\right)(s)$$

in probability for all s. Indeed, from (37) we have a constant $C_{\varepsilon,\lambda_{n_0},H_{n_0}} > 0$ such that

$$\begin{aligned}
&E\left[\left|K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b_n(r,\mathbb{B}_r^H)dr\right)(s) - K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b(r,\mathbb{B}_r^H)dr\right)(s)\right|\right]\\
&\leq C_{\varepsilon,,\lambda_{n_0},H_{n_0}} s^{\frac{1}{1+\varepsilon}-H_{n_0}-\frac{1}{2}}\left(\int_0^s |b_n(r,\mathbb{B}_r^H)-b(r,\mathbb{B}_r^H)|^{\frac{1+\varepsilon}{\varepsilon}}dr\right)^{\frac{\varepsilon}{1+\varepsilon}} \to 0
\end{aligned}$$

as $n\to\infty$ by Lemma 4.2.

Moreover, $\left\{K_{H_{n_0}}^{-1}(\frac{1}{\lambda_{n_0}}\int_0^\cdot b_n(r,\mathbb{B}_r^H)dr)\right\}_{n\geq 0}$ is bounded in $L^2([0,t]\times\Omega;\mathbb{R}^d)$. This is directly seen from (38) in Proposition 4.4.

Consequently

$$\int_0^t K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b_n(r,\mathbb{B}_r^H)dr\right)^*(s)dW_s^{n_0} \to \int_0^t K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b(r,\mathbb{B}_r^H)dr\right)^*(s)dW_s^{n_0}$$

and

$$\int_0^t \left|K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b_n(r,\mathbb{B}_r^H)dr\right)(s)\right|^2 ds \to \int_0^t \left|K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^\cdot b(r,\mathbb{B}_r^H)dr\right)(s)\right|^2 ds$$

in $L^2(\Omega)$ since the latter is bounded $L^p(\Omega)$ for any $p \geq 1$, see Proposition 4.4.

By applying the estimate $|e^x - e^y| \leq e^{x+y}|x-y|$, Hölder's inequality and the bounds in Proposition 4.4 in connection with Lemma 4.2 we see that (40) holds.

Similarly, one finds that

$$\exp\left\{\left\langle \alpha, \int_s^t b_n(r, \mathbb{B}_r^H)dr \right\rangle\right\} \to \exp\left\{\left\langle \alpha, \int_s^t b(r, \mathbb{B}_r^H)dr \right\rangle\right\}$$

in $L^p(\Omega)$ for all $p \geq 1, 0 \leq s \leq t \leq T, \alpha \in \mathbb{R}^d$.

In order to complete the proof, we note that the set

$$\Sigma_t := \left\{ \exp\left\{ \sum_{j=1}^k \langle \alpha_j, \mathbb{B}_{t_j}^H - \mathbb{B}_{t_{j-1}}^H \rangle \right\} : \{\alpha_j\}_{j=1}^k \subset \mathbb{R}^d, 0 = t_0 < \cdots < t_k = t, k \geq 1 \right\}$$

is a total subspace of $L^2(\Omega, \mathcal{F}_t, P)$ and therefore it is sufficient to prove the convergence

$$\lim_{n\to\infty} E\left[\left(\varphi(X_t^n) - E[\varphi(X_t)|\mathcal{F}_t]\right)\xi\right] = 0$$

for all $\xi \in \Sigma_t$. In doing so, we notice that φ is of linear growth and hence $\varphi(\mathbb{B}_t^H)$ has all moments. Thus, we obtain the following convergence

$$\begin{aligned}
&E\left[\varphi(X_t^n)\exp\left\{\sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}_{t_j}^H - \mathbb{B}_{t_{j-1}}^H \right\rangle\right\}\right] \\
&= E\left[\varphi(X_t^n)\exp\left\{\sum_{j=1}^k \langle \alpha_j, X_{t_j}^n - X_{t_{j-1}}^n - \int_{t_{j-1}}^{t_j} b_n(s, X_s^n)ds \rangle\right\}\right] \\
&= E\Bigg[\varphi(\mathbb{B}_t^H)\exp\Bigg\{\sum_{j=1}^k \langle \alpha_j, \mathbb{B}_{t_j}^H - \mathbb{B}_{t_{j-1}}^H \\
&\quad - \int_{t_{j-1}}^{t_j} b_n(s, \mathbb{B}_s^H)ds \rangle\Bigg\} \mathcal{E}\left(\int_0^t K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^{\cdot} b_n(r, \mathbb{B}_r^H)dr\right)^*(s)dW_s^{n_0}\right)\Bigg] \\
&\to E\Bigg[\varphi(\mathbb{B}_t^H)\exp\Bigg\{\sum_{j=1}^k \langle \alpha_j, \mathbb{B}_{t_j}^H - \mathbb{B}_{t_{j-1}}^H \\
&\quad - \int_{t_{j-1}}^{t_j} b(s, \mathbb{B}_s^H)ds \rangle\Bigg\} \mathcal{E}\left(\int_0^t K_{H_{n_0}}^{-1}\left(\frac{1}{\lambda_{n_0}}\int_0^{\cdot} b(r, \mathbb{B}_r^H)dr\right)^*(s)dW_s^{n_0}\right)\Bigg] \\
&= E\left[\varphi(X_t)\exp\left\{\sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}_{t_j}^H - \mathbb{B}_{t_{j-1}}^H \right\rangle\right\}\right] \\
&= E\left[E[\varphi(X_t)|\mathcal{F}_t]\exp\left\{\sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}_{t_j}^H - \mathbb{B}_{t_{j-1}}^H \right\rangle\right\}\right].
\end{aligned}$$

□

We now turn to step (3) of our program. For its completion we need to derive some crucial estimates.

In preparation of those estimates, we introduce some notation and definitions:

Let m be an integer and let the function $f : [0,T]^m \times (\mathbb{R}^d)^m \to \mathbb{R}$ be of the form

$$f(s,z) = \prod_{j=1}^m f_j(s_j, z_j), \quad s = (s_1, \dots, s_m) \in [0,T]^m, \quad z = (z_1, \dots, z_m) \in (\mathbb{R}^d)^m, \tag{41}$$

where $f_j : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $j = 1, \dots, m$ are smooth functions with compact support. Further, let $\varkappa : [0,T]^m \to \mathbb{R}$ a function of the form

$$\varkappa(s) = \prod_{j=1}^m \varkappa_j(s_j), \quad s \in [0,T]^m, \tag{42}$$

where $\varkappa_j : [0,T] \to \mathbb{R}$, $j = 1, \dots, m$ are integrable functions.

Let α_j be a multi-index and denote by D^{α_j} its corresponding differential operator. For $\alpha = (\alpha_1, \dots, \alpha_m)$ viewed as an element of $\mathbb{N}_0^{d\times m}$ we define $|\alpha| = \sum_{j=1}^m \sum_{l=1}^d \alpha_j^{(l)}$ and write

$$D^\alpha f(s,z) = \prod_{j=1}^m D^{\alpha_j} f_j(s_j, z_j).$$

The objective of this section is to establish an integration by parts formula of the form

$$\int_{\Delta_{\theta,t}^m} D^\alpha f(s, \mathbb{B}_s) ds = \int_{(\mathbb{R}^d)^m} \Lambda_\alpha^f(\theta, t, z) dz, \tag{43}$$

where $\mathbb{B} := \mathbb{B}_\cdot^H$, for a random field Λ_α^f. In fact, we can choose Λ_α^f to be

$$\Lambda_\alpha^f(\theta, t, z) = (2\pi)^{-dm} \int_{(\mathbb{R}^d)^m} \int_{\Delta_{\theta,t}^m} \prod_{j=1}^m f_j(s_j, z_j)(-iu_j)^{\alpha_j} \exp\{-i\langle u_j, \mathbb{B}_{s_j} - z_j\rangle\} ds du. \tag{44}$$

Let us start by *defining* $\Lambda_\alpha^f(\theta, t, z)$ as above and show that it is a well-defined element of $L^2(\Omega)$.

We also need the following notation: Given $(s,z) = (s_1, \dots, s_m, z_1 \dots, z_m) \in [0,T]^m \times (\mathbb{R}^d)^m$ and a shuffle $\sigma \in S(m,m)$ we define

$$f_\sigma(s,z) := \prod_{j=1}^{2m} f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})$$

and

$$\varkappa_\sigma(s) := \prod_{j=1}^{2m} \varkappa_{[\sigma(j)]}(s_j),$$

where $[j]$ is equal to j if $1 \le j \le m$ and $j - m$ if $m+1 \le j \le 2m$.

For a multiindex α, define

$$\Psi_\alpha^f(\theta, t, z, H_r) := \prod_{l=1}^d \sqrt{(2\left|\alpha^{(l)}\right|)!} \sum_{\sigma \in S(m,m)} \int_{\Delta_{0,t}^{2m}} |f_\sigma(s,z)| \prod_{j=1}^{2m} \frac{1}{\left|s_j - s_{j-1}\right|^{H_r(d + 2\sum_{l=1}^d \alpha_{[\sigma(j)]}^{(l)})}} ds_1 \dots ds_{2m}$$

respectively,

$$\Psi_{\alpha}^{\varkappa}(\theta, t, H_r) := \prod_{l=1}^{d} \sqrt{(2\left|\alpha^{(l)}\right|)!} \sum_{\sigma \in S(m,m)} \int_{\Delta_{0,t}^{2m}} |\varkappa_{\sigma}(s)| \prod_{j=1}^{2m} \frac{1}{\left|s_j - s_{j-1}\right|^{H_r(d+2\sum_{l=1}^{d} \alpha_{[\sigma(j)]}^{(l)})}} ds_1 \ldots ds_{2m}.$$

Theorem 4.7 *Suppose that* $\Psi_{\alpha}^{f}(\theta, t, z, H_r), \Psi_{\alpha}^{\varkappa}(\theta, t, H_r) < \infty$ *for some* $r \geq r_0$*. Then,* $\Lambda_{\alpha}^{f}(\theta, t, z)$ *as in (44) is a random variable in* $L^2(\Omega)$*. Further, there exists a universal constant* $C_r = C(T, H_r, d) > 0$ *such that*

$$E[\left|\Lambda_{\alpha}^{f}(\theta, t, z)\right|^2] \leq \frac{1}{\lambda_r^{2md}} C_r^{m+|\alpha|} \Psi_{\alpha}^{f}(\theta, t, z, H_r). \tag{45}$$

Moreover, we have

$$\left|E[\int_{(\mathbb{R}^d)^m} \Lambda_{\alpha}^{f}(\theta, t, z) dz]\right| \leq \frac{1}{\lambda_r^{md}} C_r^{m/2+|\alpha|/2} \prod_{j=1}^{m} \| f_j \|_{L^1(\mathbb{R}^d; L^{\infty}([0,T]))} (\Psi_{\alpha}^{\varkappa}(\theta, t, H_r))^{1/2}. \tag{46}$$

Proof For notational simplicity we consider $\theta = 0$ and set $\mathbb{B}_{\cdot} = \mathbb{B}_{\cdot}^{H}$, $\Lambda_{\alpha}^{f}(t, z) = \Lambda_{\alpha}^{f}(0, t, z)$.

For an integrable function $g : (\mathbb{R}^d)^m \longrightarrow \mathbb{C}$ we get that

$$\begin{aligned}
&\left|\int_{(\mathbb{R}^d)^m} g(u_1, \ldots, u_m) du_1 \ldots du_m\right|^2 \\
&= \int_{(\mathbb{R}^d)^m} g(u_1, \ldots, u_m) du_1 \ldots du_m \int_{(\mathbb{R}^d)^m} \overline{g(u_{m+1}, \ldots, u_{2m})} du_{m+1} \ldots du_{2m} \\
&= \int_{(\mathbb{R}^d)^m} g(u_1, \ldots, u_m) du_1 \ldots du_m (-1)^{dm} \int_{(\mathbb{R}^d)^m} \overline{g(-u_{m+1}, \ldots, -u_{2m})} du_{m+1} \ldots du_{2m},
\end{aligned}$$

where we employed the change of variables $(u_{m+1}, \ldots, u_{2m}) \longmapsto (-u_{m+1}, \ldots, -u_{2m})$ in the last equality.

This yields

$$\begin{aligned}
&\left|\Lambda_{\alpha}^{f}(t, z)\right|^2 \\
&= (2\pi)^{-2dm} (-1)^{dm} \int_{(\mathbb{R}^d)^{2m}} \int_{\Delta_{0,t}^{m}} \prod_{j=1}^{m} f_j(s_j, z_j)(-iu_j)^{\alpha_j} e^{-i\left\langle u_j, \mathbb{B}_{s_j} - z_j \right\rangle} ds_1 \ldots ds_m \\
&\quad \times \int_{\Delta_{0,t}^{m}} \prod_{j=m+1}^{2m} f_{[j]}(s_j, z_{[j]})(-iu_j)^{\alpha_{[j]}} e^{-i\left\langle u_j, \mathbb{B}_{s_j} - z_{[j]} \right\rangle} ds_{m+1} \ldots ds_{2m} du_1 \ldots du_{2m} \\
&= (2\pi)^{-2dm} (-1)^{dm} \sum_{\sigma \in S(m,m)} \int_{(\mathbb{R}^d)^{2m}} \left(\prod_{j=1}^{m} e^{-i\langle z_j, u_j + u_{j+m} \rangle} \right) \\
&\quad \times \int_{\Delta_{0,t}^{2m}} f_{\sigma}(s, z) \prod_{j=1}^{2m} u_{\sigma(j)}^{\alpha_{[\sigma(j)]}} \exp\left\{ -\sum_{j=1}^{2m} \left\langle u_{\sigma(j)}, \mathbb{B}_{s_j} \right\rangle \right\} ds_1 \ldots ds_{2m} du_1 \ldots du_{2m},
\end{aligned}$$

where we applied shuffling in connection with Sect. 2.2 in the last step.

