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Period-Doubling Bifurcation of Cycles in Retarded Functional Differential Equations

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Abstract
A rigorous description of a period-doubling bifurcation of limit cycles in retarded functional differential equations based on tools of functional analysis and singularity theory is presented. Particularly, sufficient conditions for its occurrence and its normal form coefficients are expressed in terms of derivatives of the operator defining given equations. We also prove the exchange of stability in the case of a non-degenerate period-doubling bifurcation. The approach concerns Fredholm operators, Lyapunov–Schmidt reduction and recognition problem for pitchfork bifurcation.

Keywords Retarded functional differential equation · Delayed differential equation · Period-doubling bifurcation · Fredholm operator · Lyapunov–Schmidt reduction

Mathematics Subject Classification 34K18 · 37G15

1 Introduction

In the context of continuous-time dynamical systems, period-doubling bifurcation is a qualitative change of dynamics characterized by a splitting of a limit cycle which leads to an emergence of a new limit cycle whose period is approximately two times bigger than the original one. This phenomenon can be studied by tools of the theory of discrete-time dynamical systems using Poincaré maps. If this discrete-time dynamical system overcomes a flip bifurcation, then period-doubling occurs in the original dynamical system, Kuznetsov [10].

The mentioned discretization has its limitations which is demonstrated, for example, by the fact that the method does not provide an easily computable formula for normal form coefficients, Kuznetsov et al. [11]. Calculation of normal form coefficients of codimension one bifurcations of limit cycles in ordinary differential equations (ODE) was done in Kuznetsov et al. [11]. Their procedure depend on the normal form of a vector field in a neighbourhood of a limit cycle deduced in Iooss [8]. Recently, the existence of a smooth periodic center mani-

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fold in a neighbourhood of a non-hyperbolic periodic orbit in retarded functional differential equations (RFDE) has been proved, Lentjes et al. [12, Preprint]. Moreover, the existence of a suitable coordinate system on the manifold enables calculation of normal form coefficients of codimension one bifurcations of limit cycles in RFDE. These results has been accomplished using sun-star calculus, Diekmann et al. [4].

In this paper an alternative approach is presented. The following text is inspired by the eighth chapter of Golubitsky and Schaeffer [6]. In fact, we have adjusted techniques applied to investigation of Hopf bifurcation so that they can be used in the case of period-doubling bifurcation in RFDE. The first step of our approach consists of regarding periodic solutions of the original RFDE as zeros of an operator Φ acting between suitable Banach spaces. Subsequently, Lyapunov–Schmidt reduction can be applied in order to obtain a reduced map ϕ operating between finite-dimensional spaces with the property that solutions of the equation $\phi = 0$ are in one to one correspondence with solutions of the equation $\Phi = 0$. According to singularity theory and the mechanism of the reduction, derivatives of the map ϕ at the bifurcation point determines a shape of its solution set near the point and they can be computed from derivatives of the original operator Φ.

Sun-star calculus studies bifurcations in RFDE from the classical viewpoint of dynamical systems theory. On the other hand, the approach used by us consists in implementing very simple and general results from functional analysis and singularity theory. The computation of normal form coefficients of period-doubling bifurcation in this text can be considered as a demonstration of the robustness of these tools and a prospect for the successful application of this method in other and more general contexts. Although Lentjes et al. [12, Preprint], announced the computation of these coefficients and Szalai and Stépán [16], derived them for the special case where delay and period are equal, we provide explicit formulae in full generality.

We consider an autonomous retarded functional differential equation of the form

$$\dot{x}(t) = F(x_t, \lambda) \tag{1}$$

where $F\colon C \times \mathbb{R} \to \mathbb{R}^n$ is supposed to have enough continuous derivatives. The state space $C = C([-r, 0], \mathbb{R}^n)$ where r is a given non-negative real number is endowed with the supremum norm $\|\cdot\|$. The history x_t at time t is an element of the space C given by $x_t(\theta) = x(t + \theta)$, where $\theta \in [-r, 0]$. The symbol λ represents the bifurcation parameter. We assume that this equation has a p-periodic solution u_0 for $\lambda = \lambda_0$.

The rest of the paper is organized as follows. In Sect. 2 we reduce the problem of finding periodic solutions of (1) to a problem of finding zeros of a function of the type $\mathbb{R}^2 \to \mathbb{R}$. We derive sufficient conditions for an occurrence of period-doubling bifurcation and expressions for its normal form coefficients. In Sect. 3, the exchange of stability of solutions is proved in the case of the non-degenerate period-doubling bifurcation. We should emphasize that the theory is presented in the terminology of germs. Especially, most of the equalities occurring after Proposition 4 are considered as relations between germs rather than functions.

2 Normal form Coefficients

First of all, we want to transform equation (1) to an algebraic equation. Let us define the following sets of functions:

$$C^1_q{:=}\{f\colon \mathbb{R}\to\mathbb{R}^n \mid f\in C^1(\mathbb{R}) \text{ and } f(t+q)=f(t) \text{ for all } t\in\mathbb{R}\},$$
$$C_q{:=}\{f\colon \mathbb{R}\to\mathbb{R}^n \mid f\in C(\mathbb{R}) \text{ and } f(t+q)=f(t) \text{ for all } t\in\mathbb{R}\}.$$

In words, the set C^1_q contains all continuously differentiable q-periodic functions, meanwhile the symbol C_q represents all continuous q-periodic functions. In order to simplify subsequent expressions, we reserve the adjective q-periodic for functions satisfying the common defining condition of sets C^1_q and C_q so q is not necessarily their period. Obviously, C^1_q is a subset of C_q. It is easy to see, that the equations

$$\|u\|_1 = \max_{t\in\mathbb{R}} |u(t)| + \max_{t\in\mathbb{R}} \left|u'(t)\right|,$$
$$\|u\| = \max_{t\in\mathbb{R}} |u(t)|,$$

where $|\cdot|$ is a fixed norm in $\mathbb{R}^n$, define Banach spaces $(C^1_q, \|\cdot\|_1)$ and $(C_q, \|\cdot\|)$. Next we omit to mention the norms explicitly.

Since we suppose that u_0 is a p-periodic solution of equation (1) and we are going to study period-doubling bifurcation, it is natural to choose C^1_{2p} and C_{2p} as a domain and range of the operator Φ, respectively. However, there is a little complication. Generically, a period of a solution varies with a change in a bifurcation parameter. Hence, if λ is close to λ_0, in general there will be solutions with periods close to p or $2p$ but not equal to these values. For this, we introduce an extra parameter to the studied equation, which enable us to solve this problem. Let us introduce a new time variable $s=(1+\tau)t$, where $\tau\in\mathbb{R}$ is the mentioned parameter. This leads to the following equation:

$$(1+\tau)\dot{u}(s) = F(u_{s,\tau},\lambda), \tag{2}$$

where $u_{s,\tau}(\theta)=u(s+(1+\tau)\theta)$. If $u(s)$ is a $2p$-periodic solution of (2), then $u((1+\tau)t)$ is a $2p/(1+\tau)$-periodic solution of (1). On the other hand, if $x(t)$ is a $2p/(1+\tau)$-periodic solution of (1), then $x(s/(1+\tau))$ is a $2p$-periodic solution of (2). Now we can define the operator $\Phi\colon C^1_{2p}\times\mathbb{R}\times\mathbb{R}\to C_{2p}$ by the following equation:

$$\Phi(u,\lambda,\tau){:=}(1+\tau)\dot{u}(s) - F(u_{s,\tau},\lambda). \tag{3}$$

According to the previous analysis, solutions of the equation $\Phi = 0$ are in one-to-one correspondence with periodic solutions of (1) whose period is approximately $2p$.

Now we briefly mention an action of the group $(\mathbb{R},+)$ on the set C_{2p} called a phase shift. An element $\alpha\in\mathbb{R}$ acts on a function $u\in C_{2p}$ in the following way:

$$(\alpha\cdot u)(s) = u(s-\alpha).$$

It is easy to see that the previous formula defines a group action correctly. Since

$$(\alpha\cdot u)_{s,\tau}(\theta) = (\alpha\cdot u)(s+(1+\tau)\theta) = u(s-\alpha+(1+\tau)\theta) = u_{s-\alpha,\tau}(\theta)$$

and (1) is autonomous, the operator Φ commutes with a phase shift:

$$\begin{aligned}\Phi(\alpha\cdot u,\lambda,\tau)(s) &= (1+\tau)\frac{\mathrm{d}(\alpha\cdot u)}{\mathrm{d}s}(s) - F((\alpha\cdot u)_{s,\tau},\lambda)\\ &= (1+\tau)\dot{u}(s-\alpha) - F(u_{s-\alpha,\tau},\lambda)\\ &= \Phi(u,\lambda,\tau)(s-\alpha) = (\alpha\cdot\Phi(u,\lambda,\tau))(s).\end{aligned}$$

Note that each element of the subgroup $2p\mathbb{Z}\subseteq\mathbb{R}$ is a symmetry of all functions in C_{2p} which means that these elements represent transformations acting trivially on C_{2p}. Of course, there are functions, such as p-periodic functions, which have more symmetries. Every element of the subgroup $p\mathbb{Z}$ is a symmetry of the solution u_0, for example.

The Fréchet differential, Chow and Hale [1], of $\Phi\colon C^1_{2p} \times \mathbb{R} \times \mathbb{R} \to C_{2p}$ with respect to the first variable at $(u_0, \lambda_0, 0)$ is given by

$$(Lv)(s) = ((\mathrm{d}\Phi)_{u_0,\lambda_0,0} \cdot v)(s) = \dot{v}(s) - (\mathrm{d}F)_{(u_0)_s,\lambda_0} \cdot v_s, \tag{4}$$

where $(\mathrm{d}F)_{(u_0)_s,\lambda_0}$ is the derivative of F with respect to the first variable evaluated at $((u_0)_s, \lambda_0)$. The operator $K\colon \mathbb{R} \times C \to \mathbb{R}^n$ defined by $K(s,\phi) = (\mathrm{d}F)_{(u_0)_s,\lambda_0} \cdot \phi$ is p-periodic in the first variable and continuous and linear in the second variable. Hence it can be expressed as follows

$$K(s,\phi) = \int_{-r}^{0} \mathrm{d}_\theta \eta(s,\theta)\phi(\theta),$$

where η is continuous from the left in θ on $(-r, 0)$, has bounded variation in θ on $[-r, 0]$ and is normalized so that $\eta(s,\theta) = 0$ for $\theta \geq 0$ and $\eta(s,\theta) = \eta(s,-r)$ for $\theta \leq -r$, see Hale [7].

Proposition 1 *The linearization $L\colon C^1_{2p} \to C_{2p}$ of the operator Φ is a Fredholm operator of index zero.*

Proof It is easily seen that $\dot{(\cdot)}\colon C^1_{2p} \to C_{2p}$ is a Fredholm operator of index zero. According to Schechter [14], it is enough to justify compactness of the operator $B\colon C^1_{2p} \to C_{2p}$ defined by $(Bv)(s) = K(s, v_s)$. Let $k = \max_{s\in\mathbb{R}} \|K(s,\cdot)\|_{\mathcal{L}(C,\mathbb{R}^n)}$ which exists since $(\mathrm{d}F)_{(u_0)_\cdot,\lambda_0}\colon \mathbb{R} \to \mathcal{L}(C,\mathbb{R}^n)$ is continuous and periodic. If $M \subseteq C^1_{2p}$ is bounded, then there is $m \in \mathbb{R}$ such that $\|v\|_1 \leq m$ for every $v \in M$. Since

$$\|Bv\| = \max_{s\in\mathbb{R}} |Bv(s)| \leq \max_{s\in\mathbb{R}} \|K(s,\cdot)\|_{\mathcal{L}(C,\mathbb{R}^n)} \cdot \|v_s\| \leq km,$$

the set BM is uniformly bounded. It remains to verify uniform continuity. We start with the following calculations:

$$\begin{aligned}
|Bv(s) - Bv(t)| &= |K(s, v_s) - K(t, v_t)| \\
&= \left|(\mathrm{d}F)_{(u_0)_s,\lambda_0} \cdot v_s - (\mathrm{d}F)_{(u_0)_t,\lambda_0} \cdot v_t\right| \\
&= \left|(\mathrm{d}F)_{(u_0)_s,\lambda_0} \cdot v_s - (\mathrm{d}F)_{(u_0)_s,\lambda_0} \cdot v_t + (\mathrm{d}F)_{(u_0)_s,\lambda_0} \cdot v_t - (\mathrm{d}F)_{(u_0)_t,\lambda_0} \cdot v_t\right| \\
&\leq \left\|(\mathrm{d}F)_{(u_0)_s,\lambda_0}\right\|_{\mathcal{L}(C,\mathbb{R}^n)} \|v_s - v_t\| + \left\|(\mathrm{d}F)_{(u_0)_s,\lambda_0} - (\mathrm{d}F)_{(u_0)_t,\lambda_0}\right\|_{\mathcal{L}(C,\mathbb{R}^n)} \|v_t\| \\
&\leq k\,\|v_s - v_t\| + m\,\left\|(\mathrm{d}F)_{(u_0)_s,\lambda_0} - (\mathrm{d}F)_{(u_0)_t,\lambda_0}\right\|_{\mathcal{L}(C,\mathbb{R}^n)}.
\end{aligned}$$

