Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-023-10316-9

COMMENTARY

Comment on: Criteria for Strong and Weak Random Attractors

Hans Crauel[1] · Sarah Geiss[2] · Michael Scheutzow[2]

Received: 1 September 2023 / Revised: 6 September 2023 / Accepted: 25 September 2023

Abstract
In the article 'Criteria for Strong and Weak Random Attractors' necessary and sufficient conditions for strong attractors and weak attractors are studied. In this note we correct two of its theorems on strong attractors.

Keywords Random attractor · Pullback attractor · Weak attractor · Omega limit set · Compact random set

Mathematics Subject Classification 37B25 · 37C70 · 37G35 · 37H99 · 37L55 · 60D05 · 60H10 · 60H15 · 60H25

We correct two theorems which provide criteria for strong attractors given in [1].

We use the same assumptions and notation as in [1], i.e. let φ be a continuous random dynamical system on a Polish space (E, d) over a metric dynamical system $(\Omega, \mathscr{F}, (\vartheta_t)_{t\in\mathbb{R}}, P)$. We use the same letter d for the complete metric on E and the Hausdorff semi-distance on subsets of E. For a subset A of E we denote the closed δ-neighborhood of A by A^δ.

In the article the following two types of strong attractors are studied:

- B-attractors, i.e. attractors that attract all bounded subsets of E,
- C-attractors, i.e. attractors that attract all compact subsets of E.

In [1, Theorem 3.1, Theorem 3.2] the following two theorems have been stated:

Theorem 1 (Original erroneous formulation) *The following are equivalent:*

This comment refers to the article available online at https://doi.org/10.1007/s10884-009-9135-8.

✉ Sarah Geiss
geiss@math.tu-berlin.de

Hans Crauel
hans.crauel@posteo.de

Michael Scheutzow
ms@math.tu-berlin.de

1 Wätjenstraße 122, 28213 Bremen, Germany

2 Institute of Mathematics, TU Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

(i) *φ has a strong B-attractor.*
(ii) *For every $\varepsilon > 0$ there exists a compact subset C_ε such that for each $\delta > 0$ and each bounded and closed subset B of E it holds that*

$$P\left\{\bigcup_{s\geq 0}\bigcap_{t\geq s}\varphi(t,\vartheta_{-t}\omega)B \subseteq C_\varepsilon^\delta\right\} \geq 1-\varepsilon.$$

(iii) *There exists a compact strongly B-attracting set $\omega \mapsto K(\omega)$.*

Theorem 2 (Original erroneous formulation) *The following are equivalent:*

(i) *φ has a strong C-attractor.*
(ii) *For every $\varepsilon > 0$ there exists a compact subset C_ε such that for each $\delta > 0$ and each compact subset B of E it holds that*

$$P\left\{\bigcup_{s\geq 0}\bigcap_{t\geq s}\varphi(t,\vartheta_{-t}\omega)B \subseteq C_\varepsilon^\delta\right\} \geq 1-\varepsilon.$$

(iii) *There exists a compact strongly C-attracting set $\omega \mapsto K(\omega)$.*

The following example shows that the original formulations of Theorem 1 and Theorem 2 are incorrect.

Example 1 Choose $E = \mathbb{R}$, $\Omega = \{0\}$ and consider $\varphi(t,\omega)x := x + t$ for all $t \geq 0$, $x \in E$, $\omega \in \Omega$. This continuous RDS satisfies for all bounded subsets $B \subset \mathbb{R}$

$$\begin{aligned}\bigcup_{T\geq 0}\bigcap_{t\geq T}\varphi(t,\vartheta_{-t}\omega)B &\subseteq \bigcap_{T\geq 0}\bigcup_{t\geq T}\varphi(t,\vartheta_{-t}\omega)B\\ &\subseteq \Omega_B(\omega) := \bigcap_{T\geq 0}\overline{\bigcup_{t\geq T}\varphi(t,\vartheta_{-t}\omega)B} = \emptyset.\end{aligned}$$

This RDS has no C-attractor and hence also no B-attractor.

In particular, (i) and (ii) of Theorems 1 and 2 of the original formulation are not equivalent. This example shows in particular that also the following stronger property is not sufficient to ensure strong B-attractors:

(ii)' For every $\varepsilon > 0$ there exists a compact subset C_ε such that for each $\delta > 0$ and each bounded and closed subset B of E it holds that

$$P\left\{\Omega_B(\omega) \subseteq C_\varepsilon^\delta\right\} \geq 1-\varepsilon.$$

The following is a corrected version of Theorem 1: The condition (ii) is modified. In addition, condition (iii) is formulated more precisely than in the original formulation.

