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Stability of Some Anticipating Semilinear Stochastic Differential Equations of Skorohod Type

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Abstract
In the present paper, we study different types of stability of the solution of a semi-linear anticipating stochastic differential equation driven by a Brownian motion, with a random variable as initial condition. The involved stochastic integral is the Skorohod one. Being the initial condition random, we need to redefine the stability concepts. The new stability criteria depend on the derivative of the initial condition in the Malliavin calculus sense.

Keywords Anticipating stochastic differential equations · Stability · Malliavin calculus · Anticipative Girsanov transformations

Mathematics Subject Classification Primary 60H10; Secondary 60H05 · 60H07 · 34D23

1 Introduction

Let $W := \{W_t, t \geq 0\}$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Consider the stochastic differential equation

$$X_t = X_0 + \int_0^t b(u, X_u)du + \int_0^t a_u X_u \delta W_u, \quad t \geq 0. \tag{1.1}$$

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Here $X_0 : \Omega \to \mathbb{R}$ is an $\mathcal{F}-$measurable random variable, $b : [0, \infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$ is an $\mathbb{F}$-adapted random field and $a : [0, \infty) \times \Omega \to \mathbb{R}$ is an $\mathbb{F}$-adapted random process. Since the initial condition X_0 is a random variable, then the stochastic integral has to be an anticipating one that allows us to integrate processes that are not necessarily adapted to the underlying filtration $\mathbb{F}$. Here we use the well-known Skorohod integral, introduced by Skorohod in [17], which is an extension of the classical Itô integral. The existence and uniqueness of the solution, and other properties, of anticipating stochastic differential equations like (1.1) have been studied in [4, 5, 12]. See also [14]. This type of equation has proven to be useful in quantitative finance, for instance in insider trading modeling. See, for example, the recent paper [6] and the references therein, and [13].

The purpose of the present paper is to study different types of stability of the solution of Eq. (1.1). Being X_0 a random variable we need to extend the concept of stability. Concretely we introduce three types of stochastic stability: weak stability in probability, exponential $p-$stability and exponential stability in probability. We prove that the solution of equation (1.1) satisfies all these types of stability under suitable conditions. The case in which X_0 is a constant, where anticipative calculus is not necessary, is treated by Khasminskii in [9] (Sections 1.5−1.8 and Chapter 5). See also Arnold [2] (Chapter 11) and Gard [7] (Chapter 5).

Stability means insensitivity of a system to small changes in the initial state. In a stable system, the trajectories that are close to each other at a specific instant continuous to be close to each other at the subsequent instants. Lyapunov developed in 1892 a method to determine stability of a system despite not knowing its explicit solution. For deterministic dynamical systems a theory of stability of solutions is very well developed, see for example [3]. On other hand, it is clear that stability is a very important property in applications. For example, for stable systems such that explicit solutions are not known we can try to find approximated solutions using numerical methods.

Stochastic stability has been developed much more recently. To generalize deterministic stability to stochastic stability it is not straightforward. Different definitions have been considered in the literature. During the last decades many results based on the Lyapunov point of view have been obtained for Itô stochastic differential equations, see Chapters 1 and 5 of Khasminskii [9] as a main reference. As it is pointed out by Khasminskii, the study of stability is important in many applications of stochastic dynamical systems (see also the references in [9]). In particular, the stability of linear systems has applications in automatic control (for instance, [11, 16]).

In the present paper, as far as we know, we extend for the first time Khasminskii notions of weak stability in probability and exponential stability in probability to the case of random initial condition, and therefore, to the case of an anticipating stochastic differential equation. Different results of stability of the solution are obtained. Malliavin differentiability hypotheses are naturally required.

In Sect. 2 we recall some preliminary results about Malliavin calculus and anticipative Girsanov transformations, following essentially Buckdhan [5] and Nualart [14]. In Sect. 3 we establish the existence and uniqueness of the solution of Eq. (1.1), extending the result for the linear case ($b(u, x) = b(u) \cdot x$) proved in Buckdhan [4]; see also Buckdhan [5] (Theorem 3.2.1). The proof of this existence and uniqueness result is sketched in the Appendix (Section 6.1). The solution of Eq. (1.1) is written in terms of an auxiliary process Z, whose properties are analyzed in Sect. 4. Finally, in Sect. 5, we introduce the three new types of stochastic stability, suitable to our context, and prove different stability results for the solution of Eq. (1.1).

2 Preliminaries

In the present paper we assume $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical Wiener space. That is, Ω is the family of all continuous functions from $[0, \infty)$ to $\mathbb{R}$ null at 0, $\mathcal{F}$ is the Borel σ-algebra of Ω, when this is equipped with the topology of uniform convergence on compact sets, and $\mathbb{P}$ is the probability measure such that the canonical process $W_t(\omega) = \omega(t)$ is a standard Brownian motion. Moreover, $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ is the completed natural filtration of W. We denote by $\mathcal{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$, and, for any $T > 0$, we denote by $\mathcal{P}_T$, the progressive σ-algebra on $\Omega \times [0, T]$.

2.1 Malliavin calculus and Sobolev spaces

Let $C_b^\infty(\mathbb{R}^n)$ be the family of all the C^∞-functions from $\mathbb{R}^n$ to $\mathbb{R}$ that are bounded together with all their partial derivatives. Consider the class $\mathcal{S}$ of smooth random variables F of the form

$$F = f(W_{t_1}, \dots, W_{t_n}), \tag{2.1}$$

with $f \in C_b^\infty(\mathbb{R}^n)$ and $t_1, \dots, t_n \in \mathbb{R}_+$. For the smooth functional F given in (2.1), we define its derivative in the Malliavin calculus sense as the process

$$D_s F = \sum_{j=1}^{n} \partial_{x_j} f(W_{t_1}, \dots, W_{t_n}) \mathbb{1}_{[0,t_j]}(s), \quad s \geq 0.$$

More generally, we define the k-th derivative of F as $D^k_{s_1,\dots,s_k} F = D_{s_k} \cdots D_{s_1} F$.

Now, we introduce the spaces $\mathbb{D}_T^{k,p}$, where $k \in \mathbb{N}$, $T > 0$ and $p \geq 1$. On $\mathcal{S}$, consider the semi-norm

$$||F||_{k,p,T} := ||F||_p + \sum_{i=1}^{k} ||(\int_{[0,T]^i} |D_z^i F|^2 dz)^{\frac{1}{2}}||_p,$$

where $||\cdot||_p$ stands for the norm in $L^p(\Omega)$. It is well-known that the operator D^k is closable from $\mathcal{S} \subset L^p(\Omega)$ into $L^p(\Omega; L^2([0, T]^k))$, see Nualart [14] (Section 1.2). Thus, the space $\mathbb{D}_T^{k,p}$ is defined as the completion of the family $\mathcal{S}$ with respect to the semi-norm $||\cdot||_{k,p,T}$. Note that if $0 < \tilde{T} < T$, we have $\mathbb{D}_T^{k,p} \subset \mathbb{D}_{\tilde{T}}^{k,p}$.

As in Buckdhan [5], $\mathbb{D}_T^{k,\infty}$ (resp. $\tilde{\mathbb{D}}_T^{k,\infty}$) denotes the family of all random variables $F \in \mathbb{D}_T^{k,2}$ such that $F \in L^\infty(\Omega)$ and $D^m F \in L^\infty(\Omega; L^2([0, T]^m))$ (resp. $D^m F \in L^\infty(\Omega \times [0, T]^m)$), for $m = 1, \dots, k$.

For $T > 0$, the Skorohod integral with respect to W, denoted by δ_T, is the adjoint of the derivative operator $D : \tilde{\mathbb{D}}_T^{1,\infty} \subset L^\infty(\Omega) \to L^\infty(\Omega \times [0, T])$. That is, u is in $Dom\, \delta_T$ if and only if $u \in L^1(\Omega \times [0, T])$ and there exists a random variable $\delta_T(u) \in L^1(\Omega)$ satisfying the duality relation

$$\mathbb{E}\left[\int_0^T u_t D_t F dt\right] = \mathbb{E}[\delta_T(u)F], \quad \text{for every} \quad F \in \tilde{\mathbb{D}}_T^{1,\infty}. \tag{2.2}$$

Sometimes, when $u \in L^2(\Omega \times [0, T])$, we consider the Skorohod integral as the adjoint of $D : \mathbb{D}_T^{1,2} \subset L^2(\Omega) \to L^2(\Omega \times [0, T])$. That is, $u \in Dom\, \delta_T$, if and only if, there exists

$\delta_T(u) \in L^2(\Omega)$ such that (2.2) holds for any $F \in \mathbb{D}_T^{1,2}$. Note that the first definition of δ_T is an extension of the second one.

The operator δ_T is an extension of the Itô integral in the sense that the set $L_a^2(\Omega \times [0,T])$ of all square-integrable and adapted processes with respect to the filtration generated by W is included in $Dom\ \delta_T$ and the operator δ_T restricted to $L_a^2(\Omega \times [0,T])$ coincides with the Itô stochastic integral with respect to W. For $u \in Dom\ \delta_T$, we make use of the notation $\delta_T(u) = \int_0^T u_t \delta W_t$ and for $t \in [0,T]$ and $u 1\!\!1_{[0,t]}$ in $Dom\ \delta_T$, we write $\delta_T(u 1\!\!1_{[0,t]}) = \int_0^t u_s \delta W_s$. Observe also that for $0 < \tilde{T} < T$, if $u \in Dom\ \delta_{\tilde{T}}$, then $u 1\!\!1_{[0,\tilde{T}]} \in Dom\ \delta_T$ and in this case, $\delta_{\tilde{T}}(u) = \delta_T(u 1\!\!1_{[0,\tilde{T}]}) = \int_0^{\tilde{T}} u_s \delta W_s$.

Let $\mathcal{S}_T$ be the family of processes of the form $u(\cdot) = \sum_{j=1}^n F_j h_j(\cdot)$, where for any $j = 1, \dots, n$, F_j is a random variable in $\mathcal{S}$ and $h_j : [0,T] \to \mathbb{R}$ is a bounded measurable function. We denote by $\mathbb{L}_T^{1,2,f}$ the closure of $\mathcal{S}_T$ with respect to the semi-norm

$$||u||_{1,2,f,T}^2 = \mathbb{E}\left(\int_{[0,T]} u_s^2 ds + \int_{\Delta_1^T} (D_s u_t)^2 ds dt \right),$$

where $\Delta_1^T = \left\{(s,t) \in [0,T]^2 : s \geq t\right\}$, and by $\mathbb{L}_T^F$, the closure of $\mathcal{S}_T$ with respect to the semi-norm

$$||u||_{F,T}^2 = ||u||_{1,2,f,T}^2 + \mathbb{E}\left(\int_{\Delta_2^T} (D_r D_s u_t)^2 dr ds dt \right),$$

with $\Delta_2^T = \{(r,s,t) \in [0,T]^3 : r \vee s \geq t\}$. Observe that $L_a^2(\Omega \times [0,T]) \subseteq \mathbb{L}_T^F$ for any $T > 0$, with $D_s u_t = D_r D_s u_t = 0$ for $s > t$ and $(r,s,t) \in \Delta_2^T$.

Finally, for a process $X \in \mathbb{L}_T^{1,2,f}$ and given $p \geq 1$, we denote by $D^- X$ the process in $L^p(\Omega \times [0,T])$ such that

$$\lim_{n \to \infty} \int_0^T \sup_{(s - \frac{1}{n}) \vee 0 < t < s} \mathbb{E}\left(\left| D_s X_t - (D^- X)_s \right|^p \right) ds = 0 \tag{2.3}$$

if such a process $D^- X$ exists. Henceforth, the space $\mathbb{L}_{T,p-}^{1,2,f}$ represents the family of processes $X \in \mathbb{L}_T^{1,2,f}$ such that (2.3) is satisfied.

2.2 Anticipative Girsanov transformations

Following Buckdhan [5], and in order to establish the existence of a unique solution to Eq. (1.1), we introduce two families $A = \{A_{s,t},\ 0 \leq s \leq t\}$ and $\{T_t,\ t \geq 0\}$ of transformations on the Wiener space Ω through the equations

$$(A_{s,t}\omega)_\cdot = \omega_\cdot - \int_{s \wedge \cdot}^{t \wedge \cdot} a_r(A_{r,t}\omega) dr \tag{2.4}$$

and

$$(T_t \omega)_\cdot = \omega_\cdot + \int_0^{t \wedge \cdot} a_r(T_r \omega) dr, \quad \omega \in \Omega. \tag{2.5}$$

Define $A_t := A_{0,t}$. Notice that, from Buckdhan [5] (Section 2.2), if $a \in L^2([0,T]; \mathbb{D}_T^{1,\infty})$, Eqs. (2.4) and (2.5) have a unique solution for $0 \leq s \leq t \leq T$, and moreover, $A_{s,t} = T_s A_t$.

Additionally, if a is also an adapted process, the Girsanov theorem (see Buckdhan [5], Proposition 2.2.3) implies

$$\mathbb{E}(F(A_{s,t})L_{s,t}) = \mathbb{E}(F), \tag{2.6}$$

for $F \in L^{\infty}(\Omega)$, where

$$L_{s,t} := \exp\left\{\int_s^t a_r dW_r - \frac{1}{2}\int_s^t a_r^2 dr\right\}. \tag{2.7}$$

In the following, we use frequently the fact that if F is $\mathcal{F}_s-$measurable, $t \geq s$ and a is an adapted process, then $F(A_t) = F(A_s)$ and $F(T_t) = F(T_s)$.

3 Anticipating semi-linear equations

In this section, for $T > 0$ fixed, we consider the anticipating semi-linear stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t a_s X_s \delta W_s, \quad 0 \leq t \leq T, \tag{3.1}$$

where the random variable X_0 and the coefficients a and b satisfy suitable conditions.

The following will be the hypotheses used in the paper. Some hypotheses are stronger than another ones, but we introduce them in this way not to ask for conditions stronger than we need in some results.

(X1) $X_0 \in L^{\infty}(\Omega)$.

(X2T) For any $T > 0$, $X_0 \in \tilde{\mathbb{D}}_T^{2,\infty}$. (Note that this implies (X1)).

(X3T) X_0 satisfies (X2T) and there exists a constant $\eta > 0$ such that $X_0 > \eta$ for all ω or $X_0 < -\eta$ for all ω.

(A1T) $a \in L_a^2([0, T]; \mathbb{D}_T^{1,\infty})$, that is, a is an $\mathbb{F}-$adapted process in $L^2([0, T]; \mathbb{D}_T^{1,\infty})$.

(A2T) a satisfies (A1T) and moreover $a \in L^{\infty}([0, T] \times \Omega)$ and $Da \in L^{\infty}(\Omega \times [0, T]^2)$.

(B1T) $b : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})-$measurable random field such that there exist an adapted non-negative process $\gamma \in L^{\infty}(\Omega \times [0, T])$ and a constant $L > 0$ satisfying

$$|b(t, x) - b(t, y)| \leq \gamma_t |x - y|, \qquad \sup_{t\in[0,T]} ||b(t, 0)||_{\infty} \leq L,$$

for all $x, y \in \mathbb{R}$ and $t \in [0, T]$ w.p.1. Recall that $|| \cdot ||_{\infty}$ stands for the essential supremum of a random variable. Let's denote

$$c_1 := \int_0^T ||\gamma_s||_{\infty} ds.$$

(B2T) b satisfies (B1T), $b(t, 0) = 0$ for all $t \in [0, T]$ and any fixed $T > 0$. b has almost surely continuous trajectories in t and x, and $\partial_x b(t, x)$ exists and it is continuous in t and x.

(B3T) b satisfies (B2T), $b(\cdot, x) \in L^p([0, T]; \mathbb{D}_T^{1,p})$ for all $p \geq 2$ and $x \in \mathbb{R}$, $b(t, x) \in \mathbb{D}_T^{1,\infty}$ for all $t \in [0, T]$ and $x \in \mathbb{R}$, $D_t b(s, \cdot)$ is a measurable random field continuous on x for any s and t, and there exists a non-negative process $M \in L^1([0, T]^2, L^{\infty}(\Omega))$ such that $|D_s b(t, x, \omega)| \leq M(s, t)\,|x|$ and

$$c_2 := \sup_{0\leq r\leq T} \int_r^T ||M(r, s)||_{\infty} ds < \infty.$$

(B4T) Assume that b satisfy (B3T) for any $T > 0$ and has the form

$$b(t, x) = \bar{b}_t x + \phi(t, x),$$

where $\bar{b} \in L^\infty(\Omega \times [0, T])$, $D\bar{b} \in L^\infty(\Omega \times [0, T]^2)$ and ϕ satisfies $(B3T)$ with a certain process δ in the role of process γ in (B2T). Moreover, the function $\partial_x^2 \phi(t, x)$ exists, it is continuous in t and x and it is bounded uniformly on $\Omega \times [0, T] \times \mathbb{R}$.

