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Oseledets Splitting and Invariant Manifolds on Fields of Banach Spaces

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Abstract
We prove a semi-invertible Oseledets theorem for cocycles acting on measurable fields of Banach spaces, i.e. we only assume invertibility of the base, not of the operator. As an application, we prove an invariant manifold theorem for nonlinear cocycles acting on measurable fields of Banach spaces.

Keywords Semi-invertible multiplicative ergodic theorem · Oseledets splitting · Fields of Banach spaces · Invariant manifolds

Mathematics Subject Classification 37H15 · 37L55 · 37B55

Introduction

The multiplicative ergodic theorem (MET) is a powerful tool with various applications in different fields of mathematics, including analysis, probability theory, and geometry, and a cornerstone in smooth ergodic theory. It was first proved by Oseledets [18] for matrix cocycles. Since then, the theorem attracted many researchers to provide new proofs and formulations with increasing generality [2,6,11,15,17,19–23].

In [12], the authors gave a proof for an MET for cocycles acting on *measurable fields of Banach spaces*. Let us quickly recall the setting here: If $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space, we call a family of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$ a measurable field if there exists a linear subspace Δ of all sections $\Pi_{\omega\in\Omega} E_\omega$ and a countable subset $\Delta_0 \subset \Delta$ such that $\{g(\omega) : g \in \Delta_0\}$ is dense in E_ω for every $\omega \in \Omega$ and $\omega \mapsto \|g(\omega)\|_{E_\omega}$ is measurable for every $g \in \Delta$. Note that this definition implies that every Banach space E_ω is separable. On the other hand, every separable Banach space defines a field of Banach spaces by simply setting $E_\omega = E$. This

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structure is similar to a measurable version of a Banach bundle with base Ω and total space $\Pi_{\omega\in\Omega}E_\omega$ in which every space E_ω is a fiber. However, the fundamental difference is that we do *not* put any measurable (or topological) structure on the bundle $\Pi_{\omega\in\Omega}E_\omega$ itself! In fact, the existence of the set Δ is a substitute for the measurable structure and will help to prove measurability for functionals defined on $\Pi_{\omega\in\Omega}E_\omega$ as we will see many times in this work. If $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is a measure preserving dynamical systems, a *cocycle* acting on the field $\{E_\omega\}_{\omega\in\Omega}$ consists of a family of maps $\varphi_\omega\colon E_\omega\to E_{\theta\omega}$. Setting $\varphi^n_\omega := \varphi_{\theta^{n-1}\omega}\circ\cdots\circ\varphi_\omega$, we furthermore claim that $\omega\mapsto\|\varphi^n_\omega(g(\omega))\|_{E_{\theta^n\omega}}$ is measurable for every $g\in\Delta$ and every $n\in\mathbb{N}$.

There are numerous examples in which it is natural to study cocycles on random spaces. In [12], our motivation was to study dynamical properties of singular stochastic delay differential equations in which the spaces E_ω are (essentially) spaces of controlled Brownian paths known in rough paths theory [8]. In the finite dimensional case, linearizing a C^1-cocycle on a manifold yields a linear cocycle acting on the tangent bundle [1, Chapter 4.2]. In the context of stochastic partial differential equations (SPDE), cocycles on random metric spaces were studied, for instance, when uniqueness of the equation is unknown and one has to work with a measurable selection instead, cf. [9] in the case of the 3D stochastic Navier–Stokes equation. Other examples in the situation of SPDE can be found in [3,4]. In the deterministic case, a similar structure appears when studying the flow on time-dependent domains [14]. More recently, scales of time-dependent Banach spaces where introduced to study dynamical properties of non-autonomous PDEs in [5,7].

We will now restate the MET [12, Theorem 4.17] in a slightly simplified version.

Theorem 0.1 *Let $(\Omega,\mathcal{F},\mathbb{P},\theta)$ be an ergodic measurable metric dynamical system and φ be a compact linear cocycle acting on a measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$. For $\mu\in\mathbb{R}\cup\{-\infty\}$ and $\omega\in\Omega$, define*

$$F_\mu(\omega) := \Big\{x\in E_\omega \,:\, \limsup_{n\to\infty}\frac{1}{n}\log\|\varphi^n_\omega(x)\|\le\mu\Big\}.$$