By taking the expectation on both sides in connection with the assumption that the fractional Brownian motions $B_{\cdot}^{i,H_i}$, $i \geq 1$ are independent we find that

$$
\begin{aligned}
&E\left[\left|\Lambda_{\alpha}^{f}(t,z)\right|^{2}\right] \\
&= (2\pi)^{-2dm}(-1)^{dm} \sum_{\sigma \in S(m,m)} \int_{(\mathbb{R}^d)^{2m}} \left(\prod_{j=1}^{m} e^{-i\langle z_j, u_j + u_{j+m}\rangle} \right) \\
&\quad \times \int_{\Delta_{0,t}^{2m}} f_{\sigma}(s,z) \prod_{j=1}^{2m} u_{\sigma(j)}^{\alpha_{[\sigma(j)]}} \exp\left\{ -\frac{1}{2} Var\left[\sum_{j=1}^{2m} \langle u_{\sigma(j)}, \mathbb{B}_{s_j} \rangle \right] \right\} ds_1 \dots ds_{2m} du_1 \dots du_{2m} \\
&= (2\pi)^{-2dm}(-1)^{dm} \sum_{\sigma \in S(m,m)} \int_{(\mathbb{R}^d)^{2m}} \left(\prod_{j=1}^{m} e^{-i\langle z_j, u_j + u_{j+m}\rangle} \right) \\
&\quad \times \int_{\Delta_{0,t}^{2m}} f_{\sigma}(s,z) \prod_{j=1}^{2m} u_{\sigma(j)}^{\alpha_{[\sigma(j)]}} \\
&\exp\left\{ -\frac{1}{2} \sum_{n \geq 1} \lambda_n^2 \sum_{l=1}^{d} Var\left[\sum_{j=1}^{2m} u_{\sigma(j)}^{(l)} B_{s_j}^{(l),n,H_n} \right] \right\} ds_1 \dots ds_{2m} du_1^{(1)} \dots du_{2m}^{(1)} \\
&\quad \dots du_1^{(d)} \dots du_{2m}^{(d)} \\
&= (2\pi)^{-2dm}(-1)^{dm} \sum_{\sigma \in S(m,m)} \int_{(\mathbb{R}^d)^{2m}} \left(\prod_{j=1}^{m} e^{-i\langle z_j, u_j + u_{j+m}\rangle} \right) \\
&\quad \times \int_{\Delta_{0,t}^{2m}} f_{\sigma}(s,z) \prod_{j=1}^{2m} u_{\sigma(j)}^{\alpha_{[\sigma(j)]}} \prod_{n \geq 1} \prod_{l=1}^{d} \exp\left\{ -\frac{1}{2} \lambda_n^2 ((u_{\sigma(j)}^{(l)})_{1 \leq j \leq 2m})^{*} Q_n ((u_{\sigma(j)}^{(l)})_{1 \leq j \leq 2m}) \right\} ds_1 \dots ds_{2m} \\
&\quad du_{\sigma(1)}^{(1)} \dots du_{\sigma(2m)}^{(1)} \dots du_{\sigma(1)}^{(d)} \dots du_{\sigma(2m)}^{(d)},
\end{aligned}
\tag{47}
$$

where $*$ stands for transposition and where

$$
Q_n = Q_n(s) := (E[B_{s_i}^{(1)} B_{s_j}^{(1)}])_{1 \leq i,j \leq 2m}.
$$

Further, we get that

$$
\begin{aligned}
&\int_{\Delta_{0,t}^{2m}} |f_{\sigma}(s,z)| \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^{2m} \prod_{l=1}^{d} \left| u_{\sigma(j)}^{(l)} \right|^{\alpha_{[\sigma(j)]}^{(l)}} \prod_{n \geq 1} \prod_{l=1}^{d} \exp\left\{ -\frac{1}{2} \lambda_n^2 ((u_{\sigma(j)}^{(l)})_{1 \leq j \leq 2m})^{*} Q_n ((u_{\sigma(j)}^{(l)})_{1 \leq j \leq 2m}) \right\} \\
&du_{\sigma(1)}^{(1)} \dots du_{\sigma(2m)}^{(1)} \dots du_{\sigma(1)}^{(d)} \dots du_{\sigma(2m)}^{(d)} ds_1 \dots ds_{2m} \\
&\leq \int_{\Delta_{0,t}^{2m}} |f_{\sigma}(s,z)| \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^{2m} \prod_{l=1}^{d} \left| u_j^{(l)} \right|^{\alpha_{[\sigma(j)]}^{(l)}} \\
&\quad \times \prod_{l=1}^{d} \exp\left\{ -\frac{1}{2} \lambda_r^2 \left\langle Q_r u^{(l)}, u^{(l)} \right\rangle \right\} \\
&\quad du_1^{(1)} \dots du_{2m}^{(1)} \dots du_1^{(d)} \dots du_{2m}^{(d)} ds_1 \dots ds_{2m} \\
&= \int_{\Delta_{0,t}^{2m}} |f_{\sigma}(s,z)| \prod_{l=1}^{d} \int_{\mathbb{R}^{2m}} \left(\prod_{j=1}^{2m} \left| u_j^{(l)} \right|^{\alpha_{[\sigma(j)]}^{(l)}} \right) \exp\left\{ -\frac{1}{2} \lambda_r^2 \left\langle Q_r u^{(l)}, u^{(l)} \right\rangle \right\} du_1^{(l)} \dots du_{2m}^{(l)} ds_1 \dots ds_{2m},
\end{aligned}
\tag{48}
$$

where

$$u^{(l)} := (u_j^{(l)})_{1\leq j\leq 2m}.$$

We obtain that

$$\int_{\mathbb{R}^{2m}} \left(\prod_{j=1}^{2m} \left|u_j^{(l)}\right|^{\alpha^{(l)}_{[\sigma(j)]}}\right) \exp\left\{-\frac{1}{2}\lambda_r^2 \left\langle Q_r u^{(l)}, u^{(l)}\right\rangle\right\} du_1^{(l)} \ldots du_{2m}^{(l)}$$
$$= \frac{1}{\lambda_r^{2m}} \frac{1}{(\det Q_r)^{1/2}} \int_{\mathbb{R}^{2m}} \left(\prod_{j=1}^{2m} \left|\left\langle Q_r^{-1/2} u^{(l)}, e_j\right\rangle\right|^{\alpha^{(l)}_{[\sigma(j)]}}\right) \exp\left\{-\frac{1}{2}\left\langle u^{(l)}, u^{(l)}\right\rangle\right\} du_1^{(l)} \ldots du_{2m}^{(l)},$$

where e_i, $i = 1, \ldots, 2m$ is the standard ONB of $\mathbb{R}^{2m}$.

We also have that

$$\int_{\mathbb{R}^{2m}} \left(\prod_{j=1}^{2m} \left|\left\langle Q_r^{-1/2} u^{(l)}, e_j\right\rangle\right|^{\alpha^{(l)}_{[\sigma(j)]}}\right) \exp\left\{-\frac{1}{2}\left\langle u^{(l)}, u^{(l)}\right\rangle\right\} du_1^{(l)} \ldots du_{2m}^{(l)}$$
$$= (2\pi)^m E\left[\prod_{j=1}^{2m} \left|\left\langle Q_r^{-1/2} Z, e_j\right\rangle\right|^{\alpha^{(l)}_{[\sigma(j)]}}\right],$$

where

$$Z \sim \mathcal{N}(\mathcal{O}, I_{2m\times 2m}).$$

On the other hand, it follows from Lemma B.6, which is a type of Brascamp–Lieb inequality, that

$$E\left[\prod_{j=1}^{2m} \left|\left\langle Q_r^{-1/2} Z, e_j\right\rangle\right|^{\alpha^{(l)}_{[\sigma(j)]}}\right]$$
$$\leq \sqrt{perm\left(\sum\right)} = \sqrt{\sum_{\pi\in S_{2\left|\alpha^{(l)}\right|}} \prod_{i=1}^{2\left|\alpha^{(l)}\right|} a_{i\pi(i)}},$$

where $perm(\sum)$ is the permanent of the covariance matrix $\sum = (a_{ij})$ of the Gaussian random vector

$$(\underbrace{\langle Q^{-1/2}Z, e_1\rangle, \ldots, \langle Q^{-1/2}Z, e_1\rangle}_{\alpha^{(l)}_{[\sigma(1)]} \text{ times}}, \underbrace{\langle Q^{-1/2}Z, e_2\rangle, \ldots, \langle Q^{-1/2}Z, e_2\rangle}_{\alpha^{(l)}_{[\sigma(2)]} \text{ times}}, \ldots, \underbrace{\langle Q^{-1/2}Z, e_{2m}\rangle, \ldots, \langle Q^{-1/2}Z, e_{2m}\rangle}_{\alpha^{(l)}_{[\sigma(2m)]} \text{ times}}),$$

$\left|\alpha^{(l)}\right| := \sum_{j=1}^m \alpha_j^{(l)}$ and where S_n denotes the permutation group of size n.

Furthermore, using an upper bound for the permanent of positive semidefinite matrices (see [7]) or direct computations, we find that

$$perm\left(\sum\right) = \sum_{\pi\in S_{2\left|\alpha^{(l)}\right|}} \prod_{i=1}^{2\left|\alpha^{(l)}\right|} a_{i\pi(i)} \leq \left(2\left|\alpha^{(l)}\right|\right)! \prod_{i=1}^{2\left|\alpha^{(l)}\right|} a_{ii}. \tag{49}$$

Let now $i \in [\sum_{k=1}^{j-1} \alpha_{[\sigma(k)]}^{(l)} + 1, \sum_{k=1}^{j} \alpha_{[\sigma(k)]}^{(l)}]$ for some arbitrary fixed $j \in \{1, \ldots, 2m\}$. Then

$$a_{ii} = E\left[\left\langle Q_r^{-1/2} Z, e_j\right\rangle \left\langle Q_r^{-1/2} Z, e_j\right\rangle\right].$$

Further, substitution yields

$$\begin{aligned}
&E\left[\left\langle Q_r^{-1/2} Z, e_j\right\rangle \left\langle Q_r^{-1/2} Z, e_j\right\rangle\right] \\
&= (\det Q_r)^{1/2} \frac{1}{(2\pi)^m} \int_{\mathbb{R}^{2m}} \langle u, e_j\rangle^2 \exp\left(-\frac{1}{2}\langle Q_r u, u\rangle\right) du_1 \ldots du_{2m} \\
&= (\det Q_r)^{1/2} \frac{1}{(2\pi)^m} \int_{\mathbb{R}^{2m}} u_j^2 \exp\left(-\frac{1}{2}\langle Q_r u, u\rangle\right) du_1 \ldots du_{2m}
\end{aligned}$$

In the next step, we want to apply Lemma B.7. Then we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^{2m}} u_j^2 \exp\left(-\frac{1}{2}\langle Q_r u, u\rangle\right) du_1 \ldots du_m \\
&= \frac{(2\pi)^{(2m-1)/2}}{(\det Q_r)^{1/2}} \int_{\mathbb{R}} v^2 \exp\left(-\frac{1}{2} v^2\right) dv \frac{1}{\sigma_j^2} \\
&= \frac{(2\pi)^m}{(\det Q_r)^{1/2}} \frac{1}{\sigma_j^2},
\end{aligned}$$

where $\sigma_j^2 := Var[B_{s_j}^{H_r} \left| B_{s_1}^{H_r}, \ldots, B_{s_{2m}}^{H_r} \text{ without } B_{s_j}^{H_r}\right]$.

We now aim at using strong local non-determinism of the form (see (24)): For all $t \in [0, T]$, $0 < r < t$:

$$Var[B_t^{H_r} \left| B_s^{H_r}, |t-s| \geq r\right] \geq K r^{2H_r}$$

for a constant K depending on H_r and T.

The latter entails that

$$(\det Q_r(s))^{1/2} \geq K^{(2m-1)/2} |s_1|^{H_r} |s_2 - s_1|^{H_r} \ldots |s_{2m} - s_{2m-1}|^{H_r}$$

as well as

$$\sigma_j^2 \geq K \min\{\left|s_j - s_{j-1}\right|^{2H_r}, \left|s_{j+1} - s_j\right|^{2H_r}\}.$$

Hence

$$\begin{aligned}
\prod_{j=1}^{2m} \sigma_j^{-2\alpha_{[\sigma(j)]}^{(l)}} &\leq K^{-2m} \prod_{j=1}^{2m} \frac{1}{\min\{\left|s_j - s_{j-1}\right|^{2H_r \alpha_{[\sigma(j)]}^{(l)}}, \left|s_{j+1} - s_j\right|^{2H_r \alpha_{[\sigma(j)]}^{(l)}}\}} \\
&\leq C^{m+|\alpha^{(l)}|} \prod_{j=1}^{2m} \frac{1}{\left|s_j - s_{j-1}\right|^{4H_r \alpha_{[\sigma(j)]}^{(l)}}}
\end{aligned}$$

for a constant C only depending on H_r and T.

So we conclude from (49) that

$$
\begin{aligned}
perm\left(\sum\right) &\leq \left(2\left|\alpha^{(l)}\right|\right)! \prod_{i=1}^{2\left|\alpha^{(l)}\right|} a_{ii} \\
&\leq \left(2\left|\alpha^{(l)}\right|\right)! \prod_{j=1}^{2m} \left((\det Q_r)^{1/2} \frac{1}{(2\pi)^m} \frac{(2\pi)^m}{(\det Q_r)^{1/2}} \frac{1}{\sigma_j^2}\right)^{\alpha^{(l)}_{[\sigma(j)]}} \\
&\leq \left(2\left|\alpha^{(l)}\right|\right)! C^{m+\left|\alpha^{(l)}\right|} \prod_{j=1}^{2m} \frac{1}{\left|s_j - s_{j-1}\right|^{4H_r\alpha^{(l)}_{[\sigma(j)]}}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
E\left[\prod_{j=1}^{2m}\left|\left\langle Q_r^{-1/2}Z, e_j\right\rangle\right|^{\alpha^{(l)}_{[\sigma(j)]}}\right] &\leq \sqrt{perm\left(\sum\right)} \\
&\leq \sqrt{(2\left|\alpha^{(l)}\right|)! C^{m+\left|\alpha^{(l)}\right|}} \prod_{j=1}^{2m} \frac{1}{\left|s_j - s_{j-1}\right|^{2H_r\alpha^{(l)}_{[\sigma(j)]}}}.
\end{aligned}
$$

Therefore we see from (47) and (48) that

$$
\begin{aligned}
&E\left[\left|\Lambda_\alpha^f(\theta, t, z)\right|^2\right] \\
&\leq C^m \sum_{\sigma\in S(m,m)} \int_{\Delta_{0,t}^{2m}} |f_\sigma(s,z)| \prod_{l=1}^{d} \int_{\mathbb{R}^{2m}} \left(\prod_{j=1}^{2m}\left|u_j^{(l)}\right|^{\alpha^{(l)}_{[\sigma(j)]}}\right) \\
&\quad \exp\left\{-\frac{1}{2}\left\langle Q_r u^{(l)}, u^{(l)}\right\rangle\right\} du_1^{(l)} \dots du_{2m}^{(l)} ds_1 \dots ds_{2m} \\
&\leq M^m \sum_{\sigma\in S(m,m)} \int_{\Delta_{0,t}^{2m}} |f_\sigma(s,z)| \frac{1}{\lambda_r^{2md}} \frac{1}{(\det Q(s))^{d/2}} \\
&\quad \prod_{l=1}^{d} \sqrt{(2\left|\alpha^{(l)}\right|)! C^{m+\left|\alpha^{(l)}\right|}} \prod_{j=1}^{2m} \frac{1}{\left|s_j - s_{j-1}\right|^{2H_r\alpha^{(l)}_{[\sigma(j)]}}} ds_1 \dots ds_{2m} \\
&= \frac{1}{\lambda_r^{2md}} M^m C^{md+|\alpha|} \prod_{l=1}^{d} \sqrt{(2\left|\alpha^{(l)}\right|)!} \sum_{\sigma\in S(m,m)} \int_{\Delta_{0,t}^{2m}} |f_\sigma(s,z)| \\
&\quad \prod_{j=1}^{2m} \frac{1}{\left|s_j - s_{j-1}\right|^{H_r(d+2\sum_{l=1}^{d}\alpha^{(l)}_{[\sigma(j)]})}} ds_1 \dots ds_{2m}
\end{aligned}
$$

for a constant M depending on d.