There is $\delta_1 > 0$ such that for every $v \in M$ and $s, t \in \mathbb{R}$ satisfying $|s - t| < \delta_1$ the following inequalities hold:

$$|v(s) - v(t)| < \frac{\varepsilon}{2k} \quad \Rightarrow \quad |v_s(\theta) - v_t(\theta)| < \frac{\varepsilon}{2k} \quad \Rightarrow \quad \|v_s - v_t\| < \frac{\varepsilon}{2k}.$$

On the other hand, continuity and periodicity of $(\mathrm{d}F)_{(u_0)_\cdot,\lambda_0}\colon \mathbb{R} \to \mathcal{L}(C,\mathbb{R}^n)$ implies its uniform continuity. Therefore, there is $\delta_2 > 0$ such that for every $s, t \in \mathbb{R}$ satisfying $|s - t| < \delta_2$ we have

$$\left\|(\mathrm{d}F)_{(u_0)_s,\lambda_0} - (\mathrm{d}F)_{(u_0)_t,\lambda_0}\right\|_{\mathcal{L}(C,\mathbb{R}^n)} < \frac{\varepsilon}{2m}.$$

We can conclude that for a given $\varepsilon > 0$ and $v \in M$ an appropriate $\delta = \min\{\delta_1, \delta_2\}$ can be found so that for every $s, t \in \mathbb{R}$ satisfying $|s - t| < \delta$ the following inequality hold:

$$|Bv(s) - Bv(t)| \leq k \cdot \frac{\varepsilon}{2k} + m \cdot \frac{\varepsilon}{2m} = \varepsilon.$$

The precompactness of the set BM has been proved. □

Let us continue with solutions of the equation $L = 0$. According to Hale [7], for any $\sigma \in \mathbb{R}$ and $\phi \in C$ there is a unique solution $x\colon [\sigma - r, \infty) \to \mathbb{R}^n$ of the equation $L = 0$ satisfying the initial condition $x_\sigma = \phi$. The solution operator $T(s, \sigma)\colon C \to C$ for $s \geq \sigma$ is defined by $T(s, \sigma)\phi = x_s$. Recall that non-zero eigenvalues of the monodromy operator $T(\sigma + p, \sigma)$ do not depend on σ and they are called Floquet multipliers of u_0. From now on, we will adopt the following assumption:

(C1) The only Floquet multipliers of u_0 satisfying $\gamma^2 = 1$ are $\gamma_1 = 1$ and $\gamma_2 = -1$ and these two eigenvalues are simple.

Proposition 2 *If u_0 satisfies the condition (C1), then* $\dim\ker L = 2$.

Proof According to the assumption, 1 is an eigenvalue of the operator

$$T(2p, 0) = T(2p, p)T(p, 0) = T(p, 0)T(p, 0) = T(p, 0)^2,$$

of which both algebraic and geometric multiplicity are equal to two. Let $(\dot{u}_0)_0$ and ϕ be corresponding eigenvectors such that $T(p, 0)\phi = -\phi$. Obviously, the $2p$-periodic function $\dot{u}_0(s)$ and the $2p$-periodic extension of the function $v_0(s) = T(s, 0)\phi(0)$ span the kernel of the operator L, which leads to $\dim\ker L = 2$. □

It should be emphasized that the function v_0 has the following property: $v_0(s + p) = -v_0(s)$ where $s \in \mathbb{R}$ is arbitrary.

Next, we are going to use the concept of the formal adjoint equation given by the following relation:

$$y(s) + \int_s^\infty y(\beta)\eta(\beta, s - \beta)\,\mathrm{d}\beta = \text{constant}, \tag{5}$$

where $y(s) \in \mathbb{R}^{n*}$. Let B_0 denote the Banach space of functions $\psi\colon [-r, 0] \to \mathbb{R}^{n*}$ of bounded variation on $[-r, 0]$, continuous from the left on $(-r, 0)$ and vanishing at zero with norm $\mathrm{Var}_{[-r,0]}\psi$. According to Hale [7], for every $t \in \mathbb{R}$ and $\psi \in B_0$ there is a unique function $y\colon \mathbb{R} \to \mathbb{R}^{n*}$ which vanishes on $[t, \infty)$, satisfies the equation (5) on $(-\infty, t - r]$ and $y_t = \psi$. The solution operator $\widetilde{T}(s, t)\colon B_0 \to B_0$ for $s \leq t$ is defined by $\widetilde{T}(s, t)\psi = y_s^0$ where $y_s^0(0) = 0$ and $y_s^0(\theta) = y(s + \theta)$ for $-r \leq \theta < 0$.

Recall that spectra of $T(p, 0)$ and $T(2p, 0) = T(p, 0)^2$ are the same as those of $\widetilde{T}(0, p)$ and $\widetilde{T}(0, 2p) = \widetilde{T}(0, p)^2$, respectively. Moreover, generalized eigenspaces of these operators can be used to decompose spaces C and B_0 in the following way:

$$\begin{aligned} C &= \ker(I - T(p, 0)) \oplus \mathrm{Im}(I - T(p, 0)) = \ker(I + T(p, 0)) \oplus \mathrm{Im}(I + T(p, 0)) \\ &= \ker(I - T(2p, 0)) \oplus \mathrm{Im}(I - T(2p, 0)), \\ B_0 &= \ker(I - \widetilde{T}(0, p)) \oplus \mathrm{Im}(I - \widetilde{T}(0, p)) = \ker(I + \widetilde{T}(0, p)) \oplus \mathrm{Im}(I + \widetilde{T}(0, p)) \\ &= \ker(I - \widetilde{T}(0, 2p)) \oplus \mathrm{Im}(I - \widetilde{T}(0, 2p)), \end{aligned}$$

where

$$\dim\ker(I - T(p,0)) = \dim\ker(I - \widetilde{T}(0,p)) = 1,$$
$$\dim\ker(I + T(p,0)) = \dim\ker(I + \widetilde{T}(0,p)) = 1,$$
$$\dim\ker(I - T(2p,0)) = \dim\ker(I - \widetilde{T}(0,2p)) = 2.$$

Let ψ_1 and ψ_2 be eigenvectors of the operator $\widetilde{T}(0,p)$ corresponding to eigenvalues 1 and -1, respectively. Obviously, these vectors span the eigenspace of the operator $\widetilde{T}(0,2p)$ corresponding to 1. Let v_1 and v_2 be $2p$-periodic extensions of solutions of (5) satisfying the initial conditions $(v_1)_{2p} = \psi_1$ and $(v_2)_{2p} = \psi_2$. These functions span the space of $2p$-periodic solutions of (5). Moreover, it can be easily seen that v_1 and v_2 are continuous.

The formal adjoint equation is necessary for decomposition of spaces C^1_{2p} and C_{2p} so that Lyapunov–Schimdt reduction can be used. This is the right time to recall an important theorem from Hale [7] (Corollary 5.1 from Chapter 6 or Theorem 1.2 from Chapter 9):

Theorem 1 *If $f \in C_{2p}$, then the equation $Lv = f$ has a $2p$-periodic solution v if and only if $\langle u, f\rangle = 0$ for all $2p$-periodic solutions u of the formal adjoint equation.*

An important consequence of this statement is the inclusion $\operatorname{Im} L \subseteq [v_1, v_2]^{\perp}$. We can conclude the following proposition.

Proposition 3 *If the solution u_0 satisfies the condition (C1), then the spaces C^1_{2p} and C_{2p} can be expressed as the following sums:*

$$C^1_{2p} = \ker L \oplus M, \qquad C_{2p} = N \oplus \operatorname{Im} L,$$

where $M := (\ker L)^{\perp}$ and $N := [L(s\dot{u}_0(s)), L(sv_0(s))]$. Moreover, spaces M and N are invariant under the action $v(\cdot) \mapsto v(\cdot + p)$.

Proof The first splitting is obvious. According to Propositions 2 and 1 it is enough to show that $\operatorname{Im} L \cap N = 0$ in order to verify the second equality. However, it is a simple consequence of the previous theorem and the following relations

$$\langle v_2, L(s\dot{u}_0(s))\rangle = \langle v_1, L(sv_0(s))\rangle = 0,$$
$$\langle v_1, L(s\dot{u}_0(s))\rangle \neq 0 \neq \langle v_2, L(sv_0(s))\rangle, \tag{6}$$

which will be proven latter. □

Before we proceed to Lyapunov–Schmidt reduction, we implement a little modification of the operator Φ. Let us define an operator $\widetilde{\Phi}\colon C^1_{2p} \times \mathbb{R} \times \mathbb{R} \to C_{2p} \times \mathbb{R}$ by the following rule:

$$\widetilde{\Phi}(u, \lambda, \tau) := \left(\Phi(u, \lambda, \tau), \int_0^{2p} \langle u(s), \dot{u}_0(s)\rangle\, \mathrm{d}s\right).$$

The invariance of the operator Φ under a time shift adds one more unnecessary dimension to the reduced problem. The reduced map can be simplified by the second component of the operator $\widetilde{\Phi}$ which fixes a phase of solutions, Kuznetsov [10]. The Fréchet differential of $\widetilde{\Phi}$ with respect to the first variable at the point $(u_0, \lambda_0, 0)$ can be expressed in the following way:

$$\widetilde{L}v = (\mathrm{d}\widetilde{\Phi})_{u_0,\lambda_0,0} \cdot v = \left(Lv, \int_0^{2p} \langle v(s), \dot{u}_0(s)\rangle\, \mathrm{d}s\right).$$

Since $\ker \widetilde{L} = [v_0]$ and $\operatorname{Im} \widetilde{L} = \operatorname{Im} L \times \mathbb{R}$, spaces C^1_{2p} and $C_{2p} \times \mathbb{R}$ can be decomposed in a way similar to the one in the last proposition:

$$C^1_{2p} = \ker \widetilde{L} \oplus \widetilde{M}, \qquad C_{2p} \times \mathbb{R} = \operatorname{Im} \widetilde{L} \oplus (N \times \{0\}),$$

where $\widetilde{M} := M + [\dot{u}_0]$.

Lyapunov–Schmidt reduction is clearly described in Golubitsky and Schaeffer [6], so we mention only necessary aspects of the procedure. Let $E\colon C_{2p} \to C_{2p}$ denote the projection of C_{2p} onto $\operatorname{Im} L$ with $\ker E = N$. The range and the kernel of the complementary projection $I - E$ are N and $\operatorname{Im} L$, respectively. Obviously, the equation $\widetilde{\Phi}(u, \lambda, \tau) = 0$ is equivalent to the following set of equations:

$$E\Phi(u, \lambda, \tau) = 0, \quad (I - E)\Phi(u, \lambda, \tau) = 0, \quad \langle u, \dot{u}_0 \rangle = 0. \tag{7}$$

The left-hand sides of the first and last equations can be considered as an operator $(E\Phi, \langle \cdot, \dot{u}_0 \rangle)\colon \ker \widetilde{L} \times \widetilde{M} \times \mathbb{R}^2 \to \operatorname{Im} \widetilde{L}$. The differential of this operator with respect to the second variable evaluated at $(u_0, \lambda_0, 0)$ is invertible. The implicit function theorem in Banach spaces implies existence of a function $W\colon \ker \widetilde{L} \times \mathbb{R}^2 \to \widetilde{M}$ defined in a neighbourhood of $(u_0, \lambda_0, 0)$ which is uniquely determined by equations

$$E\Phi(v + W(v, \lambda, \tau), \lambda, \tau) = 0, \;\; \langle v + W(v, \lambda, \tau), \dot{u}_0 \rangle = 0, \;\; W(0, \lambda_0, 0) = u_0. \tag{8}$$

where $v \in \ker \widetilde{L}$. In the last equation we have used the fact that $u_0 \in \widetilde{M}$. If we substitute $u = v + W(v, \lambda, \tau)$ into the left-hand side of the second equation of (7), then we obtain the reduced mapping $\phi\colon \ker \widetilde{L} \times \mathbb{R}^2 \to N$ defined by

$$\phi(v, \lambda, \tau) := (I - E)\Phi(v + W(v, \lambda, \tau), \lambda, \tau).$$

An important consequence of the previous analysis is a one-to-one correspondence between solutions of the equations $\widetilde{\Phi} = 0$ and $\phi = 0$.