Assume that

$$\log^+\|\varphi_\omega\|\in L^1(\Omega).$$

Then there is a measurable forward invariant set $\tilde{\Omega}\subset\Omega$ of full measure and a decreasing sequence $\{\mu_i\}_{i\ge1}$, $\mu_i\in[-\infty,\infty)$ with the properties that $\lim_{n\to\infty}\mu_n=-\infty$ and either $\mu_i>\mu_{i+1}$ or $\mu_i=\mu_{i+1}=-\infty$ such that for every $\omega\in\tilde{\Omega}$,

$$x\in F_{\mu_i}(\omega)\backslash F_{\mu_{i+1}}(\omega) \quad \textit{if and only if} \quad \lim_{n\to\infty}\frac{1}{n}\log\|\varphi^n_\omega(x)\|=\mu_i. \tag{0.1}$$

Moreover, there are numbers $m_1,m_2,\ldots$ such that $\operatorname{codim} F_{\mu_j}(\omega)=m_1+\ldots+m_{j-1}$ *for every $\omega\in\tilde{\Omega}$.*

Let us mention here that, motivated by our example of a stochastic delay equation, we proved this theorem for compact cocycles only, but it should be straightforward to generalize it to the quasi-compact case as Thieullen did in [22]. Consequently, we believe that all our results in this work will hold for quasi-compact cocycles, too.

The numbers $\{\mu_i\}$ are the *Lyapunov exponents*, the subspaces $F_\mu(\omega)$ are sometimes called *slow-growing subspaces* and the resulting filtration

$$E_\omega = F_{\mu_1}(\omega)\supset F_{\mu_2}(\omega)\supset\cdots$$

is called *Oseledets filtration*. Is is easily seen that the slow-growing spaces are *equivariant*, meaning that $\varphi_\omega(F_{\mu_i}(\omega)) \subset F_{\mu_i}(\theta\omega)$. In the proof of this theorem, no invertibility of θ or φ is assumed, in which case a filtration of slow-growing subspaces is the best one can hope for. However, things change when we assume that the base θ is invertible. In this case, it is possible to deduce a *splitting* of the spaces E_ω consisting of *fast-growing subspaces* which are invariant under φ. Such a splitting is called *Oseledets splitting*, and the corresponding theorem is called *semi-invertible MET*. Let us emphasize that we only need to assume invertibility of the base θ and no invertibility of the cocyle φ. In the context of SPDE or stochastic delay equations, these assumptions are quite natural: θ usually denotes the shift of a random trajectory (which can be shifted forward and backward in time) and the cocycle denotes the solution map, which is not injective if the equation can be solved forward in time only.

Our first main result is a semi-invertible MET on a measurable field of Banach spaces. We state a simplified version here, the full statement can be found in Theorem 1.21 below.

Theorem 0.2 *In addition to the assumptions made in Theorem 0.1, assume that θ is invertible with measurable inverse $\sigma := \theta^{-1}$ and that Assumption 1.1 holds. Then there is a θ-invariant set $\tilde{\Omega}$ of full measure such that for every $i \geq 1$ with $\mu_i > \mu_{i+1}$ and $\omega \in \tilde{\Omega}$, there is an m_i-dimensional subspace H^i_ω with the following properties:*

(i) (Invariance) $\varphi^k_\omega(H^i_\omega) = H^i_{\theta^k\omega}$ *for every* $k \geq 0$.

(ii) (Splitting) $H^i_\omega \oplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega)$. *In particular,*

$$E_\omega = H^1_\omega \oplus \cdots \oplus H^i_\omega \oplus F_{\mu_{i+1}}(\omega).$$

(iii) ('Fast-growing' subspace) For each $h_\omega \in H^i_\omega \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\varphi^n_\omega(h_\omega)\| = \mu_j$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\varphi^n_{\sigma^n\omega})^{-1}(h_\omega)\| = -\mu_j.$$

Moreover, the spaces are uniquely determined by properties (i), (ii) and (iii).

Clearly, the Oseledets splitting provides much more information about the cocycle than the filtration.

Let us discuss some important preceeding results. In the finite dimensional case, an MET for cocycles acting on measurable bundles can be found in the monograph [1, 4.2.6 Theorem] by L. Arnold. In [17], Mañé proved an MET with Oseledets splitting on a Banach bundle, assuming a topological structure on Ω and continuity of the map $\omega \mapsto \varphi_\omega$. He also assumed injectivity of φ. Besides these results, we are not aware of any METs for cocycles acting on a bundle-type structure. Lian and Lu [15] proved an MET for cocycles acting on a fixed Banach space, assuming only a measurable structure on Ω, but injectivity of the cocycle. This assumption was later removed by Doan in [6] without giving an Oseledets splitting, however. In [10], González-Tokman and Quas used this result as a "black-box" and proved that an Oseledets splitting holds in this case, too.