In the final step, we want to prove estimate (46). Using the inequality (45), we get that

$$
\begin{aligned}
&\left|E\left[\int_{(\mathbb{R}^d)^m} \Lambda_\alpha^{\varkappa f}(\theta, t, z) dz\right]\right| \\
&\leq \int_{(\mathbb{R}^d)^m} \left(E\left|\Lambda_\alpha^{\varkappa f}(\theta, t, z)\right|^2\right)^{1/2} dz \leq \frac{1}{\lambda_r^{md}} C^{m/2+|\alpha|/2} \int_{(\mathbb{R}^d)^m} (\Psi_\alpha^{\varkappa f}(\theta, t, z, H_r))^{1/2} dz.
\end{aligned}
$$

By taking the supremum over $[0, T]$ with respect to each function f_j, i.e.

$$\left|f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})\right| \leq \sup_{s_j\in[0,T]} \left|f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})\right|, \; j = 1, \dots, 2m$$

we find that

$$\begin{aligned}
&\left|E\left[\int_{(\mathbb{R}^d)^m} \Lambda_\alpha^{\varkappa f}(\theta, t, z)dz\right]\right| \\
&\leq \frac{1}{\lambda_r^{md}} C^{m/2+|\alpha|/2} \max_{\sigma\in S(m,m)} \int_{(\mathbb{R}^d)^m} \left(\prod_{l=1}^{2m} \left\| f_{[\sigma(l)]}(\cdot, z_{[\sigma(l)]})\right\|_{L^\infty([0,T])}\right)^{1/2} dz \\
&\quad \times \left(\prod_{l=1}^{d}\sqrt{(2\left|\alpha^{(l)}\right|)!} \sum_{\sigma\in S(m,m)} \int_{\Delta_{0,t}^{2m}} |\varkappa_\sigma(s)| \prod_{j=1}^{2m} \frac{1}{\left|s_j - s_{j-1}\right|^{H(d+2\sum_{l=1}^d \alpha_{[\sigma(j)]}^{(l)})}} ds_1 \dots ds_{2m}\right)^{1/2} \\
&= \frac{1}{\lambda_r^{md}} C^{m/2+|\alpha|/2} \max_{\sigma\in S(m,m)} \int_{(\mathbb{R}^d)^m} \left(\prod_{l=1}^{2m} \left\| f_{[\sigma(l)]}(\cdot, z_{[\sigma(l)]})\right\|_{L^\infty([0,T])}\right)^{1/2} dz \cdot (\Psi_\alpha^{\varkappa}(\theta, t, H_r))^{1/2} \\
&= \frac{1}{\lambda_r^{md}} C^{m/2+|\alpha|/2} \int_{(\mathbb{R}^d)^m} \prod_{j=1}^{m} \left\| f_j(\cdot, z_j)\right\|_{L^\infty([0,T])} dz \cdot (\Psi_\alpha^{\varkappa}(\theta, t, H_r))^{1/2} \\
&= \frac{1}{\lambda_r^{md}} C^{m/2+|\alpha|/2} \prod_{j=1}^{m} \left\| f_j(\cdot, z_j)\right\|_{L^1(\mathbb{R}^d; L^\infty([0,T]))} \cdot (\Psi_\alpha^{\varkappa}(\theta, t, H_r))^{1/2}.
\end{aligned}$$

□

Using Theorem 4.7 we obtain the following crucial estimate (compare [4, 5, 9, 10]):

Proposition 4.8 *Let the functions f and $\varkappa$ be as in (62), respectively as in (42). Further, let $\theta, \theta\prime, t \in [0, T], \theta\prime < \theta < t$ and*

$$\varkappa_j(s) = (K_{H_{r_0}}(s, \theta) - K_{H_{r_0}}(s, \theta\prime))^{\varepsilon_j}, \theta < s < t$$

for every $j = 1, \dots, m$ with $(\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$ for $\theta, \theta\prime \in [0, T]$ with $\theta\prime < \theta$. Let $\alpha \in (\mathbb{N}_0^d)^m$ be a multi-index. If for some $r \geq r_0$

$$H_r < \frac{\frac{1}{2} - \gamma_{r_0}}{(d - 1 + 2\sum_{l=1}^d \alpha_j^{(l)})}$$

holds for all j, where $\gamma_{r_0} \in (0, H_{r_0})$ is sufficiently small, then there exists a universal constant C_{r_0} (depending on H_{r_0}, T and d, but independent of m, $\{f_i\}_{i=1,\dots,m}$ and α) such that for any $\theta, t \in [0, T]$ with $\theta < t$ we have

$$\begin{aligned}
&\left|E\int_{\Delta_{\theta,t}^m} \left(\prod_{j=1}^m D^{\alpha_j} f_j(s_j, \mathbb{B}_{s_j})\varkappa_j(s_j)\right) ds\right| \\
&\leq \frac{1}{\lambda_r^{md}} C_{r_0}^{m+|\alpha|} \prod_{j=1}^{m} \left\| f_j(\cdot, z_j)\right\|_{L^1(\mathbb{R}^d; L^\infty([0,T]))} \left(\frac{\theta - \theta\prime}{\theta\theta\prime}\right)^{\gamma_{r_0}\sum_{j=1}^m \varepsilon_j} \theta^{(H_{r_0} - \frac{1}{2} - \gamma_{r_0})\sum_{j=1}^m \varepsilon_j} \\
&\quad \times \frac{\left(\prod_{l=1}^d (2\left|\alpha^{(l)}\right|)!\right)^{1/4} (t-\theta)^{-H_r(md+2|\alpha|) + (H_{r_0} - \frac{1}{2} - \gamma_{r_0})\sum_{j=1}^m \varepsilon_j + m}}{\Gamma\left(-H_r(2md + 4|\alpha|) + 2\left(H_{r_0} - \frac{1}{2} - \gamma_{r_0}\right)\sum_{j=1}^m \varepsilon_j + 2m\right)^{1/2}}.
\end{aligned}$$

Proof From the definition of $\Lambda_\alpha^{\varkappa f}$ (44) we see that the integral in our proposition can be expressed as

$$\int_{\Delta_{\theta,t}^m} \left(\prod_{j=1}^m D^{\alpha_j} f_j(s_j, B_{s_j}^H) \varkappa_j(s_j) \right) ds = \int_{\mathbb{R}^{dm}} \Lambda_\alpha^{\varkappa f}(\theta, t, z) dz.$$

By taking expectation and using Theorem 4.7 we get that

$$\left| E \int_{\Delta_{\theta,t}^m} \left(\prod_{j=1}^m D^{\alpha_j} f_j(s_j, B_{s_j}^H) \varkappa_j(s_j) \right) ds \right| \\ \leq \frac{1}{\lambda_r^{md}} C_r^{m/2+|\alpha|/2} \prod_{j=1}^m \left\| f_j(\cdot, z_j) \right\|_{L^1(\mathbb{R}^d; L^\infty([0,T]))} \cdot (\Psi_\alpha^\varkappa(\theta, t, H_r))^{1/2},$$

where in this case

$$\Psi_k^\varkappa(\theta, t, H_r) \\ := \prod_{l=1}^d \sqrt{(2\left|\alpha^{(l)}\right|)!} \sum_{\sigma \in S(m,m)} \int_{\Delta_{0,t}^{2m}} \prod_{j=1}^{2m} (K_{H_r}(s_j, \theta) - K_{H_r}(s_j, \theta\prime))^{\varepsilon_{[\sigma(j)]}} \\ \frac{1}{\left|s_j - s_{j-1}\right|^{H_r(d+2\sum_{l=1}^d \alpha_{[\sigma(j)]}^{(l)})}} ds_1 \ldots ds_{2m}.$$

We wish to use Lemma B.2. For this purpose, we need that $-H_r(d + 2\sum_{l=1}^d \alpha_{[\sigma(j)]}^{(l)}) + (H_{r_0} - \frac{1}{2} - \gamma_{r_0})\varepsilon_{[\sigma(j)]} > -1$ for all $j = 1, \ldots, 2m$. The worst case is, when $\varepsilon_{[\sigma(j)]} = 1$ for all j. So $H_r < \frac{\frac{1}{2} - \gamma_r}{(d-1+2\sum_{l=1}^d \alpha_{[\sigma(j)]}^{(l)})}$ for all j, since $H_{r_0} \geq H_r$. Therfore, we get that

$$\Psi_\alpha^\varkappa(\theta, t, H_r) \leq C_{r_0}^{2m} \sum_{\sigma \in S(m,m)} \left(\frac{\theta - \theta\prime}{\theta\theta\prime} \right)^{\gamma_{r_0} \sum_{j=1}^{2m} \varepsilon_{[\sigma(j)]}} \theta^{(H_{r_0} - \frac{1}{2} - \gamma_{r_0}) \sum_{j=1}^{2m} \varepsilon_{[\sigma(j)]}} \\ \times \prod_{l=1}^d \sqrt{(2\left|\alpha^{(l)}\right|)!} \Pi_\gamma(2m)(t-\theta)^{-H_r(2md+4|\alpha|) + (H_r - \frac{1}{2} - \gamma_r)\sum_{j=1}^{2m} \varepsilon_{[\sigma(j)]} + 2m},$$

where $\Pi_\gamma(m)$ is defined as in Lemma B.2 and where C_{r_0} is a constant, which only depends on H_{r_0} and T. The factor $\Pi_\gamma(m)$ has the following upper bound:

$$\Pi_\gamma(2m) \leq \frac{\prod_{j=1}^{2m} \Gamma\left(1 - H_r\left(d + 2\sum_{l=1}^d \alpha_{[\sigma(j)]}^{(l)}\right)\right)}{\Gamma\left(-H_r(2md + 4|\alpha|) + \left(H_{r_0} - \frac{1}{2} - \gamma_{r_0}\right)\sum_{j=1}^{2m} \varepsilon_{[\sigma(j)]} + 2m\right)}.$$

Note that $\sum_{j=1}^{2m} \varepsilon_{[\sigma(j)]} = 2\sum_{j=1}^{m} \varepsilon_j$. Hence, it follows that

$$
\begin{aligned}
&(\Psi_k^{\varkappa}(\theta, t, H_r))^{1/2} \\
&\le C_{r_0}^m \left(\frac{\theta-\theta\prime}{\theta\theta\prime}\right)^{\gamma_{r_0}\sum_{j=1}^m \varepsilon_j} \theta^{(H_r-\frac{1}{2}-\gamma_{r_0})\sum_{j=1}^m \varepsilon_j} \\
&\quad\times \frac{\left(\prod_{l=1}^d (2\left|\alpha^{(l)}\right|)!\right)^{1/4} (t-\theta)^{-H_r(md+2|\alpha|)-(H_{r_0}-\frac{1}{2}-\gamma_{r_0})\sum_{j=1}^m \varepsilon_j+m}}{\Gamma\left(-H_r(2md+4|\alpha|)+2\left(H_{r_0}-\frac{1}{2}-\gamma_{r_0}\right)\sum_{j=1}^m \varepsilon_j+2m\right)^{1/2}},
\end{aligned}
$$

where we used $\prod_{j=1}^{2m} \Gamma(1-H_r(d+2\sum_{l=1}^d \alpha_{[\sigma(j)]}^{(l)}) \le K^m$ for a constant $K = K(\gamma_{r_0}) > 0$ and $\sqrt{a_1+\ldots+a_m} \le \sqrt{a_1}+\ldots\sqrt{a_m}$ for arbitrary non-negative numbers $a_1, \ldots, a_m$. □

Proposition 4.9 *Let the functions f and $\varkappa$ be as in (62), respectively as in (42). Let $\theta, t \in [0, T]$ with $\theta < t$ and*

$$\varkappa_j(s) = (K_{H_{r_0}}(s, \theta))^{\varepsilon_j}, \theta < s < t$$

for every $j = 1, \ldots, m$ with $(\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m$. Let $\alpha \in (\mathbb{N}_0^d)^m$ be a multi-index. If for some $r \ge r_0$

$$H_r < \frac{\frac{1}{2}-\gamma_{r_0}}{\left(d-1+2\sum_{l=1}^d \alpha_j^{(l)}\right)}$$

holds for all j, where $\gamma_{r_0} \in (0, H_{r_0})$ is sufficiently small, then there exists a universal constant C_{r_0} (depending on H_{r_0}, T and d, but independent of m, $\{f_i\}_{i=1,\ldots,m}$ and α) such that for any $\theta, t \in [0, T]$ with $\theta < t$ we have

$$
\begin{aligned}
&\left| E \int_{\Delta_{\theta,t}^m} \left(\prod_{j=1}^m D^{\alpha_j} f_j(s_j, \mathbb{B}_{s_j})\varkappa_j(s_j)\right) ds \right| \\
&\le \frac{1}{\lambda_r^{md}} C_{r_0}^{m+|\alpha|} \prod_{j=1}^m \left\| f_j(\cdot, z_j)\right\|_{L^1(\mathbb{R}^d; L^\infty([0,T]))} \theta^{(H_{r_0}-\frac{1}{2})\sum_{j=1}^m \varepsilon_j} \\
&\quad\times \frac{\left(\prod_{l=1}^d (2\left|\alpha^{(l)}\right|)!\right)^{1/4} (t-\theta)^{-H_r(md+2|\alpha|)+(H_{r_0}-\frac{1}{2}-\gamma_{r_0})\sum_{j=1}^m \varepsilon_j+m}}{\Gamma\left(-H_r(2md+4|\alpha|)+2\left(H_{r_0}-\frac{1}{2}-\gamma_{r_0}\right)\sum_{j=1}^m \varepsilon_j+2m\right)^{1/2}}.
\end{aligned}
$$

Proof The proof is similar to the previous proposition. □

Remark 4.10 We mention that

$$\prod_{l=1}^d \left(2\left|\alpha^{(l)}\right|\right)! \le (2|\alpha|)! C^{|\alpha|}$$

for a constant C depending on d. Later on in the paper, when we deal with the existence of strong solutions, we will consider the case

$$\alpha_j^{(l)} \in \{0, 1\} \text{ for all } j, l$$

with

$$|\alpha| = m.$$

The next proposition is a verification of the sufficient condition needed to guarantee relative compactness of the approximating sequence $\{X_t^n\}_{n\geq 1}$.

Proposition 4.11 *Let $b_n : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $n\geq 1$, be a sequence of compactly supported smooth functions converging a.e. to b such that $\sup_{n\geq 1}\|b_n\|_{\mathcal{L}_{2,p}^q} < \infty$, $p,q\in(2,\infty]$. Let $X_\cdot^n$ denote the solution of (31) when we replace b by b_n. Further, let C_i for $r_0 = i$ be the (same) constant (depending only on H_i, T and d) in the estimates of Proposition 4.8 and 4.9. Then there exist sequences $\{\alpha_i\}_{i=1}^\infty$, $\beta=\{\beta_i\}_{i=1}^\infty$ (depending only on $\{H_i\}_{i=1}^\infty$) with $0<\alpha_i<\beta_i<\frac{1}{2}$, $\delta=\{\delta_i\}_{i=1}^\infty$ as in Theorem A.3 and $\lambda=\{\lambda_i\}_{i=1}^\infty$ in (28), which satisfies (25), (26), (29) and which is of the form $\lambda_i=\varphi_i\cdot\varphi(C_i)$ being independent of the size of $\sup_{n\geq 1}\|b_n\|_{\mathcal{L}_{2,p}^q}$ for a sequence $\{\varphi_i\}_{i=1}^\infty$ and a bounded function φ, such that*

$$\sum_{i=1}^{\infty}\frac{|\varphi_i|^2}{1-2^{-2(\beta_i-\alpha_i)}\delta_i^2}<\infty, \tag{50}$$

$$\sup_{n\geq 1}E[\|X_t^n\|^2]<\infty,$$

$$\sup_{n\geq 1}\sum_{i=1}^{\infty}\frac{1}{\delta_i^2}\int_0^t E[\|D_{t_0}^i X_t^n\|^2]dt_0 \leq C_1\left(\sup_{n\geq 1}\|b_n\|_{\mathcal{L}_{2,p}^q}\right)<\infty,$$

and

$$\sup_{n\geq 1}\sum_{i=1}^{\infty}\frac{1}{(1-2^{-2(\beta_i-\alpha_i)})\delta_i^2}\int_0^t\int_0^t\frac{E[\|D_{t_0}^i X_t^n - D_{t_0'}^i X_t^n\|^2]}{|t_0-t_0'|^{1+2\beta_i}}dt_0dt_0'$$
$$\leq C_2\left(\sup_{n\geq 1}\|b_n\|_{\mathcal{L}_{2,p}^q}\right)<\infty$$

for all $t\in[0,T]$, where $C_j : [0,\infty)\longrightarrow[0,\infty)$, $j=1,2$ are continuous functions depending on $\{H_i\}_{i=1}^\infty$, p, q, d, T and where D^i denotes the Malliavin derivative in the direction of the standard Brownian motion W^i, $i\geq 1$. Here, $\|\cdot\|$ denotes any matrix norm.

Remark 4.12 The proof of Proposition 4.11 shows that one may for example choose $\lambda_i = \varphi_i\cdot\varphi(C_i)$ in (28) for $\varphi(x)=\exp(-x^{100})$ and $\{\varphi_i\}_{i=1}^\infty$ satisfying (50).