This is the right time to discuss how symmetry enters our problem. Since Φ commutes with the action of $(\mathbb{R}, +)$, the equality $\Phi(\alpha u, \lambda, \tau) = \alpha \Phi(u, \lambda, \tau)$ holds for every $\alpha \in \mathbb{R}$ and $(u, \lambda, \tau) \in C^1_{2p} \times \mathbb{R} \times \mathbb{R}$. Differentiation with respect to C^1_{2p} and evaluation at $(u_0, \lambda_0, 0)$ gives the following equation:

$$(\mathrm{d}\Phi)_{\alpha u_0, \lambda_0, 0} \cdot \alpha = \alpha \cdot (\mathrm{d}\Phi)_{u_0, \lambda_0, 0}.$$

Generally, we can not deduce $L\alpha = \alpha L$, because u_0 is a non-trivial function which means that the equation $\alpha u_0 = u_0$ does not hold for every $\alpha \in \mathbb{R}$. However, the function u_0 is p-periodic, hence it has more symmetries than a general element of the space C_{2p}. Particularly, the isotropy subgroup of u_0 is $p\mathbb{Z}$, unlike the isotropy subgroup $2p\mathbb{Z}$ of a general function in C_{2p}. We can factor $\mathbb{R}$ by its subgroup $2p\mathbb{Z}$ and think about the action of the circle group $S^1 \cong \mathbb{R}/2p\mathbb{Z}$ on the space C_{2p}. In this context, the isotropy subgroup of u_0 is $\{0, p\} \subseteq S^1$, which is isomorphic to $\mathbb{Z}_2$. We can conclude that the equation $L\alpha = \alpha L$ holds for every $\alpha \in \mathbb{Z}_2$, so the operator L commutes with the action of $\mathbb{Z}_2$ on the space C_{2p}. This result has the following important consequences, whose proofs can be found in Golubitsky and Schaeffer [6]. Spaces $\ker L$ and $\operatorname{Im} L$ are $\mathbb{Z}_2$-invariant and the projection E also commutes with the action of $\mathbb{Z}_2$. Moreover, $\mathbb{Z}_2$-invariance of spaces $\ker \widetilde{L}$ and $\widetilde{M}$ and the fact that W commutes with the action of $\mathbb{Z}_2$ can be also verified by a simple calculation. Finally, $\mathbb{Z}_2$-equivariance of the reduced mapping ϕ can be concluded:

$$\phi(\alpha v, \lambda, \tau) = \alpha \phi(v, \lambda, \tau).$$

This fact enforces a special form of the reduced mapping which will be seen in a moment.

In place of abstract subspaces of a Banach space, it is more comfortable to work with Euclidean spaces, hence we try to find an appropriate coordinate representation of ϕ. We can use ordered bases (v_0) and $(L(s\dot{u}_0(s)), L(sv_0(s)))$ of spaces $\ker \widetilde{L}$ and N, respectively, and consider W and ϕ as mappings $W\colon \mathbb{R} \times \mathbb{R}^2 \to \widetilde{M}$ and $\phi\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, respectively: $W(x, \lambda, \tau) = W(xv_0, \lambda, \tau)$,

$$\begin{aligned}\phi(x, \lambda, \tau) &= \begin{pmatrix}\phi_1(x, \lambda, \tau)\\ \phi_2(x, \lambda, \tau)\end{pmatrix}\\ &= \begin{pmatrix}\langle v_1, (I-E)\Phi(xv_0 + W(x, \lambda, \tau), \lambda, \tau)\rangle / \langle v_1, L(s\dot{u}_0(s))\rangle\\ \langle v_2, (I-E)\Phi(xv_0 + W(x, \lambda, \tau), \lambda, \tau)\rangle / \langle v_2, L(sv_0(s))\rangle\end{pmatrix}\\ &= \begin{pmatrix}\langle v_1, \Phi(xv_0 + W(x, \lambda, \tau), \lambda, \tau)\rangle / \langle v_1, L(s\dot{u}_0(s))\rangle\\ \langle v_2, \Phi(xv_0 + W(x, \lambda, \tau), \lambda, \tau)\rangle / \langle v_2, L(sv_0(s))\rangle\end{pmatrix},\end{aligned}$$

where the penultimate equality can be derived from $\phi = \phi_1 \cdot L(s\dot{u}_0(s)) + \phi_2 \cdot L(sv_0(s))$ and (6) and the last equality is a consequence of $\operatorname{Im} L \subseteq [v_1, v_2]^\perp$ and $\operatorname{Im} E = \operatorname{Im} L$. The subsequent calculations can be simplified by involving the linear transformation

$$S = \begin{pmatrix}\langle v_1, L(s\dot{u}_0(s))\rangle & 0\\ 0 & \langle v_2, L(sv_0(s))\rangle\end{pmatrix},$$

which leads to the mapping

$$g(x, \lambda, \tau) = \begin{pmatrix}g_1(x, \lambda, \tau)\\ g_2(x, \lambda, \tau)\end{pmatrix} = S\phi(x, \lambda, \tau) = \begin{pmatrix}\langle v_1, \Phi(xv_0 + W(x, \lambda, \tau), \lambda, \tau)\rangle\\ \langle v_2, \Phi(xv_0 + W(x, \lambda, \tau), \lambda, \tau)\rangle\end{pmatrix}.$$

This function has qualitatively the same solution set as ϕ and, moreover, the transformation S has no influence on $\mathbb{Z}_2$-equivariance. Therefore, we can deduce the following properties of components of the reduced map: $g_1(-x, \lambda, \tau) = g_1(x, \lambda, \tau)$ and $g_2(-x, \lambda, \tau) = -g_2(x, \lambda, \tau)$. This simply says that g_1 and g_2 are respectively even and odd functions of x. We can find germs $a(z, \lambda, \tau)$ and $b(z, \lambda, \tau)$ satisfying $g_1(x, \lambda, \tau) = a(x^2, \lambda, \tau)$ and $g_2(x, \lambda, \tau) = xb(x^2, \lambda, \tau)$. A proof of this statement can be found, for example, in Golubitsky and Schaeffer [6].

The following proposition enables us to leave out the parameter τ.

Proposition 4 *The germ a satisfies the following relations:*

$$a(0, \lambda_0, 0) = 0, \qquad a_\tau(0, \lambda_0, 0) \neq 0.$$

Proof The first statement can be simply proved by substitution:

$$a(0, \lambda_0, 0) = \langle v_1, \Phi(W(0, \lambda_0, 0), \lambda_0, 0)\rangle = \langle v_1, \Phi(u_0, \lambda_0, 0)\rangle = \langle v_1, 0\rangle = 0.$$

The inequality requires differentiation:

$$\begin{aligned}a_\tau(0, \lambda_0, 0) &= \langle v_1, (\mathrm{d}\Phi)_{u_0, \lambda_0, 0} \cdot W_\tau(0, \lambda_0, 0) + \Phi_\tau(u_0, \lambda_0, 0)\rangle\\ &= \langle v_1, \Phi_\tau(u_0, \lambda_0, 0)\rangle,\end{aligned}$$

where the inclusion $\operatorname{Im} L \subseteq [v_1, v_2]^\perp$ is used in the last equality. The idea behind verification of the inequality $\langle v_1, \Phi_\tau(u_0, \lambda_0, 0)\rangle \neq 0$ comes from the proof of Theorem 2.2 in Chapter 10 in Hale [7].

Firstly, we evaluate the derivative Φ_τ at the point $(u_0, \lambda_0, 0)$:

$$\Phi_\tau(u_0, \lambda_0, 0)(s) = \dot{u}_0(s) - (\mathrm{d}F)_{(u_0)_s} \cdot (\cdot)(\dot{u}_0)_s(\cdot) = \dot{u}_0(s) - K(s, (\cdot)(\dot{u}_0)_s(\cdot)).$$

Note that $\langle v_2, \Phi_\tau(u_0, \lambda_0, 0)\rangle = 0$, since $\Phi_\tau(u_0, \lambda_0, 0)$ is a p-periodic function. Let us suppose that $\langle v_1, \Phi_\tau(u_0, \lambda_0, 0)\rangle = 0$. Consequently, the equation

$$\dot{z}(s) = K(s, z_s) + \Phi_\tau(u_0, \lambda_0, 0)(s) \tag{9}$$

has a non-trivial $2p$-periodic solution $z_0(s)$. Let $x_0(s) = z_0(s) - s\dot{u}_0(s)$. The following calculations show that x_0 is a solution of the equation $L = 0$. Since

$$\begin{aligned}(z_0)_s(\theta) &= z_0(s+\theta) = x_0(s+\theta) + s\dot{u}_0(s+\theta) + \theta\dot{u}_0(s+\theta)\\ &= (x_0)_s(\theta) + s(\dot{u}_0)_s(\theta) + \theta(\dot{u}_0)_s(\theta),\end{aligned}$$

we can conclude that

$$\begin{aligned}\dot{x}_0(s) &= \dot{z}_0(s) - \dot{u}_0(s) - s\ddot{u}_0(s)\\ &= K(s, (z_0)_s) + \Phi_\tau(u_0, \lambda_0, 0)(s) - \dot{u}_0(s) - sK(s, (\dot{u}_0)_s)\\ &= K(s, (z_0)_s) + \dot{u}_0(s) - K(s, (\cdot)(\dot{u}_0)_s(\cdot)) - \dot{u}_0(s) - sK(s, (\dot{u}_0)_s)\\ &= K(s, (x_0)_s) + sK(s, (\dot{u}_0)_s) + K(s, (\cdot)(\dot{u}_0)_s(\cdot)) - K(s, (\cdot)(\dot{u}_0)_s(\cdot))\\ &\quad - sK(s, (\dot{u}_0)_s)\\ &= K(s, (x_0)_s).\end{aligned}$$

This fact together with the equation

$$\begin{aligned}T(s+2p, s)(x_0)_s = (x_0)_{s+2p} &= (z_0)_s - (\cdot)(\dot{u}_0)_s - s(\dot{u}_0)_s - 2p(\dot{u}_0)_s\\ &= (x_0)_s - 2p(\dot{u}_0)_s\end{aligned}$$

implies that $(T(2p, 0) - I)(x_0)_0 = -2p(\dot{u}_0)_0$, which contradicts the fact that the geometric multiplicity of 1 is equal to its algebraic multiplicity as an eigenvalue of the operator $T(2p, 0)$. □

We can conclude that the equation $a(x^2, \lambda, \tau) = 0$ implicitly defines a function $\tau(x^2, \lambda)$ in a neighbourhood of the point $(0, \lambda_0, 0)$. Consequently, a solution set of the equation $g = 0$ coincides with a solution set of the equation $g_2(x, \lambda, \tau(x^2, \lambda)) = 0$.

Heretofore, we have discussed only the hypothesis (C1) which is an essential part of the set of sufficient conditions for an occurrence of period-doubling bifurcation. However, we need more information about the equation $\Phi = 0$ in order to describe situation around the bifurcation point. Particularly, we should focus on the so called non-degeneracy conditions. For this purpose, we look at derivatives of g_2 at $(0, \lambda_0)$ which can be computed from derivatives of Φ. Singularity theory enables us to use this data to determine the type of a bifurcation up to an equivalence of germs. This is the aim of the following proposition taken from Golubitsky and Schaeffer [6] (Proposition 2.14 from Chapter 6):

Proposition 5 *A germ $xr(x^2, \lambda)$ is strongly $\mathbb{Z}_2$-equivalent to the germ $x(\varepsilon x^2 + \delta(\lambda - \lambda_0))$ if and only if $r(0, \lambda_0) = 0$, $\operatorname{sgn} r_z(0, \lambda_0) = \varepsilon$ and $\operatorname{sgn} r_\lambda(0, \lambda_0) = \delta$, where $r = r(z, \lambda)$ and $\varepsilon, \delta \neq 0$.*

It is useful to introduce the following notation: $r(x^2, \lambda) := b(x^2, \lambda, \tau(x^2, \lambda))$, hence $g_2(x, \lambda, \tau(x^2, \lambda)) = xr(x^2, \lambda)$. The following identities can be achieved:

$$r_\lambda(0,\lambda_0) = \langle v_2, (\mathrm{d}\Phi_\lambda)_{u_0,\lambda_0,0}\cdot v_0 + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_\lambda(0,\lambda_0,0), v_0)\rangle - \frac{\langle v_1, \Phi_\lambda(u_0,\lambda_0,0)\rangle}{\langle v_1, \Phi_\tau(u_0,\lambda_0,0)\rangle}\langle v_2, (\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}\cdot v_0 + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_\tau(0,\lambda_0,0), v_0)\rangle, \tag{10}$$

$$r_z(0,\lambda_0) = \frac{\langle v_2, (\mathrm{d}^3\Phi)_{u_0,\lambda_0,0}(v_0,v_0,v_0) + 3(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_{xx}(0,\lambda_0,0), v_0)\rangle}{6} - \frac{\langle v_1, (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0,v_0)\rangle}{2\langle v_1, \Phi_\tau(u_0,\lambda_0,0)\rangle}\langle v_2, (\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}\cdot v_0 + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_\tau(0,\lambda_0,0), v_0)\rangle. \tag{11}$$

These values are in fact normal form coefficients. All calculations which are necessary for derivation of the above formulae together with calculations of derivatives of the implicitly defined function W are included in the Appendix. Sufficient conditions for an occurrence of a non-degenerate period-doubling bifurcation are simple consequences of these formulae and Proposition 5:

Theorem 2 *If u_0 is a periodic solution of the equation (1) for $\lambda = \lambda_0$ with period p such that the condition (C1) is satisfied and values (10) and (11) are non-zero, then the system (1) overcomes a non-degenerate period-doubling bifurcation for $\lambda = \lambda_0$.*

It remains to uncover the relation between the derivative $r_\lambda(0,\lambda_0)$ and the derivative of the critical multiplier. We need to construct a family of periodic solutions parametrized by λ to which a family $\rho(\lambda)$ of Floquet multipliers can be assigned. A possible choice is $W(0,\lambda,\tau(0,\lambda))$. The definition of W implies annihilation of $E\Phi$. The germ a is also annihilated, according to the definition of the function τ. Finally, $g_2 = 0$ also holds because $x = 0$. Consequently, $\Phi(W(0,\lambda,\tau(0,\lambda)),\lambda,\tau(0,\lambda)) = 0$ which implies that $W(0,\lambda,\tau(0,\lambda))$ is a family of solutions.