Let us mention that our result is not only the first which provides a splitting on a bundle structure of Banach spaces without using a topological structure on Ω, it also weakens the measurability assumption on φ significantly in case we are dealing with a single Banach space E only. In fact, the standard measurability assumption, for instance in [11], is *strong* *measurability* of φ, meaning that for fixed $x \in E$, the map

$$\Omega \ni \omega \mapsto \varphi_\omega(x) \in E \tag{0.2}$$

should be measurable. In contrast, our assumption means that the maps

$$\Omega \ni \omega \mapsto \|\varphi_\omega^{k+n}(x) - \varphi_{\theta^n\omega}^k(\tilde{x})\|_E \in \mathbb{R}$$

should be measurable for every $n, k \in \mathbb{N}_0$ and $x, \tilde{x} \in S$ where S is a countable and dense subset of E. This assumption is clearly implied by (0.2).

The proof of Theorem 0.2 pushes forward the volume growth-approach advocated by Blumenthal [2] and González-Tokman, Quas [11] which provides a clear growth interpretation of the Lyapunov exponents. In a way, our result complements these two works in case of a single Banach space E. In particular, we are not imposing any further assumptions on E like reflexivity or separability of the dual as in [11].

A typical application for an MET is the construction of stable and unstable manifolds, cf. [17,20,21]. Here, the existence of the Oseledets splitting is crucial. Our second main contribution is an invariant manifold theorem for nonlinear cocycles acting on fields of Banach spaces. We state an informal version here, the precise statements are formulated in Theorems 2.10 and 2.17.

Theorem 0.3 *Let φ be a nonlinear, differentiable cocycle acting on a measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$. Assume that Y_ω is a random fixed point of φ, in particular $\varphi_\omega(Y_\omega) = Y_{\theta\omega}$. Then, under the same measurability and integrability assumptions as in Theorem 0.2, the linearized cocycle $D_{Y_\omega}\varphi_\omega$ has a Lyapunov spectrum $\{\mu_n\}_{n\geq 1}$. Under further assumptions on φ and Y, there is a θ-invariant set $\tilde{\Omega}$ of full measure, closed subspaces S_ω and U_ω of E_ω and immersed submanifolds $S_{loc}(\omega)$ and $U_{loc}(\omega)$ of E_ω such that for every $\omega \in \tilde{\Omega}$,*

$$T_{Y(\omega)}S_{loc}(\omega) = S_\omega \quad \text{and} \quad T_{Y(\omega)}U_{loc}(\omega) = U_\omega$$

and the properties that for every $Z_\omega \in S_{loc}(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| \leq \mu_{j_0} < 0$$

and for every $Z_\omega \in U_{loc}(\omega)$ one has $\varphi_{\sigma^n\omega}^n(Z_{\sigma^n\omega}) = Z_\omega$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Z_{\sigma^n\omega} - Y_{\sigma^n\omega}\| \leq -\mu_{k_0} < 0.$$

Here we have set $\mu_{j_0} = \max\{\mu_j : \mu_j < 0\}$ and $\mu_{k_0} = \min\{\mu_k : \mu_k > 0\}$. In the hyperbolic case, i.e. if all Lyapunov exponents are non-zero, the submanifolds $S_{loc}^\upsilon(\omega)$ and $U_{loc}^\upsilon(\omega)$ are transversal, i.e.

$$E_\omega = T_{Y_\omega}U_{loc}^\upsilon(\omega) \oplus T_{Y_\omega}S_{loc}^\upsilon(\omega).$$

The structure of the paper is as follows. In Sect. 1, we prove a semi-invertible MET for cocycles acting on measurable fields of Banach spaces. This result is applied in Sect. 2 to deduce the existence of local stable and unstable manifolds for nonlinear cocycles.

Notation

– For Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, $L(X, Y)$ denotes the space of bounded linear functions from X to Y equipped with usual operator norm. We will often not explicitly write a subindex for Banach space norms and use the symbol $\|\cdot\|$ instead. *Differentiability* of a function $f\colon X \to Y$ will always mean Fréchet-differentiability. A C^m *function* denotes an m-times Fréchet-differentiable function. If $A, B \subseteq X$, we denote by $d(A, B) := \inf_{a\in A, b\in B} \|a-b\|$ the distance between two sets A and B. We also set $d(x, B) := d(B, x) := d(\{x\}, B)$ for $x \in X$, $B \subseteq X$.