Proof The most challenging estimate is the last one, the two others can be proven easily. Take $t_0, t_0' > 0$ such that $0<t_0'<t_0<t$. Using the chain rule for the Malliavin derivative, see [55, Proposition 1.2.3], we have

$$D_{t_0}^i X_t^n = \lambda_i K_{H_i}(t,t_0)I_d + \int_{t_0}^t b_n'(t_1, X_{t_1}^n)D_{t_0}X_{t_1}^n dt_1$$

P-a.s. for all $0 \leq t_0 \leq t$ where $b_n'(t,z) = \left(\frac{\partial}{\partial z_j} b_n^{(i)}(t,z)\right)_{i,j=1,\dots,d}$ denotes the Jacobian matrix of b_n at a point (t,z) and I_d the identity matrix in $\mathbb{R}^{d\times d}$. Thus we have

$$\begin{aligned}
&D_{t_0}^i X_t^n - D_{t_0'}^i X_t^n \\
&\quad = \lambda_i (K_{H_i}(t,t_0) I_d - K_{H_i}(t,t_0') I_d) \\
&\qquad + \int_{t_0}^t b_n'(t_1, X_{t_1}^n) D_{t_0}^i X_{t_1}^n dt_1 - \int_{t_0'}^t b_n'(t_1, X_{t_1}^n) D_{t_0'}^i X_{t_1}^n dt_1 \\
&\quad = \lambda_i (K_{H_i}(t,t_0) I_d - K_{H_i}(t,t_0') I_d) \\
&\qquad - \int_{t_0'}^{t_0} b_n'(t_1, X_{t_1}^n) D_{t_0'}^i X_{t_1}^n dt_1 + \int_{t_0}^t b_n'(t_1, X_{t_1}^n)(D_{t_0}^i X_{t_1}^n - D_{t_0'}^i X_{t_1}^n) dt_1 \\
&\quad = \lambda_i \mathcal{K}_{t_0,t_0'}^{H_i}(t) I_d - (D_{t_0'}^i X_{t_0}^n - \lambda_i K_{H_i}(t_0,t_0') I_d) \\
&\qquad + \int_{t_0}^t b_n'(t_1, X_{t_1}^n)(D_{t_0}^i X_{t_1}^n - D_{t_0'}^i X_{t_1}^n) dt_1,
\end{aligned}$$

where as in Proposition 4.8 we define

$$\mathcal{K}_{t_0,t_0'}^{H_i}(t) = K_{H_i}(t,t_0) - K_{H_i}(t,t_0').$$

Iterating the above equation we arrive at

$$\begin{aligned}
D_{t_0}^i X_t^n - D_{t_0'}^i X_t^n = \lambda_i \mathcal{K}_{t_0,t_0'}^{H_i}(t) I_d \\
&+ \lambda_i \sum_{m=1}^{\infty} \int_{\Delta_{t_0,t}^m} \prod_{j=1}^m b_n'(t_j, X_{t_j}^n) \mathcal{K}_{t_0,t_0'}^{H_i}(t_m) I_d dt_m \cdots dt_1 \\
&- \left(I_d + \sum_{m=1}^{\infty} \int_{\Delta_{t_0,t}^m} \prod_{j=1}^m b_n'(t_j, X_{t_j}^n) dt_m \cdots dt_1 \right) \left(D_{t_0'}^i X_{t_0}^n - \lambda_i K_{H_i}(t_0,t_0') I_d \right).
\end{aligned}$$

On the other hand, observe that one may again write

$$D_{t_0'}^i X_{t_0}^n - \lambda_i K_{H_i}(t_0,t_0') I_d = \lambda_i \sum_{m=1}^{\infty} \int_{\Delta_{t_0',t_0}^m} \prod_{j=1}^m b_n'(t_j, X_{t_j}^n)(K_{H_i}(t_m,t_0') I_d)\, dt_m \cdots dt_1.$$

In summary,

$$D_{t_0}^i X_t^n - D_{t_0'}^i X_t^n = \lambda_i I_1(t_0',t_0) + \lambda_i I_2^n(t_0',t_0) + \lambda_i I_3^n(t_0',t_0),$$

where

$$I_1(t_0',t_0) := \mathcal{K}^{H_i}_{t_0,t_0'}(t)I_d = K_{H_i}(t,t_0)I_d - K_{H_i}(t,t_0')I_d$$

$$I_2^n(t_0',t_0) := \sum_{m=1}^{\infty}\int_{\Delta^m_{t_0,t}}\prod_{j=1}^m b_n'(t_j,X^n_{t_j})\mathcal{K}^{H_i}_{t_0,t_0'}(t_m)I_d\,dt_m\cdots dt_1$$

$$I_3^n(t_0',t_0) := -\left(I_d + \sum_{m=1}^{\infty}\int_{\Delta^m_{t_0,t}}\prod_{j=1}^m b_n'(t_j,X^n_{t_j})dt_m\cdots dt_1\right)$$

$$\times\left(\sum_{m=1}^{\infty}\int_{\Delta^m_{t_0',t_0}}\prod_{j=1}^m b_n'(t_j,X^n_{t_j})(K_{H_i}(t_m,t_0')I_d)dt_m\cdots dt_1.\right).$$

Hence,

$$E[\|D^i_{t_0}X^n_t - D^i_{t_0'}X^n_t\|^2] \le C\lambda_i^2\left(E[\|I_1(t_0',t_0)\|^2] + E[\|I_2^n(t_0',t_0)\|^2] + E[\|I_3^n(t_0',t_0)\|^2]\right).$$

It follows from Lemma B.1 and condition (50) that

$$\sum_{i=1}^{\infty}\frac{\lambda_i^2}{1-2^{-2(\beta_i-\alpha_i)}\delta_i^2}\int_0^t\int_0^t\frac{\|I_1(t_0',t_0)\|^2_{L^2(\Omega)}}{|t_0-t_0'|^{1+2\beta_i}}dt_0dt_0'$$

$$\le \sum_{i=1}^{\infty}\frac{\lambda_i^2}{1-2^{-2(\beta_i-\alpha_i)}\delta_i^2}t^{4H_i-6\gamma_i-2\beta_i-1} < \infty$$

for a suitable choice of sequence $\{\beta_i\}_{i\ge 1}\subset(0,1/2)$.

Let us continue with the term $I_2^n(t_0',t_0)$. Then Theorem 3.2, Cauchy–Schwarz inequality and Lemma 4.4 imply

$$E[\|I_2^n(t_0',t_0)\|^2]$$
$$\le C(\|b_n\|_{L_p^q})E\left[\left\|\sum_{m=1}^{\infty}\int_{\Delta^m_{t_0,t}}\prod_{j=1}^m b_n'(t_j,x+\mathbb{B}^H_{t_j})\mathcal{K}^{H_i}_{t_0,t_0'}(t_m)I_d\,dt_m\cdots dt_1\right\|^4\right]^{1/2},$$

where $C:[0,\infty)\to[0,\infty)$ is the function from Lemma 4.4. Taking the supremum over n we have

$$\sup_{n\ge 0}C(\|b_n\|_{L_p^q}) =: C_1 < \infty.$$

Let $\|\cdot\|$ from now on denote the matrix norm in $\mathbb{R}^{d\times d}$ such that $\|A\| = \sum_{i,j=1}^d |a_{ij}|$ for a matrix $A=\{a_{ij}\}_{i,j=1,\dots,d}$, then we have

$$E[\|I_2^n(t_0',t_0)\|^2]$$
$$\le C_1\left(\sum_{m=1}^{\infty}\sum_{j,k=1}^d\sum_{l_1,\dots,l_{m-1}=1}^d\left\|\int_{\Delta^m_{t_0,t}}\frac{\partial}{\partial x_{l_1}}b_n^{(j)}\left(t_1,x+\mathbb{B}^H_{t_1}\right)\right.\right.$$
$$\left.\left.\times\frac{\partial}{\partial x_{l_2}}b_n^{(l_1)}\left(t_2,x+\mathbb{B}^H_{t_2}\right)\cdots\frac{\partial}{\partial x_k}b_n^{(l_{m-1})}\left(t_m,x+\mathbb{B}^H_{t_m}\right)\mathcal{K}^{H_i}_{t_0,t_0'}(t_m)dt_m\cdots dt_1\right\|_{L^4(\Omega,\mathbb{R})}\right)^2. \tag{51}$$

Now, the aim is to shuffle the four integrals above. Denote

$$J_2^n(t_0', t_0) := \int_{\Delta^m_{t_0,t}} \frac{\partial}{\partial x_{l_1}} b_n^{(j)}\left(t_1, x + \mathbb{B}^H_{t_1}\right) \cdots \frac{\partial}{\partial x_k} b_n^{(l_{m-1})}\left(t_m, x + \mathbb{B}^H_{t_m}\right) \mathcal{K}^{H_i}_{t_0,t_0'}(t_m) dt. \tag{52}$$

Then, shuffling $J_2^n(t_0', t_0)$ as shown in (17), one can write $(J_2^n(t_0', t_0))^2$ as a sum of at most 2^{2m} summands of length $2m$ of the form

$$\int_{\Delta^{2m}_{t_0,t}} g_1^n\left(t_1, x + \mathbb{B}^H_{t_1}\right) \cdots g_{2m}^n\left(t_{2m}, x + \mathbb{B}^H_{t_{2m}}\right) dt_{2m} \cdots dt_1,$$

where for each $l = 1, \ldots, 2m$,

$$g_l^n(\cdot, x + \mathbb{B}^H_\cdot) \in \left\{ \frac{\partial}{\partial x_k} b_n^{(j)}(\cdot, x + \mathbb{B}^H_\cdot), \frac{\partial}{\partial x_k} b_n^{(j)}(\cdot, x + \mathbb{B}^H_\cdot)\mathcal{K}^{H_i}_{t_0,t_0'}(\cdot),\ j, k = 1, \ldots, d \right\}.$$

Repeating this argument once again, we find that $J_2^n(t_0', t_0)^4$ can be expressed as a sum of, at most, 2^{8m} summands of length $4m$ of the form

$$\int_{\Delta^{4m}_{t_0,t}} g_1^n\left(t_1, x + \mathbb{B}^H_{t_1}\right) \cdots g_{4m}^n\left(t_{4m}, x + \mathbb{B}^H_{t_{4m}}\right) dt_{4m} \cdots dt_1, \tag{53}$$

where for each $l = 1, \ldots, 4m$,

$$g_l^n(\cdot, x + \mathbb{B}^H_\cdot) \in \left\{ \frac{\partial}{\partial x_k} b_n^{(j)}(\cdot, x + \mathbb{B}^H_\cdot), \frac{\partial}{\partial x_k} b_n^{(j)}(\cdot, x + \mathbb{B}^H_\cdot)\mathcal{K}^{H_i}_{t_0,t_0'}(\cdot),\ j, k = 1, \ldots, d \right\}.$$

It is important to note that the function $\mathcal{K}^{H_i}_{t_0,t_0'}(\cdot)$ appears only once in term (52) and hence only four times in term (53). So there are indices $j_1, \ldots, j_4 \in \{1, \ldots, 4m\}$ such that we can write (53) as

$$\int_{\Delta^{4m}_{t_0,t}} \left(\prod_{j=1}^{4m} b_j^n(t_j, x + \mathbb{B}^H_{t_j}) \right) \prod_{l=1}^{4} \mathcal{K}^{H_i}_{t_0,t_0'}(t_{j_l}) dt_{4m} \cdots dt_1,$$

where

$$b_l^n(\cdot, x + \mathbb{B}^H_\cdot) \in \left\{ \frac{\partial}{\partial x_k} b_n^{(j)}(\cdot, x + \mathbb{B}^H_\cdot),\ j, k = 1, \ldots, d \right\}, \quad l = 1, \ldots, 4m.$$

The latter enables us to use the estimate from Proposition 4.8 for $\sum_{r=1}^{4m} \varepsilon_r = 4$, $|\alpha| = 4m$, $\sum_{l=1}^{d} \alpha_j^{(l)} = 1$ for all l, $H_r < \frac{1}{2(d+2)}$ for some $r \geq i$ combined with Remark 4.10. Thus we obtain that

$$\begin{aligned} &\left(E(J_2^n(t_0', t_0))^4\right)^{1/4} \\ &\leq \frac{1}{\lambda_r^{md}} C_i^{2m} \|b_n\|^m_{L^1(\mathbb{R}^d; L^\infty([0,T]))} \left| \frac{t_0 - t_0'}{t_0 t_0'} \right|^{\gamma_i} t_0^{(H_i - \frac{1}{2} - \gamma_i)} \\ &\times \frac{C(d)^m ((8m)!)^{1/16} |t - t_0|^{-H_r(md+2m) + (H_i - \frac{1}{2} - \gamma_i) + m}}{\Gamma(-H_r(2 \cdot 4md + 4 \cdot 4m) + 2(H_i - \frac{1}{2} - \gamma_i) + 8m)^{1/8}} \end{aligned}$$

for a constant $C(d)$ depending only on d.

Then the series in (51) is summable over $j, k, l_1, \ldots, l_{m-1}$ and m. Hence, we just need to verify that the double integral is finite for suitable γ_i's and β_i's. Indeed,

$$\int_0^t \int_0^t \frac{|t_0 - t_0'|^{2\gamma_i - 1 - 2\beta_i}}{|t_0 t_0'|^{2\gamma_i}} t_0^{2\left(H_i - \frac{1}{2} - \gamma_i\right)} |t - t_0|^{-2\left(H_i - \frac{1}{2} - \gamma_i\right)} dt_0 dt_0' < \infty,$$

whenever $2\left(H_i - \frac{1}{2} - \gamma_i\right) > -1$, $2\gamma_i - 1 - 2\beta_i > -1$ and $2\left(H_i - \frac{1}{2} - \gamma_i\right) - 2\gamma_i > -1$ which is fulfilled if for instance $\gamma_i < H_i/4$ and $0 < \beta_i < \gamma_i$.

Now we may choose for example a function φ with $\varphi(x) = \exp(-x^{100})$. In this case, we find that

$$C_i^{2m} \lambda_i = \varphi_i C_i^{2m} \varphi(C_i) \le \varphi_i \left(\frac{1}{50}\right)^{\frac{m}{50}} m^{\frac{m}{50}}$$

So, finally, if H_r for a fixed $r \ge i$ is sufficiently small, the sums over $i \ge 1$ also converge since we have φ_i satisfying (50).

For the term I_3^n we may use Theorem 3.2, Cauchy–Schwarz inequality twice and observe that the first factor of I_3^n is bounded uniformly in $t_0, t \in [0, T]$ by a simple application of Proposition 4.9 with $\varepsilon_j = 0$ for all j. Then, the remaining estimate is fairly similar to the case of I_2^n by using Proposition 4.9 again. As for the estimate for the Malliavin derivative the reader may agree that the arguments are analogous. □

The following is a consequence of combining Lemma 4.6 and Proposition 4.11.

Corollary 4.13 *For every $t \in [0, T]$ and continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with at most linear growth we have*

$$\varphi(X_t^n) \xrightarrow{n \to \infty} \varphi(E[X_t|\mathcal{F}_t])$$

strongly in $L^2(\Omega)$. In addition, $E[X_t|\mathcal{F}_t]$ is Malliavin differentiable along any direction W^i, $i \ge 1$ of $\mathbb{B}_\cdot^H$. Moreover, the solution X is $\mathcal{F}$-adapted, thus being a strong solution.

Proof This is a direct consequence of the relative compactness from Theorem A.3 combined with Proposition 4.11 and by Lemma 4.6, we can identify the limit as $E[X_t|\mathcal{F}_t]$. Then the convergence holds for any bounded continuous functions as well. The Malliavin differentiability of $E[X_t|\mathcal{F}_t]$ is verified by taking $\varphi = I_d$ and the second estimate in Proposition 4.11 in connection with [55, Proposition 1.2.3]. □

Finally, we can complete step (4) of our scheme.