Lemma 1 *Functions $W(0,\lambda,\tau(0,\lambda))$ are p-periodic.*

Proof If we restrict to p-periodic functions only, then the space $\ker L$ would be one-dimensional. This degeneracy is caused by the action of $\mathbb{R}$ defined by a phase shift. Hence, for every $\lambda \approx \lambda_0$ there should be a p-periodic solution of $\Phi = 0$ which is unique up to a phase shift. Since the variable x corresponds to a direction leading into the space C^1_{2p}, it is intuitive to choose $x = 0$ in order to reach these solutions.

Recall, that $W\colon \mathbb{R}\times\mathbb{R}^2 \to \widetilde{M}$ is defined implicitly by the equation

$$E\Phi(xv_0 + W(x,\lambda,\tau),\lambda,\tau) = 0, \qquad \langle xv_0 + W(x,\lambda,\tau), \dot{u}_0\rangle = 0 \tag{12}$$

in a neighbourhood of $(0,\lambda_0,0)$. Let us fix $x = 0$ and consider the following equation:

$$E\Phi(w,\lambda,\tau) = 0, \qquad \langle w, \dot{u}_0\rangle = 0,$$

where $w \in \widetilde{M}$. The differential of left-hand sides with respect to w evaluated at $(u_0,\lambda_0,0)$ is again invertible, hence the equation uniquely determines a function $w(\lambda,\tau)$. However, since $W(0,\lambda,\tau)$ also satisfies the equation, we can conclude $w(\lambda,\tau) = W(0,\lambda,\tau)$. Let us consider Φ as an operator $\Phi\colon C^1_p\times\mathbb{R}\times\mathbb{R} \to C_p$ for a moment. The differential of left-hand sides of the equations

$$\widetilde{E}\Phi(\widetilde{w},\lambda,\tau) = 0, \qquad \frac{1}{p}\int_0^p \langle \widetilde{w}(s), \dot{u}_0(s)\rangle\,\mathrm{d}s = 0$$

with respect to $\widetilde{w} \in C_p^1$ evaluated at $(u_0, \lambda_0, 0)$ is also invertible. The symbol $\widetilde{E}$ denotes the projection of C_p onto the space $\operatorname{Im} L|_{C_p^1}$ with the kernel $[L(s\dot{u}_0(s))]$. Therefore, the equation uniquely determines a function

$$\widetilde{w}(\lambda, \tau)\colon \mathbb{R}^2 \to C_p^1 \subseteq \widetilde{M}.$$

The uniqueness of the function w implies $\widetilde{w}(\lambda, \tau) = w(\lambda, \tau) = W(0, \lambda, \tau)$. Consequently, $W(0, \lambda, \tau)$ is p-periodic. □

Since a multiplier of $W(0, \lambda, \tau(0, \lambda))$ is an eigenvalue of the monodromy operator $T(\lambda, p, 0)$, where $T(\lambda, s, \sigma)$ is the solution operator of the equation $(\mathrm{d}\Phi)_{W(0,\lambda,\tau(0,\lambda)),\lambda,\tau(0,\lambda)} = 0$, it can be defined implicitly in the following way. Let $\Sigma\colon \mathbb{R} \times \operatorname{Im}(I + T(p, 0)) \times \mathbb{R} \to C$ be defined by

$$\Sigma(\rho, \upsilon, \lambda) := T(\lambda, p, 0)(\phi_0 + \upsilon) - \rho(\phi_0 + \upsilon).$$

Obviously, the point $(-1, 0, \lambda_0)$ satisfies the equation $\Sigma = 0$. The differential of Σ with respect to the first two variables evaluated at $(-1, 0, \lambda_0)$ is given by

$$\Sigma_\rho = -\phi_0, \qquad \Sigma_\upsilon = T(p, 0) + I.$$

Since $C = \ker(I + T(p, 0)) \oplus \operatorname{Im}(I + T(p, 0))$, the differential is invertible. Consequently, the equation $\Sigma = 0$ defines smooth functions $\rho(\lambda)$ and $\upsilon(\lambda)$ in a neighbourhood of the point $(-1, 0, \lambda_0)$. Therefore, $\rho(\lambda)$ is the critical Floquet multiplier of the solution $W(0, \lambda, \tau(0, \lambda))$ with the corresponding eigenvector $\phi_0 + \upsilon(\lambda)$. We can formulate the following theorem:

Theorem 3 *The following equality holds:*

$$r_\lambda(0, \lambda_0) = \frac{\rho'(\lambda_0)}{p} \langle v_2, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(s v_0(s))\rangle,$$

where $\langle v_2, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(s v_0(s))\rangle \neq 0$.

Proof Obviously, the function

$$u(s) = \mathrm{e}^{\frac{\log \rho(\lambda)^2}{2p} s} \mathrm{e}^{-\frac{\log \rho(\lambda)^2}{2p} s} T(\lambda, s, 0)(\phi_0 + \upsilon(\lambda))(0) = \mathrm{e}^{\frac{\log \rho(\lambda)^2}{2p} s} h(\lambda, s)$$

is a solution of the equation $(\mathrm{d}\Phi)_{W(0,\lambda,\tau(0,\lambda)),\lambda,\tau(0,\lambda)} = 0$. Moreover, the function $h(\lambda, s)$ is $2p$-periodic:

$$\begin{aligned} h(\lambda, s + 2p) &= \mathrm{e}^{-\frac{\log \rho(\lambda)^2}{2p}(s+2p)} T(\lambda, s + 2p, 0)(\phi_0 + \upsilon(\lambda))(0) \\ &= \mathrm{e}^{-\frac{\log \rho(\lambda)^2}{2p} s} \frac{1}{\rho(\lambda)^2} T(\lambda, s + 2p, 2p) T(\lambda, p, 0)^2 (\phi_0 + \upsilon(\lambda))(0) \\ &= \mathrm{e}^{-\frac{\log \rho(\lambda)^2}{2p} s} \frac{1}{\rho(\lambda)^2} T(\lambda, s, 0) \rho(\lambda)^2 (\phi_0 + \upsilon(\lambda))(0) \\ &= \mathrm{e}^{-\frac{\log \rho(\lambda)^2}{2p} s} T(\lambda, s, 0)(\phi_0 + \upsilon(\lambda))(0) = h(\lambda, s). \end{aligned}$$

Differentiation of the equation

$$(\mathrm{d}\Phi)_{W(0,\lambda,\tau(0,\lambda)),\lambda,\tau(0,\lambda)} \exp(s \log \rho(\lambda)^2 / 2p) h(\lambda, s) = 0$$

with respect to λ and evaluation at λ_0 leads to the following equation:

$$(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_\lambda + W_\tau \cdot \tau_\lambda, v_0) + (\mathrm{d}\Phi_\lambda)_{u_0,\lambda_0,0}(v_0) + (\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}(v_0) \cdot \tau_\lambda + (\mathrm{d}\Phi)_{u_0,\lambda_0,0}\left(-\frac{\rho'(\lambda_0)}{p} s v_0(s) + h_\lambda(\lambda_0, s)\right) = 0.$$

Multiplication by the function v_2 gives us the desired result.

The inequality $\langle v_2, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(s v_0(s))\rangle \neq 0$ can be deduced in the same way as it was done for $\langle v_1, \Phi_\tau(u_0, \lambda_0, 0)\rangle \neq 0$. The following lines are spent by verification of this inequality. Firstly, we show that $L(s v_0(s))$ is periodic:

$$\begin{aligned} L(s v_0(s)) &= v_0(s) + s\dot{v}_0(s) - K(s, ((\cdot)v_0(\cdot))_s) \\ &= v_0(s) + s\dot{v}_0(s) - sK(s, (v_0)_s) - K(s, (\cdot)(v_0)_s(\cdot)) \\ &= v_0(s) + sLv_0(s) - K(s, (\cdot)(v_0)_s(\cdot)) = v_0(s) - K(s, (\cdot)(v_0)_s(\cdot)). \end{aligned}$$

Moreover, this function has the following property: $L((s+p)v_0(s+p)) = -L(s v_0(s))$ for every $s \in \mathbb{R}$. Consequently, the scalar product $\langle v_1, L(s v_0(s))\rangle$ is equal to zero. Suppose that $\langle v_2, L(s v_0(s))\rangle = 0$. According to Theorem 1, there is a $2p$-periodic solution $z(s)$ of the equation $\dot{z}(s) = K(s, z_s) + L(s v_0(s))$. The function $x(s) = z(s) - s v_0(s)$ satisfies the following equation:

$$\begin{aligned} \dot{x}(s) &= K(s, z_s) + L(s v_0(s)) - v_0(s) - s\dot{v}_0(s) \\ &= K(s, x_s) + sK(s, (v_0)_s) + K(s, (\cdot)(v_0)_s(\cdot)) - K(s, (\cdot)(v_0)_s(\cdot)) - s\dot{v}_0(s) \\ &= K(s, x_s). \end{aligned}$$

This fact together with the equation

$$T(s+2p, s)(x)_s = (x)_{s+2p} = (z)_s - (\cdot)(v_0)_s(\cdot) - s(v_0)_s - 2p(v_0)_s = (x)_s - 2p(v_0)_s$$

implies that $(T(2p, 0) - I)(x)_0 = -2p(v_0)_0$, which contradicts the fact that the geometric multiplicity of 1 is equal to its algebraic multiplicity as an eigenvalue of the operator $T(2p, 0)$. □

Note that the condition $\rho'(\lambda_0) \neq 0$ together with the last theorem implies existence of doubled periodic orbits. If $r_\lambda(0, \lambda_0) \neq 0$, then the equation $r(x^2, \lambda) = 0$ implicitly defines a function $\lambda(x^2)$ satisfying $r(x^2, \lambda(x^2)) = 0$ in a neighbourhood of the point $(0, \lambda_0)$. However, if there are no other non-degeneracy conditions, we are not able to tell anything about this branch of solutions. The non-degenerate period-doubling bifurcation which resembles pitchfork bifurcation is achieved by including the condition $r_z(0, \lambda_0) \neq 0$.

3 Stability

The aim of this section is verification of exchange of stability in the case of a non-degenerate period-doubling bifurcation. Alongside the hypothesis (C1) we will assume that

(C2) All Floquet multipliers of the $2p$-periodic solution u_0 lies inside the unit circle except of $\gamma_1 = 1$ and $\gamma_2 = (-1)^2$.

Since the only possible accumulation point of the set of Floquet multipliers is zero, there is $0 < \delta < 1$ such that all Floquet multipliers γ except of γ_1 and γ_2 satisfy $|\gamma| \leq \delta$.

Let us introduce the idea behind the following procedure shortly. Firstly, we define an auxiliary function $\mu(x, \lambda)$ implicitly in virtue of the following proposition taken from Hale [7].

Proposition 6 *A linear q-periodic RFDE has a characteristic multiplier $\mathrm{e}^{\mu q}$ if and only if it has a solution of the form $u(s)\mathrm{e}^{\mu s}$ where $u(s+q) = u(s)$.*

The function μ enable us to connect derivatives of the reduced mapping with derivatives of the critical multiplier which is going to be defined implicitly as an eigenvalue of the monodromy operator. We adopt the following notation: $\Omega(x, \lambda) := x v_0 + W(x, \lambda, \tau(x^2, \lambda))$ and $f(x, \lambda) := g_2(x, \lambda, \tau(x^2, \lambda))$.

According to Proposition 6, if $\Omega(x, \lambda)$ is a solution of the equation $\Phi = 0$, then its Floquet exponents satisfy the equation

$$(\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}(\mathrm{e}^{\mu s} u(s)) = 0$$

for a suitable $2p$-periodic function u. In order to use this equation for an implicit definition of a function, it is necessary to extend the operator defined by the left-hand side of this equation and to find an appropriate range for it. The purpose of the following lemma is just to motivate the choice of the range. Since it can be easily verified by tools built in Hale [7] (Chapter 8), we state the lemma without a proof.