Now we proceed as in Nualart [14] (Theorem 3.3.6). Consider $L_{0,t}$ defined in (2.7). Remember that Hypothesis (A1T) implies that for $0 \le s \le t$, $L_{0,s}(T_t) = L_{0,s}(T_s)$. Also notice that Hypotheses (A1T) and (B1T) imply that for all $x \in \mathbb{R}$ and almost all $\omega \in \Omega$, the equation

$$Z_t(\omega, x) = x + \int_0^t L_{0,s}^{-1}(T_t\omega) b(s, L_{0,s}(T_t\omega) Z_s(\omega, x), T_s\omega) ds, \quad t \in [0, T], \tag{3.2}$$

has a unique solution. The relation between this equation and Eq. (3.1) is given by the following theorem:

Theorem 3.1 *Assume (X1), (A1T) and (B1T) hold. Define*

$$X_t = L_{0,t} Z_t(A_t, X_0(A_t)). \tag{3.3}$$

Then, the process $X = \{X_t,\ 0 \le t \le T\}$ satisfies $1\!\!1_{[0,t]}(\cdot)\, a.X. \in \mathrm{Dom}\, \delta_T$ for all $t \in [0, T]$, belongs to $L^1(\Omega \times [0, T])$ and is a solution of Eq. (3.1). Conversely, if $Y \in L^1(\Omega \times [0, T])$ is a solution of Eq. (3.1) and a satisfies (A2T) then Y agrees with the right hand side of (3.3).

Remarks 3.2

1. Note that in the linear case (i.e. $b(s, x) = \bar{b}_s \cdot x$), Hypothesis (B1T) has to be applied to $\gamma := |\bar{b}|$ This is treated in Buckdhan [5] (Theorem 3.2.1). In this case, (3.3) has the form

$$X_t = L_{0,t} \cdot \exp\left\{\int_0^t \bar{b}_s ds\right\} \cdot X_0(A_t),\ t \ge 0.$$

2. The semi-linear Eq. (3.1), when a is a deterministic function of $L^2([0, T])$, is considered in Nualart [14] (Theorem 3.3.6). In the present paper, following ideas stated in Buckdhan [5] (Chapter 3), we extend the result in Nualart [14] to the case that a is a process that satisfies Hypothesis (A1T) and γ is random.
3. Assume (X1), (A2T) and (B1T) are satisfied for any $T > 0$. Note that in this case, Theorem 3.1 says that equation (3.1) has a unique solution on $\Omega \times [0, \infty)$ given by (3.3). For the existence we only need to assume that, for each $T > 0$, $\gamma \in L^1([0, T], L^\infty(\Omega))$. Condition $\gamma \in L^\infty(\Omega \times [0, T])$ is needed for the uniqueness.

Proof Since the proof of this theorem is long and similar to those in Nualart [14] or Buckdhan [5], we only sketch it in the Appendix (Subsection 6.1). For details, the reader can see the references [4, 5, 14]. □

4 Some properties of process *Z*

In this section we establish some properties of process Z introduced in Eq. (3.2).

Lemma 4.1 *Let $T > 0$ and assume (A2T) and (B2T) hold. Then, the solution Z of Eq. (3.2) satisfies*

$$|Z_t(\omega, x)| \leq |x| \exp\left(\int_0^t \|\gamma_s\|_\infty ds\right) \leq |x| e^{c_1} \tag{4.1}$$

and

$$|\partial_x Z_t(\omega, x)| \leq \exp\left(\int_0^t \|\gamma_s\|_\infty ds\right) \leq e^{c_1}, \tag{4.2}$$

for $(t, x) \in [0, T] \times \mathbb{R}$ and for almost all $\omega \in \Omega$.

Proof Let $\omega \in \Omega$ be such that (B2T) is satisfied. Note that being $L_{0,s}$ adapted to the underlying filtration $\mathbb{F}$, Eq. (3.2) can be written as

$$Z_t(\omega, x) = x + \int_0^t L_{0,s}^{-1}(T_s\omega) b(s, L_{0,s}(T_s\omega) Z_s(\omega, x), T_s\omega) ds, \quad t \in [0, T]. \tag{4.3}$$

Inequality (4.1) is an immediate consequence of Gronwall's lemma, (A1T) and (B2T). Taking partial derivatives with respect to x in Eq. (4.3) and using Hartman [8] (Section 5.3) and (A2T) we obtain that $\partial_x Z_t(\omega, x)$ exists and satisfies the equation

$$\partial_x Z_t(\omega, x) = 1 + \int_0^t (\partial_x b)(s, L_{0,s}(T_s\omega) Z_s(\omega, x), T_s\omega) \partial_x Z_s(\omega, x) ds, \quad t \in [0, T], \tag{4.4}$$

whose explicit solution is given by

$$\partial_x Z_t(\omega, x) = \exp\left(\int_0^t (\partial_x b)(s, L_{0,s}(T_s\omega) Z_s(\omega, x), T_s\omega) ds\right). \tag{4.5}$$

Finally (B1T) and (A2T) give that inequality (4.2) is true. □

Lemma 4.2 *Fix $T > 0$. Assume (A1T) and (B2T) hold. Then, for $x \in \mathbb{R}$,*

$$(t, \omega) \mapsto Z_t(A_t\omega, x)$$

is $\mathcal{P}_T$-measurable and belong to $\mathbb{L}^F$.

Proof Let $t_0 \in (0, T]$. Then, (3.2) implies

$$Z_t(A_{t_0}\omega, x) = x + \int_0^t L_{0,s}^{-1} b(s, L_{0,s} Z_s(A_{t_0}\omega, x)) ds, \quad t \in [0, t_0]. \tag{4.6}$$

Note that thanks the fact

$$\left| L_{0,s}^{-1} b(s, L_{0,s}\, y) - L_{0,s}^{-1} b(s, L_{0,s}\, \bar{y}) \right| \leq \|\gamma\|_{L^\infty([0,T]\times\Omega)} \, |y - \bar{y}|,$$

we have the previous equation has a unique solution. Moreover, this solution is adapted since b is $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable. So, $t \to Z_t(A_{t_0}, x)$ is $\mathcal{F}_t$-measurable for all $t \in [0, t_0]$. In particular, $Z_t(A_t, x)$ is $\mathcal{F}_t$-measurable for all $t \in [0, t_0]$, and consequently, for all $t \in [0, T]$.

Finally, for $s < t$,

$$Z_s(A_t, x) = Z_s(A_s T_s A_t, x) = Z_s(A_s T_t A_t, x) = Z_s(A_s, x),$$

where the second equality is a consequence of the fact that $Z_t(A_t, x)$ is $\mathcal{F}_t$-measurable. Moreover, thanks to inequality (4.1), the solution belongs to $L^2(\Omega \times [0, T])$. So, it belongs to $\mathbb{L}_T^F$. □

Lemma 4.3 *Let $T > 0$. Assume that (A1T) and (B2T) are satisfied. Then, for $x > 0$ (resp. $x < 0$), we have*

$$Z_t(A_t, x) \geq x \exp\Big\{ -\int_0^t \gamma_s ds \Big\} \quad (\textit{resp. } Z_t(A_t, x) \leq x \exp\Big\{ -\int_0^t \gamma_s ds \Big\}),$$

for any $t \in [0, T]$ and $\omega \in \Omega$ for which (B2T) is true.

Proof We know that $Z_t(A_t, x)$ satisfies Eq. (4.6). Assume $x > 0$. The negative case is analogous. Fix $\omega \in \Omega$ satisfying (B2T). Assume there exists t_0 such that $Z_u(A_u, x) > 0$ for all $u < t_0$ and $Z_{t_0}(A_{t_0}, x) = 0$. On $[0, t_0]$, using (B2T), we have

$$-\gamma_u L_{0,u} Z_u(A_u, x) \leq b(u, L_{0,u} Z_u(A_u, x)) \leq \gamma_u L_{0,u} Z_u(A_u, x).$$

Therefore, for $t \in [0, t_0]$,

$$\partial_t Z_t(A_t, x) = L_{0,t}^{-1}\, b(t, L_{0,t} Z_t(A_t, x)) \geq -\gamma_t Z_t(A_t, x). \tag{4.7}$$

Hence, by Hartman [8] (Remark 1 of Theorem 4.1 in Section 3.3), we have

$$Z_t(A_t, x) \geq x \exp\Big\{ -\int_0^t \gamma_s ds \Big\}, \quad t \leq t_0.$$

In particular, for $t = t_0$,

$$0 \geq x \exp\Big\{ -\int_0^{t_0} \gamma_s ds \Big\}.$$

and this is a contradiction. Therefore, $Z_t(A_t, x)$ is positive for all $t \in [0, T]$ and, consequently, (4.7) is satisfied for $t \in [0, T]$, which gives that the result holds. □

Lemma 4.4 *Let $T > 0$. Assume that (A2T) and (B3T) hold. Then, for all $p \geq 2$ and $x \in \mathbb{R}$, the process $Z_t(A_t, x)$ belongs to $L^p([0, T], \mathbb{D}_T^{1,p})$, and for $r, t \in [0, T]$ we have*

$$\begin{aligned} D_r Z_t(A_t, x) = \int_{r\wedge t}^t U(t, s) \Big[&(D_r L_{0,s}^{-1}) b(s, L_{0,s} Z_s(A_s, x)) \\ &+ L_{0,s}^{-1}\, (\partial_x b)(s, L_{0,s} Z_s(A_s, x))(D_r L_{0,s}) Z_s(A_s, x) \\ &+ L_{0,s}^{-1}\, D_r b(s, z)|_{z = L_{0,s} Z_s(A_s, x)} \Big] ds, \end{aligned} \tag{4.8}$$

where

$$U(t, s) := \exp\Big\{ \int_s^t (\partial_x b)(u, L_{0,u} Z_u(A_u, x)) du \Big\}.$$

Remark 4.5 Note that (4.8) is the solution of the linear stochastic differential equation

$$\begin{aligned} D_r Z_t(A_t, x) = &\int_{r\wedge t}^t (D_r L_{0,s}^{-1}) b(s, L_{0,s} Z_s(A_s, x)) ds \\ &+ \int_{r\wedge t}^t L_{0,s}^{-1} (\partial_x b)(s, L_{0,s} Z_s(A_s, x))(D_r L_{0,s}) Z_s(A_s, x) ds \\ &+ \int_{r\wedge t}^t L_{0,s}^{-1} D_r b(s, z)|_{z = L_{0,s} Z_s(A_s, x)} ds \\ &+ \int_{r\wedge t}^t (\partial_x b)(s, L_{0,s} Z_s(A_s, x)) D_r Z_s(A_s, x) ds, \ t \in [0, T]. \end{aligned}$$

See for example [2], Section 8.2.

Proof of Lemma 4.4 The proof is inspired in [14] (Section 2.2). Let $c := c_1 \vee c_2 \vee 1$. Recall that c_1 and c_2 are finite constants thanks $(B1T)$ and $(B3T)$, and $Z_t(A_t, x)$ satisfies Eq. (4.6).

We consider the Picard approximations of $Z_t(A_t, x)$. For $n = 0$ we define

$$Z_{t,(0)}(A_t, x) = x$$

and we apply induction on n to define, for $n \geq 1$, the adapted and continuous process

$$Z_{t,(n)}(A_t, x) = x + \int_0^t L_{0,s}^{-1} b(s, L_{0,s} Z_{s,(n-1)}(A_s, x))ds. \tag{4.9}$$

We divide the proof in two steps.

1. In this first step we prove that for any $n \geq 0$, $Z_{\cdot,(n)}(A_\cdot, x)$ is a continuous and adapted process bounded in $L^p([0, T] \times \Omega)$ for any $p \geq 1$, uniformly in n. Moreover, $Z_{t,(n)}(A_t, x)$ converges to $Z_t(A_t, x)$ with probability one, uniformly in $t \in [0, T]$, and in $L^p([0, T] \times \Omega)$, for any $p \geq 1$.
   Note that from (4.9) and (B2T), we have
   $$|Z_{t,(n+1)}(A_t, x)| \leq |x| + \int_0^t \gamma_s \, |Z_{s,(n)}(A_s, x)|ds, \quad t \in [0, T].$$
   Therefore, iterating this inequality, we have $|Z_{t,(n)}(A_t, x)| \leq |x|e^{c_1}$ for all $n \in \mathbb{N}$ and $t \in [0, T]$.
   On the other hand, we have
   $$|Z_{t,(1)} - Z_{t,(0)}| \leq \int_0^t L_{0,s}^{-1} \, |b(s, L_{0,s} Z_{s,(0)}(A_s, x))|ds \leq |x| \int_0^t \gamma_s ds,$$
   and iterating again, we obtain
   $$\begin{aligned} |Z_{t,(n+1)} - Z_{t,(n)}| &\leq \int_0^t L_{0,s}^{-1} \, |b(s, L_{0,s} Z_{s,(n)}(A_s, x)) - b(s, L_{0,s} Z_{s,(n-1)}(A_s, x))|ds \\ &\leq \int_0^t \gamma_s |Z_{s,(n)}(A_s, x) - Z_{s,(n-1)}(A_s, x)|ds \\ &\leq \frac{|x|}{(n+1)!} (\int_0^t \gamma_s ds)^{n+1}. \end{aligned}$$
   Thus,
   $$\sum_{n=0}^{\infty} |Z_{t,(n+1)} - Z_{t,(n)}| \leq c_1 |x| \sum_{n=0}^{\infty} \frac{c_1^n}{n!} = c_1 e^{c_1} |x| < \infty$$
   implies the statement.
2. Now we want to check the differentiability of $Z_t(A_t, x)$ in the Malliavin calculus sense. Using Lemma 1.5.3 in [14], it is enough to check that, for any $n \geq 0$ and $p \geq 2$,
   $$Z_{t,(n)}(A_t, x) \in L^p([0, T], \mathbb{D}_T^{1,p})$$
   and
   $$\sup_{n \geq 0} \sup_{0 \leq r \leq T} \mathbb{E} \left(\sup_{r \leq t \leq T} |D_r Z_{t,(n)}(A_t, x)|^p \right) < \infty.$$

Note that $Z_{t,(0)}(A_t, x) = x \in L^p([0, T], \mathbb{D}_T^{1,p})$ for all $p \geq 2$. Now, assume that $Z_{t,(n)}(A_t, x) \in L^p([0, T], \mathbb{D}_T^{1,p})$ for all $p \geq 2$. Then, using (4.9), (A2T), (B3T) and [10] (Lemma 2.2), we have, for $r \leq t$,

$$
\begin{aligned}
D_r Z_{t,(n+1)}(A_t, x) = & \int_r^t (D_r L_{0,s}^{-1})\, b(s, L_{0,s} Z_{s,(n)}(A_s, x)) ds \\
& + \int_r^t L_{0,s}^{-1} (\partial_x b)(s, L_{0,s} Z_{s,(n)}(A_s, x))(D_r L_{0,s}) Z_{s,(n)}(A_s, x) ds \\
& + \int_r^t (\partial_x b)(s, L_{0,s} Z_{s,(n)}(A_s, x)) D_r Z_{s,(n)}(A_s, x) ds \\
& + \int_r^t L_{0,s}^{-1} D_r b(s, z, \omega)|_{z = L_{0,s} Z_{s,(n)}(A_s, x)} ds.
\end{aligned}
$$

On the other hand, being $Z_{\cdot,(n)}(A_\cdot, x)$ an adapted process, for any $r > t$ we have

$$
D_r Z_{t,(n)}(A_t, x) = 0.
$$

Now putting together the first two terms on the right hand side we have

$$
\begin{aligned}
|D_r Z_{t,(n+1)}(A_t, x)| \leq & \int_r^t \gamma_s \cdot \left(|D_r L_{0,s}^{-1}|\, |L_{0,s}| + |L_{0,s}^{-1}|\, |D_r L_{0,s}| \right) \cdot |Z_{s,(n)}(A_s, x)| ds \\
& + \int_r^t \gamma_s |D_r Z_{s,(n)}(A_s, x)| ds \\
& + \int_r^t |L_{0,s}^{-1}| \cdot M(r, s) \cdot |L_{0,s}| \cdot |Z_{s,(n)}(A_s, x)| ds, \quad t \in [r, T].
\end{aligned}
$$