Corollary 4.14 *The constructed solution $X_\cdot$ of (31) is strong.*

Proof We have to show that X_t is $\mathcal{F}_t$-measurable for every $t \in [0, T]$ and by Remark 4.5 we see that there exists a strong solution in the usual sense, which is Malliavin differentiable. In proving this, let φ be a globally Lipschitz continuous function. Then it follows from Corollary 4.13 that there exists a subsequence n_k, $k \ge 0$, that

$$\varphi(X_t^{n_k}) \to \varphi(E[X_t|\mathcal{F}_t]), \quad P - a.s.$$

as $k \to \infty$.

Further, by Lemma 4.6 we also know that

$$\varphi(X_t^n) \to E\left[\varphi(X_t)|\mathcal{F}_t\right]$$

weakly in $L^2(\Omega)$. By the uniqueness of the limit we immediately obtain that

$$\varphi\left(E[X_t|\mathcal{F}_t]\right) = E\left[\varphi(X_t)|\mathcal{F}_t\right], \quad P-a.s.$$

which implies that X_t is $\mathcal{F}_t$-measurable for every $t \in [0, T]$. □

Finally, we turn to step (5) and complete this Section by showing pathwise uniqueness. Following the same argument as in [60, Chapter IX, Exercise (1.20)] we see that strong existence and uniqueness in law implies pathwise uniqueness. The argument does not rely on the process being a semimartingale. Hence, uniqueness in law is enough. The following Lemma actually implies the desired uniqueness by estimate (37) in connection with [40, Theorem 7.7].

Lemma 4.15 *Let X be a strong solution of (31) where $b \in L_p^q$, $p, q \in (2, \infty]$. Then the estimates (33) and (34) hold for X in place of $\mathbb{B}^H_\cdot$. As a consequence, uniqueness in law holds for Eq. (31) and since X strong, pathwise uniqueness follows.*

Proof Assume first that b is bounded. Fix any $n \geq 1$ and set

$$\eta_s^n = K_{H_n}^{-1}\left(\frac{1}{\lambda_n}\int_0^{\cdot} b(r, X_r)dr\right)(s).$$

Since b is bounded it is easy to see from (36) by changing $\mathbb{B}^H_\cdot$ with X and bounding b that for every $\kappa \in \mathbb{R}$,

$$E_{\widetilde{P}}\left[\exp\left\{-2\kappa\int_0^T (\eta_s^n)^* dW_s^n - 2\kappa^2\int_0^T |\eta_s^n|^2 ds\right\}\right] = 1, \tag{54}$$

where

$$\frac{d\widetilde{P}}{dP} = \exp\left\{-\int_0^T (\eta_s^n)^* dW_s^n - \frac{1}{2}\int_0^T |\eta_s^n|^2 ds\right\}.$$

Hence, $X_t - x$ is a regularizing fractional Brownian motion with Hurst sequence H under $\widetilde{P}$. Define

$$\xi_T^\kappa := \exp\left\{-\kappa\int_0^T (\eta_s^n)^* dW_s^n - \frac{\kappa}{2}\int_0^T |\eta_s^n|^2 ds\right\}.$$

Then,

$$\begin{aligned}
E_{\widetilde{P}}\left[\xi_T^\kappa\right] &= E_{\widetilde{P}}\left[\exp\left\{-\kappa\int_0^T (\eta_s^n)^* dW_s^n - \frac{\kappa}{2}\int_0^T |\eta_s^n|^2 ds\right\}\right]\\
&= E_{\widetilde{P}}\left[\exp\left\{-\kappa\int_0^T (\eta_s^n)^* dW_s^n - \kappa^2\int_0^T |\eta_s^n|^2 ds\right\}\exp\left\{\left(\kappa^2 + \frac{\kappa}{2}\right)\int_0^T |\eta_s^n|^2 ds\right\}\right]\\
&\leq \left(E_{\widetilde{P}}\left[\exp\left\{2\left|\kappa^2 + \frac{\kappa}{2}\right|\int_0^T |\eta_s^n|^2 ds\right\}\right]\right)^{1/2}
\end{aligned}$$

in view of (54).

On the other hand, using (38) with X in place of $\mathbb{B}^H_\cdot$ we have

$$\int_0^T |\eta_s|^2 ds \leq C_{\varepsilon,\lambda_n,H_n,T}\left(1 + \int_0^T |b(r, X_r)|^{\frac{1+\varepsilon}{\varepsilon}} dr\right), \quad P-a.s.$$

for any $\varepsilon \in (0, 1)$. Hence, applying Lemma 4.2 we get

$$E_{\widetilde{P}}\left[\xi_T^\kappa\right] \le e^{\left|\kappa^2+\frac{\kappa}{2}\right| C_{\varepsilon,\lambda_n,H_n,T}} \left(A\left(C_{\varepsilon,\lambda_n,H_n,T}\left|\kappa^2+\frac{\kappa}{2}\right| \||b|^{\frac{1+\varepsilon}{\varepsilon}}\|_{L_p^q}\right)\right)^{1/2},$$

where A is the analytic function from Lemma 4.2.

Furthermore, observe that for every $\kappa \in \mathbb{R}$ we have

$$E_P[\xi_T^\kappa] = E_{\widetilde{P}}[\xi_T^{\kappa-1}]. \tag{55}$$

In fact, (55) holds for any $b \in L_p^q$ by considering $b_n := b\mathbf{1}_{\{|b|\le n\}}$, $n \ge 1$ and then letting $n \to \infty$.

Finally, let $\delta \in (0, 1)$ and apply Hölder's inequality in order to get

$$E_P\left[\int_0^T h(t, X_t)dt\right] \le T^\delta \left(E_{\widetilde{P}}\left[(\xi_T^1)^{\frac{1+\delta}{\delta}}\right]\right)^{\frac{\delta}{1+\delta}} \left(E_{\widetilde{P}}\left[\int_0^T h(t, X_t)^{1+\delta}dt\right]\right)^{\frac{1}{1+\delta}},$$

and

$$E_P\left[\exp\left\{\int_0^T h(t, X_t)dt\right\}\right] \le T^\delta \left(E_{\widetilde{P}}\left[(\xi_T^1)^{\frac{1+\delta}{\delta}}\right]\right)^{\frac{\delta}{1+\delta}} \left(E_{\widetilde{P}}\left[\exp\left\{(1+\delta)\int_0^T h(t, X_t)dt\right\}\right]\right)^{\frac{1}{1+\delta}},$$

for every Borel measurable function. Since we know that $X_t - x$ is a regularizing fractional Brownian motion with Hurst sequence H under $\widetilde{P}$, the result follows by Lemma 4.2 by choosing δ close enough to 0. □

Using all the previous intermediate results, we are now able to state the main result of this Section:

Theorem 4.16 *Retain the conditions for $\lambda = \{\lambda_i\}_{i\ge1}$ with respect to $\mathbb{B}_\cdot^H$ in Theorem 4.11. Let $b \in \mathcal{L}_{2,p}^q$, $p, q \in (2, \infty]$. Then there exists a unique (global) strong solution X_t, $0 \le t \le T$ of Eq. (31). Moreover, for every $t \in [0, T]$, X_t is Malliavin differentiable in each direction of the Brownian motions W^n, $n \ge 1$ in (27).*

5 Infinitely Differentiable Flows for Irregular Vector Fields

From now on, we denote by $X_t^{s,x}$ the solution to the following SDE driven by a regularizing fractional Brownian motion $\mathbb{B}_\cdot^H$ with Hurst sequence H:

$$dX_t^{s,x} = b(t, X_t^{s,x})dt + d\mathbb{B}_t^H, \quad s, t \in [0, T], \quad s \le t, \quad X_s^{s,x} = x \in \mathbb{R}^d.$$

We will then assume the hypotheses from Theorem 4.16 on b and H.

The next estimate essentially tells us that the stochastic mapping $x \mapsto X_t^{s,x}$ is P-a.s. infinitely many times continuously differentiable. In particular, it shows that the strong solution constructed in the former section, in addition to being Malliavin differentiable, is also smooth in x and, although we will not prove it explicitly here, it is also smooth in the Malliavin sense, and since Hörmander's condition is met then implies that the densities of the marginals are also smooth.

Theorem 5.1 *Let $b \in C_c^\infty((0, T) \times \mathbb{R}^d)$. Fix integers $p \ge 2$ and $k \ge 1$. Choose a r such that $H_r < \frac{1}{(d-1+2k)}$. Then there exists a continuous function $C_{k,d,H_r,p,\overline{p},\overline{q},T} : [0, \infty)^2 \to [0, \infty)$,* *depending on $k, d, H_r, p, \overline{p}, \overline{q}$ and T.*

$$\sup_{s,t\in[0,T]}\sup_{x\in\mathbb{R}^d} \mathrm{E}\left[\left\|\frac{\partial^k}{\partial x^k}X_t^{s,x}\right\|^p\right] \leq C_{k,d,H_r,p,\overline{p},\overline{q},T}\left(\|b\|_{L_{\overline{p}}^{\overline{q}}}, \|b\|_{L_\infty^1}\right).$$

Proof For notational simplicity, let $s = 0$, $\mathbb{B}_\cdot = \mathbb{B}_\cdot^H$ and let X_t^x, $0 \leq t \leq T$ be the solution with respect to the vector field $b \in C_c^\infty((0,T)\times\mathbb{R}^d)$. We know that the stochastic flow associated with the smooth vector field b is smooth, too (compare to e.g. [38]). Hence, we get that

$$\frac{\partial}{\partial x}X_t^x = I_d + \int_s^t Db(u, X_u^x)\cdot\frac{\partial}{\partial x}X_u^x du, \tag{56}$$

where $Db(u, \dot{)} : \mathbb{R}^d \longrightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ is the derivative of b with respect to the space variable.

By using Picard iteration, we see that

$$\frac{\partial}{\partial x}X_t^x = I_d + \sum_{m\geq 1}\int_{\Delta_{0,t}^m} Db(u, X_{u_1}^x)\ldots Db(u, X_{u_m}^x)du_m\ldots du_1, \tag{57}$$

where

$$\Delta_{s,t}^m = \{(u_m, \ldots u_1) \in [0,T]^m : \theta < u_m < \cdots < u_1 < t\}.$$

By applying dominated convergence, we can differentiate both sides with respect to x and find that

$$\frac{\partial^2}{\partial x^2}X_t^x = \sum_{m\geq 1}\int_{\Delta_{0,t}^m}\frac{\partial}{\partial x}\left[Db(u, X_{u_1}^x)\ldots Db(u, X_{u_m}^x)\right]du_m\ldots du_1.$$

Further, the Leibniz and chain rule yield

$$\begin{aligned}&\frac{\partial}{\partial x}\left[Db(u_1, X_{u_1}^x)\ldots Db(u_m, X_{u_m}^x)\right]\\&= \sum_{r=1}^m Db(u_1, X_{u_1}^x)\ldots D^2b(u_r, X_{u_r}^x)\frac{\partial}{\partial x}X_{u_r}^x\ldots Db(u_m, X_{u_m}^x),\end{aligned}$$

where $D^2b(u,\cdot) = D(Db(u,\cdot)) : \mathbb{R}^d \longrightarrow L(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d))$.

Therefore (57) entails

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} X_t^x &= \sum_{m_1 \geq 1} \int_{\Delta_{0,t}^{m_1}} \sum_{r=1}^{m_1} Db(u_1, X_{u_1}^x) \dots D^2 b(u_r, X_{u_r}^x) \\
&\quad \times \left(I_d + \sum_{m_2 \geq 1} \int_{\Delta_{0,u_r}^{m_2}} Db(v_1, X_{v_1}^x) \dots Db(v_{m_2}, X_{v_{m_2}}^x) dv_{m_2} \dots dv_1 \right) \\
&\quad \times Db(u_{r+1}, X_{u_{r+1}}^x) \dots Db(u_{m_1}, X_{u_{m_1}}^x) du_{m_1} \dots du_1 \\
&= \sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \int_{\Delta_{0,t}^{m_1}} Db(u_1, X_{u_1}^x) \dots D^2 b(u_r, X_{u_r}^x) \dots Db(u_{m_1}, X_{u_{m_1}}^x) du_{m_1} \dots du_1 \\
&\quad + \sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \sum_{m_2 \geq 1} \int_{\Delta_{0,t}^{m_1}} \int_{\Delta_{0,u_r}^{m_2}} Db(u_1, X_{u_1}^x) \dots D^2 b(u_r, X_{u_r}^x) \\
&\quad \times Db(v_1, X_{v_1}^x) \dots Db(v_{m_2} X_{v_{m_2}}^x) Db(u_{r+1}, X_{u_{r+1}}^x) \dots Db(u_{m_1}, X_{u_{m_1}}^x) \\
&\quad dv_{m_2} \dots dv_1 du_{m_1} \dots du_1 \\
&=: I_1 + I_2. \qquad (58)
\end{aligned}$$

In the next step, we wish to employ Lemma B.8 (in connection with shuffling in Sect. 2.2) to the term I_2 in (58) and get that

$$I_2 = \sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \sum_{m_2 \geq 1} \int_{\Delta_{0,t}^{m_1+m_2}} \mathcal{H}_{m_1+m_2}^X(u) du_{m_1+m_2} \dots du_1 \qquad (59)$$

for $u = (u_1, \dots, u_{m_1+m_2})$, where the integrand $\mathcal{H}_{m_1+m_2}^X(u) \in \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$ has entries given by sums of at most $C(d)^{m_1+m_2}$ terms, which are products of length m_1+m_2 of functions being elements of the set

$$\left\{ \frac{\partial^{\gamma^{(1)}+\ldots+\gamma^{(d)}}}{\partial^{\gamma^{(1)}} x_1 \dots \partial^{\gamma^{(d)}} x_d} b^{(r)}(u, X_u^x),\ r = 1, \dots, d,\ \gamma^{(1)} + \dots + \gamma^{(d)} \leq 2,\ \gamma^{(l)} \in \mathbb{N}_0,\ l = 1, \dots, d \right\}.$$

Here it is important to mention that second order derivatives of functions in those products of functions on $\Delta_{0,t}^{m_1+m_2}$ in (59) only occur once. Hence the total order of derivatives $|\alpha|$ of those products of functions in connection with Lemma B.8 in the Appendix is

$$|\alpha| = m_1 + m_2 + 1. \qquad (60)$$

Let us now choose $p, c, r \in [1, \infty)$ such that $cp = 2^q$ for some integer q and $\frac{1}{r} + \frac{1}{c} = 1$. Then we can employ Hö lder's inequality and Girsanov's theorem (see Theorem 2.2) combined with Lemma 4.4 and obtain that

$$\begin{aligned}
&E[\|I_2\|^p] \\
&\leq C(\|b\|_{L_{\overline{p}}^{\overline{q}}}) \left(\sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \sum_{m_2 \geq 1} \sum_{i \in I} \left\| \int_{\Delta_{0,t}^{m_1+m_2}} \mathcal{H}_i^{\mathbb{B}}(u) du_{m_1+m_2} \dots du_1 \right\|_{L^{2^q}(\Omega;\mathbb{R})} \right)^p \qquad (61)
\end{aligned}$$

where $C : [0,\infty) \longrightarrow [0,\infty)$ is a continuous function depending on $p, \overline{p}$ and $\overline{q}$. Here $\#I \le K^{m_1+m_2}$ for a constant $K = K(d)$ and the integrands $\mathcal{H}_i^{\mathbb{B}}(u)$ are of the form

$$\mathcal{H}_i^{B^H}(u) = \prod_{l=1}^{m_1+m_2} h_l(u_l), h_l \in \Lambda, l = 1, \ldots, m_1+m_2$$

where

$$\Lambda := \left\{ \begin{array}{l} \frac{\partial^{\gamma^{(1)}+\cdots+\gamma^{(d)}}}{\partial^{\gamma^{(1)}} x_1 \ldots \partial^{\gamma^{(d)}} x_d} b^{(r)}(u, x+\mathbb{B}_u),\ r = 1, \ldots, d, \\ \gamma^{(1)} + \cdots + \gamma^{(d)} \le 2,\ \gamma^{(l)} \in \mathbb{N}_0,\ l = 1, \cdots, d \end{array} \right\}.$$

As above we observe that functions with second order derivatives only occur once in those products.