Lemma 2 *Let $\Lambda u(s) = \dot{u}(s) - K(s, u_s)$ be a linear q-periodic RFDE. If $\mu \in \mathbb{R}$ and $u \in C_q^1$, then the equation $\Lambda(\mathrm{e}^{\mu \cdot} u(\cdot))(s) = 0$ holds for every $s \in \mathbb{R}$ if and only if it holds for every $s \in [0, q]$.*

Note that the space $C([0, 2p], \mathbb{R}^n)$ has the following obvious property:

$$C([0, 2p], \mathbb{R}^n) \cong C_{2p} \oplus [s e_1, \ldots, s e_n], \tag{13}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ and 1 is on the i-th position. Now we can define an operator $\Psi : \mathbb{R}^n \times \mathbb{R}^2 \times \widetilde{M} \times \mathbb{R}^2 \to C([0, 2p], \mathbb{R}^n)$ by the following equation:

$$\begin{aligned} &\Psi(c, \mu, \eta, w, x, \lambda) \\ &\quad := (\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}(\mathrm{e}^{\mu s}(v_0(s) + w(s))) + \eta \Phi_\tau(\Omega(x, \lambda), \lambda, \tau(x^2, \lambda)) + cs, \end{aligned}$$

where the domain of the function defined by the right-hand side is reduced to the interval $[0, 2p]$. Finally, in order to reduce the kernel of the linearization of the previous operator, we define an operator $\widetilde{\Psi} : \mathbb{R}^n \times \mathbb{R}^2 \times \widetilde{M} \times \mathbb{R}^2 \to C([0, 2p], \mathbb{R}^n) \times \mathbb{R}$ with the larger range:

$$\widetilde{\Psi}(c, \mu, \eta, w, x, \lambda) := (\Psi(c, \mu, \eta, w, x, \lambda), \langle w, \dot{u}_0 \rangle).$$

It is easily seen that $\widetilde{\Psi}(0, 0, 0, 0, 0, \lambda_0) = (0, 0)$.

Lemma 3 *The equation $\widetilde{\Psi} = 0$ implicitly defines a smooth function $\mu(x, \lambda)$.*

Proof The derivatives of $\widetilde{\Psi}$ with respect to the first four variables are given by

$$\begin{aligned} \widetilde{\Psi}_{c_i} &= (s e_i, 0), & \widetilde{\Psi}_\mu &= (L(s v_0(s)), 0), \\ \widetilde{\Psi}_\eta &= (\Phi_\tau(u_0, \lambda_0, 0), 0), & \mathrm{d}_w \widetilde{\Psi} &= (L, \langle \cdot, \dot{u}_0 \rangle). \end{aligned}$$

The relations $\langle v_1, L(s v_0(s)) \rangle = \langle v_2, \Phi_\tau(u_0, \lambda_0, 0) \rangle = 0$ and $\langle v_1, \Phi_\tau(u_0, \lambda_0, 0) \rangle \neq 0 \neq \langle v_2, L(s v_0(s)) \rangle$ together with (13) implies invertibility of the differential of $\widetilde{\Psi}$ with respect to the first four variables. According to the implicit function theorem, the equation $\widetilde{\Psi} = 0$ defines smooth functions $c(x, \lambda)$, $\mu(x, \lambda)$, $\eta(x, \lambda)$ and $w(x, \lambda)$ in a neighbourhood of the point $(0, 0, 0, 0, 0, \lambda_0)$. □

Now we are going to define a function which coincides with the critical Floquet multiplier. Let $T(x,\lambda,t,s)$ be the solution operator of the equation $(\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}=0$, where $t\geq s$ and the corresponding monodromy operator will be denoted by $U(x,\lambda){:=}T(x,\lambda,2p,0)$. Finally, let $\phi=(\dot{u}_0)_0$ and $\phi_0=(v_0)_0$ be eigenvectors of the monodromy operator $U{:=}U(0,\lambda_0)$ corresponding to the eigenvalue 1. We define the operator $\Xi:\mathbb{R}^2\times\mathrm{Im}(I-U)\times\mathbb{R}^2\to C$ by the following expression:

$$\Xi(\sigma,\nu,\omega,x,\lambda){:=}U(x,\lambda)(\phi_0+\omega)+\nu(\dot{\Omega}(x,\lambda))_0-\sigma(\phi_0+\omega).$$

Obviously, the point $(1,0,0,0,\lambda_0)$ satisfies the equation $\Xi=0$.

Lemma 4 *The equation $\Xi=0$ implicitly defines a smooth function $\sigma(x,\lambda)$.*

Proof The differential of Ξ with respect to the first three variables is given by

$$\Xi_\sigma=-\phi_0,\qquad \Xi_\nu=(\dot{\Omega}(0,\lambda_0))_0=(\dot{u}_0)_0=\phi,\qquad \Xi_\omega=U(0,\lambda_0)-I=U-I.$$

Since $C=\ker(I-U)\oplus\mathrm{Im}(I-U)$, the differential is invertible. Consequently, the equation $\Xi=0$ defines smooth functions $\sigma(x,\lambda)$, $\nu(x,\lambda)$ and $\omega(x,\lambda)$ in a neighbourhood of the point $(1,0,0,0,\lambda_0)$. □

Lemma 5 *If $f(x,\lambda)=0$, then $\sigma(x,\lambda)$ is the critical Floquet multiplier of the solution $\Omega(x,\lambda)$ of the equation $\Phi=0$.*

Proof The assumption implies $(\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}\dot{\Omega}(x,\lambda)=0$ which leads to the equality $U(x,\lambda)(\dot{\Omega}(x,\lambda))_0=(\dot{\Omega}(x,\lambda))_{2p}=(\dot{\Omega}(x,\lambda))_0$. Let us use the notation $V(x,\lambda)=\phi_0+\omega(x,\lambda)$. If the arguments are omitted, then the equations have the following forms:

$$(U-I)\dot{\Omega}_0=0,\qquad (U-\sigma I)V=-\nu\dot{\Omega}_0.$$

Suppose that $\sigma(x,\lambda)\neq 1$. The first equation can be expressed in the following way:

$$(U-I)\dot{\Omega}_0=0\;\Leftrightarrow\;(U-\sigma I)\dot{\Omega}_0=(1-\sigma)\dot{\Omega}_0\;\Leftrightarrow\;(U-\sigma I)\left(\frac{\nu}{1-\sigma}\dot{\Omega}_0\right)=\nu\dot{\Omega}_0.$$

Summing of the two equations gives

$$(U-\sigma I)\left(V+\frac{\nu}{1-\sigma}\dot{\Omega}_0\right)=0,$$

so $\sigma(x,\lambda)$ is a Floquet multiplier of $\Omega(x,\lambda)$. If $\sigma(x,\lambda)=1$, then the equation $\Xi=0$ has the form $(U-I)V=-\nu\dot{\Omega}_0$. Consequently, $(U-I)^2V=0$ which implies that $\sigma(x,\lambda)=1$ is a Floquet multiplier. It remains to connect the eigenvalue σ with the critical multiplier γ_2. According to (C2) and upper semicontinuity of separated parts of a spectrum of a continuous linear operator, Kato [9], all multipliers except γ_1 and γ_2 satisfy $|\gamma|\leq\delta<1$ for some $\delta>0$ so these multipliers do not come into play. Moreover, the sum of generalized eigenspaces corresponding to γ_1 and γ_2 is two-dimensional and the multiplier γ_1 is associated with the eigenfunction $\dot{\Omega}$. Thus, the second multiplier is the only possibility. □

The proof of the previous lemma gives us more than just information about the critical Floquet multiplier. If $\Omega(x,\lambda)$ is a solution of the equation $\Phi=0$, then the pair $\mathcal{B}=(\dot{\Omega}(x,\lambda),\phi_0+\omega(x,\lambda))$ forms a basis of the sum of generalized eigenspaces corresponding to eigenvalues of $U(x,\lambda)$ which are close to 1. Moreover, $U(x,\lambda)\mathcal{B}=\mathcal{B}M$ where

$$M=\begin{pmatrix}1 & -\nu(x,\lambda)\\ 0 & \sigma(x,\lambda)\end{pmatrix}.$$

Since M is invertible, we can calculate its logarithm:

$$\log M = \begin{pmatrix} 0 & -\dfrac{\nu(x,\lambda)\log\sigma(x,\lambda)}{\sigma(x,\lambda)-1} \\ 0 & \log\sigma(x,\lambda) \end{pmatrix},$$

where it is assumed that $\log\sigma/(\sigma-1)=1$ for $\sigma=1$. If we define $B=\log M/2p$, then

$$\mathrm{e}^{Bs} = \begin{pmatrix} 1 & -\dfrac{\nu(x,\lambda)}{\sigma(x,\lambda)-1}\left(\exp\left(\dfrac{s}{2p}\log\sigma(x,\lambda)\right)-1\right) \\ 0 & \exp\left(\dfrac{s}{2p}\log\sigma(x,\lambda)\right) \end{pmatrix}.$$

The vector $P(s)=T(x,\lambda,s,0)\mathcal{B}\mathrm{e}^{-Bs}$ with elements in C is $2p$-periodic, Hale [7]:

$$\begin{aligned} P(s+2p) &= T(x,\lambda,s+2p,0)\mathcal{B}\mathrm{e}^{-B(s+2p)} \\ &= T(x,\lambda,s+2p,2p)T(x,\lambda,2p,0)\mathcal{B}\mathrm{e}^{-2pB}\mathrm{e}^{-Bs} \\ &= T(x,\lambda,s,0)\mathcal{B}MM^{-1}\mathrm{e}^{-Bs} = T(x,\lambda,s,0)\mathcal{B}\mathrm{e}^{-Bs} = P(s). \end{aligned} \tag{14}$$

This is going to be used in a moment.

Before we accept fulfillment of non-degeneracy conditions, we formulate one more statement giving us a useful identity.

Lemma 6 *If $\Omega(x,\lambda)$ is a solution of the equation $\Phi=0$, then*

$$(\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}(s\dot{\Omega}(x,\lambda)(s)) = (1+\tau(x^2,\lambda))\Phi_\tau(\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)).$$

Proof We start with evaluation of the following expression:

$$\begin{aligned} ((\cdot)u(\cdot))_{s,\tau}(\theta) &= (s+(1+\tau)\theta)u(s+(1+\tau)\theta) \\ &= su(s+(1+\tau)\theta)+(1+\tau)\theta u(s+(1+\tau)\theta) \\ &= su_{s,\tau}(\theta)+(1+\tau)((\cdot)u_{s,\tau}(\cdot))(\theta). \end{aligned}$$

This together with $(\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}\dot{\Omega}(x,\lambda)=0$ give us the following set of equalities:

$$\begin{aligned} &(\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}(s\dot{\Omega}(x,\lambda)(s)) \\ &\quad= (1+\tau(x^2,\lambda))\left(\dot{\Omega}(x,\lambda)(s)+s\ddot{\Omega}(x,\lambda)(s)\right) \\ &\qquad - (\mathrm{d}F)_{\Omega(x,\lambda)_{s,\tau(x^2,\lambda)},\lambda}((\cdot)\dot{\Omega}(x,\lambda)(\cdot))_{s,\tau(x^2,\lambda)} \\ &\quad= (1+\tau(x^2,\lambda))\left(\dot{\Omega}(x,\lambda)(s)+s\ddot{\Omega}(x,\lambda)(s)\right) \\ &\qquad - s(\mathrm{d}F)_{\Omega(x,\lambda)_{s,\tau(x^2,\lambda)},\lambda}\dot{\Omega}(x,\lambda)_{s,\tau(x^2,\lambda)} \\ &\qquad - (1+\tau(x^2,\lambda))(\mathrm{d}F)_{\Omega(x,\lambda)_{s,\tau(x^2,\lambda)},\lambda}((\cdot)\dot{\Omega}(x,\lambda)_{s,\tau(x^2,\lambda)}(\cdot)) \\ &\quad= s(\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}\dot{\Omega}(x,\lambda)+(1+\tau(x^2,\lambda))\Phi_\tau(\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)) \\ &\quad= (1+\tau(x^2,\lambda))\Phi_\tau(\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)) \end{aligned}$$

which finishes the proof. □

Henceforth, we assume that $r_\lambda(0,\lambda_0)\neq 0$. This implies existence of a function $\lambda(x^2)$ satisfying $r(x^2,\lambda(x^2))=0$. Especially, $\Omega(x,\lambda(x^2))$ is a family of solutions bifurcating from the point (u_0, λ_0). Let us denote the second component of $P(s)(0) = T(x, \lambda(x^2), s, 0)\mathcal{B}(0)\mathrm{e}^{-Bs}$ by $\widetilde{w}(x)(s)$ and $\widetilde{\mu}(x) := \log(\sigma(x, \lambda(x^2))/2p)$:

$$\widetilde{w} := \frac{\nu(x, \lambda(x^2))}{\mathrm{e}^{2p\widetilde{\mu}(x)} - 1}(1 - \mathrm{e}^{-\widetilde{\mu}(x)s})\dot{\Omega}(x, \lambda(x^2))(s) \\ + \mathrm{e}^{-\widetilde{\mu}(x)s} T(x, \lambda(x^2), s, 0)(\phi_0 + \omega(x, \lambda(x^2)))(0).$$

According to (14), the function $\widetilde{w}(x)(s)$ is periodic. Obviously, both components of $T(x, \lambda, s, 0)\mathcal{B}(0)$ are solutions of the equation $(\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)} = 0$ so we can conclude the following equations:

$$(\mathrm{d}\Phi)_{\Omega(x,\lambda(x^2)),\lambda(x^2),\tau(x^2,\lambda(x^2))}\left(T(x, \lambda(x^2), s, 0)\mathcal{B}(0)\mathrm{e}^{-Bs}\mathrm{e}^{Bs}\begin{pmatrix}0\\1\end{pmatrix}\right) = 0,$$

$$(\mathrm{d}\Phi)_{\Omega(x,\lambda(x^2)),\lambda(x^2),\tau(x^2,\lambda(x^2))}\left((\dot{\Omega}(x, \lambda(x^2))(s), \widetilde{w}(x)(s))\mathrm{e}^{Bs}\begin{pmatrix}0\\1\end{pmatrix}\right) = 0,$$

$$(\mathrm{d}\Phi)_{\ldots}\left(\frac{-\nu(x, \lambda(x^2))}{\mathrm{e}^{2p\widetilde{\mu}(x)} - 1}(\mathrm{e}^{\widetilde{\mu}(x)s} - 1)\dot{\Omega}(x, \lambda(x^2))(s) + \mathrm{e}^{\widetilde{\mu}(x)s}\widetilde{w}(x)(s)\right) = 0.$$

The last equation can be used for calculation of derivatives of the function $\widetilde{\mu}(x)$ which is the critical Floquet exponent of the solution $\Omega(x, \lambda(x^2))$.