Defining

$$
K(r, s) := |D_r L_{0,s}^{-1}|\, |L_{0,s}| + |L_{0,s}^{-1}|\, |D_r L_{0,s}|
$$

and joining the first and the third term on the right hand side we obtain

$$
\begin{aligned}
|D_r Z_{t,(n+1)}(A_t, x)| \leq & \int_r^t [\gamma_s K(r, s) + M(r, s)] |Z_{s,(n)}(A_s, x)| ds \\
& + \int_r^t \gamma_s |D_r Z_{s,(n)}(A_s, x)| ds, \quad t \in [r, T].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|D_r Z_{t,(n+1)}(A_t, x)| \leq & \left(\left(\sup_{r \leq s \leq T} K(r, s) \right) \int_r^T \gamma_s ds + \int_r^T M(r, s) ds \right) \\
& \sup_{0 \leq s \leq T} |Z_{s,(n)}(A_s, x)| \\
& + \int_r^t \gamma_s |D_r Z_{s,(n)}(A_s, x)| ds, \quad t \in [0, T].
\end{aligned}
$$

Consequently, using (B3T), Lemma 4.1 and Step 1, we have

$$
\begin{aligned}
|D_r Z_{t,(n+1)}(A_t, x)| \leq & |x| c e^c \left(1 + \sup_{r \leq s \leq T} K(r, s) \right) \\
& + \int_r^t \gamma_s |D_r Z_{s,(n)}(A_s, x)| ds, \quad t \in [r, T].
\end{aligned}
$$

Applying Gronwall's Lemma with r and ω fixed, we obtain

$$|D_r Z_{t,(n+1)}(A_t, x)| \leq |x| c e^c g(r, \omega, T) \exp\left\{ \int_r^t \gamma_s ds \right\} \leq |x| c e^{2c} g(r, \omega, T), \quad (4.10)$$

where $g(r, \omega, T) := 1 + \sup_{r \leq s \leq T} K(r, s)$. Note that the right-hand side of (4.10) is independent of n and $t \in [r, T]$.
We know by Step 1 that $Z_{t,(n+1)}(A_t, x) \in L^p(\Omega)$. So, it remains only to check

$$\sup_{0 \leq r \leq T} \mathbb{E}\left(\sup_{r \leq t \leq T} |D_r Z_{t,(n+1)}(A_t, x)|^p \right) < \infty,$$

uniformly in $n \geq 0$.
Note that, by (4.10), we have

$$\mathbb{E}\left(\sup_{r \leq t \leq T} |D_r Z_{t,(n+1)}(A_t, x)|^p \right) \leq |x|^p c^p e^{2cp} \mathbb{E}\left(|g(r, \omega, T)|^p \right),$$

for all $n \in \mathbb{N}$. Therefore, the problem reduces to check

$$\sup_{0 \leq r \leq T} \mathbb{E}\left(\left[1 + \sup_{r \leq s \leq T} K(r, s) \right]^p \right) < \infty.$$

Using Hölder inequality, it is enough to see

$$\sup_{0 \leq r \leq T} \mathbb{E}\left(\sup_{r \leq s \leq T} K(r, s)^p \right) < \infty,$$

which, by applying Hölder inequality again, is equivalent to check

$$\sup_{0 \leq r \leq T} \mathbb{E}\left(|a_r|^{2p} \right) < \infty,$$

$$\sup_{0 \leq r \leq T} \mathbb{E}\left(\sup_{r \leq s \leq T} \left| \int_r^s D_r a_u dW_u \right|^{2p} \right) < \infty$$

and

$$\sup_{0 \leq r \leq T} \mathbb{E} \int_r^T |a_u|^p \cdot |D_r a_u|^p du < \infty.$$

The first and third statements are obvious from (A2T). The second one is true thanks to Burkholder-Davis-Gundy inequality and (A2T). Therefore, $Z_{t,(n)}(A_t, x)$ is a well defined object in $\mathbb{D}^{1,p}$ and $Z_\cdot(A_\cdot, x) \in L^p([0, T], \mathbb{D}_T^{1,p})$.

Finally, (4.8) follows from (4.6) and Remark 4.5. □

5 Stability of the solution

Remember that Theorem 3.1, under Hypotheses (X1T), (A2T) and (B1T), implies that there exists a unique solution of Eq. (3.1) in $L^1(\Omega \times [0, T])$ for any $T > 0$.

5.1 Auxiliary results

In this section we establish some auxiliary tools that we need to study the stability of the solution of Eq. (3.1).

Lemma 5.1 *Let $T > 0$. Assume (X2T), (A2T) and (B3T) hold. Then, $Z_t(A_t, X_0(A_t))$ belongs to $\mathbb{L}_T^{1,2,f}$ and for $s > t$ we have*

$$D_s Z_t(A_t, X_0(A_t)) = \partial_x Z_t(A_t, X_0(A_t))(D_s X_0)(A_t).$$

Proof By Lemma 4.2 the process $t \mapsto Z_t(A_t, x)$ is in the space $\mathbb{L}_T^{1,2,f}$. Assume first that $X_0 \in \mathcal{S}$. Proceeding as in Ocone and Pardoux [15] (proofs of Lemmas 2.3 and 2.4), together with (4.1) and (4.2), we obtain that for $s > t$,

$$D_s Z_t(A_t, X_0(A_t)) = \partial_x Z_t(A_t, X_0(A_t)) D_s(X_0(A_t)) = \partial_x Z_t(A_t, X_0(A_t))(D_s X_0)(A_t),$$

where the last equality is a consequence of Buckdhan [5] (equality (2.2.26)). Hence, the result is satisfied due to Buckdhan [5] (Proposition 2.1.2) and (4.4). □

Lemma 5.2 *Let $T > 0$. Assume (X2T), (A2T) and (B3T) hold. Let X be the solution of (3.1). Then, $X \in L^p([0, T], \mathbb{D}_T^{1,p})$ for all $p \geq 1$.*

Proof Observe that (2.7), Propositions 1.3.8 and 1.5.5 in [14] and $(A2T)$ establish that $L_{0,\cdot} \in L^p([0, T], \mathbb{D}_T^{1,p})$ for any $p \geq 1$. Hence, by Theorem 3.1 it is enough to show that $Z_\cdot(A_\cdot, X_0(A_\cdot)) \in L^p([0, T], \mathbb{D}_T^{1,p})$ for all $p \geq 1$. Toward this end we first assume $X_0 \in \mathcal{S}$. In this case, Lemmas 4.1 and 4.4, (4.8) together with the dominated convergence theorem, (A2T) and (B2T) yield that we can proceed as in the proof of Lemma 2.1 in [10] to see that $Z_\cdot(A_\cdot, X_0(A_\cdot)) \in L^p([0, T], \mathbb{D}_T^{1,p})$ with

$$D_s Z_t(A_t, X_0(A_t)) = D_s Z_t(A_t, x)|_{x=X_0(A_t)} + (\partial_x Z_t)(A_t, X_0(A_t)) D_s(X_0(A_t)).$$

Hence, Buckdhan [5] (Proposition 2.1.2, Lemma 2.2.13, and (2.2.26)) yield, for any $s, t \in [0, T]$,

$$D_s Z_t(A_t, X_0(A_t)) = D_s Z_t(A_t, x)|_{x=X_0(A_t)} + (\partial_x Z_t)(A_t, X_0(A_t))(D_s X_0)(A_t). \quad (5.1)$$

Finally, the result follows from (A2T), (B2T), (4.4), (4.8), Lemma 4.1, Buckdhan [5] (Proposition 2.1.2) and the dominated convergence theorem. □

For any $\nu \in (0, 1]$, we consider the Lyapunov function

$$F(x, y) = |x|^\nu e^{-\nu y}, \quad x, y \in \mathbb{R}. \quad (5.2)$$

The following result is the main tool for the study of the stability of the solution to (3.1).

Theorem 5.3 *Let $T > 0$. Assume Hypotheses (X3T), (A2T) and (B4T) hold. Let X be the solution of (3.1) given by (3.3) and*

$$Y_t := \int_0^t \left(\bar{b}_s - \frac{a_s^2}{2} + \varepsilon_s \right) ds, \ \ t \in [0, T],$$

where $\varepsilon := \{\varepsilon_s, s \in [0, T]\}$ is a positive adapted process belonging to $L^\infty(\Omega \times [0, T])$. Then, for any $t \in [0, T]$ we get

$$\mathbb{E}\left(F(X_t, Y_t)\right) = \mathbb{E}\left(|X_0|^\nu\right) + \nu \mathbb{E}\left(\int_0^t F(X_s, Y_s) \left[\nu \frac{a_s^2}{2} + \frac{\phi(s, X_s)}{X_s} - \varepsilon_s \right] ds \right)$$

$$+\nu(\nu-1)\mathbb{E}\left(\int_0^t F(X_s,Y_s)a_s\frac{\partial_x Z_s(A_s,X_0(A_s))}{Z_s(A_s,X_0(A_s))}\cdot(D_sX_0)(A_s)ds\right). \tag{5.3}$$

For simplicity we will write

$$\mathbb{E}\left(F(X_t,Y_t)\right)=\mathbb{E}\left(|X_0|^\nu\right)+\nu\mathbb{E}\left(\int_0^t F(X_s,Y_s)\eta(s)ds\right)$$

with

$$\eta(s):=\nu\frac{a_s^2}{2}+\frac{\phi(s,X_s)}{X_s}-\varepsilon_s+(\nu-1)a_s\frac{\partial_x Z_s(A_s,X_0(A_s))}{Z_s(A_s,X_0(A_s))}(D_sX_0)(A_s). \tag{5.4}$$

Remark 5.4 Note that as a consequence of Lemmas 4.1 and 4.3 we have, for either $X_0>0$ a.s. or $X_0<0$ a.s.,

$$\left|\frac{\partial_x Z_s(A_s,X_0(A_s))}{Z_s(A_s,X_0(A_s))}\right|\leq\frac{e^{2c_1}}{|X_0(A_s)|},\ a.s.$$

Proof of Theorem 5.3 Note that by $(X2T)$ the random variable $X_0\in\mathbb{D}_T^{1,2}$ and then, there exists a sequence $\{X_0^{(n)}\in\mathcal{S},n\geq1\}$ that converges to X_0 in $\mathbb{D}_T^{1,2}$. By $(X3T)$ we can assume that $|X_0^{(n)}|>\frac{\eta}{2}$ for all $n\in\mathbb{N}$, where $\eta>0$ is given by Hypothesis (X3T).

Being $a,\bar{b}$ and ε processes in $L_a^2(\Omega\times[0,T])$, it is well-known that we can consider three sequences $\{a^{(n)},n\geq1\}$, $\{\bar{b}^{(n)},n\geq1\}$ and $\{\varepsilon^{(n)},n\geq1\}$ of adapted processes of the form

$$a_t^{(n)}=\sum_{i=0}^{m_n-1}F_{i,n}1\!\!1_{(t_i,t_{i+1}]}(t),\quad \bar{b}_t^{(n)}=\sum_{i=0}^{m_n-1}G_{i,n}1\!\!1_{(t_i,t_{i+1}]}(t),\quad \varepsilon_t^{(n)}=\sum_{i=0}^{m_n-1}E_{i,n}1\!\!1_{(t_i,t_{i+1}]}(t),$$

with $F_{i,n},G_{i,n},E_{i,n}\in\mathcal{S}$, such that

$$\lim_{n\to+\infty}\mathbb{E}\int_0^T\left[a_t^{(n)}-a_t\right]^2dt=0,\qquad \lim_{n\to+\infty}\mathbb{E}\int_0^T\left[\bar{b}_t^{(n)}-\bar{b}_t\right]^2dt=0$$

and

$$\lim_{n\to\infty}\mathbb{E}\int_0^T|\varepsilon_t^{(n)}-\varepsilon_t|^2dt=0.$$

Moreover, observe that given (X2T), (A2T) and (B4T), it is straightforward to prove that $||X_0^{(n)}||_\infty$, $||a^{(n)}||_{L^\infty(\Omega\times[0,T])}$, $||\bar{b}^{(n)}||_{L^\infty(\Omega\times[0,T])}$ and $||\varepsilon^{(n)}||_{L^\infty(\Omega\times[0,T])}$ are bounded respectively by the norms $c||X_0||_\infty$, $c||a||_{L^\infty(\Omega\times[0,T])}$, $c||\bar{b}||_{L^\infty(\Omega\times[0,T])}$ and $c||\varepsilon||_{L^\infty(\Omega\times[0,T])}$ for a certain generic constant $c\geq1$.

But we are interested in approximating $a,\bar{b}$ and ε by continuous processes. Towards this end, let $n\in\mathbb{N}$. For each n, define

$$t_i=\frac{iT}{n},\qquad i=0,\ldots,n$$

and

$$x_j=\frac{j}{n},\qquad j=-n^2,\ldots,-1,0,1,\ldots,n^2.$$

We can consider functions $k_i \in \mathcal{C}_c^\infty(\mathbb{R})$, with values in $[0, 1]$, that approximate indicator functions in the following sense:

$$k_i^{(n)}(t) = \begin{cases} 1, \ t \in [t_i, t_{i+1}], \\ 0, \ t \notin \left[t_i - \frac{1}{n^2}, \left(t_{i+1} + \frac{1}{n^2}\right) \wedge T\right]. \end{cases}$$

Then, we can change the previous processes $\{a^{(n)}, n \geq 1\}$, $\{\bar{b}^{(n)}, n \geq 1\}$ and $\{\varepsilon^{(n)}, n \geq 1\}$ by continuous and adapted versions of the form

$$a_t^{(n)} = \sum_{i=1}^{m_n} F_{i,n} k_i^{(n)}(t), \qquad \bar{b}_t^{(n)} = \sum_{i=1}^{m_n} G_{i,n} k_i^{(n)}(t), \qquad \varepsilon_t^{(n)} = \sum_{i=1}^{m_n} E_{i,n} k_i^n(t),$$

with $F_{i,n}, G_{i,n}, E_{i,n} \in \mathcal{S}$, such that

$$\lim_{n\to+\infty} \mathbb{E} \int_0^T \left[a_t^{(n)} - a_t\right]^2 dt = 0, \qquad \lim_{n\to+\infty} \mathbb{E} \int_0^T \left[\bar{b}_t^{(n)} - \bar{b}_t\right]^2 dt = 0$$

and

$$\lim_{n\to+\infty} \mathbb{E} \int_0^T |\varepsilon_t^{(n)} - \varepsilon_t|^2 dt = 0.$$

The fact that we can change $\mathbb{1}_{(t_i,t_{i+1}]}$ by $k_i^{(n)}(t)$ is proved by the arguments used in Sect. 6.2.

The approximation of ϕ is slightly more complicated. See Sect. 6.2 for details. In Sect. 6.2, we consider the adapted random field

$$\phi^{(n)}(t, x) = \int_0^x \psi^{(n)}(t, y) dy, \qquad n \geq 1,$$

where

$$\psi^{(n)}(t, x) = \partial_x \phi(t, 0) + g\left(\int_0^x \bar{\psi}^{(n)}(t, y) dy\right),$$

with g smooth enough,

$$\bar{\psi}^{(n)}(t, x) = \sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} H_{i,j}^{(n)} k_i^{(n)}(t) \ \mathbb{1}_{(x_j, x_{j+1}]}(x), \qquad n \geq 1,$$

and $H_{i,j}^{(n)} \in \mathcal{S}$. Taking into account the construction in Appendix (Sect. 6.2) we can prove that $\bar{\psi}^{(n)}(t, x)$ is bounded and that $\bar{\psi}^{(n)}(t, x) \longrightarrow \partial_x^2 \phi(t, x)$ when n tends to $+\infty$ for almost all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}$. Moreover, we can also check that $\psi^{(n)}(t, x) \longrightarrow \partial_x \phi(t, x)$ almost surely when n tends to $+\infty$, the function $\bar{\psi}^{(n)}(t, x)$ is uniformly bounded with respect all the parameters (including n) and

$$\lim_{n\to+\infty} \mathbb{E} \int_K \int_0^T \left[\partial_x \phi(t, x) - \psi^{(n)}(t, x)\right]^2 dt dx = 0,$$

for any compact $K \subset \mathbb{R}$. As a consequence, we also have

$$\lim_{n\to+\infty} \sup_{x\in K} \mathbb{E} \int_0^T \left[\phi(t, x) - \phi^{(n)}(t, x)\right]^2 dt = 0. \tag{5.5}$$

Now, we divide the proof into two steps. First, we prove the result using the simple processes defined above and then, for the general case.