Let

$$J = \left(\int_{\Delta_{0,t}^{m_1+m_2}} \mathcal{H}_i^{\mathbb{B}}(u) du_{m_1+m_2} \ldots du_1 \right)^{2^q}.$$

By using shuffling (see Sect. 2.2) once more, successively, we find that J has a reprsentation as a sum of, at most of length $K(q)^{m_1+m_2}$ with summands of the form

$$\int_{\Delta_{0,t}^{2^q(m_1+m_2)}} \prod_{l=1}^{2^q(m_1+m_2)} f_l(u_l) du_{2^q(m_1+m_2)} \ldots du_1, \tag{62}$$

where $f_l \in \Lambda$ for all l.

Note that the number of factors f_l in the above product, which have a second order derivative, is exactly 2^q. Hence the total order of the derivatives in (62) in connection with Lemma B.8 (where one in that Lemma formally replaces X_u^x by $x + \mathbb{B}_u$ in the corresponding terms) is

$$|\alpha| = 2^q(m_1 + m_2 + 1). \tag{63}$$

We now aim at using Theorem 4.9 for $m = 2^q(m_1+m_2)$ and $\varepsilon_j = 0$ and find that

$$\begin{aligned} &\left| E\left[\int_{\Delta_{0,t}^{2^q(m_1+m_2)}} \prod_{l=1}^{2^q(m_1+m_2)} f_l(u_l) du_{2^q(m_1+m_2)} \ldots du_1 \right] \right| \\ &\le C^{m_1+m_2} (\|b\|_{L_\infty^1})^{2^q(m_1+m_2)} \\ &\quad\times \frac{((2(2^q(m_1+m_2+1))!)^{1/4}}{\Gamma(-H_r(2d2^q(m_1+m_2) + 42^q(m_1+m_2+1)) + 22^q(m_1+m_2))^{1/2}} \end{aligned}$$

for a constant C depending on H_r, T, d and q.

Therefore the latter combined with (61) implies that

$$\begin{aligned} &E[\|I_2\|^p] \\ &\le C(\|b\|_{L_{\overline{p}}^{\overline{q}}}) \left(\sum_{m_1 \ge 1} \sum_{m_2 \ge 1} K^{m_1+m_2} (\|b\|_{L_\infty^1})^{2^q(m_1+m_2)} \right. \\ &\quad \left. \times \frac{((2(2^q(m_1+m_2+1))!)^{1/4}}{\Gamma(-H_r(2d2^q(m_1+m_2) + 42^q(m_1+m_2+1)) + 22^q(m_1+m_2))^{1/2}})^{1/2^q} \right)^p \end{aligned}$$

for a constant K depending on H_r, T, d, p and q.

Since $\frac{1}{2(d+3)} \leq \frac{1}{2(d+2\frac{m_1+m_2+1}{m_1+m_2})}$ for $m_1,\ m_2 \geq 1$, one concludes that the above sum converges, whenever $H_r < \frac{1}{2(d+3)}$.

Further, one gets an estimate for $E[\|I_1\|^p]$ by using similar reasonings as above. In summary, we obtain the proof for $k = 2$.

We now give an explanation how we can generalize the previous line of reasoning to the case $k \geq 2$: In this case, we we have that

$$\frac{\partial^k}{\partial x^k} X_t^x = I_1 + \cdots + I_{2^{k-1}}, \tag{64}$$

where each I_i, $i = 1, \ldots, 2^{k-1}$ is a sum of iterated integrals over simplices of the form $\Delta_{0,u}^{m_j}$, $0 < u < t$, $j = 1, \ldots, k$ with integrands having at most one product factor $D^k b$, while the other factors are of the form $D^j b$, $j \leq k-1$.

In the following we need the following notation: For multi-indices $m. = (m_1, \ldots, m_k)$ and $r := (r_1, \ldots, r_{k-1})$, set

$$m_j^- := \sum_{i=1}^{j} m_i$$

and

$$\sum_{\substack{m\geq 1\\ r_l \leq m_l^-\\ l=1,\ldots,k-1}} := \sum_{m_1\geq 1}\sum_{r_1=1}^{m_1}\sum_{m_2\geq 1}\sum_{r_2=1}^{m_2^-}\cdots\sum_{r_{k-1}=1}^{m_{k-1}^-}\sum_{m_k\geq 1}.$$

In what follows, without loss of generality we confine ourselves to deriving an estimate with respect to the summand $I_{2^{k-1}}$ in (64). Just as in the case $k = 2$, we obtain by employing Lemma B.8 (in connection with shuffling in Sect. 2.2) that

$$I_{2^{k-1}} = \sum_{\substack{m\geq 1\\ r_l \leq m_l^-\\ l=1,\ldots,k-1}} \int_{\Delta_{0,t}^{m_1+\ldots+m_k}} \mathcal{H}_{m_1+\ldots+m_k}^X(u) du_{m_1+m_2} \ldots du_1 \tag{65}$$

for $u = (u_{m_1+\ldots+m_k}, \ldots, u_1)$, where the integrand $\mathcal{H}_{m_1+\ldots+m_k}^X(u) \in \otimes_{j=1}^{k+1}\mathbb{R}^d$ has entries, which are given by sums of at most $C(d)^{m_1+\ldots+m_k}$ terms. Those terms are given by products of length $m_1 + \ldots m_k$ of functions, which are elements of the set

$$\left\{\begin{array}{c}\frac{\partial^{\gamma^{(1)}+\cdots+\gamma^{(d)}}}{\partial^{\gamma^{(1)}}x_1\ldots\partial^{\gamma^{(d)}}x_d} b^{(r)}(u, X_u^x), r = 1, \ldots, d,\\ \gamma^{(1)} + \cdots + \gamma^{(d)} \leq k, \gamma^{(l)} \in \mathbb{N}_0, l = 1, \ldots, d\end{array}\right\}.$$

Exactly as in the case $k = 2$ we can invoke Lemma B.8 in the Appendix and get that the total order of derivatives $|\alpha|$ of those products of functions is

$$|\alpha| = m_1 + \ldots + m_k + k - 1. \tag{66}$$

Then we can adopt the line of reasoning as before and choose $p, c, r \in [1, \infty)$ such that $cp = 2^q$ for some integer q and $\frac{1}{r} + \frac{1}{c} = 1$ and find by applying Hölder's inequality and

Girsanov's theorem (see Theorem 2.2) combined with Lemma 4.4 that

$$E[\| I_{2^{k-1}} \|^p] \leq C(\|b\|_{L^{\overline{q}}_{\overline{p}}}) \left(\sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1,\ldots,k-1}} \sum_{i \in I} \left\| \int_{\Delta_{0,t}^{m_1+m_2}} \mathcal{H}_i^{\mathbb{B}}(u) du_{m_1+\cdots+m_k} \ldots du_1 \right\|_{L^{2^q}(\Omega;\mathbb{R})} \right)^p, \quad (67)$$

where $C : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function depending on $p, \overline{p}$ and $\overline{q}$. Here $\#I \leq K^{m_1+\ldots+m_k}$ for a constant $K = K(d)$ and the integrands $\mathcal{H}_i^{\mathbb{B}}(u)$ take the form

$$\mathcal{H}_i^{\mathbb{B}}(u) = \prod_{l=1}^{m_1+\cdots+m_k} h_l(u_l), \ h_l \in \Lambda, \ l = 1, \ldots, m_1 + \cdots + m_k,$$

where

$$\Lambda := \left\{ \begin{array}{l} \frac{\partial^{\gamma^{(1)}+\cdots+\gamma^{(d)}}}{\partial^{\gamma^{(1)}} x_1 \ldots \partial^{\gamma^{(d)}} x_d} b^{(r)}(u, x + \mathbb{B}_u), \ r = 1, \ldots, d, \\ \gamma^{(1)} + \cdots + \gamma^{(d)} \leq k, \ \gamma^{(l)} \in \mathbb{N}_0, \ l = 1, \ldots, d \end{array} \right\}.$$

Define

$$J = \left(\int_{\Delta_{0,t}^{m_1+\ldots+m_k}} \mathcal{H}_i^{\mathbb{B}}(u) du_{m_1+\cdots+m_k} \ldots du_1 \right)^{2^q}.$$

Once more, repeated shuffling (see Sect. 2.2) shows that J can be represented as a sum of, at most of length $K(q)^{m_1+\ldots m_k}$ with summands of the form

$$\int_{\Delta_{0,t}^{2^q(m_1+\cdots+m_k)}} \prod_{l=1}^{2^q(m_1+\cdots+m_k)} f_l(u_l) du_{2^q(m_1+\cdots+m_k)} \ldots du_1, \quad (68)$$

where $f_l \in \Lambda$ for all l.

By applying Lemma B.8 again (where one in that Lemma formally replaces X_u^x by $x + B_u^H$ in the corresponding expressions) we obtain that the total order of the derivatives in the products of functions in (68) is given by

$$|\alpha| = 2^q(m_1 + \cdots + m_k + k - 1). \quad (69)$$

Then Proposition 4.9 for $m = 2^q(m_1 + \ldots + m_k)$ and $\varepsilon_j = 0$ yields that

$$\left| E\left[\int_{\Delta_{0,t}^{2^q(m_1+\cdots+m_k)}} \prod_{l=1}^{2^q(m_1+\cdots+m_k)} f_l(u_l) du_{2^q(m_1+\cdots+m_k)} \ldots du_1 \right] \right|$$
$$\leq C^{m_1+\cdots+m_k} (\|b\|_{L^1_\infty})^{2^q(m_1+\cdots+m_k)}$$
$$\times \frac{((2(2^q(m_1 + \ldots + m_k + k - 1))!)^{1/4}}{\Gamma(-H_r(2d2^q(m_1 + \cdots + m_k) + 42^q(m_1 + \cdots + m_k + k - 1)) + 22^q(m_1 + \cdots + m_k))^{1/2}}$$

for a constant C depending on H_r, T, d and q.

Thus we can conclude from (67) that

$$E[\| I_{2k-1} \|^p] \le C(\|b\|_{L^{\frac{q}{p}}}) \left(\sum_{m_1 \ge 1} \cdots \sum_{m_k \ge 1} K^{m_1+\cdots+m_k} (\|b\|_{L^1_\infty})^{2^q(m_1+\cdots+m_k)} \right.$$
$$\left. \times \frac{((2(2^q(m_1+\cdots+m_k+k-1))!)^{1/4}}{\Gamma(-H_r(2d2^q(m_1+\cdots+m_k)+42^q(m_1+\cdots+m_k+k-1))+22^q(m_1+\cdots+m_k))^{1/2}})^{1/2^q} \right)^p$$
$$\le C(\|b\|_{L^{\frac{q}{p}}}) \left(\sum_{m \ge 1} \sum_{\substack{l_1,\ldots,l_k \ge 0: \\ l_1+\cdots+l_k=m}} K^m (\|b\|_{L^1_\infty})^{2^q m} \right.$$
$$\left. \times \frac{((2(2^q(m+k-1))!)^{1/4}}{\Gamma(-H_r(2d2^q m+42^q(m+k-1))+22^q m)^{1/2}})^{1/2^q} \right)^p$$

for a constant K depending on H_r, T, d, p and q.

Since $H_r < \frac{1}{2(d-1+2k)}$ by assumption, we see that the above sum converges. Hence the proof follows. □

Finally, we are coming to the proof of the main result of this paper (Theorem 1.1), which shows that the regularizing fractional Brownian motion $\mathbb{B}^H_\cdot$ "produces" an infinitely continuously differentiable stochastic flow $x \mapsto X^x_t$, when b merely belongs to $\mathcal{L}^q_{2,p}$ for any $p, q \in (2, \infty]$.

Proof of Theorem 1.1 First, we approximate the irregular drift vector field b by a sequence of functions $b_n : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $n \ge 0$ in $C_c^\infty((0, T) \times \mathbb{R}^d, \mathbb{R}^d)$ in the sense of (32). Let $X^{n,x} = \{X^{n,x}_t, t \in [0, T]\}$ be the solution to (31) with initial value $x \in \mathbb{R}^d$ associated with b_n.

We find that for any test function $\varphi \in C_c^\infty(U, \mathbb{R}^d)$ and fixed $t \in [0, T]$ the set of random variables

$$\langle X^{n,\cdot}_t, \varphi \rangle := \int_U \langle X^{n,x}_t, \varphi(x) \rangle_{\mathbb{R}^d} dx, \quad n \ge 0$$

is relatively compact in $L^2(\Omega)$. In proving this, we want to apply the compactness criterion Theorem A.3 in terms of the Malliavin derivative in the Appendix. Using the sequence $\{\delta_i\}_{i=1}^\infty$ in Proposition 4.11, we get that

$$\sum_{i=1}^\infty \frac{1}{\delta_i^2} E[\int_0^T |D_s^{i,(j)} \langle X^{n,\cdot}_t, \varphi \rangle|^2 ds]$$
$$= \sum_{i=1}^\infty \frac{1}{\delta_i^2} E[\int_0^T \left(\int_U \sum_{l=1}^d D_s^{i,(j)} X_t^{n,x,(l)} \varphi_l(x) dx \right)^2 ds]$$
$$\le 2^{d-1} \|\varphi\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \lambda\{\text{supp}\,(\varphi)\} \sup_{x \in U} \sum_{i=1}^\infty \frac{1}{\delta_i^2} E\left[\int_0^T \|D_s^i X_t^{n,x}\|^2 ds \right],$$

where $D^{i,(j)}$ denotes the Malliavin derivative in the direction of $W^{i,(j)}$ where W^i is the d-dimensional standard Brownian motion defining $B^{H_i,i}$ and $W^{i,(j)}$ its j-th component, λ the Lebesgue measure on $\mathbb{R}^d$, supp (φ) the support of φ and $\| \cdot \|$ a matrix norm. So it follows from the estimates in Proposition 4.11 that

$$\sup_{n\geq 0}\sum_{i=1}^{\infty}\frac{1}{\delta_i^2}\|D_\cdot^i\langle X_t^{n,\cdot},\varphi\rangle\|^2_{L^2(\Omega\times[0,T])}\leq C\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^d)}\lambda\{\text{supp }(\varphi)\}.$$

Similarly, we get that

$$\sup_{n\geq 0}\sum_{i=1}^{\infty}\frac{1}{(1-2^{-2(\beta_i-\alpha_i)})\delta_i^2}\int_0^T\int_0^T\frac{E[\|D_{s'}^i\langle X_t^{n,\cdot},\varphi\rangle-D_s^i\langle X_t^{n,\cdot},\varphi\rangle\|^2]}{|s'-s|^{1+2\beta_i}}<\infty$$

for some sequences $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ as in Proposition 4.11. Hence $\langle X_t^{n,\cdot},\varphi\rangle$, $n\geq 0$ is relatively compact in $L^2(\Omega)$. Denote by $Y_t(\varphi)$ its limit after taking (if necessary) a subsequence.

By adopting the same reasoning as in Lemma 4.6 one proves that

$$\langle X_t^{n,\cdot},\varphi\rangle\xrightarrow{n\to\infty}\langle X_t^{\cdot},\varphi\rangle$$

weakly in $L^2(\Omega)$. Then by uniqueness of the limit we see that

$$\langle X_t^{n,\cdot},\varphi\rangle\underset{n\longrightarrow\infty}{\longrightarrow}Y_t(\varphi)=\langle X_t^{\cdot},\varphi\rangle$$

in $L^2(\Omega)$ for all t (without using a subsequence).