– Let X, Y be Banach spaces. For $x_1, \dots, x_k \in X$, set

$$\mathrm{Vol}(x_1, x_2, \dots, x_k) := \|x_1\| \prod_{i=2}^{k} d(x_i, \langle x_j \rangle_{1\le j<i}). \tag{0.3}$$

For a given bounded linear function $T : X \to Y$ and $k \ge 1$, we define

$$D_k(T) := \sup_{\|x_i\|=1; i=1,\dots,k} \mathrm{Vol}\big(T(x_1), T(x_2), \dots, T(x_k)\big).$$

– Let E be a vector space. If we can write E as a direct sum $E = F \oplus H$ of vector spaces, we have an *algebraic splitting*. We also say that F *is a complement of* H and vice versa. The projection operator $\Pi_{F\|H}(e) = f$ with $e = f + h$, $f \in F$, $h \in H$, is called the *projection operator onto* F *parallel to* H. If E is a normed space and $\Pi_{F\|H}$ is bounded linear, i.e.

$$\|\Pi_{F\|H}\| = \sup_{f\in F, h\in H, f+h\neq 0} \frac{\|f\|}{\|f+h\|} < \infty,$$

we call $E = F \oplus H$ a *topological splitting*. For normed spaces, a *splitting* will always mean a topological splitting.

– Let $(\Omega, \mathcal{F})$ be a measurable space. We call a family of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$ *a measurable field of Banach spaces* if there is a set of sections

$$\Delta \subset \prod_{\omega\in\Omega} E_\omega$$

with the following properties:

(i) Δ is a linear subspace of $\prod_{\omega\in\Omega} E_\omega$.
(ii) There is a countable subset $\Delta_0 \subset \Delta$ such that for every $\omega \in \Omega$, the set $\{g(\omega) : g \in \Delta_0\}$ is dense in E_ω.
(iii) For every $g \in \Delta$, the map $\omega \mapsto \|g(\omega)\|_{E_\omega}$ is measurable.

– Let $(\Omega, \mathcal{F})$ be a measurable space. If there exists a measurable map $\theta\colon \Omega \to \Omega, \omega \mapsto \theta\omega$, with a measurable inverse θ^{-1}, we call $(\Omega, \mathcal{F}, \theta)$ a *measurable dynamical system*. We will use the notation $\theta^n\omega$ for n-times applying θ to an element $\omega \in \Omega$. We also set $\theta^0 := \mathrm{Id}_\Omega$ and $\theta^{-n} := (\theta^n)^{-1}$. If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ that is invariant under θ, i.e. $\mathbb{P}(\theta^{-1}A) = \mathbb{P}(A) = \mathbb{P}(\theta A)$ for every $A \in \mathcal{F}$, we call the tuple $\big(\Omega, \mathcal{F}, \mathbb{P}, \theta\big)$ a *measure-preserving dynamical system*. The system is called *ergodic* if every θ-invariant set has probability 0 or 1.

– Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a measure-preserving dynamical system and $(\{E_\omega\}_{\omega\in\Omega}, \Delta)$ a measurable field of Banach spaces. A *continuous cocycle on* $\{E_\omega\}_{\omega\in\Omega}$ consists of a family of continuous maps

$$\varphi_\omega\colon E_\omega \to E_{\theta\omega}. \tag{0.4}$$

If φ is a continuous cocycle, we define $\varphi^n_\omega\colon E_\omega \to E_{\theta^n\omega}$ as

$$\varphi^n_\omega := \varphi_{\theta^{n-1}\omega} \circ \cdots \circ \varphi_\omega.$$

We also set $\varphi_\omega^0 := \mathrm{Id}_{E_\omega}$. We say that φ acts on $\{E_\omega\}_{\omega\in\Omega}$ if the maps

$$\omega \mapsto \|\varphi(n, \omega, g(\omega))\|_{E_{\theta^n\omega}}, \quad n \in \mathbb{N}$$

are measurable for every $g \in \Delta$. In this case, we will speak of a *continuous random dynamical system on a field of Banach spaces*. If the map (0.4) is bounded linear/compact, we call φ a bounded linear/compact cocycle.