1. Here, we fix $n \in \mathbb{N}$. Let $Z^{(n)}$ be the solution to (3.2) when we change a and b by $a^{(n)}$ and $b^{(n)}$, respectively. Note that the change of a by $a^{(n)}$ implies the change of operators $A_{s,t}$ and T_t. Here, we also change X_0 by and $X_0^{(n)}$.
By Lemma 4.4, we have that $Z_t^{(n)}(A_t^{(n)}, x) \in L^p([0,T]; \mathbb{D}_T^{1,p})$ for any $p > 1$ and $x \in \mathbb{R}$. Moreover, from Lemma 5.1, we also have, for $s > t$,

$$\begin{aligned} D_s Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)})) &= \partial_x Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)})) \left(D_s X_0^{(n)}\right)(A_t^{(n)}) \\ &= \exp\left\{\int_0^t \partial_x b^{(n)}(u, L_{0,u}^{(n)} Z_u^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)})))du\right\} \\ &\quad \times \left(D_s X_0^{(n)}\right)(A_t^{(n)}), \end{aligned} \tag{5.6}$$

where the last equality follows from (4.5). Remember that, as a consequence of the definition of $b^{(n)}$ we have that $\partial_x b^{(n)}$ is bounded on $\Omega \times [0,T] \times \mathbb{R}$. Hence we have that $Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)}))$ is a bounded process because of Hypothesis (B2T), (3.2) and (4.1). The fact that $Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)})) \in \mathbb{L}^F \cap L^\beta(\Omega \times [0,T])$, for any $\beta > 2$, is not obvious since in Lemma 4.2 the initial condition is deterministic. The fact of belonging to $\mathbb{L}^F$ can be proved by considering the approximation

$$\sum_{j=-m}^{m} \partial_x Z_t^{(n)}(A_t^{(n)}, x_j) \int_0^{X_0^{(n)}(A_t^{(n)})} \mathbb{1}_{(x_j, x_{j+1}]}(x)dx, \tag{5.7}$$

and taking into account (4.5), Lemmas 4.2 and 4.4 and the assumptions on the coefficients. Now it is easy to see that $X_t^{(n)} = L_{0,t}^{(n)} Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)}))$ belongs to $\mathbb{L}^F$ with

$$\begin{aligned} D_s X_t^{(n)} &= L_{0,t}^{(n)} D_s Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)})), \\ D_r D_s X_t^{(n)} &= \left(D_r L_{0,t}^{(n)}\right) D_s Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)})) \\ &\quad + L_{0,t}^{(n)} D_r D_s Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)})), \end{aligned} \tag{5.8}$$

$s > t$ and for any $r \in [0,T]$.
Now our goal is to use Remark 4 of Theorem 3 in Alòs-Nualart [1] in order to apply the Itô formula (3.2) of that paper.
Note first of all that hypotheses on $a^{(n)}$ implies

$$L_{0,t}^{(n)} \in L^p(\Omega \times [0,T]), \quad \text{for any } p > 1, \tag{5.9}$$

and (4.1) and the hypotheses on $X_0^{(n)}$ imply that

$$\mathbb{E}\left(\int_0^T \left|a_s^{(n)} L_{0,s}^{(n)} Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))\right|^2 ds\right)^2 < \infty. \tag{5.10}$$

From (5.6) and the hypotheses on $\bar{b}^{(n)}$, $\phi^{(n)}$ and $X_0^{(n)}$ it is clear that

$$\left|D_s Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)}))\right| \le C, \qquad s > t. \tag{5.11}$$

So, hypotheses on $a^{(n)}$ and $X_0^{(n)}$, (5.9) and (4.2) implies that

$$\int_0^T \left(\int_0^s \left| \mathbb{E} D_s \left(a_r^{(n)} L_{0,r}^{(n)} Z_r^{(n)}(A_r^{(n)}, X_0^{(n)}(A_r^{(n)})) \right) \right|^2 dr \right)^2 ds < \infty. \tag{5.12}$$

We also need to study (5.8). We divide it into three parts. Hypotheses on $X_0^{(n)}$ and $a^{(n)}$ (in particular $a^{(n)}$ is adapted), (5.9) and (5.11) give

$$\int_0^T \mathbb{E} \left(\int_0^T \int_0^s \left| \left(D_u a_r^{(n)} \right) L_{0,r}^{(n)} D_s Z_r^{(n)}(A_r^{(n)}, X_0^{(n)}(A_r^{(n)})) \right|^2 dr du \right)^2 ds < \infty, \tag{5.13}$$

and

$$\int_0^T \mathbb{E} \left(\int_0^T \int_0^s \left| a_r^{(n)} \left(D_u L_{0,r}^{(n)} \right) D_s Z_r^{(n)}(A_r^{(n)}, X_0^{(n)}(A_r^{(n)})) \right|^2 dr du \right)^2 ds < \infty. \tag{5.14}$$

In order to deal with the remaining term we need to take into account the following

$$\begin{aligned} & a_r^{(n)} L_{0,r}^{(n)} D_u D_s Z_r^{(n)}(A_r^{(n)}, X_0^{(n)}(A_r^{(n)})) \\ & \quad = a_r^{(n)} L_{0,r}^{(n)} D_u \left[\exp \left\{ \int_0^r \partial_x b^{(n)}(v, L_{0,v}^{(n)} Z_v^{(n)}(A_r^{(n)}, X_0^{(n)}(A_r^{(n)}))) dv \right\} \left(D_s X_0^{(n)} \right) (A_r^{(n)}) \right]. \end{aligned}$$

The factor with $D_u \left(D_s X_0^{(n)} \right) (A_r^{(n)})$ is bounded as before thanks hypotheses on $a^{(n)}$ and $X_0^{(n)}$. On the other hand, we have

$$\begin{aligned} & a_r^{(n)} L_{0,r}^{(n)} D_u \left[\exp \left\{ \int_0^r \partial_x b^{(n)}(v, L_{0,v}^{(n)} Z_v^{(n)}(A_r^{(n)}, X_0^{(n)}(A_r^{(n)}))) dv \right\} \right] \left(D_s X_0^{(n)} \right) (A_r^{(n)}) \\ & \quad = A(r,u,s) + B(r,u,s), \end{aligned}$$

with

$$\begin{aligned} A(r,u,s) &= a_r^{(n)} L_{0,r}^{(n)} D_u \left[\exp \left\{ \int_0^r \bar{b}_v^{(n)} dv \right\} \right] \\ & \qquad \times \exp \left\{ \int_0^r \partial_x \phi^{(n)}(v, L_{0,v}^{(n)} Z_v^{(n)}(A_r^{(n)}, X_0^{(n)}(A_r^{(n)}))) dv \right\} \left(D_s X_0^{(n)} \right) (A_r^{(n)}), \\ B(r,u,s) &= a_r^{(n)} L_{0,r}^{(n)} \exp \left\{ \int_0^r \bar{b}_v^{(n)} dv \right\} \left(D_s X_0^{(n)} \right) (A_r^{(n)}) \\ & \qquad \times D_u \left[\exp \left\{ \int_0^r \partial_x \phi^{(n)}(v, L_{0,v}^{(n)} Z_v^{(n)}(A_r^{(n)}, X_0^{(n)}(A_v^{(n)}))) dv \right\} \right]. \end{aligned}$$

Using similar arguments as before we can show

$$\int_0^T \mathbb{E} \left(\int_0^T \int_0^s |A(r,u,s)|^2 dr du \right)^2 ds < \infty. \tag{5.15}$$

Hypotheses on $a^{(n)}$, $\bar{b}^{(n)}$ and $X_0^{(n)}$, the construction of $\phi^{(n)}$, (5.9), (5.11) and arguing as in (5.12), we can obtain

$$\int_0^T \mathbb{E} \left(\int_0^T \int_0^s |B(r,u,s)|^2 dr du \right)^2 ds < \infty. \tag{5.16}$$

Notice that using $a^{(n)}$, $\bar{b}^{(n)}$ and $\varepsilon^{(n)}$ we can also define

$$Y_t^{(n)} := \int_0^t \left(\bar{b}_s^{(n)} - \frac{(a_s^{(n)})^2}{2} + \varepsilon_s^{(n)} \right) ds, \ \ t \in [0, T].$$

Moreover we can consider $F_m(x, y) := \alpha_m(x)^\nu e^{-\nu y}$ where α_m is an infinite derivable function such that $\alpha_m(x) = |x|$ on $(-\frac{1}{m}, \frac{1}{m})^c$ and $\frac{1}{2m} \leq \alpha_m(x) \leq \frac{1}{m}$ on $(-\frac{1}{m}, \frac{1}{m})$. The expression (5.8) together with the bounds (5.10) and (5.12)–(5.16) (we can argue in a similar way for the points (ii) and (iii) in Remark 4 of [1]) allow us to apply the Itô formula for the Skorohod integral (see [1]) and to obtain, for $\frac{1}{m} \leq \frac{\eta}{2}$,

$$\begin{aligned} F_m\left(X_t^{(n)}, Y_t^{(n)}\right) &= |X_0^{(n)}|^\nu + \int_0^t \partial_x F_m(X_s^{(n)}, Y_s^{(n)}) \bar{b}_s^{(n)} X_s^{(n)} ds \\ &+ \int_0^t \partial_x F_m(X_s^{(n)}, Y_s^{(n)}) \phi^{(n)}(s, X_s^{(n)}) ds + \int_0^t \partial_x F_m(X_s^{(n)}, Y_s^{(n)}) a_s^{(n)} X_s^{(n)} \delta W_s \\ &+ \int_0^t \partial_y F_m(X_s^{(n)}, Y_s^{(n)}) \left(\bar{b}_s^{(n)} - \frac{(a_s^{(n)})^2}{2} + \varepsilon_s^{(n)} \right) ds \\ &+ \frac{1}{2} \int_0^t \partial_{x,x}^2 F_m(X_s^{(n)}, Y_s^{(n)}) \left(a_s^{(n)} X_s^{(n)} \right)^2 ds \\ &+ \int_0^t \partial_{x,x}^2 F_m(X_s^{(n)}, Y_s^{(n)}) a_s^{(n)} X_s^{(n)} D_s^- X_s^{(n)} ds. \end{aligned}$$

Taking into account the definition of F_m and (B4T) we have

$$\begin{aligned} F_m\left(X_t^{(n)}, Y_t^{(n)}\right) &= |X_0^{(n)}|^\nu + \nu \int_0^t \partial_x \alpha_m(X_s^{(n)}) \, \alpha_m(X_s^{(n)})^{\nu-1} \, e^{-\nu Y_s^{(n)}} \, \bar{b}_s^{(n)} X_s^{(n)} ds \\ &+ \nu \int_0^t \partial_x \alpha_m(X_s^{(n)}) \, \alpha_m(X_s^{(n)})^{\nu-1} \, e^{-\nu Y_s^{(n)}} \, \phi^{(n)}(s, X_s^{(n)}) ds \\ &+ \nu \int_0^t \partial_x \alpha_m(X_s^{(n)}) \, \alpha_m(X_s^{(n)})^{\nu-1} \, e^{-\nu Y_s^{(n)}} \, a_s^{(n)} X_s^{(n)} \delta W_s \\ &- \nu \int_0^t \alpha_m(X_s^{(n)})^\nu \, e^{-\nu Y_s^{(n)}} \left(\bar{b}_s^{(n)} - \frac{(a_s^{(n)})^2}{2} + \varepsilon_s^{(n)} \right) ds \\ &+ \frac{\nu}{2} \int_0^t \partial_{x,x}^2 \alpha_m(X_s^{(n)}) \, \alpha_m(X_s^{(n)})^{\nu-1} \, e^{-\nu Y_s^{(n)}} \left(a_s^{(n)} X_s^{(n)} \right)^2 ds \\ &+ \frac{\nu(\nu-1)}{2} \int_0^t \left(\partial_x \alpha_m(X_s^{(n)}) \right)^2 \alpha_m(X_s^{(n)})^{\nu-2} \, e^{-\nu Y_s^{(n)}} \left(a_s^{(n)} X_s^{(n)} \right)^2 ds \\ &+ \nu \int_0^t \partial_{x,x}^2 \alpha_m(X_s^{(n)}) \, \alpha_m(X_s^{(n)})^{\nu-1} \, e^{-\nu Y_s^{(n)}} \, a_s^{(n)} X_s^{(n)} D_s^- X_s^{(n)} ds \\ &+ \nu(\nu-1) \int_0^t \left(\partial_x \alpha_m(X_s^{(n)}) \right)^2 \alpha_m(X_s^{(n)})^{\nu-2} \, e^{-\nu Y_s^{(n)}} \, a_s^{(n)} X_s^{(n)} D_s^- X_s^{(n)} ds. \end{aligned} \tag{5.17}$$

Multiplying the two sides of (5.17) by $1\!\!1_{A_m}$ with

$$A_m = \left\{ x \in \Omega; \inf_{r \in [0,T]} \left| X_r^{(n)} \right| > \frac{1}{m} \right\},$$

and thanks the definition of α_m, and the local property of the Lebesgue and Skorohod integrals (see Lemma 5.2 and Proposition 1.3.15 in [14]) we get

$$\begin{aligned}
1\!\!1_{A_m} F\left(X_t^{(n)}, Y_t^{(n)}\right) &= 1\!\!1_{A_m}\Bigg\{|X_0^{(n)}|^\nu + \nu \int_0^t |X_s^{(n)}|^\nu\, e^{-\nu Y_s^{(n)}}\, \bar{b}_s^{(n)} ds \\
&+\nu \int_0^t |X_s^{(n)}|^\nu\, e^{-\nu Y_s^{(n)}}\, \frac{\phi^{(n)}(s, X_s^{(n)})}{X_s^{(n)}} ds + \nu \int_0^t |X_s^{(n)}|^\nu\, e^{-\nu Y_s^{(n)}}\, a_s^{(n)} \delta W_s \\
&-\nu \int_0^t |X_s^{(n)}|^\nu\, e^{-\nu Y_s^{(n)}} \left(\bar{b}_s^{(n)} - \frac{(a_s^{(n)})^2}{2} + \varepsilon_s^{(n)}\right) ds \\
&+\frac{\nu(\nu-1)}{2} \int_0^t |X_s^{(n)}|^\nu\, e^{-\nu Y_s^{(n)}} \left(a_s^{(n)}\right)^2 ds \\
&+\nu(\nu-1) \int_0^t |X_s^{(n)}|^\nu\, e^{-\nu Y_s^{(n)}}\, a_s^{(n)} \frac{D_s^- X_s^{(n)}}{X_s^{(n)}} ds \Bigg\}.
\end{aligned}$$

Using that $a^{(n)}$, $\bar{b}^{(n)}$ and $\phi^{(n)}$ are adapted to the underlying filtration $\mathbb{F}$, (3.3), Lemmas 5.1 and 4.3, Hypothesis (B4T), (4.5), Lemma 2.6 in [12], Proposition 2.1.4 in [5] and the fact that $D_s^- L_{0,s}^{(n)} = 0$, we have

$$D_s^- X_s^{(n)} = X_s^{(n)} \cdot \frac{\partial_x Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))}{Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))} (D_s X_0^{(n)})(A_s^{(n)}). \tag{5.18}$$

Noting that Lemma 4.3 and (3.3) imply

$$\lim_{m\to+\infty} 1\!\!1_{\{\inf_{r\in[0,T]} |X_r^{(n)}| > \frac{1}{m}\}} = 1\!\!1_{\{\inf_{r\in[0,T]} |X_r^{(n)}| > 0\}} = 1,$$