The whole theoretical background has been presented so it remains to perform necessary calculations the results of which are summarized in the following statements. Since their proofs are quite technical, they are postponed to the Appendix.

Lemma 7 $\mu_{xx}(0, \lambda_0) = -\dfrac{f_{xxx}(0, \lambda_0)}{\langle v_2, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(sv_0(s))\rangle}.$

In order to simplify notation, we write $m(x) := \mu(x, \lambda(x^2))$.

Lemma 8 $m_{xx}(0) = \widetilde{\mu}_{xx}(0)$.

Theorem 4 $\widetilde{\mu}_{xx}(0) = -\dfrac{4r_z(0, \lambda_0)}{\langle v_2, L(sv_0(s))\rangle}.$

Now we are prepared to verify the main result of this section.

Theorem 5 *In the case of a non-degenerate period-doubling bifurcation the exchange of stability occurs.*

Proof Let us suppose that $\langle v_2, L(sv_0(s))\rangle > 0$, the opposite case can be solved analogously. If

$$0 < r_\lambda(0, \lambda_0) = \frac{\rho'(\lambda_0)}{p}\langle v_2, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(sv_0(s))\rangle,$$

then $\rho(\lambda)$ is increasing which means that p-periodic solutions are stable for $\lambda > \lambda_0$ and unstable for $\lambda < \lambda_0$. According to Proposition 5, in the case $r_z(0, \lambda_0) > 0$ the doubled solutions appear for $\lambda < \lambda_0$ and they are stable because $\widetilde{\mu}$ is negative in a neighbourhood of 0. Otherwise, these solutions appear for $\lambda > \lambda_0$ and they are unstable. The case $r_\lambda(0, \lambda_0) < 0$ can be investigated in the same way. This proves the exchange of stability for non-degenerate period-doubling bifurcation. □

4 Discussion and Conclusion

Generally, normal form coefficients are considered to be the quantities a and c in the normal form of a period-doubling bifurcation:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1 + a\xi^2 + \mathcal{O}(\xi^4),$$
$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = c\xi^3 + \mathcal{O}(\xi^4),$$

where τ and ξ are coordinates on the center manifold with ξ being transverse to the orbit of $\dot{u}_0$, Lentjes et al. [12, Preprint]. On the other hand, normal form coefficients derived in this work correspond to Taylor coefficients of the reduced mapping ϕ. The definition of a and c together with our analysis imply that a, c and $r_\lambda(0, \lambda_0)$, $r_z(0, \lambda_0)$ have the same sign, respectively. Since formulae for these coefficients coincide in the case of ODE, see Kuznetsov et al. [11], we can expect their agreement also in the case of RFDE.

A next step would be to actually implement the obtained expressions and see how well they behave numerically. Additionally, the following equations could potentially be used for the continuation of period-doubling bifurcations of limit cycles in RFDE:

$$\widetilde{\Phi}(u, \lambda, \tau) = 0, \quad (\mathrm{d}\Phi)_{u,\lambda,\tau} \cdot v = 0, \quad \langle v, v \rangle = 1, \quad v(p) = -v(0).$$

This should then be compared with the current implementation in DDE-BifTool and Knut, see Engelborghs et al. [5] and Szalai [15]. Moreover, the left-hand sides of (10) and (11) could play the role of a test function for detection of singularities, for the case of ODE see Dhooge et al. [2], Dhooge et al. [3] and Kuznetsov [10]. The development of appropriate numerical methods can lead to a strong tool for investigation of real-life models incorporating delays which overcome period-doubling bifurcation. Examples of such models include, for instance, the system of coupled Fitzhugh-Nagumo oscillators with two delays studied by Saha and Feudel [13], and the time-periodic model of machining analyzed by Szalai and Stépán [16], where authors used sun-star calculus.

The existence of a smooth periodic center manifold in a neighbourhood of a limit cycle together with a suitable coordinate system is a base of the investigation of period-doubling bifurcation in Iooss [8], Kuznetsov et al. [11] and Lentjes et al. [12, Preprint]. We have not needed such information, since the main idea of the presented theory lies in considering a given differential equation as an algebraic equation. This can be expressed rigorously as the possibility to find suitable Banach spaces and an operator between them with the property that its zero set coincides with solutions of the original equation. The key to success is an appropriate choice of an operator of which linearization satisfies conditions imposed on Fredholm operators so that Lyapunov–Schmidt reduction can be used. In general, this approach could be applied to another bifurcations and to another types of differential equations, for example, bifurcations in neutral functional differential equations or Arnold tongues. The second area would probably lead to equivariant singularity theory.

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Declarations

Conflict of interest The author declares that he has no competing interests.

Ethical Approval This declaration is not applicable.

5 Appendix

5.1 Calculation of r_λ

Differentiation of the defining equation $r(z,\lambda) = b(z,\lambda,\tau(z,\lambda))$ with respect to λ leads to

$$r_\lambda(0,\lambda_0) = b_\lambda(0,\lambda_0,0) + b_\tau(0,\lambda_0,0)\cdot\tau_\lambda(0,\lambda_0). \tag{15}$$

If it is not necessary to write down arguments of functions, we will omit them. We can continue with evaluation of b_λ, b_τ and τ_λ. These values are going to be calculated from equations $a(x^2,\lambda,\tau(x^2,\lambda)) = 0$ and $g_2(x,\lambda,\tau) = xb(x^2,\lambda,\tau)$. Let us recall that $z = x^2$ is the first variable on which the germs a, b and τ depend.

$$\begin{aligned}
a = 0 \quad \Big/\tfrac{\partial}{\partial\lambda} \quad &\Rightarrow \quad a_\lambda(0,\lambda_0,0) + a_\tau(0,\lambda_0,0)\cdot\tau_\lambda(0,\lambda_0) = 0\\
&\Rightarrow \quad \tau_\lambda = -a_\lambda/a_\tau\\
g_2 = xb \quad \Big/\tfrac{\partial}{\partial x} \quad &\Rightarrow \quad (g_2)_x = b + 2x^2 b_z \quad \Big/\tfrac{\partial}{\partial\lambda},\tfrac{\partial}{\partial\tau}\\
&\Rightarrow \quad (g_2)_{x\lambda} = b_\lambda + 2x^2 b_{z\lambda}, \quad (g_2)_{x\tau} = b_\tau + 2x^2 b_{z\tau}\\
&\Rightarrow \quad b_\lambda(0,\lambda_0,0) = (g_2)_{x\lambda}(0,\lambda_0,0)\\
&\Rightarrow \quad b_\tau(0,\lambda_0,0) = (g_2)_{x\tau}(0,\lambda_0,0).
\end{aligned}$$

Since a_τ is computed in Proposition 4, it is enough to evaluate a_λ in order to calculate τ_λ:

$$\begin{aligned}
a_\lambda(0,\lambda_0,0) = (g_1)_\lambda(0,\lambda_0,0) &= \langle v_1, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}\cdot W_\lambda(0,\lambda_0,0) + \Phi_\lambda(u_0,\lambda_0,0)\rangle\\
&= \langle v_1, \Phi_\lambda(u_0,\lambda_0,0)\rangle.
\end{aligned}$$

Hence, τ_λ is given by

$$\tau_\lambda(0,\lambda_0) = -\frac{\langle v_1, \Phi_\lambda(u_0,\lambda_0,0)\rangle}{\langle v_1, \Phi_\tau(u_0,\lambda_0,0)\rangle}.$$

The derivative $(g_2)_x(x,\lambda,\tau)$ is given by

$$\begin{aligned}
(g_2)_x &= (\langle v_2, \Phi(xv_0 + W(x,\lambda,\tau),\lambda,\tau)\rangle)_x\\
&= \langle v_2, (\mathrm{d}\Phi)_{xv_0+W(x,\lambda,\tau),\lambda,\tau}(v_0 + W_x(x,\lambda,\tau))\rangle.
\end{aligned}$$

Now we can calculate $(g_2)_{x\lambda}$ and $(g_2)_{x\tau}$:

$$\begin{aligned}(g_2)_{x\lambda}(0,\lambda_0,0) &= \langle v_2, (\mathrm{d}\Phi_\lambda)_{W(0,\lambda_0,0),\lambda_0,0}(v_0 + W_x(0,\lambda_0,0))\rangle \\ &\quad + \langle v_2, (\mathrm{d}^2\Phi)_{W(0,\lambda_0,0),\lambda_0,0}(W_\lambda(0,\lambda_0,0), v_0 + W_x(0,\lambda_0,0))\rangle \\ &\quad + \langle v_2, (\mathrm{d}\Phi)_{W(0,\lambda_0,0),\lambda_0,0}(W_{x\lambda}(0,\lambda_0,0))\rangle \\ (g_2)_{x\tau}(0,\lambda_0,0) &= \langle v_2, (\mathrm{d}\Phi_\tau)_{W(0,\lambda_0,0),\lambda_0,0}(v_0 + W_x(0,\lambda_0,0))\rangle \\ &\quad + \langle v_2, (\mathrm{d}^2\Phi)_{W(0,\lambda_0,0),\lambda_0,0}(W_\tau(0,\lambda_0,0), v_0 + W_x(0,\lambda_0,0))\rangle \\ &\quad + \langle v_2, (\mathrm{d}\Phi)_{W(0,\lambda_0,0),\lambda_0,0}(W_{x\tau}(0,\lambda_0,0))\rangle\end{aligned}$$

The lower indices can be simplified by the third equality in (8). Actually, the differentials are calculated at the point $(u_0, \lambda_0, 0)$. Since $\operatorname{Im} L \subseteq [v_1, v_2]^\perp$, the last summands are equal to zero in both cases. The value $W_x(0,\lambda_0,0)$ can be deduced from the defining equations (8) or (12):

$$\begin{aligned} E\Phi(xv_0 + W(x,\lambda,\tau),\lambda,\tau) &= 0 \qquad \Big/ \tfrac{\partial}{\partial x}\Big|_{(0,\lambda_0,0)} \\ E(\mathrm{d}\Phi)_{u_0,\lambda_0,0}(v_0 + W_x(0,\lambda_0,0)) &= 0 \\ EL(v_0 + W_x(0,\lambda_0,0)) &= 0 \\ LW_x(0,\lambda_0,0) &= 0.\end{aligned}$$

The last equation is a consequence of $v_0 \in \ker L$ and $EL = L$. Note that $W_x \in \widetilde{M}$ and $L\colon M \to \operatorname{Im} L$ is invertible. We can deduce $W_x(0,\lambda_0,0) \in [\dot{u}_0]$. Since $\langle W_x(0,\lambda_0,0), \dot{u}_0\rangle = 0$, the equality $W_x(0,\lambda_0,0) = 0$ can be concluded. We can put together all recent calculations in order to simplify expressions for b_λ and b_τ:

$$\begin{aligned} b_\tau(0,\lambda_0,0) &= \langle v_2, (\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}\cdot v_0 + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_\tau(0,\lambda_0,0), v_0)\rangle, \\ b_\lambda(0,\lambda_0,0) &= \langle v_2, (\mathrm{d}\Phi_\lambda)_{u_0,\lambda_0,0}\cdot v_0 + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_\lambda(0,\lambda_0,0), v_0)\rangle.\end{aligned}$$

If we substitute these expressions into the equation (15), then we get (10) which can be written in the following alternative form:

$$\begin{aligned} r_\lambda(0,\lambda_0) &= \langle v_2, (\mathrm{d}\Phi_\lambda)_{u_0,\lambda_0,0}\cdot v_0\rangle - \frac{\langle v_1, \Phi_\lambda(u_0,\lambda_0,0)\rangle}{\langle v_1, \Phi_\tau(u_0,\lambda_0,0)\rangle}\cdot\langle v_2, (\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}\cdot v_0\rangle \\ &\quad + \langle v_2, (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_\lambda(0,\lambda_0,0) + \tau_\lambda(0,\lambda_0)W_\tau(0,\lambda_0,0), v_0), v_0)\rangle.\end{aligned}$$

We finish these calculations by specifying the function $W_\lambda + \tau_\lambda W_\tau$. We can use the fact that $W(0,\lambda,\tau(0,\lambda))$ is a family of solutions of the equation $\Phi = 0$. Therefore, the equation

$$\Phi(W(0,\lambda,\tau(0,\lambda)),\lambda,\tau(0,\lambda)) = 0$$

can be differentiated with respect to λ and subsequently evaluated at $\lambda = \lambda_0$ which leads to the fact that $W_\lambda + \tau_\lambda W_\tau$ is a $2p$-periodic solution of the following equation:

$$L(W_\lambda + \tau_\lambda W_\tau) + \Phi_\lambda(u_0,\lambda_0,0) + \tau_\lambda(0,\lambda_0)\Phi_\tau(u_0,\lambda_0,0) = 0.$$

Moreover, we can say that $\langle W_\lambda + \tau_\lambda W_\tau, \dot{u}_0\rangle = 0$ which is a consequence of (8) or (12).