We observe that $X_t^{n,\cdot}$, $n\geq 0$ is bounded in the Sobolev norm $L^2(\Omega,W^{k,\alpha}(U))$ for each $n\geq 0$ and $k\geq 1$. Indeed, from Proposition 5.1 it follows that

$$\begin{aligned}\sup_{n\geq 0}\|X_t^{n,\cdot}\|^2_{L^2(\Omega,W^{k,\alpha}(U))}&=\sup_{n\geq 0}\sum_{i=0}^{k}E\left[\left\|\frac{\partial^i}{\partial x^i}X_t^{n,\cdot}\right\|^2_{L^\alpha(U)}\right]\\&\leq\sum_{i=0}^{k}\int_U\sup_{n\geq 0}E\left[\left\|\frac{\partial^i}{\partial x^i}X_t^{n,x}\right\|^\alpha\right]^{\frac{2}{\alpha}}dx\\&<\infty.\end{aligned}$$

The space $L^2(\Omega,W^{k,\alpha}(U))$, $\alpha\in(1,\infty)$ is reflexive. So the set $\{X_t^{n,x}\}_{n\geq 0}$ is (relatively) weakly compact in $L^2(\Omega,W^{k,\alpha}(U))$ for every $k\geq 1$. Hence, there exists a subsequence $n(j)$, $j\geq 0$ such that

$$X_t^{n(j),\cdot}\underset{j\to\infty}{\xrightarrow{w}}Y\in L^2(\Omega,W^{k,\alpha}(U)).$$

We als know that $X_t^{n,x}\to X_t^x$ strongly in $L^2(\Omega)$ for all t.

So for all $A\in\mathcal{F}$ and $\varphi\in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$ we have for all multi-indices γ with $|\gamma|\leq k$ that

$$\begin{aligned}E[1_A\langle X_t^{\cdot},D^\gamma\varphi\rangle]&=\lim_{j\to\infty}E[1_A\langle X_t^{n(j),\cdot},D^\gamma\varphi\rangle]\\&=\lim_{j\to\infty}(-1)^{|\gamma|}E[1_A\langle D^\gamma X_t^{n(j),\cdot},\varphi\rangle]=(-1)^{|\gamma|}E[1_A\langle D^\gamma Y,\varphi\rangle]\end{aligned}$$

Using the latter, we can conclude that

$$X_t^{\cdot}\in L^2(\Omega,W^{k,\alpha}(U)),\quad P-a.s.$$

Since $k\geq 1$ is arbitrary, the proof follows. □

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Appendix A: A Compactness Criterion for Subsets of $L^2(\Omega)$

The following result which is originally due to [19] in the finite dimensional case and which can be e.g. found in [13], provides a compactness criterion of square integrable functionals of cylindrical Wiener processes on a Hilbert space:

Theorem A.1 *Let $B_t, 0 \le t \le T$ be a cylindrical Wiener process on a separable Hilbert space H with respect to a complete probability space $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is generated by B_t, $0 \le t \le T$. Further, let $\mathcal{L}_{HS}(H, \mathbb{R})$ be the space of Hilbert-Schmidt operators from H to $\mathbb{R}$ and let $D : \mathbb{D}^{1,2} \longrightarrow L^2(\Omega; L^2([0, T]) \otimes \mathcal{L}_{HS}(H, \mathbb{R}))$ be the Malliavin derivative in the direction of B_t, $0 \le t \le T$, where $\mathbb{D}^{1,2}$ is the space of Malliavin differentiable random variables in $L^2(\Omega)$.*

Suppose that C is a self-adjoint compact operator on $L^2([0, T]) \otimes \mathcal{L}_{HS}(H, \mathbb{R})$ with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbb{D}^{1,2} : \|G\|_{L^2(\Omega)} + \left\|C^{-1} DG\right\|_{L^2(\Omega; L^2([0,T]) \otimes \mathcal{L}_{HS}(H,\mathbb{R}))} \le c \right\}$$

is relatively compact in $L^2(\Omega)$.

In this paper we aim at using a special case of the the previous theorem, which is more suitable for explicit estimations. To this end we need the following auxiliary result from [19].

Lemma A.2 *Denote by v_s,$s \ge 0$ with $v_0 = 1$ the Haar basis of $L^2([0, 1])$. Define for any $0 < \alpha < \frac{1}{2}$ the operator A_α on $L^2([0, 1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \quad \textit{if} \quad s = 2^k + j, \quad k \ge 0, \quad 0 \le j \le 2^k$$

and

$$A_\alpha 1 = 1.$$

Then for $\alpha < \beta < \frac{1}{2}$ we have that

$$\|A_\alpha f\|^2_{L^2([0,1])} \le 2 \left(\|f\|^2_{L^2([0,1])} + \frac{1}{1 - 2^{-2(\beta-\alpha)}} \int_0^1 \int_0^1 \frac{|f(t) - f(u)|^2}{|t-u|^{1+2\beta}} dt du \right).$$

Theorem A.3 *Let D^i be the Malliavin derivative in the direction of the i-th component of B_t, $0 \le t \le 1$, $i \ge 1$. In addition, let $0 < \alpha_i < \beta_i < \frac{1}{2}$ and $\delta_i > 0$ for all $i \ge 1$. Define the sequence $\lambda_{s,i} = 2^{-k\alpha_i} \delta_i$, if $s = 2^k + j$, $k \ge 0, 0 \le j \le 2^k$, $i \ge 1$. Assume that $\lambda_{s,i} \longrightarrow 0$ for $s, i \longrightarrow \infty$. Let $c > 0$ and $\mathcal{G}$ the collection of all $G \in \mathbb{D}^{1,2}$ such that*

$$\|G\|_{L^2(\Omega)} \le c,$$

$$\sum_{i \ge 1} \delta_i^{-2} \left\|D^i G\right\|^2_{L^2(\Omega; L^2([0,1]))} \le c$$

and

$$\sum_{i\geq 1}\frac{1}{(1-2^{-2(\beta_i-\alpha_i)})\delta_i^2}\int_0^1\int_0^1\frac{\left\|D_t^iG-D_u^iG\right\|_{L^2(\Omega)}^2}{|t-u|^{1+2\beta_i}}dtdu\leq c.$$

Then $\mathcal{G}$ is relatively compact in $L^2(\Omega)$.

Proof As before denote by v_s, $s\geq 0$ with $v_0=1$ the Haar basis of $L^2([0,1])$ and by $e_i^*=\langle e_i,\cdot\rangle_H$, $i\geq 1$ an orthonormal basis of $\mathcal{L}_{HS}(H,\mathbb{R})$ ($\cong H^*$) where e_i, $i\geq 1$ is an orthonormal basis of H. Define a self-adjoint compact operator C on $L^2([0,1])\otimes\mathcal{L}_{HS}(H,\mathbb{R})$ with dense image by

$$C(v_s\otimes e_i^*)=\lambda_{s,i}v_s\otimes e_i^*,\quad s\geq 0,\quad i\geq 1.$$

Then it follows for $G\in\mathbb{D}^{1,2}$ from Lemma A.2 that

$$\begin{aligned}
&\left\|C^{-1}DG\right\|^2_{L^2(\Omega;L^2([0,1])\otimes\mathcal{L}_{HS}(H,\mathbb{R}))}\\
&=\sum_{i\geq 1}\sum_{s\geq 0}\lambda_{s,i}^{-2}E\left[\langle DG,v_s\otimes e_i^*\rangle^2_{L^2([0,1])\otimes\mathcal{L}_{HS}(H,\mathbb{R}))}\right]\\
&=\sum_{i\geq 1}\delta_i^{-2}\left\|A_{\alpha_i}D^iG\right\|^2_{L^2(\Omega;L^2([0,1]))}\\
&\leq 2\sum_{i\geq 1}\delta_i^{-2}\left\|D^iG\right\|^2_{L^2(\Omega;L^2([0,1]))}\\
&\quad+2\sum_{i\geq 1}\frac{1}{(1-2^{-2(\beta_i-\alpha_i)})\delta_i^2}\int_0^1\int_0^1\frac{\left\|D_t^iG-D_u^iG\right\|^2_{L^2(\Omega)}}{|t-u|^{1+2\beta_i}}dtdu\\
&\leq M
\end{aligned}$$

for a constant $M<\infty$. So using Theorem A.1 we obtain the result. □

Appendix B: Technical Estimates

The following technical estimate is used in the course of the paper.

Lemma B.1 *Let $H\in(0,1/2)$ and $t\in[0,T]$ be fixed. Then, there exists a $\beta\in(0,1/2)$ such that*

$$\int_0^t\int_0^t\frac{|K_H(t,t_0')-K_H(t,t_0)|^2}{|t_0'-t_0|^{1+2\beta}}dt_0dt_0'<\infty. \tag{70}$$

Proof Let $t_0,t_0'\in[0,t]$, $t_0'<t_0$ be fixed. Write

$$K_H(t,t_0)-K_H(t,t_0')=c_H\left[f_t(t_0)-f_t(t_0')+\left(\frac{1}{2}-H\right)\left(g_t(t_0)-g_t(t_0')\right)\right],$$

where $f_t(t_0):=\left(\frac{t}{t_0}\right)^{H-\frac{1}{2}}(t-t_0)^{H-\frac{1}{2}}$ and $g_t(t_0):=\int_{t_0}^t\frac{f_u(t_0)}{u}du$, $t_0\in[0,t]$.

We will proceed to estimating $K_H(t,t_0)-K_H(t,t_0')$. First, observe the following fact,

$$\frac{y^{-\alpha}-x^{-\alpha}}{(x-y)^{\gamma}}\leq Cy^{-\alpha-\gamma}$$

for every $0 < y < x < \infty$ and $\alpha := (\frac{1}{2} - H) \in (0, 1/2)$ and $\gamma < \frac{1}{2} - \alpha$. This implies

$$
\begin{aligned}
f_t(t_0) - f_t(t_0') &= \left(\frac{t}{t_0}(t - t_0)\right)^{H-\frac{1}{2}} - \left(\frac{t}{t_0'}(t - t_0')\right)^{H-\frac{1}{2}} \\
&\leq C\left(\frac{t}{t_0}(t - t_0)\right)^{H-\frac{1}{2}-\gamma} t^{2\gamma} \frac{(t_0 - t_0')^\gamma}{(t_0 t_0')^\gamma} \\
&\leq C\frac{(t_0 - t_0')^\gamma}{(t_0 t_0')^\gamma}(t - t_0)^{H-\frac{1}{2}-\gamma} \\
&\leq C\frac{(t_0 - t_0')^\gamma}{(t_0 t_0')^\gamma} t_0^{H-\frac{1}{2}-\gamma}(t - t_0)^{H-\frac{1}{2}-\gamma}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
g_t(t_0) - g_t(t_0') &= \int_{t_0}^{t} \frac{f_u(t_0) - f_u(t_0')}{u} du - \int_{t_0'}^{t_0} \frac{f_u(t_0')}{u} du \\
&\leq \int_{t_0}^{t} \frac{f_u(t_0) - f_u(t_0')}{u} du \\
&\leq C\frac{(t_0 - t_0')^\gamma}{(t_0 t_0')^\gamma} \int_{t_0}^{t} \frac{(u - t_0)^{H-\frac{1}{2}-\gamma}}{u} du \\
&\leq C\frac{(t_0 - t_0')^\gamma}{(t_0 t_0')^\gamma} t_0^{H-\frac{1}{2}-\gamma} \int_{1}^{\infty} \frac{(u - 1)^{H-\frac{1}{2}-\gamma}}{u} du \\
&\leq C\frac{(t_0 - t_0')^\gamma}{(t_0 t_0')^\gamma} t_0^{H-\frac{1}{2}-\gamma} \\
&\leq C\frac{(t_0 - t_0')^\gamma}{(t_0 t_0')^\gamma} t_0^{H-\frac{1}{2}-\gamma}(t - t_0)^{H-\frac{1}{2}-\gamma}.
\end{aligned}
$$

As a result, we have for every $\gamma \in (0, H)$, $0 < t_0' < t_0 < t < T$,

$$
K_H(t, t_0) - K_H(t, t_0') \leq C_{H,T}\frac{(t_0 - t_0')^\gamma}{(t_0 t_0')^\gamma} t_0^{H-\frac{1}{2}-\gamma}(t - t_0)^{H-\frac{1}{2}-\gamma}, \tag{71}
$$

for some constant $C_{H,T} > 0$ depending only on H and T.

Thus

$$\begin{aligned}
&\int_0^t \int_0^{t_0} \frac{(K_H(t,t_0)-K_H(t,t_0'))^2}{|t_0-t_0'|^{1+2\beta}} dt_0' dt_0 \\
&\quad \le C \int_0^t \int_0^{t_0} \frac{|t_0-t_0'|^{-1-2\beta+2\gamma}}{(t_0 t_0')^{2\gamma}} t_0^{2H-1-2\gamma} (t-t_0)^{2H-1-2\gamma} dt_0' dt_0 \\
&\quad = C \int_0^t t_0^{2H-1-4\gamma} (t-t_0)^{2H-1-2\gamma} \int_0^{t_0} |t_0-t_0'|^{-1-2\beta+2\gamma} (t_0')^{-2\gamma} dt_0' dt_0 \\
&\quad = C \int_0^t t_0^{2H-1-4\gamma} (t-t_0)^{2H-1-2\gamma} \frac{\Gamma(-2\beta+2\gamma)\Gamma(-2\gamma+1)}{\Gamma(-2\beta+1)} t_0^{-2\beta} dt_0 \\
&\quad \le C \int_0^t t_0^{2H-1-4\gamma-2\beta} (t-t_0)^{2H-1-2\gamma} dt_0 \\
&\quad = C \frac{\Gamma(2H-2\gamma)\Gamma(2H-4\gamma-2\beta)}{\Gamma(4H-6\gamma-2\beta)} t^{4H-6\gamma-2\beta-1} < \infty,
\end{aligned}$$

for appropriately chosen small γ and β.

On the other hand, we have that

$$\begin{aligned}
&\int_0^t \int_{t_0}^{t} \frac{(K_H(t,t_0)-K_H(t,t_0'))^2}{|t_0-t_0'|^{1+2\beta}} dt_0' dt_0 \\
&\quad \le C \int_0^t t_0^{2H-1-4\gamma} (t-t_0)^{2H-1-2\gamma} \int_{t_0}^t \frac{|t_0-t_0'|^{-1-2\beta+2\gamma}}{(t_0')^{2\gamma}} dt_0' dt_0 \\
&\quad \le C \int_0^t t_0^{2H-1-6\gamma} (t-t_0)^{2H-1-2\gamma} \int_{t_0}^t |t_0-t_0'|^{-1-2\beta+2\gamma} dt_0' dt_0 \\
&\quad = C \int_0^t t_0^{2H-1-6\gamma} (t-t_0)^{2H-1-2\beta} dt_0 \\
&\quad \le C t^{4H-6\gamma-2\beta-1}.
\end{aligned}$$

Hence

$$\int_0^t \int_0^t \frac{(K_H(t,t_0)-K_H(t,t_0'))^2}{|t_0-t_0'|^{1+2\beta}} dt_0' dt_0 < \infty.$$

□

Lemma B.2 *Let $H \in (0, 1/2)$, $\theta, t \in [0, T]$, $\theta < t$ and $(\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m$ be fixed. Assume $w_j + \left(H - \frac{1}{2} - \gamma\right) \varepsilon_j > -1$ for all $j = 1, \ldots, m$. Then exists a finite constant $C = C(H, T) > 0$ such that*

$$\begin{aligned}
&\int_{\Delta^m_{\theta,t}} \prod_{j=1}^m (K_H(s_j,\theta)-K_H(s_j,\theta'))^{\varepsilon_j} |s_j - s_{j-1}|^{w_j} ds \\
&\quad \le C^m \left(\frac{\theta-\theta'}{\theta\theta'}\right)^{\gamma \sum_{j=1}^m \varepsilon_j} \theta^{\left(H-\frac{1}{2}-\gamma\right)\sum_{j=1}^m \varepsilon_j} \Pi_\gamma(m)\,(t-\theta)^{\sum_{j=1}^m w_j + \left(H-\frac{1}{2}-\gamma\right)\sum_{j=1}^m \varepsilon_j + m}
\end{aligned}$$

for $\gamma \in (0, H)$, where

$$\Pi_\gamma(m) := \prod_{j=1}^{m-1} \frac{\Gamma\left(\sum_{l=1}^j w_l + \left(H-\frac{1}{2}-\gamma\right)\sum_{l=1}^j \varepsilon_l + j\right)\Gamma\left(w_{j+1}+1\right)}{\Gamma\left(\sum_{l=1}^{j+1} w_l + \left(H-\frac{1}{2}-\gamma\right)\sum_{l=1}^j \varepsilon_l + j + 1\right)}. \tag{72}$$

Observe that if $\varepsilon_j = 0$ for all $j = 1, \ldots, m$ we obtain the classical formula.