(5.18) leads to write

$$\begin{aligned}
F\left(X_t^{(n)}, Y_t^{(n)}\right) &= |X_0^{(n)}|^\nu + \nu \int_0^t F(X_s^{(n)}, Y_s^{(n)}) \bar{b}_s^{(n)} ds \\
&+\nu \int_0^t F(X_s^{(n)}, Y_s^{(n)}) \frac{\phi^{(n)}(s, X_s^{(n)})}{X_s^{(n)}} ds + \nu \int_0^t F(X_s^{(n)}, Y_s^{(n)}) a_s^{(n)} \delta W_s \\
&-\nu \int_0^t F(X_s^{(n)}, Y_s^{(n)}) \left(\bar{b}_s^{(n)} - \frac{(a_s^{(n)})^2}{2} + \varepsilon_s^{(n)}\right) ds \\
&+\frac{\nu(\nu-1)}{2} \int_0^t F(X_s^{(n)}, Y_s^{(n)}) \Bigg[2a_s^{(n)} \frac{\partial_x Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))}{Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))} (D_s X_0^{(n)})(A_s^{(n)}) \\
&\quad + \left(a_s^{(n)}\right)^2 \Bigg] ds.
\end{aligned}$$

So,

$$F\left(X_t^{(n)}, Y_t^{(n)}\right) = |X_0^{(n)}|^\nu + \nu \int_0^t F(X_s^{(n)}, Y_s^{(n)}) a_s^{(n)} \delta W_s$$
$$+\nu \int_0^t F(X_s^{(n)}, Y_s^{(n)}) \left[\nu \frac{\left(a_s^{(n)}\right)^2}{2} + \frac{\phi^{(n)}(s, X_s^{(n)})}{X_s^{(n)}} - \varepsilon_s^{(n)} \right] ds$$
$$+\nu(\nu-1) \int_0^t F(X_s^{(n)}, Y_s^{(n)}) a_s^{(n)} \frac{\partial_x Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))}{Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))} (D_s X_0^{(n)})(A_s^{(n)}) ds.$$

Note that

$$F(X_s^{(n)}, Y_s^{(n)})\, a_s^{(n)} = \left| Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)})) \right|^\nu (L_{0,s}^{(n)})^\nu e^{-\nu Y_s^{(n)}} a_s^{(n)}$$

is an element of $L_T^{1,2,f}$ (using the same argument applied to (5.7), together with Lemmas 4.3, 4.4 and 5.2). But it is not enough to show that the expectation of the Skorohod integral is zero. In order to prove it, we have that $a^{(n)}, L_{0,s}^{(n)}, e^{-Y_s^{(n)}} \in L^p([0,T]; \mathbb{D}_T^{1,p})$ for all $p > 1$. From Lemma 4.3,

$$\left| Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)})) \right| \geq \frac{\eta}{2} \exp\left(-\int_0^T \|\gamma_s\|_\infty ds \right).$$

As before, with $\frac{1}{m} \leq \frac{\eta}{2} \exp\left(-\int_0^T \|\gamma_s\|_\infty ds\right)$, we have

$$\left| Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)})) \right|^\nu = \alpha_m \left(Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)})) \right)^\nu.$$

Note that $\partial_x \left(\alpha_m(x)^\nu\right) = \nu\, \alpha_m(x)^{\nu-1} \partial_x \alpha_m(x)$. For the case of positive initial condition, we obtain $\alpha_m(x)^{\nu-1} \leq (2m)^{1-\nu}$ and we also know that $\partial_x \alpha_m$ is bounded. So, the proof of Lemma 5.2 implies that $Z^{(n)}_\cdot(A^{(n)}_\cdot, X_0^{(n)}(A^{(n)}_\cdot)) \in L^p([0,T]; \mathbb{D}_T^{1,p})$ for all $p > 1$. Therefore, taking expectations in the penultimate equality, we prove the result for the particular case of simple processes introduced above:

$$\mathbb{E}\left[F\left(X_t^{(n)}, Y_t^{(n)}\right) \right] = \mathbb{E}|X_0^{(n)}|^\nu + \nu\, \mathbb{E} \int_0^t F(X_s^{(n)}, Y_s^{(n)}) \eta^{(n)}(s) ds$$

where

$$\eta^{(n)}(s) = \nu \frac{\left(a_s^{(n)}\right)^2}{2} + \frac{\phi^{(n)}(s, X_s^{(n)})}{X_s^{(n)}} - \varepsilon_s^{(n)}$$
$$+(\nu-1) a_s^{(n)} \frac{\partial_x Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))}{Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))} (D_s X_0^{(n)})(A_s^{(n)}).$$

2. In order to prove the general case we will take limits on the last equality as $n \to \infty$ to show

$$\mathbb{E}\left[F\left(X_t, Y_t\right)\right] = \mathbb{E}\left(|X_0|^\nu\right) + \nu\, \mathbb{E} \int_0^t F(X_s, Y_s) \eta(s) ds,$$

where η is introduced in (5.4). This claim is detailed as follows.
First of all, we have the convergence of $\mathbb{E}(|X_0^{(n)}|^\nu)$ to $\mathbb{E}(|X_0|^\nu)$ as a consequence of the

fact that by construction $X_0^{(n)}$ converges in $L^2(\Omega)$ to X_0.
In relation with the second term it is enough to show that

$$\lim_{n\to\infty} \mathbb{E} \int_0^T |F(X_s, Y_s)\eta(s) - F(X_s^{(n)}, Y_s^{(n)})\eta^{(n)}(s)|ds = 0$$

in order to finish the proof. To do so, we utilize the inequality

$$\mathbb{E} \int_0^T |F(X_s, Y_s)\eta(s) - F(X_s^{(n)}, Y_s^{(n)})\eta^{(n)}(s)|ds \leq B_{1,n} + B_{2,n},$$

with

$$B_{1,n} = \mathbb{E} \int_0^T |F(X_s, Y_s) - F(X_s^{(n)}, Y_s^{(n)})||\eta(s)|ds$$

and

$$B_{2,n} = \mathbb{E} \int_0^T F(X_s^{(n)}, Y_s^{(n)})|\eta(s) - \eta^{(n)}(s)|ds.$$

We first deal with $B_{1,n}$. Note that from Lemma 4.3, Remark 5.4 and Hypotheses (X3T), (A2T) and (B4T), the process η is bounded. That is, there exists a constant $C > 0$ such that

$$\sup_{(\omega,t)\in\Omega\times[0,T]} |\eta(t)| \leq C.$$

Therefore, (3.3), (4.3), Lemma 4.1, (6.20), the hypothesis on the coefficients a, ε and b, the definition of their approximations $a^{(n)}$, $\varepsilon^{(n)}$ and $b^{(n)}$, and the Cauchy-Schwartz inequality yield

$$\begin{aligned}
B_{1,n} &\leq C\,\mathbb{E} \int_0^T |F(X_s, Y_s) - F(X_s^{(n)}, Y_s^{(n)})|ds \\
&\leq C\,\mathbb{E} \int_0^T \left|L_{0,s}^{\nu} - (L_{0,s}^{(n)})^{\nu}\right| ds + C\,\mathbb{E} \int_0^T \left(L_{0,s}^{(n)}\right)^{\nu} \left|e^{-\nu Y_s} - e^{\nu Y_s^{(n)}}\right| ds \\
&\quad + C\,\mathbb{E} \int_0^T \left(L_{0,s}^{(n)}\right)^{\nu} \left||Z_s(A_s, X_0(A_s))|^{\nu} - |Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))|^{\nu}\right| ds \\
&\leq C\,\mathbb{E} \int_0^T \left|L_{0,s}^{\nu} - (L_{0,s}^{(n)})^{\nu}\right| ds \\
&\quad + C \left(\mathbb{E} \int_0^T \left(L_{0,s}^{(n)}\right)^{2\nu} ds\right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T \left|e^{-\nu Y_s} - e^{\nu Y_s^{(n)}}\right|^2 ds\right)^{\frac{1}{2}} \\
&\quad + C \left(\mathbb{E} \int_0^T \left(L_{0,s}^{(n)}\right)^{2\nu} ds\right)^{\frac{1}{2}} \\
&\quad \times \left(\mathbb{E} \int_0^T \left||Z_s(A_s, X_0(A_s))|^{\nu} - |Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))|^{\nu}\right|^2 ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.19}$$

We claim that our hypotheses allow to show that all these terms converge to zero. Indeed, the last summand goes to zero as $n \to +\infty$ due to (6.12) in Sect. 6.3 and the integral of $|L_{0,s}^{\nu} - (L_{0,s}^{(n)})^{\nu}|$ also tends zero because of the properties of Itô's integral. Moreover, it

is easy to see that the second summand converges to zero.
Concerning the second term we can write

$$B_{2,n} = \mathbb{E} \int_0^T \left(L_{0,s}^{(n)}\right)^{\nu} \left|Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))\right|^{\nu} e^{-\nu Y_s^{(n)}} \left|\eta(s) - \eta^{(n)}(s)\right| ds.$$

Note that by Lemma 4.1, the hypotheses on the coefficients and applying Cauchy-Schwarz inequality we have

$$B_{2,n} \le C \left(\mathbb{E} \int_0^T \left(L_{0,s}^{(n)}\right)^{2\nu} ds\right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T \left|\eta(s) - \eta^{(n)}(s)\right|^2 ds\right)^{\frac{1}{2}},$$

because $|Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))|^{\nu} \, e^{-\nu Y_s^{(n)}} \le C$ (see (6.20)). The first factor in the right hand side is also bounded. Finally, in a similar way as in Sect. 6.2 we may take a subsequence of $\eta^{(n)}(\cdot)$ that is denoted with the same subindex for simplicity and then,

$$\lim_{n\to\infty} \mathbb{E} \int_0^T \left|\eta(s) - \eta^{(n)}(s)\right|^2 ds = 0 \tag{5.20}$$

thanks (3.3) and (4.5), Remark 5.4, assumptions on the coefficient $a^{(n)}$ and the initial condition $X_0^{(n)}$, the definition of $\phi^{(n)}$, and Sects. 6.2 and 6.3. Namely, we obtain that there is a constant $C > 0$ such that

$$\left|\eta(s) - \eta^{(n)}(s)\right|^2 \le C \sum_{i=1}^{4} \eta_i^{(n)}(s),$$

where

$$\eta_1^{(n)}(s) = \left|a_s^2 - \left(a_s^{(n)}\right)^2\right|^2,$$

$$\eta_2^{(n)}(s) = \left|\frac{\phi^{(n)}(s, X_s^{(n)})}{X_s^{(n)}} - \frac{\phi(s, X_s)}{X_s}\right|^2,$$

$$\eta_3^{(n)}(s) = \left|\varepsilon_s - \varepsilon_s^{(n)}\right|^2$$

and

$$\eta_4^{(n)}(s) = \Bigg|a_s \frac{\partial_x Z_s(A_s, X_0(A_s))}{Z_s(A_s, X_0(A_s))}(D_s X_0)(A_s) - a_s^{(n)} \frac{\partial_x Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))}{Z_s^{(n)}(A_s^{(n)}, X_0^{(n)}(A_s^{(n)}))}(D_s X_0^{(n)})(A_s^{(n)})\Bigg|^2.$$

Using the construction of $a^{(n)}$ and $\varepsilon^{(n)}$, it is obvious that

$$\lim_{n\to\infty} \mathbb{E} \int_0^T \left[\eta_1^{(n)}(s) + \eta_3^{(n)}(s)\right] ds = 0.$$

In order to deal with the $\eta_2^{(n)}$, we divide it into three parts as follows:

$$\eta_2^{(n)}(s) = \frac{1}{|X_s^{(n)} X_s|^2} \left[X_s \, \phi^{(n)}(s, X_s^{(n)}) - X_s^{(n)} \, \phi(s, X_s)\right]^2$$

$$\leq \frac{C}{|X_s^{(n)}X_s|^2}\left[|X_s|^2 \left(\phi^{(n)}(s, X_s^{(n)}) - \phi(s, X_s)\right)^2 + \phi(s, X_s)^2 \left(X_s - X_s^{(n)}\right)^2 \right]$$

$$\leq \frac{C}{|X_s^{(n)}X_s|^2}\left[|X_s|^2 \left\{\left(\phi^{(n)}(s, X_s^{(n)}) - \phi(s, X_s^{(n)})\right)^2 + \left(\phi(s, X_s^{(n)}) - \phi(s, X_s)\right)^2\right\} + \phi(s, X_s)^2 \left(X_s - X_s^{(n)}\right)^2 \right].$$

Now, (3.3), (X3T), the construction of $X_s^{(n)}$, Lemmas 4.1 and 4.3, and the arguments used in the study of $B_{1,n}$ in (5.19), (6.11), (6.25) and (6.26) imply that

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \eta_2^{(n)}(s)ds = 0.$$

The convergence of the fourth term is more complicated. It is a consequence of the construction of $a^{(n)}$ and $X_0^{(n)}$, Remark 5.4, Equality (4.5), the approximation of Z by $Z^{(n)}$ given in Sect. 6.3, a similar argument used in (6.14) involving the second derivative of X_0 in order to study the difference between $(D_s X_0)(A_s)$ and $(D_s X_0^{(n)})(A_s^{(n)})$ and, finally, the more delicate aspect is to state

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left|\partial_x\phi(s, X_s) - \partial_x\phi^{(n)}(s, X_s^{(n)})\right|^2 ds\right] = 0.$$

This fact is true because of (4.5) and the mean value theorem. We can see that it holds applying Sect. 6.2. Indeed,

$$\mathbb{E}\left[\int_0^T \left|\partial_x\phi(s, X_s) - \partial_x\phi^{(n)}(s, X_s^{(n)})\right|^2 ds\right] \leq 2\mathbb{E}\int_0^T \left(\eta_{4,1}^{(n)}(s) + \eta_{4,2}^{(n)}(s)\right) ds.$$

Here

$$\mathbb{E}\int_0^T \eta_{4,1}^{(n)}(s)ds = \mathbb{E}\left[\int_0^T \left|\partial_x\phi(s, X_s) - \partial_x\phi^{(n)}(s, X_s)\right|^2 ds\right],$$
$$\mathbb{E}\int_0^T \eta_{4,2}^{(n)}(s)ds = \mathbb{E}\left[\int_0^T \left|\partial_x\phi^{(n)}(s, X_s) - \partial_x\phi^{(n)}(s, X_s^{(n)})\right|^2 ds\right].$$

The construction of $\phi^{(n)}$, together with (B4T), yields

$$\mathbb{E}\int_0^T \eta_{4,2}^{(n)}(s)ds \leq \mathbb{E}\left[\int_0^T \left|\int_{X_s^{(n)}}^{X_s} \partial_x^2\phi^{(n)}(s, y)dy\right|^2 ds\right] \leq C\mathbb{E}\left[\int_0^T \left|X_s - X_s^{(n)}\right|^2 ds\right].$$

Section 6.3 implies that this quantity converges to zero as n tends to $+\infty$. In order to study the other term we observe

$$\mathbb{E}\int_0^T \eta_{4,1}^{(n)}(s)ds = \mathbb{E}\left[\int_0^T \left|\int_0^{X_s} \left[\partial_x^2\phi(s, y) - \partial_x^2\phi^{(n)}(s, y)\right] dy\right|^2 ds\right], \quad (5.21)$$

since $\partial_x^2\phi^{(n)}(s, 0) = \partial_x^2\phi(s, 0)$ by definition. Also by definition, $\left|\partial_x\phi(s, y) - \partial_x\phi^{(n)}(s, y)\right|$ converges to zero almost surely in $(\omega, s, y) \in \Omega \times [0, T] \times \mathbb{R}$, and it is bounded by a constant (see (B4T) and Section 6.2). Consequently, the dominated convergence theorem leads to

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \eta_4^{(n)}(s)ds = 0.$$

Thus, the proof of the theorem is finished. □

5.2 Main results

In this section $X(\cdot, X_0)$ stands for the unique solution to Eq. (3.1), under Hypotheses (X3T), (A2T) and (B4T). Now, we introduce three types of stability for this solution.

Definition 5.5 Assume that X_0 satisfies (X3T). We say that $X(\cdot, 0) \equiv 0$ is stable in probability if, for every $\rho > 0$ and $\delta > 0$, there exists $r > 0$ such that

$$\sup_{t \geq 0} \mathbb{P}\{|X(t, X_0)| > \rho\} < \delta,$$

for any X_0 satisfying

$$\sup_{(s,\omega)\in\mathbb{R}_+\times\Omega} \left[\frac{|D_s X_0|}{|X_0|} + |X_0|\right] \leq r. \tag{5.22}$$

Remark 5.6 Note that if we only consider deterministic initial conditions in Eq. (3.1), then the last definition agrees with the usual stability in probability for Eq. (3.1). See Section 1.5 in Khasminskii [9]. Note the same happens if X_0 is a random variable independent of the Brownian filtration and satisfies (X3T). In this case we can assume that X_0 is $\mathcal{F}_0$-measurable and we have $D_s X_0 = 0$ for all $s \geq 0$ and (5.22) reduces to $||X_0||_\infty \leq r$, the usual condition in the deterministic case (in addition to the condition that its absolute value is greater than a constant).