5.2 Calculation of r_z

Let us recall the defining equations $g_2(x, \lambda, \tau) = xb(x^2, \lambda, \tau)$ and $r(z, \lambda) = b(z, \lambda, \tau(z, \lambda))$. Differentiation of the second equation with respect to z leads to

$$r_z(0, \lambda_0) = b_z(0, \lambda_0, 0) + b_\tau(0, \lambda_0, 0) \cdot \tau_z(0, \lambda_0).$$

Since b_τ has been computed, we focus on evaluation of b_z and τ_z:

$$\begin{aligned} a = 0 \quad \Big/ \tfrac{\partial}{\partial z} \quad &\Rightarrow \quad a_z(0, \lambda_0, 0) + a_\tau(0, \lambda_0, 0) \cdot \tau_z(0, \lambda_0) = 0 \\ &\Rightarrow \quad \tau_z = -a_z/a_\tau. \end{aligned}$$

We digress to compute a_z:

$$\begin{aligned} g_1(x, \lambda, \tau) &= a(x^2, \lambda, \tau) && \Big/ \tfrac{\partial}{\partial x} \\ (g_1)_x(x, \lambda, \tau) &= 2xa_z(x^2, \lambda, \tau) && \Big/ \tfrac{\partial}{\partial x} \\ (g_1)_{xx}(x, \lambda, \tau) &= 2a_z(x^2, \lambda, \tau) + 4x^2 a_{zz}(x^2, \lambda, \tau) \\ (g_1)_{xx}(0, \lambda_0, 0) &= 2a_z(0, \lambda_0, 0). \end{aligned}$$

The left-hand side can be evaluated in the following way:

$$\begin{aligned} g_1(x, \lambda, \tau) &= \langle v_1, \Phi(xv_0 + W(x, \lambda, \tau), \lambda, \tau)\rangle \\ (g_1)_x(x, \lambda, \tau) &= \langle v_1, (\mathrm{d}\Phi)_{xv_0+W(x,\lambda,\tau),\lambda,\tau}(v_0 + W_x(x, \lambda, \tau))\rangle \\ (g_1)_{xx} &= \langle v_1, (\mathrm{d}^2\Phi)(v_0 + W_x, v_0 + W_x) + (\mathrm{d}\Phi)(W_{xx})\rangle \\ (g_1)_{xx}(0, \lambda_0, 0) &= \langle v_1, (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0, v_0) + (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(W_{xx}(0, \lambda_0, 0))\rangle \\ &= \langle v_1, (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0, v_0)\rangle. \end{aligned}$$

We have found that

$$\tau_z(0, \lambda_0) = -\frac{\langle v_1, (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0, v_0)\rangle}{2\langle v_1, \Phi_\tau(u_0, \lambda_0, 0)\rangle}.$$

It remains to calculate b_z:

$$\begin{aligned} g_2(x, \lambda, \tau) &= xb(x^2, \lambda, \tau) && \Big/ \tfrac{\partial}{\partial x} \\ (g_2)_x(x, \lambda, \tau) &= b(x^2, \lambda, \tau) + 2x^2 b_z(x^2, \lambda, \tau) && \Big/ \tfrac{\partial}{\partial x} \\ (g_2)_{xx} &= 2xb_z + 4xb_z + 4x^3 b_{zz} = 6xb_z + 4x^3 b_{zz} && \Big/ \tfrac{\partial}{\partial x} \\ (g_2)_{xxx} &= 6b_z + 12x^2 b_{zz} + 12x^2 b_{zz} + 8x^4 b_{zzz} \\ (g_2)_{xxx}(0, \lambda_0, 0) &= 6b_z(0, \lambda_0, 0). \end{aligned}$$

The left-hand side is given by

$$\begin{aligned} g_2(x, \lambda, \tau) &= \langle v_2, \Phi(xv_0 + W(x, \lambda, \tau), \lambda, \tau)\rangle \\ (g_2)_x(x, \lambda, \tau) &= \langle v_2, (\mathrm{d}\Phi)_{xv_0+W(x,\lambda,\tau),\lambda,\tau}(v_0 + W_x(x, \lambda, \tau))\rangle \\ (g_2)_{xx} &= \langle v_2, (\mathrm{d}^2\Phi)(v_0 + W_x, v_0 + W_x) + (\mathrm{d}\Phi)(W_{xx})\rangle \\ (g_2)_{xxx} &= \langle v_2, (\mathrm{d}^3\Phi)(v_0 + W_x, v_0 + W_x, v_0 + W_x)\rangle \\ &\quad + \langle v_2, 3(\mathrm{d}^2\Phi)(W_{xx}, v_0 + W_x) + (\mathrm{d}\Phi)(W_{xxx})\rangle \\ (g_2)_{xxx}(0, \lambda_0, 0) &= \langle v_2, (\mathrm{d}^3\Phi)_{u_0,\lambda_0,0}(v_0, v_0, v_0)\rangle \\ &\quad + \langle v_2, 3(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_{xx}(0, \lambda_0, 0), v_0)\rangle. \end{aligned}$$

We can conclude (11) which can be written in the following alternative form:

$$\begin{aligned} r_z(0,\lambda_0) &= \frac{1}{6}\cdot\langle v_2, (\mathrm{d}^3\Phi)_{u_0,\lambda_0,0}(v_0,v_0,v_0)\rangle \\ &\quad - \frac{\langle v_1, (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0,v_0)\rangle}{2\langle v_1, \Phi_\tau(u_0,\lambda_0,0)\rangle}\cdot\langle v_2, (\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}\cdot v_0\rangle \\ &\quad + \frac{1}{2}\langle v_2, (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_{xx}(0,\lambda_0,0) + 2\tau_z(0,\lambda_0)W_\tau(0,\lambda_0,0), v_0)\rangle. \end{aligned}$$

Finally, we are going to focus on the function $W_{xx} + 2\tau_z W_\tau$. The function $\Omega(x,\lambda) = xv_0 + W(x,\lambda,\tau(x^2,\lambda))$ satisfies the following equations:

$$E\Phi(\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)) = 0, \qquad \langle v_1, \Phi(\Omega(x,\lambda),\lambda,\tau(x^2,\lambda))\rangle = 0.$$

The second derivatives of the left-hand sides evaluated at $(0,\lambda_0)$ lead to

$$\begin{aligned} E((\mathrm{d}\Phi)_{u_0,\lambda_0,0}\Omega_{xx}(0,\lambda_0) + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0,v_0) + 2\tau_z(0,\lambda_0)\Phi_\tau(u_0,\lambda_0,0)) &= 0, \\ \langle v_1, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}\Omega_{xx}(0,\lambda_0) + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0,v_0) + 2\tau_z(0,\lambda_0)\Phi_\tau(u_0,\lambda_0,0)\rangle &= 0. \end{aligned}$$

Since the equation

$$\langle v_2, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}\Omega_{xx}(0,\lambda_0) + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0,v_0) + 2\tau_z(0,\lambda_0)\Phi_\tau(u_0,\lambda_0,0)\rangle = 0$$

is also valid, we can conclude

$$L\Omega_{xx}(0,\lambda_0) + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0,v_0) + 2\tau_z(0,\lambda_0)\Phi_\tau(u_0,\lambda_0,0) = 0. \tag{16}$$

Moreover, we can say that $\langle W_{xx} + 2\tau_z W_\tau, \dot{u}_0\rangle = 0$ which is a consequence of (8) or (12).

5.3 Proof of Lemma 7

If it does not cause any confusion, we omit arguments of functions. Firstly, we calculate derivatives of f:

$$\begin{aligned} f(x,\lambda) &= \langle v_2, \Phi(\Omega(x,\lambda),\lambda,\tau(x^2,\lambda))\rangle \\ f_x(x,\lambda) &= \langle v_2, (\mathrm{d}\Phi)_{\Omega(x,\lambda),\lambda,\tau(x^2,\lambda)}\Omega_x(x,\lambda) \\ &\quad + 2x\tau_z(x^2,\lambda)\Phi_\tau(\Omega(x,\lambda),\lambda,\tau(x^2,\lambda))\rangle \\ f_{xx} &= \langle v_2, (\mathrm{d}^2\Phi)(\Omega_x,\Omega_x) + 2x\tau_z(\mathrm{d}\Phi_\tau)\Omega_x + (\mathrm{d}\Phi)\Omega_{xx} \\ &\quad + 2\tau_z\Phi_\tau + 4x^2\tau_{zz}\Phi_\tau + 2x\tau_z(\mathrm{d}\Phi_\tau)\Omega_x + 4x^2\tau_z^2\Phi_{\tau\tau}\rangle \\ f_{xxx}(0,\lambda_0) &= \langle v_2, (\mathrm{d}^3\Phi)_{u_0,\lambda_0,0}(v_0,v_0,v_0) + 2(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(\Omega_{xx}(0,\lambda_0),v_0) \\ &\quad + 2\tau_z(0,\lambda_0)(\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}v_0 + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(\Omega_{xx}(0,\lambda_0),v_0) \\ &\quad + (\mathrm{d}\Phi)_{u_0,\lambda_0,0}\Omega_{xxx}(0,\lambda_0) + 4\tau_z(0,\lambda_0)(\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}v_0\rangle \\ &= \langle v_2, (\mathrm{d}^3\Phi)_{u_0,\lambda_0,0}(v_0,v_0,v_0) + 3(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(\Omega_{xx}(0,\lambda_0),v_0) \\ &\quad + 6\tau_z(0,\lambda_0)(\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}v_0\rangle. \end{aligned}$$

Let us focus on derivatives of μ. Differentiation of the equation $\Psi = 0$ with respect to x leads to

$$\begin{aligned}(\mathrm{d}^2\Phi)(\Omega_x, \mathrm{e}^{\mu s}(v_0 + w)) + 2x\tau_z(\mathrm{d}\Phi_\tau)(\mathrm{e}^{\mu s}(v_0 + w)) \\ + (\mathrm{d}\Phi)(\mu_x s\mathrm{e}^{\mu s}(v_0 + w) + \mathrm{e}^{\mu s}w_x) \\ + \eta_x\Phi_\tau + \eta(\mathrm{d}\Phi_\tau)\Omega_x + 2x\eta\tau_z\Phi_{\tau\tau} + c_x s = 0.\end{aligned}$$

If we evaluate the previous equation at $(x, \lambda) = (0, \lambda_0)$, we get the following result:

$$\begin{aligned}(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0, v_0) + (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(\mu_x(0, \lambda_0)sv_0(s) + w_x(0, \lambda_0)) \\ + \eta_x(0, \lambda_0)\Phi_\tau(u_0, \lambda_0, 0) + c_x(0, \lambda_0)s = 0.\end{aligned}$$

Since the first three summands are $2p$-periodic, we can conclude $c_x(0, \lambda_0) = 0$. Multiplication by vectors v_1 and v_2 gives the following identities:

$$\begin{aligned}&\mu_x(0, \lambda_0) = 0, \\ &\eta_x(0, \lambda_0) = -\frac{\langle v_1, (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0, v_0)\rangle}{\langle v_1, \Phi_\tau(u_0, \lambda_0, 0)\rangle} = 2\tau_z(0, \lambda_0),\end{aligned}$$

where the first equality is a consequence of p-periodicity of $(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0, v_0)$. Let us emphasize that $w_x(0, \lambda_0)$ satisfies the following equation:

$$(\mathrm{d}\Phi)_{u_0,\lambda_0,0}w_x(0, \lambda_0) + (\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0, v_0) + 2\tau_z(0, \lambda_0)\Phi_\tau(u_0, \lambda_0, 0) = 0. \tag{17}$$

Now we can proceed to the second derivative evaluated at $(0, \lambda_0)$:

$$\begin{aligned}(\mathrm{d}^3\Phi)(v_0, v_0, v_0) + (\mathrm{d}^2\Phi)(\Omega_{xx}, v_0) + (\mathrm{d}^2\Phi)(v_0, w_x) + 2\tau_z(\mathrm{d}\Phi_\tau)v_0 \\ + (\mathrm{d}^2\Phi)(v_0, w_x) + (\mathrm{d}\Phi)(\mu_{xx}sv_0(s) + w_{xx}) \\ + \eta_{xx}\Phi_\tau + \eta_x(\mathrm{d}\Phi_\tau)v_0 + \eta_x(\mathrm{d}\Phi_\tau)v_0 + c_{xx}s = 0, \\ (\mathrm{d}^3\Phi)(v_0, v_0, v_0) + (\mathrm{d}^2\Phi)(\Omega_{xx}, v_0) + 2(\mathrm{d}^2\Phi)(v_0, w_x) + 6\tau_z(\mathrm{d}\Phi_\tau)v_0 \\ + \eta_{xx}\Phi_\tau + (\mathrm{d}\Phi)(\mu_{xx}sv_0(s) + w_{xx}) + c_{xx}s = 0.\end{aligned}$$

Periodicity of all summands except the last one implies $c_{xx}(0, \lambda_0) = 0$.