Theorem 1 (Corrected formulation) *The following are equivalent:*

(i) *φ has a strong B-attractor.*
(ii) *For every $\varepsilon > 0$ there exists a compact subset C_ε such that for each $\delta > 0$ and each bounded and closed subset B of E there exists a $T > 0$ such that*

$$P\left\{\bigcup_{t\geq T}\varphi(t,\vartheta_{-t}\omega)B \subseteq C_\varepsilon^\delta\right\} \geq 1-\varepsilon.$$

(iii) There exists a random set $K \subseteq E \times \Omega$ such that $K(\omega)$ is P-a.s. compact and K attracts all bounded subsets, i.e.

$$\lim_{t\to\infty} d(\varphi(t, \vartheta_{-t}\omega)B, K(\omega)) = 0 \quad P\text{-a.s.}$$

for every bounded subset B.

Remark 1 By [2, Lemma 3.5] and its proof we have that

$$\bigcup_{t\geq T} \varphi(t, \vartheta_{-t}\omega)B \in \mathcal{B} \otimes \bar{\mathscr{F}} \quad \text{and} \quad \Omega_B(\omega) \in \mathcal{B} \otimes \bar{\mathscr{F}}$$

for all bounded closed subsets B of E. Here $\mathcal{B}$ denotes the Borel σ-algebra of E and $\bar{\mathscr{F}}$ the P-completion of $\mathscr{F}$. Therefore, we have by the measurable projection theorem that

$$\begin{aligned}\Omega \backslash &\left\{ \omega \in \Omega \;\middle|\; \bigcup_{t\geq T} \varphi(t, \vartheta_{-t}\omega)B \subseteq C_\varepsilon^\delta \right\} \\ &= \mathrm{pr}_\Omega \left(\left\{ \bigcup_{t\geq T} \varphi(t, \vartheta_{-t}\omega)B \right\} \cap \left\{ (E \backslash C_\varepsilon^\delta) \times \Omega \right\} \right) \in \bar{\mathscr{F}}\end{aligned}$$

where $\mathrm{pr}_\Omega : E \times \Omega \to \Omega$ denotes the projection onto Ω. Hence, the expression in (ii) of Theorem 1 is well-defined.

Proof The proof is similar to the proof presented in [1].

Equivalence of (i) and (iii) is proven in [3, Theorem 13], see also [4, Theorem 3.4, Remark 3.5].

We first show (i) $\implies$ (ii): Let $\varepsilon > 0$ be arbitrary. Since E is a Polish space and the attractor A is a random variable taking values in the compact sets, there exists a compact subset $C_\varepsilon \subseteq E$ such that

$$P\{A(\omega) \subseteq C_\varepsilon\} \geq 1 - \varepsilon/2 \tag{1}$$

(see Crauel [5, Proposition 2.15]). Let $B \subseteq E$ be a bounded and closed set. Then we have by (i)

$$\lim_{t\to\infty} d(\varphi(t, \vartheta_{-t}\omega)B, A(\omega)) = 0 \quad P\text{-a.s.},$$

i.e. for every $\delta > 0$ there exists a $T(\omega) > 0$ such that for all $t \geq T(\omega)$ we have $d(\varphi(t, \vartheta_{-t}\omega)B, A(\omega)) \leq \delta$ P-almost surely. Hence, there exists some deterministic $T > 0$ such that

$$P\left\{ \bigcup_{t\geq T} \varphi(t, \vartheta_{-t}\omega)B \subseteq A(\omega)^\delta \right\} \geq 1 - \varepsilon/2. \tag{2}$$

Combining (1) and (2) implies (ii).

Now we show (ii) $\implies$ (iii): Let $(B_k)_{k\in\mathbb{N}}$ be a sequence of bounded closed subsets of E such that $B_0 \subseteq B_1 \subseteq B_2 \ldots$ and such that for any bounded subset $B \subseteq E$ there exists some $k \in \mathbb{N}$ such that $B \subseteq B_k$. We modify the random attractor constructed in the proof given in [1] to ensure that it is indeed a random set: We define $A(\omega)$ to be the (unique) smallest closed random set that contains $\bigcup_{k\in\mathbb{N}} \Omega_{B_k}(\omega)$, see [3, Proposition 17]. By (ii) for all $\varepsilon > 0$

there exists a compact set $C_\varepsilon \subseteq E$ such that for every $\delta > 0$ and for every $k \in \mathbb{N}$ there exist $T(k) > 0$ such that

$$P\left\{\bigcup_{t\geq T(k)} \varphi(t, \vartheta_{-t}\omega)B_k \subseteq C_\varepsilon^\delta\right\} \geq 1-\varepsilon.$$

Using that C_ε is closed, this implies that $P\{\Omega_{B_k}(\omega) \subseteq C_\varepsilon\} \geq 1-\varepsilon$. As $\Omega_{B_k}(\omega) \subseteq \Omega_{B_{k+1}}(\omega)$ this implies $P\{\bigcup_{k\in\mathbb{N}} \Omega_{B_k}(\omega) \subseteq C_\varepsilon\} \geq 1-\varepsilon$. This implies by the properties of $A(\omega)$ given by [3, Proposition 17] that $A(\omega)$ is a compact random set.