Definition 5.7 Assume that X_0 satisfies (X3T) and $p > 0$. We say that $X(\cdot, 0) \equiv 0$ is exponentially p-stable if there are positive constants $A, r, \alpha > 0$ such that

$$\mathbb{E}\left(|X(t, X_0)|^p\right) \leq A\mathbb{E}(|X_0|^p)\exp(-\alpha t), \quad t \geq 0,$$

for any X_0 satisfying

$$\sup_{(s,\omega)\in\mathbb{R}_+\times\Omega} \left[\frac{|D_s X_0|}{|X_0|}\right] \leq r. \tag{5.23}$$

Remark 5.8 For instance, in order to have an example of an initial condition satisfying this definition, we can consider $X_0 = \eta \exp\{\phi(F)\}$ with $\eta \in \mathbb{R} - \{0\}$, $F = \int_0^\infty h(s)\delta W_s$, where $\|h\|_\infty$ is assumed to be small enough and ϕ' is bounded.

Definition 5.9 Suppose that X_0 satisfies (X3T). We say that $X(\cdot, 0) \equiv 0$ is exponentially stable in probability if, for a given $\xi > 0$, there are constants $A, r, \alpha > 0$ such that

$$\mathbb{P}(|X(t, X_0)| > \xi) \leq A\exp(-\alpha t), \quad \forall t \geq 0,$$

for any X_0 satisfying (5.23)

Remark 5.10 Note that the exponential p-stability implies

$$\lim_{t\to\infty} \mathbb{E}\left(|X(t, X_0)|^p\right) = 0$$

and the exponential stability in probability implies that

$$\lim_{t\to\infty} \mathbb{P}(|X(t, X_0)| > \xi) = 0.$$

Theorem 5.11 *Suppose that a, b and X_0 satisfy (A2T), (B4T) and (X3T) for any $T > 0$, respectively. Moreover, assume that X_0 satisfies (5.22) and that*

$$\sup_{t\geq 0} Y_t = \sup_{t\geq 0} \int_0^t \left[\bar{b}_s - \frac{a_s^2}{2} + \varepsilon_s \right] ds \leq k, \text{ for all } \omega \in \Omega, \tag{5.24}$$

for a constant $k > 0$ and some positive adapted process ε such that

$$\frac{\nu a_t^2}{2} + \delta_t + r(1-\nu)|a_t|e^{2c_1} \leq \varepsilon_t, \quad \text{for all } t \geq 0, \tag{5.25}$$

for some $\nu \in (0, 1]$, δ_t defined in (B4T), c_1 in (B1T) and r in (5.22). Then, the solution to equation (3.1) is stable in probability.

Proof. Lemmas 4.1 and 4.3, together with (5.3), (5.4) and (5.25), yield

$$\mathbb{E}F(X_t, Y_t) \leq \mathbb{E}(|X_0|^\nu). \tag{5.26}$$

Indeed, note that

$$\begin{aligned} \eta(s) &\leq \nu\frac{a_s^2}{2} + \left|\frac{\phi(s, X_s)}{X_s}\right| - \varepsilon_s + (1-\nu)|a_s| \cdot \left|\frac{\partial_x Z_s(A_s, X_0(A_s))}{Z_s(A_s, X_0(A_s))}\right| \cdot |(D_s X_0)(A_s)| \\ &\leq \nu\frac{a_s^2}{2} + \delta_s - \varepsilon_s + (1-\nu)|a_s|e^{2c_1} \cdot \left|\frac{(D_s X_0)(A_s)}{X_0(A_s)}\right| \end{aligned}$$

and consequently, (5.25) and (5.22) gives that (5.26) holds.
Therefore, for $\rho > 0$,

$$\mathbb{E}F(X_t, Y_t) \geq \int_{\{|X_t|>\rho\}} F(X_t, Y_t) d\mathbb{P} = \int_{\{|X_t|>\rho\}} |X_t|^\nu e^{-\nu Y_t} d\mathbb{P} \geq \rho^\nu\, e^{-k\nu}\, \mathbb{P}(|X_t| > \rho),$$

where we have used (5.24) and the definition of the process Y. Thus, (5.26) gives

$$\mathbb{P}(|X_t| > \rho) \leq \frac{\mathbb{E}(|X_0|^\nu)}{\rho^\nu}\, e^{k\nu}.$$

Hence, choosing $\mathbb{E}(|X_0|^\nu)$ small enough, the result holds. □

Remark 5.12 Note also that if $\varepsilon_s = \delta_s + \epsilon$ for a certain $\epsilon > 0$ and a is bounded in $(0, +\infty) \times \Omega$, we always can find positive constants ν and r small enough such that (5.26) holds. So, a sufficient condition to prove that the theorem holds is to assume $\delta_s + \epsilon$ satisfies (5.24).

We also have the following stability criterion.

Theorem 5.13 *Assume (A2T), (B4T) and (X3T) are satisfied for any $T > 0$ and (5.23) and (5.24) hold. Assume also there exists a strictly positive constant k_0 such that*

$$\frac{\nu a_t^2}{2} + \delta_t + r(1-\nu)|a_t|e^{2c_1} - \varepsilon_t \leq -k_0, \tag{5.27}$$

for all $t \geq 0$ and some $\nu \in (0, 1]$. Then, the solution to Eq. (3.1) is exponentially ν-stable.

Remark 5.14 Note that in comparison with Theorem 5.11, now the condition is (5.27) instead of (5.25).

Proof of Theorem 5.13 In order to apply Theorem 4.1 in Hartman [8] (pag 26) we may use the approximation $X_\cdot^{(n)}$ because if we use the original $X_\cdot$, the derivative of $\mathbb{E}F(X_\cdot, Y_\cdot)$ is not continuous. Using the same arguments given in the proof of Theorem 5.3, we have that there exists a sequence $\{F(X_t^{(n)}, Y_t^{(n)}), n \geq 1\}$ that converges to $F(X_t, Y_t)$ in $L^1(\Omega \times [0, T])$ (see the study of (5.19)) and satisfies

$$\mathbb{E}[F(X_t^{(n)}, Y_t^{(n)})] = \mathbb{E}(|X_0^{(n)}|^\nu) + \nu \int_0^t \mathbb{E}[F(X_s^{(n)}, Y_s^{(n)})\eta^{(n)}(s)]ds.$$

Now, the goal is to apply Theorem 4.1 in Hartman [8]. Borrowing its notation, we define a function $U(t, u) = -k_1 \nu u$, on $(0, T) \times \mathbb{R}$. Then, the solution of $u^{'}(t) = U(t, u)$ in $[0, T]$ is $u(t) = u(0)e^{-k_1 \nu t}$. Moreover, if we define $v(t) = \mathbb{E}[F(X_t^{(n)}, Y_t^{(n)})]$, since $\eta^{(n)}(\cdot)$ is continuous thanks to the definitions of all the coefficients and the constructions of $\phi^{(n)}$, $X^{(n)}$ and $Z^{(n)}$, we have

$$v^{'}(t) = \nu \mathbb{E}\left[F(X_t^{(n)}, Y_t^{(n)})\eta^{(n)}(t)\right], \qquad \text{for all } t \in [0, T].$$

Furthermore,

$$\mathbb{E}[F(X_t^{(n)}, Y_t^{(n)})\eta^{(n)}(t)] = \mathbb{E}\left[F(X_t^{(n)}, Y_t^{(n)})\left(\eta^{(n)}(t) - \eta(t)\right)\right] + \mathbb{E}\left[F(X_t^{(n)}, Y_t^{(n)})\eta(t)\right].$$

So, using Proposition 2.1.2 in [5], (5.20), (5.27) and proceeding as in the proof of Theorem 5.11 we have that there exists $0 < k_1 < k_0$ such that for any n big enough,

$$v^{'}(t) = \nu \mathbb{E}\left[F(X_t^{(n)}, Y_t^{(n)})\eta^{(n)}(t)\right] \leq -k_1 \nu v(t).$$

Then, defining $u(0) = v(0) = \mathbb{E}(|X_0^{(n)}|^\nu)$ and applying Theorem 4.1 in [8] we have

$$\mathbb{E}(F(X_t^{(n)}, Y_t^{(n)})) \leq \mathbb{E}(|X_0^{(n)}|^\nu)e^{-k_1 \nu t} \tag{5.28}$$

for any $t \in [0, T]$. Letting $n \to \infty$ in (5.28) we have

$$\mathbb{E}(F(X_t, Y_t)) \leq \mathbb{E}(|X_0|^\nu)e^{-k_1 \nu t}.$$

Finally, (5.24) allows us to get

$$e^{-k\nu}\mathbb{E}(|X_t|^\nu) \leq \mathbb{E}(|X_t|^\nu e^{-\nu Y_t}) \leq \mathbb{E}(|X_0|^\nu)e^{-k_1 \nu t},$$

which implies the desired result □

An immediate consequence of previous theorem is the following result:

Corollary 5.15 *Under the hypotheses of Theorem 5.13, the solution to equation (3.1) is exponentially stable in probability.*

Proof Observe that for any $\rho > 0$,

$$\mathbb{P}(|X_t| > \rho) \leq \frac{\mathbb{E}(|X_t|^\nu)}{\rho^\nu} \leq \left(\frac{e^k}{\rho}\right)^\nu \mathbb{E}(|X_0|^\nu)\, e^{-k_1 \nu t}.$$ □

Moreover, we also have the following result:

Theorem 5.16 *Assume that (A2T), (B4T) and (X3T) hold for any $T > 0$. Also assume that (5.24) is satisfied and that for some $\nu \in (0, 1]$ there exists $\eta < 0$ such that*

$$\frac{\nu a_t^2}{2} + \delta_t - \varepsilon_t < \eta < 0, \quad \forall t \geq 0.$$

Then, the solution to Eq. (3.1) is exponentially $\nu-$stable and exponentially stable in probability, for any initial condition $X_0 \in \mathbb{D}^{1,2}$ such that $\sup_{(s,\omega)\in\mathbb{R}_+\times\Omega}\{\frac{|D_s X_0|}{|X_0|}\}$ is small enough.

Proof The result is an immediate consequence of (5.3). □

6 Appendix

6.1 Proof of Theorem 3.1

The purpose of this section is to provide a proof of Theorem 3.1.

Here, to simplify the notation, we assume that $c_1 \leq L$ without loss of generality. As in Nualart [14] (Proof of Theorem 3.3.6), we apply Gronwall's lemma and (B1T) to equation (3.2) and then, we use (3.3) to obtain

$$|X_t| \leq L_{0,t} e^L \left[|X_0(A_t)| + L \int_0^t L_{0,s}^{-1}\, ds \right].$$

So, from (2.6), we have

$$\begin{aligned}\mathbb{E}\left(|X_t|\right) &\leq e^L \mathbb{E}\left[|X_0(A_t)| L_{0,t} + L\ L_{0,t} \int_0^t \left(L_{0,s}^{-1}\right) ds \right] \\ &= e^L \mathbb{E}[|X_0|] + L e^L \mathbb{E}\left[L_{0,t} \int_0^t \left(L_{0,s}^{-1}\right) ds \right] < \infty\end{aligned}$$

as a consequence of the fact that $\sup_{0\leq t\leq T} \mathbb{E}\left[L_{0,t}^r\right] < +\infty$ and $\sup_{0\leq t\leq T} \mathbb{E}\left[L_{0,t}^{-r}\right] < +\infty$, for any $r \geq 1$, which follows from (A1T). Moreover, we have

$$\sup_{0\leq t\leq T} \mathbb{E}|X_t| < \infty. \tag{6.1}$$

The proof that X, introduced in (3.3), is a solution to Eq. (3.1) is similar to that of Theorem 3.3.6 in Nualart [14]. Thus, using (2.6), (3.3), (6.1), Buckdhan [5] (Lemma 2.2.13), the integration by parts formula and the Girsanov's theorem, we obtain

$$\mathbb{E}\left[\int_0^t a_s\, X_s\, D_s G\, ds\right] = \mathbb{E}\left[G\left(X_t - X_0 - \int_0^t b(s, X_s) ds\right)\right], \tag{6.2}$$

for any $G \in \mathcal{S}$. Therefore the duality relation (2.2) implies that

$$\int_0^t a_s X_s \delta W_s = X_t - X_0 - \int_0^t b(s, X_s) ds$$

because the right-hand side is an integrable process due to (6.1) and Hypothesis (B1T). Consequently, (3.1) holds.

Now, we prove the uniqueness of the solution to equation (3.1). To do so, we make use of the fact that there is a sequence $\{a_s^n : s \in [0,T]\}$ of the form

$$a_s^n = \sum_{i=0}^{n-1} F_{i,n}\, \mathbb{1}_{(t_i,t_{i+1}]}(s),$$

where $F_{i,n} \in \mathcal{S}$, $i = 0,\dots,n-1$, and $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$, such that a^n goes to a in $L^2([0,T];\mathbb{D}_T^{1,2})$, $\|a^n\|_{L^\infty(\Omega\times[0,T])} \le \|a\|_{L^\infty(\Omega\times[0,T])}$, $\|Da^n\|_{L^\infty(\Omega\times[0,T]^2)} \le \|Da\|_{L^\infty(\Omega\times[0,T]^2)} + 1$ and

$$G(A_t^n) = G(A_s^n) - \int_s^t a_u^n D_u(G(A_u^n))du, \tag{6.3}$$

where $G \in \mathcal{S}$ and A^n is the solution to equation (2.4) when we change a by a^n (see Lemmas 3.2.3 and 3.2.4 in Buckdhan [5]).

Let Y be a solution to (3.1) such that Y belongs to $L^1(\Omega\times[0,T])$ and $\mathbb{1}_{[0,t]}\, a\, Y \in \text{Dom}\,\delta$, for all $t \in [0,T]$. Multiplying the members of (3.1) by $G(A_t^n)$ and taking expectations, we have

$$\mathbb{E}\left[Y_t G(A_t^n)\right] = \mathbb{E}\left[Y_0 G(A_t^n)\right] + \mathbb{E}\left[\int_0^t b(s,Y_s)G(A_t^n)ds\right] + \mathbb{E}\left[\int_0^t a_s Y_s D_s(G(A_t^n))ds\right].$$

Integrating by parts and using (6.3), we get

$$\begin{aligned}\mathbb{E}\left[Y_t G(A_t^n)\right] &= \mathbb{E}[Y_0 G] - \mathbb{E}\left[Y_0\int_0^t a_u^n D_u(G(A_u^n))du\right] + \mathbb{E}\left[\int_0^t b(s,Y_s)G(A_s^n)ds\right]\\ &\quad -\mathbb{E}\left[\int_0^t b(s,Y_s)\int_s^t a_u^n D_u(G(A_u^n))duds\right] + \mathbb{E}\left[\int_0^t a_s Y_s D_s(G(A_s^n))ds\right]\\ &\quad -\mathbb{E}\left[\int_0^t a_s Y_s \int_s^t D_s(a_u^n D_u(G(A_u^n)))duds\right].\end{aligned} \tag{6.4}$$

Consequently, by Fubini's theorem, and proceeding as in Buckdhan [5] (Proof of Theorem 3.2.1), we obtain

$$\begin{aligned}\mathbb{E}[Y_t G(A_t)] &= \mathbb{E}[Y_0 G] + \lim_{n\to\infty}\mathbb{E}\left[\int_0^t b(s,Y_s)G(A_s^n)ds\right]\\ &\quad + \lim_{n\to\infty}\mathbb{E}\left[\int_0^t Y_s\left(a_s - a_s^n\right)D_s(G(A_s^n))ds\right]\\ &= \mathbb{E}[Y_0 G] + \mathbb{E}\left[\int_0^t b(s,Y_s)G(A_s)ds\right].\end{aligned}$$

Hence, Girsanov theorem (see (2.6)) implies

$$\mathbb{E}\left[Y_t(T_t)L_{0,t}^{-1}(T_t)G\right] = \mathbb{E}[Y_0 G] + \mathbb{E}\left[G\int_0^t b(s,Y_s(T_s),T_s)L_{0,s}^{-1}(T_s)ds\right],$$

for any $G \in \mathcal{S}$. So,

$$Y_t(T_t)L_{0,t}^{-1}(T_t) = Y_0 + \int_0^t b(s,Y_s(T_s),T_s)L_{0,s}^{-1}(T_s)ds.$$

Thus, the uniqueness of the solution to Eq. (3.2) leads to establish

$$Y_t(T_t)L_{0,t}^{-1}(T_t) = Z_t(Y_0), \ \text{w.p.1}.$$

It means Y is equal to the right-hand side of (3.3). So, the proof of Theorem 3.1 is complete.