It remains to prove the identity $\langle v_2, (\mathrm{d}^2\Phi)(\Omega_{xx}, v_0)\rangle = \langle v_2, (\mathrm{d}^2\Phi)(w_x, v_0)\rangle$. According to (16) and (17), functions $\Omega_{xx}(0, \lambda_0)$ and $w_x(0, \lambda_0)$ satisfy the same equation so we can conclude $L(\Omega_{xx} - w_x) = 0$ which can be equivalently expressed by $\Omega_{xx} - w_x \in \ker L$. The last step consists of verification of the relation $\langle v_2, (\mathrm{d}^2\Phi)(v_0, u)\rangle = 0$ for $u \in \ker L$. Since $\langle v_2, (\mathrm{d}^2\Phi)(v_0, v_0)\rangle = 0$, it is enough to show that $\langle v_2, (\mathrm{d}^2\Phi)(v_0, \dot{u}_0)\rangle = 0$. If we differentiate the equation $(\mathrm{d}\Phi)\dot{\Omega}(x, \lambda(x^2)) = 0$ with respect to x and evaluate it at $(0, \lambda_0)$ we arrive to

$$(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0, \dot{u}_0) + (\mathrm{d}\Phi)_{u_0,\lambda_0,0}\dot{v}_0 = 0.$$

Multiplication by the vector v_2 gives us the desired result.

5.4 Proof of Lemma 8

Since μ is implicitly defined by the equation $\Psi = 0$, the function m satisfies the following equation:

$$\begin{aligned}(\mathrm{d}\Phi)_{\Omega(x,\lambda(x^2)),\lambda(x^2),\tau(x^2,\lambda(x^2))}(\mathrm{e}^{m(x)s}(v_0(s) + w(x, \lambda(x^2))(s))) \\ + \eta(x, \lambda(x^2))\Phi_\tau(\Omega(x, \lambda(x^2)), \lambda(x^2), \tau(x^2, \lambda(x^2))) + c(x, \lambda(x^2))t = 0.\end{aligned}$$

However, we can use Lemma 6 in order to get more appropriate form of this equation:

$$(\mathrm{d}\Phi)_{\Omega(x,\lambda(x^2)),\lambda(x^2),\tau(x^2,\lambda(x^2))}\Bigg(\mathrm{e}^{m(x)s}(v_0(s)+w(x,\lambda(x^2))(s)) \\ +\frac{\eta(x,\lambda(x^2))}{1+\tau(x,\lambda(x^2))}s\dot{\Omega}(x,\lambda(x^2))(s)\Bigg)+c(x,\lambda(x^2))s=0.$$

Differentiation with respect to x leads to

$$(\mathrm{d}^2\Phi)(\Omega_x+2x\lambda_z\Omega_\lambda,\ldots)+2x\lambda_z(\mathrm{d}\Phi_\lambda)(\ldots)+(2x\tau_z+2x\lambda_z\tau_\lambda)(\mathrm{d}\Phi_\tau)(\ldots) \\ +(\mathrm{d}\Phi)\Bigg(m_x s\mathrm{e}^{ms}(v_0+w)+\mathrm{e}^{ms}(w(x,\lambda(x^2)))_x+\left(\frac{\eta}{1+\tau}\right)_x s\dot{\Omega} \\ +\frac{\eta}{1+\tau}s(\dot{\Omega}_x+2x\lambda_z\dot{\Omega}_\lambda)\Bigg)+(c(x,\lambda(x^2)))_x s=0,$$

where the dots represent the expression between large parentheses in the previous equation. The following equalities can be deduced from the proof of Lemma 7:

$$m_x(0)=0,\qquad \frac{\partial}{\partial x}\bigg|_{x=0}\left(\frac{\eta(x,\lambda(x^2))}{1+\tau(x^2,\lambda(x^2))}\right)=2\tau_z(0,\lambda_0).$$

The second differentiation gives

$$(\mathrm{d}^3\Phi)(v_0,v_0,v_0)+(\mathrm{d}^2\Phi)(\Omega_{xx}+2\lambda_z\Omega_\lambda,v_0)+(\mathrm{d}^2\Phi)(v_0,w_x+2\tau_z s\dot{u}_0) \\ +2\lambda_z(\mathrm{d}\Phi_\lambda)v_0+2(\tau_z+\lambda_z\tau_\lambda)(\mathrm{d}\Phi_\tau)v_0+(\mathrm{d}^2\Phi)(v_0,w_x+2\tau_z s\dot{u}_0) \\ +(\mathrm{d}\Phi)(m_{xx}sv_0+(w(x,\lambda(x^2)))_{xx}+\left(\frac{\eta}{1+\tau}\right)_{xx}s\dot{u}_0+4\tau_z s\dot{v}_0) \\ +(c(x,\lambda(x^2)))_{xx}s=0. \tag{18}$$

It can be easily shown that $(c(x,\lambda(x^2)))_{xx}=0$.

Let us continue with the function $\widetilde{\mu}$. In order to simplify notation, we will use the abbreviation $j(x)=(\mathrm{e}^{\widetilde{\mu}(x)s}-1)/(\mathrm{e}^{2p\widetilde{\mu}(x)}-1)$. We emphasize that $j(0)=s/2p$.

$$(\mathrm{d}^2\Phi)(\Omega_x+2x\lambda_z\Omega_\lambda,-\nu j\dot{\Omega}+\mathrm{e}^{\widetilde{\mu}s}\widetilde{w}) \\ +2x\lambda_z(\mathrm{d}\Phi_\lambda)(-\nu j\dot{\Omega}+\mathrm{e}^{\widetilde{\mu}s}\widetilde{w})+(2x\tau_z+2x\lambda_z\tau_\lambda)(\mathrm{d}\Phi_\tau)(-\nu j\dot{\Omega}+\mathrm{e}^{\widetilde{\mu}s}\widetilde{w}) \\ +(\mathrm{d}\Phi)\Big(-((\nu(x,\lambda(x^2)))_x j+\nu j_x)\dot{\Omega}-\nu j(\dot{\Omega}_x+2x\lambda_z\dot{\Omega}_\lambda) \\ +\widetilde{\mu}_x s\mathrm{e}^{\widetilde{\mu}s}\widetilde{w}+\mathrm{e}^{\widetilde{\mu}s}\widetilde{w}_x\Big)=0.$$

If we evaluate the left-hand side at $x=0$ and multiply the equation by v_1 and v_2, then we arrive to

$$\widetilde{\mu}_x(0)=0,\qquad \frac{\partial}{\partial x}\bigg|_{x=0}\frac{\nu(x,\lambda(x^2))}{2p}=\frac{\langle v_1,(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0,v_0)\rangle}{\langle v_1,\Phi_\tau(u_0,\lambda_0,0)\rangle}=-2\tau_z(0,\lambda_0).$$

Moreover, the function $\widetilde{w}_x$ satisfies the following equation:

$$(\mathrm{d}\Phi)_{u_0,\lambda_0,0}\widetilde{w}_x+(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(v_0,v_0)+2\tau_z(0,\lambda_0)\Phi_\tau(u_0,\lambda_0,0)=0.$$

The second derivative evaluated at zero satisfies the equation

$$
\begin{aligned}
(\mathrm{d}^3\Phi)(v_0, v_0, v_0) &+ (\mathrm{d}^2\Phi)(\Omega_{xx} + 2\lambda_z\Omega_\lambda, v_0) + (\mathrm{d}^2\Phi)(v_0, 2\tau_z s\dot{u}_0 + \widetilde{w}_x) \\
&+ 2\lambda_z(\mathrm{d}\Phi_\lambda)v_0 + 2(\tau_z + \lambda_z\tau_\lambda)(\mathrm{d}\Phi_\tau)v_0 + (\mathrm{d}^2\Phi)(v_0, 2\tau_z s\dot{u}_0 + \widetilde{w}_x) \\
&+ (\mathrm{d}\Phi)(-(\nu(x, \lambda(x^2)))_{xx} j\dot{u}_0 + 4\tau_z s\dot{v}_0 + \widetilde{\mu}_{xx} s v_0 + \widetilde{w}_{xx}) = 0.
\end{aligned}
$$

If we multiply this equation and equation (18) by v_2 and we subtract the second one from the first one, then we get

$$
2\langle v_2, (\mathrm{d}^2\Phi)(v_0, w_x - \widetilde{w}_x)\rangle + \langle v_2, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(s v_0(s))\rangle(m_{xx} - \widetilde{\mu}_{xx}) = 0.
$$

The first summand is zero because of the same reasons as in the proof of Lemma 7. The lemma is proven.

5.5 Proof of Theorem 5

This is a simple consequence of the previous lemmas, so we just need to untangle the established notation. We start with the derivative m_{xx}:

$$
\begin{aligned}
m(x) &= \mu(x, \lambda(x^2)), \\
m_x(x) &= \mu_x(x, \lambda(x^2)) + 2x\lambda_z(x^2)\mu_\lambda(x, \lambda(x^2)), \\
m_{xx}(0) &= \mu_{xx}(0, \lambda_0) + 2\lambda_z(0)\mu_\lambda(0, \lambda_0).
\end{aligned}
$$

The derivative $\lambda_z(0)$ can be calculated from the definition of the function $\lambda(x^2)$:

$$
r(z, \lambda(z)) = 0 \quad \Rightarrow \quad r_z(0, \lambda_0) + \lambda_z(0) r_\lambda(0, \lambda_0) \quad \Rightarrow \quad \lambda_z(0) = -\frac{r_z(0, \lambda_0)}{r_\lambda(0, \lambda_0)}.
$$

This together with previous lemmas lead to the equation

$$
\widetilde{\mu}_{xx}(0) = m_{xx}(0) = -\frac{f_{xxx}(0, \lambda_0)}{\langle v_2, (\mathrm{d}\Phi)_{u_0,\lambda_0,0}(s v_0(s))\rangle} - 2\frac{r_z(0, \lambda_0)}{r_\lambda(0, \lambda_0)}\mu_\lambda(0, \lambda_0).
$$

The derivative $f_{xxx}(0, \lambda_0)$ can be expressed in terms of the derivative $r_z(0, \lambda_0)$:

$$
\begin{aligned}
f(x, \lambda) &= x r(x^2, \lambda), \\
f_x(x, \lambda) &= r(x^2, \lambda) + 2x^2 r_z(x^2, \lambda), \\
f_{xx}(x, \lambda) &= 2x r_z(x^2, \lambda) + 4x r_z(x^2, \lambda) + 4x^3 r_{zz}(x^2, \lambda), \\
f_{xxx}(0, \lambda_0) &= 6 r_z(0, \lambda_0).
\end{aligned}
$$

We need to express the derivative $r_\lambda(0, \lambda_0)$ in terms of $\mu_\lambda(0, \lambda_0)$. If we differentiate the defining equation for μ with respect to λ and we evaluate the result in $(0, \lambda_0)$, then we get

$$
\begin{aligned}
(\mathrm{d}^2\Phi)_{u_0,\lambda_0,0}(W_\lambda + W_\tau \cdot \tau_\lambda, v_0) &+ (\mathrm{d}\Phi_\lambda)_{u_0,\lambda_0,0}(v_0) + (\mathrm{d}\Phi_\tau)_{u_0,\lambda_0,0}(v_0) \cdot \tau_\lambda \\
&+ (\mathrm{d}\Phi)_{u_0,\lambda_0,0}\,(\mu_\lambda(0, \lambda_0) s v_0(s) + w_\lambda(0, \lambda_0)) \\
&+ \eta_\lambda(0, \lambda_0)\Phi_\tau(u_0, \lambda_0, 0) + c_\lambda(0, \lambda_0)s = 0.
\end{aligned}
$$

Since all summands except the last one are $2p$-periodic, we can conclude $c_\lambda(0, \lambda_0) = 0$. Multiplication by the function v_2 gives us the identity $r_\lambda(0, \lambda_0) = -\mu_\lambda(0, \lambda_0)\langle v_2, L(s v_0(s))\rangle$. The following equality can be deduced

$$
\widetilde{\mu}_{xx}(0) = -\frac{6 r_z(0, \lambda_0)}{\langle v_2, L(s v_0(s))\rangle} + 2\frac{r_z(0, \lambda_0)}{\langle v_2, L(s v_0(s))\rangle} = -\frac{4 r_z(0, \lambda_0)}{\langle v_2, L(s v_0(s))\rangle}.
$$

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