1 Semi-invertible MET on Fields of Banach Spaces

In this section, $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ will denote an ergodic measure-preserving dynamical system and we set $\sigma := \theta^{-1}$. Let $(\{E_\omega)_{\omega\in\Omega}, \Delta, \Delta_0)$ be a measurable field of Banach space and let $\psi_\omega\colon E_\omega \to E_{\theta\omega}$ be a compact linear cocycle acting on it. In the sequel, we will furthermore assume that the following assumption is satisfied:

Assumption 1.1 For each $g, \tilde{g} \in \Delta$ and $n, k \geq 0$,

$$\omega \to \|\psi^k_{\theta^n\omega}[\psi^n_\omega(g(\omega)) - \tilde{g}(\theta^n\omega)]\|_{E_{\theta^{n+k}\omega}}$$

is measurable.

We will always assume that

$$\log^+ \|\psi_\omega\| \in L^1(\Omega).$$

Under this condition, the Multiplicative Ergodic Theorem [12, Theorem 4.17] applies and yields the existence of Lyapunov exponents $\{\mu_1 > \mu_2 > \ldots\} \subset [-\infty, \infty)$ on a θ-invariant set of full measure $\tilde{\Omega} \subset \Omega$. More precisely, there are numbers $\Lambda_k \in [-\infty, \infty)$ such that

$$\Lambda_k = \lim_{n\to\infty} \frac{1}{n} \log D_k\big(\psi^n_\omega\big), \quad k \geq 1$$

for every $\omega \in \tilde{\Omega}$. Setting $\lambda_k = \Lambda_k - \Lambda_{k-1}$, the sequence (μ_k) is the subsequence of (λ_k) defined by removing all multiple elements. For any $\mu \in [-\infty, \infty)$, we define the closed subspace

$$F_\mu(\omega) = \left\{\xi \in E_\omega \mid \limsup_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(\xi)\| \leq \mu\right\}.$$

Note that ψ is invariant on these spaces in the sense that

$$\psi^n_\omega|_{F_\mu(\omega)}\colon F_\mu(\omega) \to F_\mu(\theta^n\omega).$$

We also saw in [12, Theorem 4.17] that there are numbers $m_i \in \mathbb{N}$ such that $m_i = \dim\big(F_{\mu_i}(\omega)/F_{\mu_{i+1}}(\omega)\big)$ for every $\omega \in \tilde{\Omega}$.

If not otherwise stated, $\tilde{\Omega} \subset \Omega$ will always denote a θ-invariant set of full measure. Note that we can always assume w.l.o.g. that a given set of full measure $\Omega_0 \subset \Omega$ is θ-invariant, otherwise we can consider

$$\bigcap_{k\in\mathbb{Z}} \theta^k(\Omega_0)$$

instead.

Next, we collect some basic Lemmas. Recall the definition of Vol and D_k.

Lemma 1.2 *Let X, Y be Banach spaces and $T : X \to Y$ a linear operator. For $k \in \mathbb{N}$, there exist positive constants c_k, C_k depending only on k such that*

$$c_k D_k(T) \le D_k(T^*) \le C_k D_k(T) \tag{1.1}$$

where by $T^* : Y^* \to X^*$ we mean the dual map of T.

Proof [11, Lemma 3]. □

Lemma 1.3 *For a Banach space X and $k \ge 1$, the map*

$$\begin{aligned} \mathrm{Vol} : X^k &\longrightarrow \mathbb{R} \\ (x_1, x_2, \ldots, x_k) &\mapsto \|x_1\| \prod_{i=2}^{k} d(x_i, \langle x_j \rangle_{1 \le j < i}) \end{aligned} \tag{1.2}$$

is continuous.

Proof [15, Lemma 4.2]. □

Lemma 1.4 *For every $g \in \Delta$ and $j \ge 1$, the map*

$$\omega \mapsto d\big(g(\omega), F_{\mu_j}(\omega)\big)$$

is measurable.