6.2 Construction of $\phi^{(n)}$

Let $n \in \mathbb{N}$. Define the partition

$$t_i = \frac{iT}{n}, \qquad i = 0, \dots, n$$

and

$$x_j = \frac{j}{n}, \qquad j = -n^2, \dots, -1, 0, 1, \dots, n^2.$$

Thanks to $(B4T)$, $\partial_x^2 \phi((t_i - \frac{1}{n^2}) \vee 0, x_j) \in L^2(\Omega)$ for any (i, j) and we can find $F_{i,j}^{(n,m)} \in \mathcal{S}$ such that $F_{i,j}^{(n,m)} \longrightarrow \partial_x^2 \phi((t_i - \frac{1}{n^2}) \vee 0, x_j)$, as $m \to +\infty$, in $L^2(\Omega)$ and a.s. So, let

$$Q = \sup_{(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}} \left|\partial_x^2 \phi(t, x)\right|,$$

which is finite due to (B4T). Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$ taking values in [0, 1] such that

$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2, \end{cases}$$

and $f_Q(x) = f(\frac{x}{2Q})$. Then, if we define

$$\tilde{F}_{i,j}^{(n,m)} = f_Q\left(F_{i,j}^{(n,m)}\right) F_{i,j}^{(n,m)},$$

we have that $\tilde{F}_{i,j}^{(n,m)} \in \mathcal{S}$, $\left|\tilde{F}_{i,j}^{(n,m)}\right| \le 4Q$, $\tilde{F}_{i,j}^{(n,m)} = F_{i,j}^{(n,m)}$ if $\left|F_{i,j}^{(n,m)}\right| \le 2Q$, and, moreover, $\tilde{F}_{i,j}^{(n,m)} \longrightarrow \partial_x^2 \phi((t_i - \frac{1}{n^2}) \vee 0, x_j)$ in $L^2(\Omega)$ and a.s., as m goes to $+\infty$. So, now we can take $H_{i,j}^{(n)} = \tilde{F}_{i,j}^{(n,n_0)}$ with $n_0 \in \mathbb{N}$ such that

$$\mathbb{E}\left[\left|\partial_x^2 \phi((t_i - \frac{1}{n^2}) \vee 0, x_j) - H_{i,j}^{(n)}\right|^2\right] \le \frac{1}{n^2}. \tag{6.5}$$

Using the function $k_i^{(n)}$ introduced in the proof of Theorem 5.3 we define the following bounded random field

$$\bar{\psi}^{(n)}(t, x) = \sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} H_{i,j}^{(n)} k_i^{(n)}(t) \;\; 1\!\!1_{(x_j, x_{j+1}]}(x), \qquad n \ge 1,$$

where we take into account that the indicator depends on n because x_j is so. The function $(t, x) \mapsto \bar{\psi}^{(n)}(t, x)$ is continuous in time with probability one satisfying $|\bar{\psi}^{(n)}(t, x)| \le 16Q$. Our next step is to show that, for any compact $K \subset \mathbb{R}$, we get

$$\lim_{n\to\infty} \mathbb{E} \int_K \int_0^T \left[\partial_x^2 \phi(t, x) - \bar{\psi}^{(n)}(t, x)\right]^2 dt dx = 0. \tag{6.6}$$

To do so, we observe

$$\mathbb{E} \int_K \int_0^T \left[\partial_x^2 \phi(t, x) - \bar{\psi}^{(n)}(t, x)\right]^2 dt \le \sum_{i=1}^4 C I_i^{(n)}, \tag{6.7}$$

with

$$I_1^{(n)} = \mathbb{E}\int_K \int_0^T \left[\partial_x^2 \phi(t,x) - \sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} \partial_x^2 \phi(t_i, x_j)\, 1\!\!1_{(t_i,t_{i+1}]}(t)\, 1\!\!1_{(x_j,x_{j+1}]}(x) \right]^2 dtdx,$$

$$I_2^{(n)} = \mathbb{E}\int_K \int_0^T \Bigg[\sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} \left(\partial_x^2 \phi(t_i, x_j) - \partial_x^2 \phi((t_i - \frac{1}{n^2}) \vee 0, x_j) \right)$$
$$\times 1\!\!1_{(t_i,t_{i+1}]}(t)\, 1\!\!1_{(x_j,x_{j+1}]}(x) \Bigg]^2 dtdx,$$

$$I_3^{(n)} = \mathbb{E}\int_k \int_0^T \left[\sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} \left(\partial_x^2 \phi((t_i - \frac{1}{n^2}) \vee 0, x_j) - H_{i,j}^{(n)} \right) 1\!\!1_{(t_i,t_{i+1}]}(t)\, 1\!\!1_{(x_j,x_{j+1}]}(x) \right]^2 dtdx,$$

$$I_4^{(n)} = \mathbb{E}\int_K \int_0^T \left[\sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} H_{i,j}^{(n)}\, 1\!\!1_{(x_j,x_{j+1}]}(x) \left(1\!\!1_{(t_i,t_{i+1}]}(t) - k_i^{(n)}(t) \right) \right]^2 dtdx.$$

We first study $I_1^{(n)}$ and $I_2^{(n)}$. Let $M > 0$ such that $K \subseteq [-M.M]$. Then, for $n > M$,

$$I_1^{(n)} = \mathbb{E}\int_K \int_0^T \sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} \left| \partial_x^2 \phi(t,x) - \partial_x^2 \phi(t_i, x_j) \right|^2 1\!\!1_{(t_i,t_{i+1}]}(t)\, 1\!\!1_{(x_j,x_{j+1}]}(x) dtdx$$
$$\leq \mathbb{E}\int_K \int_0^T \sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} \sup_{y\in K} \left[\left| \partial_x^2 \phi(t,y) - \partial_x^2 \phi(t_i, x_j) \right|^2 1\!\!1_{(t_i,t_{i+1}]}(t)\, 1\!\!1_{(x_j,x_{j+1}]}(y) \right] dtdx.$$

Now, due to the continuity of $\partial_x^2 \phi$, uniformly on $[0, T] \times K$, we have

$$\lim_{n\to\infty} \left[I_1^{(n)} + I_2^{(n)} \right] = 0. \tag{6.8}$$

Secondly, we have that (6.5) gives

$$I_3^{(n)} \leq \frac{2MT}{n^2},$$

obtaining

$$\lim_{n\to+\infty} I_3^{(n)} = 0. \tag{6.9}$$

We now study the last term. For n large enough, we have

$$I_4^{(n)} \leq CQ^2 \mathbb{E}\int_0^T \int_{-M}^M \left[\sum_{i=0}^{n-1} \sum_{j=-n^2}^{n^2-1} 1\!\!1_{(x_j,x_{j+1}]}(x) \left| 1\!\!1_{(t_i,t_{i+1}]}(t) - k_i^{(n)}(t) \right| \right]^2 dxdt$$
$$\leq CQ^2 \int_0^T \int_{-M}^M \left[\left(\sum_{j=-nM-1}^{nM} 1\!\!1_{(x_j,x_{j+1}]}(x) \right) \left(\sum_{i=0}^{n-1} \left| 1\!\!1_{(t_i,t_{i+1}]}(t) - k_i(t) \right| \right) \right]^2 dxdt.$$

It is not difficult to see that

$$\sum_{j=-nM-1}^{nM} 1\!\!1_{(x_j,x_{j+1}]}(x) \leq 1$$

and

$$\sum_{i=0}^{n-1} \left| \mathbb{1}_{(t_i,t_{i+1}]}(t) - k_i(t) \right| \le \sum_{i=0}^{n-1} \mathbb{1}_{(t_i - \frac{1}{n^2}, t_i]}(t) + \sum_{i=0}^{n-1} \mathbb{1}_{(t_{i+1}, t_{i+1}+\frac{1}{n^2}]}(t).$$

So, using these facts we get

$$\begin{aligned} I_4^{(n)} &\le CQ^2 \int_0^T \int_{-M}^M \left[\sum_{i=0}^{n-1} | \mathbb{1}_{(t_i,t_{i+1}]}(t) - k_i(t)| \right]^2 dxdt \\ &\le CQ^2 \int_0^T \int_{-M}^M \left[\sum_{i=0}^{n-1} \mathbb{1}_{(t_i - \frac{1}{n^2}, t_i]}(t) + \sum_{i=0}^{n-1} \mathbb{1}_{(t_{i+1}, t_{i+1}+\frac{1}{n^2}]}(t) \right]^2 dxdt \\ &\le CQ^2 \int_0^T \int_{-M}^M \sum_{i=0}^{n-1} \mathbb{1}_{(t_i - \frac{1}{n^2}, t_i]}(t)\, dxdt + CQ^2 \int_0^T \int_{-M}^M \sum_{i=0}^{n-1} \mathbb{1}_{(t_{i+1}, t_{i+1}+\frac{1}{n^2}]}(t)\, dxdt \\ &\le \frac{CMTQ^2}{n}, \end{aligned}$$

and, therefore

$$\lim_{n\to\infty} I_4^{(n)} = 0. \tag{6.10}$$

Now, putting together (6.8), (6.9) and (6.10) in (6.7), we get that (6.6) holds.

Finally, let $g \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $|g(x)| \le |x|$ and

$$g(x) = \begin{cases} x, & |x| \le 4\|\delta\|_\infty, \\ 0, & |x| \ge 8\|\delta\|_\infty, \end{cases}$$

where $\|\delta\|_\infty = \sup_{(\omega,t)\in\Omega\times[0,T]} |\delta_t(\omega)|$. With (6.6) in mind, we define, for any $(t,x) \in [0,T]\times\mathbb{R}$,

$$\psi^{(n)}(t,x) = \partial_x\phi(t,0) + g\left(\int_0^x \bar\psi^{(n)}(t,y)dy \right),$$

and

$$\phi^{(n)}(t,x) = \int_0^x \psi^{(n)}(t,y)dy.$$

Remember that $|\bar\psi^{(n)}(t,x)| \le 16Q$ and we observe that (6.6) implies

$$\partial_x^2\phi(t,x) = \lim_{n\to+\infty} \bar\psi^{(n)}(t,x), \quad \text{for almost all } (\omega,x,t) \in \Omega\times\mathbb{R}\times[0,T],$$

taking a subsequence if it is necessary. Furthermore, we have that, for any $x \in \mathbb{R}$,

$$\lim_{n\to+\infty} \int_0^T \left[\partial_x\phi(t,x) - \psi^{(n)}(t,x) \right] dt = 0,$$

as a consequence of the dominated convergence theorem since $|\partial_x^2\phi(t,y) - \bar\psi^{(n)}(t,y)|$ is bounded. Indeed, $\partial_x\phi^{(n)}(t,x) = \psi^{(n)}(t,x)$, the function $\psi^{(n)}(t,x)$ is bounded ($|\psi^{(n)}(t,x)| \le 9\|\delta\|_\infty$) and continuous in x, and

$$\psi^{(n)}(t,x) \longrightarrow \partial_x\phi(t,0) + g\left(\int_0^x \partial_x^2\phi(t,y)dy \right) = \partial_x\phi(t,x), \qquad \text{a.s.}$$

In a similar way we obtain that the fact that $\partial_x \phi^{(n)} = \psi^{(n)}$, for any $x \in \mathbb{R}$, yields

$$\mathbb{E}\int_0^T \left[\phi(t,x) - \phi^{(n)}(t,x)\right]^2 dt = \mathbb{E}\int_0^T \left[\int_0^x \left(\partial_x \phi(t,y) - \psi^{(n)}(t,y)\right) dy\right]^2 dt$$
$$\leq |x|\,\mathbb{E}\int_0^T \int_0^x \left(\partial_x \phi(t,y) - \psi^{(n)}(t,y)\right)^2 dydt.$$

Hence, we can find $M > 0$ such that $K \subseteq [-M, M]$ and

$$\sup_{x\in K} \mathbb{E}\int_0^T \left[\phi(t,x) - \phi^{(n)}(t,x)\right]^2 dt$$
$$\leq M\mathbb{E}\int_0^T \int_{-M}^M \left[\partial_x \phi(t,y) - \psi^{(n)}(t,y)\right]^2 dydt \longrightarrow 0,$$

as n goes to ∞.

6.3 Convergence of $Z^{(n)}$ to Z

In this subsection of the Appendix we show the convergence of $Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)}))$ to $Z_t(A_t, X_0(A_t))$ in $L^1(\Omega \times [0,T])$. It means

$$\lim_{n\to+\infty} \mathbb{E}\int_0^T \left|Z_t(A_t, X_0(A_t)) - Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)}))\right| dt = 0. \tag{6.11}$$

Note that if (6.11) is true, then we also have

$$\lim_{n\to+\infty} \mathbb{E}\int_0^T \left|Z_t(A_t, X_0(A_t)) - Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)}))\right|^2 dt = 0, \tag{6.12}$$

because $|Z_t(A_t, X_0(A_t))|$ and $|Z_t^{(n)}(A_t^{(n)}, X_0^{(n)}(A_t^{(n)}))|$ are bounded by a constant independent of n due to Lemma 4.1, Hypothesis (X1) and Sect. 6.2 (see also inequality (6.20) below).

Now we will prove that (6.11) is satisfied. For simplicity we write $Z_s^{(n)}(x)$ and $Z_s(x)$ instead of $Z_s^{(n)}(A_s^{(n)}, x)$ and $Z_s(A_s, x)$, respectively. Using the triangle inequality, we have

$$\mathbb{E}\int_0^T \left|Z_t(X_0(A_t)) - Z_t^{(n)}(X_0^{(n)}(A_t^{(n)}))\right| dt \leq \theta_1^n + \theta_2^n, \tag{6.13}$$

with

$$\theta_1^n = \mathbb{E}\int_0^T \left|Z_t(X_0(A_t)) - Z_t(X_0^{(n)}(A_t^{(n)}))\right| dt,$$
$$\theta_2^n = \mathbb{E}\int_0^T \left|Z_t(X_0^{(n)}(A_t^{(n)})) - Z_t^{(n)}(X_0^{(n)}(A_t^{(n)}))\right| dt.$$

We first study θ_1^n. For this, we observe that (3.2) allows to get

$$|Z_t(x) - Z_t(y)| \leq |x-y| + \int_0^t L_{0,s}^{-1}\, |b(s, L_{0,s}Z_s(x)) - b(s, L_{0,s}Z_s(y))| ds$$
$$\leq |x-y| + \int_0^t ||\gamma_s||_\infty\, |Z_s(x) - Z_s(y)| ds,$$

and applying Gronwall's Lemma we have, for c_1 defined in (B1T),

$$|Z_t(x) - Z_t(y)| \le e^{c_1}|x - y|.$$

Consequently, using the triangle inequality again, we can establish

$$\theta_1^n \le e^{c_1}\mathbb{E}\int_0^T \left|X_0(A_t) - X_0^{(n)}(A_t^{(n)})\right| dt = e^{c_1}\left[\theta_{1,1}^n + \theta_{1,2}^n\right], \tag{6.14}$$

with

$$\theta_{1,1}^n = \mathbb{E}\int_0^T \left|X_0(A_t) - X_0(A_t^{(n)})\right| dt,$$
$$\theta_{1,2}^n = \mathbb{E}\int_0^T \left|X_0(A_t^{(n)}) - X_0^{(n)}(A_t^{(n)})\right| dt.$$