Remark B.3 Observe that

$$\Pi_\gamma(m) \leq \frac{\prod_{j=1}^m \Gamma(w_j+1)}{\Gamma\left(\sum_{j=1}^m w_j + \left(H - \frac{1}{2} - \gamma\right)\sum_{j=1}^{m-1} \varepsilon_j + m\right)} \leq \frac{\prod_{j=1}^m \Gamma(w_j+1)}{\Gamma\left(\sum_{j=1}^m w_j + \left(H - \frac{1}{2} - \gamma\right)\sum_{j=1}^{m} \varepsilon_j + m\right)},$$

since the function Γ is increasing on $(1, \infty)$.

Proof First, we recall the following well-known formula: for given exponents $a, b > -1$ and some fixed $s_{j+1} > s_j$ we have

$$\int_\theta^{s_{j+1}} (s_{j+1} - s_j)^a (s_j - \theta)^b ds_j = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} (s_{j+1} - \theta)^{a+b+1}.$$

We recall from Lemma 70 that for every $\gamma \in (0, H)$, $0 < \theta' < \theta < s_j < T$,

$$K_H(s_j, \theta) - K_H(s_j, \theta') \leq C_{H,T} \frac{(\theta - \theta')^\gamma}{(\theta\theta')^\gamma} \theta^{H-\frac{1}{2}-\gamma} (s_j - \theta)^{H-\frac{1}{2}-\gamma},$$

for some constant $C_{H,T} > 0$ depending only on H and T. In view of the above arguments we have

$$\begin{aligned}
&\int_\theta^{s_2} |K_H(s_1, \theta) - K_H(s_1, \theta')|^{\varepsilon_1} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1} ds_1 \\
&\leq C_{H,T}^{\varepsilon_1} \frac{(\theta - \theta')^{\gamma\varepsilon_1}}{(\theta\theta')^{\gamma\varepsilon_1}} \theta^{\left(H-\frac{1}{2}-\gamma\right)\varepsilon_1} \int_\theta^{s_2} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1 + \left(H-\frac{1}{2}-\gamma\right)\varepsilon_1} ds_1 \\
&= C_{H,T}^{\varepsilon_1} \frac{(\theta - \theta')^{\gamma\varepsilon_1}}{(\theta\theta')^{\gamma\varepsilon_1}} \theta^{\left(H-\frac{1}{2}-\gamma\right)\varepsilon_1} \frac{\Gamma(\hat{w}_1)\Gamma(\hat{w}_2)}{\Gamma(\hat{w}_1 + \hat{w}_2)} (s_2 - \theta)^{w_1 + w_2 + \left(H-\frac{1}{2}-\gamma\right)\varepsilon_1 + 1},
\end{aligned}$$

where

$$\hat{w}_1 := w_1 + \left(H - \frac{1}{2} - \gamma\right)\varepsilon_1 + 1, \quad \hat{w}_2 := w_2 + 1.$$

Integrating iteratively we obtain the desired formula. □

Finally, we give a similar estimate to the previous one.

Lemma B.4 *Let $H \in (0, 1/2)$, $\theta, t \in [0, T]$, $\theta < t$ and $(\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m$ be fixed. Assume $w_j + \left(H - \frac{1}{2}\right)\varepsilon_j > -1$ for all $j = 1, \ldots, m$. Then exists a finite constant $C > 0$ such that*

$$\int_{\Delta_{\theta,t}^m} \prod_{j=1}^m (K_H(s_j, \theta))^{\varepsilon_j} |s_j - s_{j-1}|^{w_j} ds \leq C^m \theta^{\left(H-\frac{1}{2}\right)\sum_{j=1}^m \varepsilon_j} \Pi_0(m) (t - \theta)^{\sum_{j=1}^m w_j + \left(H-\frac{1}{2}\right)\sum_{j=1}^m \varepsilon_j + m}$$

for $\gamma \in (0, H)$, where Π_0 is given as in (72). Observe that if $\varepsilon_j = 0$ for all $j = 1, \ldots, m$ we obtain the classical formula.

Remark B.5 Observe that

$$\Pi_0(m) \leq \frac{\prod_{j=1}^m \Gamma(w_j+1)}{\Gamma\left(\sum_{j=1}^m w_j + \left(H-\frac{1}{2}\right)\sum_{j=1}^m \varepsilon_j + m\right)},$$

due to the fact that Γ is increasing on $(1, \infty)$.

Proof By similar arguments as in the proof of Lemma 70 it is easy to derive the following estimate

$$|K_H(s_j, \theta)| \leq C_{H,T}|s_j - \theta|^{H-\frac{1}{2}}\theta^{H-\frac{1}{2}}$$

for every $0 < \theta < s_j < T$ and some constant $C_{H,T} > 0$. This implies

$$\begin{aligned}
&\int_\theta^{s_2} (K_H(s_1, \theta))^{\varepsilon_1}|s_2 - s_1|^{w_2}|s_1-\theta|^{w_1} ds_1 \\
&\leq C_{H,T}^{\varepsilon_1}\theta^{\left(H-\frac{1}{2}\right)\varepsilon_1}\int_\theta^{s_2}|s_2-s_1|^{w_2}|s_1-\theta|^{w_1+\left(H-\frac{1}{2}\right)\varepsilon_1}ds_1 \\
&= C_{H,T}^{\varepsilon_1}\theta^{\left(H-\frac{1}{2}\right)\varepsilon_1}\frac{\Gamma\left(w_1+w_2+\left(H-\frac{1}{2}\right)\varepsilon_1+1\right)\Gamma(w_2+1)}{\Gamma\left(w_1+w_2+\left(H-\frac{1}{2}\right)\varepsilon_1+2\right)}(s_2-\theta)^{w_1+w_2+\left(H-\frac{1}{2}\right)\varepsilon_1+1}
\end{aligned}$$

Integrating iteratively one obtains the desired estimate. □

The next auxiliary result can be found in [39].

Lemma B.6 *Assume that $X_1, \ldots, X_n$ are real centered jointly Gaussian random variables, and $\Sigma = (E[X_jX_k])_{1\leq j,k\leq n}$ is the covariance matrix, then*

$$E[|X_1|\ldots|X_n|] \leq \sqrt{perm(\Sigma)},$$

where $perm(A)$ is the permanent of a matrix $A = (a_{ij})_{1\leq i,j\leq n}$ defined by

$$perm(A) = \sum_{\pi\in S_n}\prod_{j=1}^n a_{j,\pi(j)}$$

for the symmetric group S_n.

The next result corresponds to Lemma 3.19 in [16]:

Lemma B.7 *Let $Z_1, \ldots, Z_n$ be mean zero Gaussian variables which are linearly independent. Then for any measurable function $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ we have that*

$$\begin{aligned}
&\int_{\mathbb{R}^n} g(v_1)\exp\left(-\frac{1}{2}Var\left[\sum_{j=1}^n v_jZ_j\right]\right)dv_1\ldots dv_n \\
&= \frac{(2\pi)^{(n-1)/2}}{(\det Cov(Z_1,\ldots,Z_n))^{1/2}}\int_{\mathbb{R}} g\left(\frac{v}{\sigma_1}\right)\exp\left(-\frac{1}{2}v^2\right)dv,
\end{aligned}$$

where $\sigma_1^2 := Var[Z_1|Z_2,\ldots,Z_n]$.

Lemma B.8 *Let n, p and k be non-negative integers, $k \leq n$. Assume we have functions $f_j : [0,T] \to \mathbb{R}$, $j = 1, \ldots, n$ and $g_i : [0,T] \to \mathbb{R}$, $i = 1, \ldots, p$ such that*

$$f_j \in \left\{ \frac{\partial^{\alpha_j^{(1)}+\ldots+\alpha_j^{(d)}}}{\partial^{\alpha_j^{(1)}} x_1 \ldots \partial^{\alpha_j^{(d)}} x_d} b^{(r)}(u, X_u^x),\ r = 1, \ldots, d \right\},\ j = 1, \ldots, n$$

and

$$g_i \in \left\{ \frac{\partial^{\beta_i^{(1)}+\ldots+\beta_i^{(d)}}}{\partial^{\beta_i^{(1)}} x_1 \ldots \partial^{\beta_i^{(d)}} x_d} b^{(r)}(u, X_u^x),\ r = 1, \ldots, d \right\},\ i = 1, \ldots, p$$

for $\alpha := (\alpha_j^{(l)}) \in \mathbb{N}_0^{d \times n}$ and $\beta := (\beta_i^{(l)}) \in \mathbb{N}_0^{d \times p}$, where $X_\cdot^x$ is the strong solution to

$$X_t^x = x + \int_0^t b(u, X_u^x) du + B_t^H,\ 0 \leq t \leq T$$

for $b = (b^{(1)}, \ldots, b^{(d)})$ with $b^{(r)} \in C_c([0,T] \times \mathbb{R}^d)$ for all $r = 1, \ldots, d$. So (as we shall say in the sequel) the product $g_1(r_1) \cdot \cdots \cdot g_p(r_p)$ has a total order of derivatives $|\beta| = \sum_{l=1}^d \sum_{i=1}^p \beta_i^{(l)}$. We know from Sect. 2.2 that

$$\begin{aligned} &\int_{\Delta_{\theta,t}^n} f_1(s_1) \ldots f_k(s_k) \int_{\Delta_{\theta,s_k}^p} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 f_{k+1}(s_{k+1}) \ldots f_n(s_n) ds_n \ldots ds_1 \\ &\quad = \sum_{\sigma \in A_{n,p}} \int_{\Delta_{\theta,t}^{n+p}} h_1^\sigma(w_1) \ldots h_{n+p}^\sigma(w_{n+p}) dw_{n+p} \ldots dw_1, \end{aligned} \tag{73}$$

where $h_l^\sigma \in \{f_j, g_i : 1 \leq j \leq n,\ 1 \leq i \leq p\}$, $A_{n,p}$ is a subset of permutations of $\{1, \ldots, n+p\}$ such that $\# A_{n,p} \leq C^{n+p}$ for an appropriate constant $C \geq 1$, and $s_0 = \theta$. Then the products

$$h_1^\sigma(w_1) \cdot \cdots \cdot h_{n+p}^\sigma(w_{n+p})$$

have a total order of derivatives given by $|\alpha| + |\beta|$.

Proof The result is proved by induction on n. For $n = 1$ and $k = 0$ the result is trivial. For $k = 1$ we have

$$\begin{aligned} &\int_\theta^t f_1(s_1) \int_{\Delta_{\theta,s_1}^p} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 ds_1 \\ &\quad = \int_{\Delta_{\theta,t}^{p+1}} f_1(w_1) g_1(w_2) \ldots g_p(w_{p+1}) dw_{p+1} \ldots dw_1, \end{aligned}$$

where we have put $w_1 = s_1$, $w_2 = r_1, \ldots, w_{p+1} = r_p$. Hence the total order of derivatives involved in the product of the last integral is given by $\sum_{l=1}^d \alpha_1^{(l)} + \sum_{l=1}^d \sum_{i=1}^p \beta_i^{(l)} = |\alpha| + |\beta|$.

Assume the result holds for n and let us show that this implies that the result is true for $n+1$. Either $k = 0$, 1 or $2 \leq k \leq n+1$. For $k = 0$ the result is trivial. For $k = 1$ we have

$$\begin{aligned} &\int_{\Delta_{\theta,t}^{n+1}} f_1(s_1) \int_{\Delta_{\theta,s_1}^p} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 f_2(s_2) \ldots f_{n+1}(s_{n+1}) ds_{n+1} \ldots ds_1 \\ &\quad = \int_\theta^t f_1(s_1) \left(\int_{\Delta_{\theta,s_1}^n} \int_{\Delta_{\theta,s_1}^p} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 f_2(s_2) \ldots f_{n+1}(s_{n+1}) ds_{n+1} \ldots ds_2 \right) ds_1. \end{aligned}$$

Using Sect. 2.2 we obtain by employing the shuffle permutations that the latter inner double integral on diagonals can be written as a sum of integrals on diagonals of length $p+n$ with products having a total order of derivatives given by $\sum_{l=1}\sum_{j=2}^{n+1}\alpha_j^{(l)}+\sum_{l=1}^{d}\sum_{i=1}^{p}\beta_i^{(l)}$. Hence we obtain a sum of products, whose total order of derivatives is $\sum_{l=1}^{d}\sum_{j=2}^{n+1}\alpha_j^{(l)}+\sum_{l=1}^{d}\sum_{i=1}^{p}\beta_i^{(l)}+\sum_{l=1}^{d}\alpha_1^{(l)}=|\alpha|+|\beta|$.

For $k\geq 2$ we have (in connection with Sect. 2.2) from the induction hypothesis that

$$\begin{aligned}&\int_{\Delta_{\theta,t}^{n+1}} f_1(s_1)\dots f_k(s_k)\int_{\Delta_{\theta,s_k}^{p}} g_1(r_1)\dots g_p(r_p)dr_p\dots dr_1 f_{k+1}(s_{k+1})\dots f_{n+1}(s_{n+1})ds_{n+1}\dots ds_1\\&=\int_\theta^t f_1(s_1)\int_{\Delta_{\theta,s_1}^{n}} f_2(s_2)\dots f_k(s_k)\int_{\Delta_{\theta,s_k}^{p}} g_1(r_1)\dots g_p(r_p)dr_p\dots dr_1\\&\quad\times f_{k+1}(s_{k+1})\dots f_{n+1}(s_{n+1})ds_{n+1}\dots ds_2ds_1\\&=\sum_{\sigma\in A_{n,p}}\int_\theta^t f_1(s_1)\int_{\Delta_{\theta,s_1}^{n+p}} h_1^\sigma(w_1)\dots h_{n+p}^\sigma(w_{n+p})dw_{n+p}\dots dw_1ds_1,\end{aligned}$$

where each of the products $h_1^\sigma(w_1)\cdot\dots\cdot h_{n+p}^\sigma(w_{n+p})$ have a total order of derivatives given by $\sum_{l=1}\sum_{j=2}^{n+1}\alpha_j^{(l)}+\sum_{l=1}^{d}\sum_{i=1}^{p}\beta_i^{(l)}$. Thus we get a sum with respect to a set of permutations $A_{n+1,p}$ with products having a total order of derivatives which is

$$\sum_{l=1}^{d}\sum_{j=2}^{n+1}\alpha_j^{(l)}+\sum_{l=1}^{d}\sum_{i=1}^{p}\beta_i^{(l)}+\sum_{l=1}^{d}\alpha_1^{(l)}=|\alpha|+|\beta|.$$

□

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