It remains to prove that $A(\omega)$ attracts all bounded sets. To this end consider an arbitrary bounded subset B of E and let $k \in \mathbb{N}$ be such that $B \subseteq B_k$. Let $\varepsilon > 0$ be arbitrary. By (ii) there exists for every $m \in \mathbb{N}$ some $T_m > 0$ such that

$$P\left\{d\left(\bigcup_{t\geq T_m} \varphi(t, \vartheta_{-t}\omega)B_k, C_\varepsilon\right) \leq 1/m\right\} \geq 1-\varepsilon.$$

which implies

$$P\left\{\sup_{t\geq T_m} d(\varphi(t, \vartheta_{-t}\omega)B_k, C_\varepsilon) \leq 1/m \text{ for infinitely many m}\right\} \geq 1-\varepsilon.$$

To obtain the previous inequality we used for $M_m := \{\sup_{t\geq T_m} d(\varphi(t, \vartheta_{-t}\omega)B_k, C_\varepsilon) \leq 1/m\}$ that

$$P\left[\bigcap_{n\in\mathbb{N}} \bigcup_{m=n}^{\infty} M_m\right] = \lim_{n\to\infty} P\left[\bigcup_{m=n}^{\infty} M_m\right] \geq \limsup_{m\to\infty} P[M_m] \geq 1-\varepsilon.$$

Hence, we have

$$P\left\{\lim_{t\to\infty} d(\varphi(t, \vartheta_{-t}\omega)B_k, C_\varepsilon) = 0\right\} \geq 1-\varepsilon.$$

Due to $\Omega_{B_k} \subseteq A$ and compactness of C_ε this implies

$$P[\lim_{t\to\infty} d(\varphi(t, \vartheta_{-t}\omega)B_k, A(\omega)) \neq 0] < \varepsilon.$$

The assertion follows as the previous inequality holds for arbitrary $\varepsilon > 0$. □

The following is a corrected version of Theorem 2. It follows from the proof given in [1] and the corrected proof of Theorem 1. (One can use e.g. [3, Lemma 8] to verify that Ω_B is invariant.)

Theorem 2 (Corrected formulation) *The following are equivalent:*

(i) *φ has a strong C-attractor.*

(ii) *For every $\varepsilon > 0$ there exists a compact subset C_ε such that for each $\delta > 0$ and each compact subset B of E there exists a $T > 0$ such that*

$$P\left\{\bigcup_{t\geq T} \varphi(t, \vartheta_{-t}\omega)B \subset C_\varepsilon^\delta\right\} \geq 1-\varepsilon.$$

(iii) *There exists a random set $K \subseteq E \times \Omega$ such that $K(\omega)$ is P-a.s. compact and K attracts all compact subsets.*

Author Contributions All authors contributed equally.

Funding Open Access funding enabled and organized by Projekt DEAL. SG gratefully acknowledges the support by the DFG Research Unit FOR 2402.

Declarations

Conflict of interest The authors have no conflicts of interest to declare.

References

1. Crauel, H., Dimitroff, G., Scheutzow, M.: Criteria for strong and weak random attractors. J. Dynam. Differ. Equ. **21**(2), 233–247 (2009). https://doi.org/10.1007/s10884-009-9135-8
2. Crauel, H.: Global random attractors are uniquely determined by attracting deterministic compact sets. Ann. Mat. Pura Appl. **176**(4), 57–72 (1999). https://doi.org/10.1007/BF02505989
3. Crauel, H., Scheutzow, M.: Minimal random attractors. J. Differ. Equ. **265**, 702–718 (2018). https://doi.org/10.1016/j.jde.2018.03.011
4. Crauel, H.: Random point attractors versus random set attractors. J. London Math. Soc. **63**(2), 413–427 (2001). https://doi.org/10.1017/S0024610700001915
5. Crauel, H.: Random probability measures on polish spaces. In: Stochastics Monographs, vol. 11, p. 118. Taylor & Francis, London (2002)