By Propositions 2.1.4 and 2.2.12 in [5], we obtain

$$\theta_{1,1}^n \le \left\|\left(\int_0^T |D_s X_0|^2 ds\right)^{\frac{1}{2}}\right\|_{L^\infty(\Omega)} \mathbb{E}\int_0^T \left|A_t - A_t^{(n)}\right|_{CM} dt, \tag{6.15}$$

where CM means the norm of Cameron-Martin. Now, we consider the last factor of (6.15). For $s \le t$ and a certain generic constant $C \ge 1$, we can apply (2.4) and [5] (Poposition 2.1.4) to conclude

$$\begin{aligned}
\left|A_{s,t} - A_{s,t}^{(n)}\right|_{CM}^2 :&= \int_s^t \left(a_r(A_{r,t}) - a_r^{(n)}(A_{r,t}^{(n)})\right)^2 dr \\
&\le 2\int_s^t \left(a_r(A_{r,t}) - a_r(A_{r,t}^{(n)})\right)^2 dr + 2\int_s^t \left(a_r(A_{r,t}^{(n)}) - a_r^{(n)}(A_{r,t}^{(n)})\right)^2 dr \\
&\le 2\int_s^t \left\|\int_0^T (D_u a_r)^2 du\right\|_{L^\infty(\Omega\times[0,T])} \left|A_{r,t} - A_{r,t}^{(n)}\right|_{CM}^2 dr \\
&\quad +2C\|a\|_{L^\infty(\Omega\times[0,T])}\int_s^t \left|a_r(A_{r,t}^{(n)}) - a_r^{(n)}(A_{r,t}^{(n)})\right| dr \\
&\le 2T\|Da\|_{L^\infty(\Omega\times[0,T]^2)}^2 \int_s^t \left|A_{r,t} - A_{r,t}^{(n)}\right|_{CM}^2 dr \\
&\quad +2C\|a\|_{L^\infty(\Omega\times[0,T])}\int_s^t \left|a_r(A_{r,t}^{(n)}) - a_r^{(n)}(A_{r,t}^{(n)})\right| dr.
\end{aligned}$$

Hence, taking expectation and using (2.6),

$$\begin{aligned}
\mathbb{E}\left(\left|A_{s,t} - A_{s,t}^{(n)}\right|_{CM}^2\right) &\le 2T\|Da\|_{L^\infty(\Omega\times[0,T]^2)}^2 \int_s^t \mathbb{E}\left(\left|A_{r,t} - A_{r,t}^{(n)}\right|_{CM}^2\right) dr \\
&\quad +2C\|a\|_{L^\infty(\Omega\times[0,T])}\left(\mathbb{E}\int_s^t \left|a_r - a_r^{(n)}\right|^2 dr\right)^{\frac{1}{2}} \left(\mathbb{E}\int_s^t \left(L_{r,t}^{(n)}\right)^{-1} dr\right)^{\frac{1}{2}} \\
&\le 2T\|Da\|_{L^\infty(\Omega\times[0,T]^2)}^2 \int_s^t \mathbb{E}\left(\left|A_{r,t} - A_{r,t}^{(n)}\right|_{CM}^2\right) dr \\
&\quad +C\|a\|_{L^\infty(\Omega\times[0,T])}\left\|a - a^{(n)}\right\|_{L^2(\Omega\times[0,T])}.
\end{aligned}$$

Thus, Gronwall Lemma implies that, for $0 \le s \le t \le T$,

$$\mathbb{E}\left(\left|A_{s,t} - A_{s,t}^{(n)}\right|_{CM}^2\right) \le C\|a\|_{L^2(\Omega\times[0,T])}\left\|a - a^{(n)}\right\|_{L^2(\Omega\times[0,T])} \exp\left\{2T^2\|Da\|_{L^2(\Omega\times[0,T]^2)}^2\right\}.$$

(6.16)

Similarly, changing a and $a^{(n)}$ by X_0 and $X_0^{(n)}$, respectively, we are able to state

$$\begin{aligned}\theta^n_{1,2} &\leq \left(\mathbb{E}\int_0^T \left|X_0(A_t^{(n)}) - X_0^{(n)}(A_t^{(n)})\right|^2 L_{0,t}^{(n)} dt\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \left(L_{0,t}^{(n)}\right)^{-1}\right)^{\frac{1}{2}} \\ &\leq C\sqrt{T}\left\|X_0 - X_0^{(n)}\right\|_{L^2(\Omega)}. \end{aligned} \tag{6.17}$$

So, putting togheter (6.14), (6.15), (6.16) and (6.17) and considering the assumptions on a, $a^{(n)}$, X_0 and $X_0^{(n)}$, we get

$$\lim_{n\to+\infty} \theta_1^n = \lim_{n\to+\infty} \mathbb{E}\int_0^T \left|Z_t(X_0(A_t)) - Z_t(X_0^{(n)}(A_t^{(n)}))\right| dt = 0. \tag{6.18}$$

Now, we analyze θ_2^n. Because of (4.6) we have, for $t \in [0, T]$,

$$\begin{aligned}|Z_t(x) - Z_t^{(n)}(x)| &\leq \int_0^t \left|L_{0,s}^{-1}\, b(s, L_{0,s}Z_s(x)) - (L_{0,s}^n)^{-1}\, b^{(n)}(s, L_{0,s}^{(n)}Z_s^{(n)}(x))\right| ds \\ &\leq \int_0^t L_{0,s}^{-1}\left|b(s, L_{0,s}Z_s(x)) - b(s, L_{0,s}Z_s^{(n)}(x))\right| ds \\ &\quad + \int_0^t \left|L_{0,s}^{-1}\, b(s, L_{0,s}Z_s^{(n)}(x)) - (L_{0,s}^n)^{-1}\, b^{(n)}(s, L_{0,s}^{(n)}Z_s^{(n)}(x))\right| ds \\ &\leq \int_0^t ||\gamma_s||_\infty\, |Z_s(x) - Z_s^{(n)}(x)| ds \\ &\quad + \int_0^t \left|L_{0,s}^{-1}\, b(s, L_{0,s}Z_s^{(n)}(x)) - (L_{0,s}^n)^{-1}\, b^{(n)}(s, L_{0,s}^{(n)}Z_s^{(n)}(x))\right| ds.\end{aligned}$$

Applying Gronwall's Lemma we obtain

$$|Z_t(x) - Z_t^{(n)}(x)| \leq e^{c_1}\int_0^t \left|L_{0,s}^{-1}\, b(s, L_{0,s}Z_s^{(n)}(x)) - (L_{0,s}^n)^{-1}\, b^{(n)}(s, L_{0,s}^{(n)}Z_s^{(n)}(x))\right| ds.$$

So, from (B4T), we can decompose

$$\theta_2^n \leq e^{c_1}\sum_{i=1}^4 H_{i,n}, \tag{6.19}$$

with

$$\begin{aligned}H_{1,n} &= \mathbb{E}\int_0^T\int_0^t \left|\bar{b}_s - \bar{b}_s^{(n)}\right|\left|Z_s^{(n)}(X_0^{(n)}(A_t^{(n)}))\right| dsdt, \\ H_{2,n} &= \mathbb{E}\int_0^T\int_0^t L_{0,s}^{-1}\left|\phi(s, L_{0,s}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)}))) - \phi(s, L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)})))\right| dsdt, \\ H_{3,n} &= \mathbb{E}\int_0^T\int_0^t \left|L_{0,s}^{-1} - (L_{0,s}^{(n)})^{-1}\right|\left|\phi(s, L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)})))\right| dsdt, \\ H_{4,n} &= \mathbb{E}\int_0^T\int_0^t (L_{0,s}^{(n)})^{-1}\left|\phi(s, L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)}))) - \phi^{(n)}(s, L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)})))\right| dsdt.\end{aligned}$$

As in Lemma 4.1, using that $\|\bar{b}^{(n)}\|_{L^\infty(\Omega\times[0,T])} \leq c\|\bar{b}\|_{L^\infty(\Omega\times[0,T])}$ for a certain generic $c \geq 1$ (due to (B4T) and the definition of $\bar{b}^{(n)}$) and that $|\phi^{(n)}(t, x)| \leq 9|x|\|\delta\|_{L^\infty(\Omega\times[0,T])}$

(thanks the construction of $\phi^{(n)}$ in Sect. 6.2), we have

$$\left|Z_s^{(n)}(\omega, x)\right| \le C|x|, \tag{6.20}$$

for all $\omega \in \Omega$ and for $n \ge 1$. Then, Cauchy-Schwartz inequality, (6.20) and the fact that $\|X_0^{(n)}\|_{L^\infty(\Omega)} \le c\|X_0\|_{L^\infty(\Omega)}$, for a certain generic $C \ge 1$, give that

$$H_{1,n} \le CT^{\frac{3}{2}} \left(\mathbb{E}\int_0^T \left|\bar{b}_t - \bar{b}_t^{(n)}\right|^2 dt\right)^{\frac{1}{2}}. \tag{6.21}$$

Proceding as in (6.21), we obtain

$$\begin{aligned} H_{2,n} &\le \|\delta\|_{L^\infty(\Omega\times[0,T])}\mathbb{E}\int_0^T\int_0^t L_{0,s}^{-1}\left|L_{0,s} - L_{0,s}^{(n)}\right|\left|Z_s^{(n)}(X_0^{(n)}(A_t^{(n)}))\right| dsdt \\ &\le CT\|\delta\|_{L^\infty(\Omega\times[0,T])}\left(\mathbb{E}\int_0^T\left|L_{0,s} - L_{0,s}^{(n)}\right|^2 ds\right)^{\frac{1}{2}}. \end{aligned} \tag{6.22}$$

Moreover, using $\phi(s, 0) = 0$ in (B2T), we get

$$\begin{aligned} H_{3,n} &\le C\ \|\delta\|_{L^\infty(\Omega\times[0,T])}\ \mathbb{E}\int_0^T\int_0^t\left|L_{0,s}^{-1} - (L_{0,s}^{(n)})^{-1}\right| L_{0,s}^{(n)}\left|Z_s^{(n)}(X_0^{(n)}(A_t^{(n)}))\right| dsdt \\ &\le CT\ \|\delta\|_{L^\infty(\Omega\times[0,T])}\ \mathbb{E}\int_0^T\left|L_{0,s}^{-1}L_{0,s}^{(n)} - 1\right| ds \\ &\le CT\ \|\delta\|_{L^\infty(\Omega\times[0,T])}\ \mathbb{E}\int_0^T L_{0,s}^{-1}\left|L_{0,s}^{(n)} - L_{0,s}\right| ds \\ &\le CT\ \|\delta\|_{L^\infty(\Omega\times[0,T])}\left(\mathbb{E}\int_0^T\left|L_{0,s}^{(n)} - L_{0,s}\right|^2 ds\right)^{\frac{1}{2}}. \end{aligned} \tag{6.23}$$

Now we deal with the last term $H_{4,n}$. Note that $H_{4,n}$ has the form

$$H_{4,n} = H_{4,1,n}^M + H_{4,2,n}^M, \tag{6.24}$$

with

$$\begin{aligned} H_{4,1,n}^M &= \mathbb{E}\int_0^T\int_0^t \mathbb{1}_{\{L_{0,s}^{(n)} < M\}}(L_{0,s}^{(n)})^{-1} \\ &\qquad \times\left|\phi(s, L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)}))) - \phi^{(n)}(s, L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)})))\right| dsdt, \\ H_{4,2,n}^M &= \mathbb{E}\int_0^T\int_0^t \mathbb{1}_{\{L_{0,s}^{(n)} \ge M\}}(L_{0,s}^{(n)})^{-1} \\ &\qquad \times\left|\phi(s, L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)}))) - \phi^{(n)}(s, L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)})))\right| dsdt. \end{aligned}$$

On one hand, on $\{L_{0,s}^{(n)} < M\}$, we know that $L_{0,s}^{(n)}Z_s^{(n)}(X_0^{(n)}(A_t^{(n)}))$ is bounded, then, for a compact K big enough, we establish, for certain constant $C > 0$ and $L > 0$ such that $K \subset [-L, L]$,

$$\left[H_{4,1,n}^M\right]^2 \le TC\ \mathbb{E}\int_0^T \sup_{x\in K}\left|\phi(s, x) - \phi^{(n)}(s, x)\right|^2 ds$$

$$
\begin{aligned}
&= TC\,\mathbb{E}\int_0^T \sup_{x\in K}\left(\int_0^x \left[\partial_x\phi(s,y)-\psi^{(n)}(s,y)\right]dy\right)^2 ds\\
&\le TC\,\mathbb{E}\int_0^T \sup_{x\in K}\left(|x|\int_0^x \left[\partial_x\phi(s,y)-\psi^{(n)}(s,y)\right]^2 dy\right) ds\\
&\le TCL\,\mathbb{E}\int_0^T \int_{-L}^L \left[\partial_x\phi(s,y)-\psi^{(n)}(s,y)\right]^2 dyds,
\end{aligned}
\tag{6.25}
$$

and this converges to zero as a consequence of Sect. 6.2.

On the other hand, Lemma 4.1, (6.20) and (B4T) yield

$$
\begin{aligned}
H^M_{4,2,n} &\le \mathbb{E}\int_0^T\int_0^t \mathbb{1}_{\{L^{(n)}_{0,s}\ge M\}}(L^{(n)}_{0,s})^{-1}\\
&\qquad\times\left[\left|\phi(s,L^{(n)}_{0,s}Z^{(n)}_s(X^{(n)}_0(A^{(n)}_t)))\right|+\left|\phi^{(n)}(s,L^{(n)}_{0,s}Z^{(n)}_s(X^{(n)}_0(A^{(n)}_t)))\right|\right]dsdt\\
&\le CT\,\|\delta\|_{L^\infty(\Omega\times[0,T])}\,\mathbb{E}\int_0^T \mathbb{1}_{\{L^{(n)}_{0,s}\ge M\}}ds,
\end{aligned}
$$

and Txebitxeff inequality implies

$$
\lim_{M\to+\infty} H^M_{4,2,n} \le \lim_{M\to+\infty}\frac{CT^2\,\|\delta\|_{L^\infty(\Omega\times[0,T])}}{M}=0. \tag{6.26}
$$

So, last part of Section 6.2, the definitions of $a^{(n)}$ and $\bar{b}^{(n)}$, together with (6.19) and (6.21)–(6.26), allow us to obtain

$$
\lim_{n\to+\infty}\theta^n_2 = \lim_{n\to\infty}\mathbb{E}\int_0^T\left|Z_t(X^{(n)}_0(A^{(n)}_t))-Z^{(n)}_t(X^{(n)}_0(A^{(n)}_t))\right|dt=0. \tag{6.27}
$$

Finally, (6.18), (6.27) and (6.13) yield that (6.11) holds. □

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References

1. Alòs, E., Nualart, D.: An extension of Itô's formula for anticipating processes. J. Theor. Probab. **11**(2), 493–514 (1998)
2. Arnold, L.: Stochastic Differential Equations: Theory and Applications. Wiley, Hoboken (1974)
3. Bhatia, N.P., Szegö, G.P.: Stability Theory of Dynamical Systems. Springer, Berlin (1970)
4. Buckdahn, R.: Linear Skorohod stochastic differential equations. Probab. Theory Relat. Fields **90**, 223–240 (1991)
5. Buckdahn, R.: Anticipative Girsanov Transformations and Skorohod Stochastic Differential Equations. Memoirs of AMS, Volume 111 (533) (1994)
6. Escudero, C., Ranilla-Cortina, S.: Optimal portfolios for different anticipating integrals under insider information. Mathematics (MDPI) **9**, 75 (2021)
7. Gard, T.C.: Introduction to Stochastic Differential Equations. Marcel Dekker, New York (1988)
8. Hartman, P.: Ordinary Differential Equations, 2nd edn. SIAM, Philadelphia (2002)
9. Khasminskii, R.: Stochastic Stability of Differential Equations, 2nd edn. Springer, Berlin (2012)
10. Kohatsu-Higa, A., León, J.A.: Anticipating stochastic differential equation of Stratonovich type. Appl. Math. Optim. **36**, 263–289 (1997)
11. Laning, J.H., Battin, R.H.: Random Processes in Automatic Control. McGraw-Hill, New York (1956)
12. León, J.A., Márquez-Carreras, D., Vives, J.: Anticipating linear stochastic differential equations driven by a Lévy process. Electron. J. Probab. **17**(89), 1–26 (2012)
13. León, J.A., Navarro, R., Nualart, D.: An anticipating calculus approach to the utility maximization of an insider. Math. Finance **13**(1), 171–185 (2003)
14. Nualart, D.: The Malliavin Calculus and Related Topics, 2nd edn. Springer, Berlin (2006)
15. Ocone, D., Pardoux, E.: A generalized Itô–Ventzell formula. Application to a class of anticipating stochastic differential equations. Annales de l'IHP Section B **25**(1), 39–71 (1989)
16. Pugachev, V.S.: Theory of Random Functions: And Its Application to Control Problems. Fizmatgiz, Moscow (1960)
17. Skorokhod, A.V.: On a generalization of a stochastic integral. Theory Probab. Appl. **20**, 219–233 (1975)