Proof As in the proof to [12, Lemma 4.3]. □

For a Banach space X and a closed subspace $U \subset X$, the quotient space X/U is again a Banach space with norm

$$\|[x]\|_{X/U} = \inf_{u \in U} \|x - u\|.$$

For an element $x \in E_\omega$, we denote by $[x]_\mu$ its equivalence class in the quotient space $E_\omega/F_\mu(\omega)$. From the invariance property of ψ, the map

$$[\psi_\omega^n]_{\mu_{j+1}} : \frac{F_{\mu_j}(\omega)}{F_{\mu_{j+1}}(\omega)} \longrightarrow \frac{F_{\mu_j}(\theta^n\omega)}{F_{\mu_{j+1}}(\theta^n\omega)}, \quad [\psi_\omega^n]_{\mu_{j+1}}([x]) := [\psi_\omega^n(x)]_{\mu_{j+1}}$$

is well-defined for every $j \ge 1$ and $n \in \mathbb{N}$. Note also that $[\psi_\omega^n]_{\mu_{j+1}}$ is bijective for $\omega \in \tilde{\Omega}$. Indeed, injectivity is straightforward and surjectivity follows from the fact that $F_{\mu_j}(\omega)/F_{\mu_{j+1}}(\omega)$ and $F_{\mu_j}(\theta^n\omega)/F_{\mu_{j+1}}(\theta^n\omega)$ are finite-dimensional with the same dimension m_i.

Lemma 1.5 *For $m, n \in \mathbb{N}$, the maps*

$$f_1(\omega) := D_m(\psi_\omega^n |_{F_{\mu_2}(\omega)}) \quad \textit{and} \quad f_2(\omega) := D_m([\psi_\omega^n]_{\mu_2})$$

are measurable.

Proof It is not hard to see that

$$f_1(\omega) = \lim_{l \to \infty} \liminf_{k \to \infty} \left[\sup_{\{\xi_\omega^t\}_{1 \le t \le m} \subset B_\omega^{l,k}(\mu_2)} \mathrm{Vol}\left(\psi_\omega^n(\xi_\omega^1), \ldots, \psi_\omega^n(\xi_\omega^m)\right) \right] \tag{1.3}$$

where

$$B_\omega^{l,k}(\mu_2) = \Big\{\xi \in F_{\mu_1}(\omega) : \ \|\xi\| = 1, \|\psi_\omega^k(\xi)\| < \exp\big(k(\mu_2 + \frac{1}{l})\big)\Big\},$$

cf. the proof of [12, Lemma 4.3]. Let $\{g_t\}_{1\le t\le m} \subset \Delta_0$ and $C(g_t) := \{\omega : g_t(\omega) \in B_\omega^{l,k}(\mu_2)\}$. As a consequence of Lemma 1.4, these sets are measurable and we have

$$\sup_{\{\xi_\omega^t\}_{1\le t\le m}\subset B_\omega^{l,k}(\mu_2)} \operatorname{Vol}\big(\psi_\omega^n(\xi_\omega^1), \ldots, \psi_\omega^n(\xi_\omega^m)\big) = \sup_{\{g_t\}_{1\le t\le m}\subset\Delta_0} \operatorname{Vol}\left(\psi_\omega^n\big(\frac{g_1(\omega)}{\|g_1(\omega)\|}\big), \ldots, \psi_\omega^n\big(\frac{g_m(\omega)}{\|g_m(\omega)\|}\big)\right) \prod_{1\le t\le m} \chi_{C(g_t)}(\omega)$$

which implies measurability of f_1. For f_2, note first that

$$f_2(\omega) = \lim_{l\to\infty} \liminf_{k\to\infty} \left[\sup_{\{\xi_\omega^t\}_{1\le t\le m}\subset F_{\mu_1}(\omega)} \frac{\operatorname{Vol}\big([\psi_\omega^n(\xi_\omega^1)]_{\mu_2}, \ldots, [\psi_\omega^n(\xi_\omega^m)]_{\mu_2}\big)}{\prod_{1\le t\le m} \|[\xi_\omega^t]_{\mu_2}\|} \right]$$

where we set $\frac{0}{0} := 0$. Again as before

$$\sup_{\{\xi_\omega^t\}_{1\le t\le m}\subset F_{\mu_1}(\omega)} \frac{\operatorname{Vol}\big([\psi_\omega^n(\xi_\omega^1)]_{\mu_2}, \ldots, [\psi_\omega^n(\xi_\omega^m)]_{\mu_2}\big)}{\prod_{1\le t\le m} \|[\xi_\omega^t]_{\mu_2}\|} = \sup_{\{g_t\}_{1\le t\le m}\subset\Delta_0} \frac{\operatorname{Vol}\big([\psi_\omega^n(g_1(\omega))]_{\mu_2}, \ldots, [\psi_\omega^n(g_k(\omega))]_{\mu_2}\big)}{\prod_{1\le t\le m} d\big(g_t(\omega), F_{\mu_2}(\omega)\big)}.$$

It remains to show that for $g \in \Delta$, $d\big(\psi_\omega^n\big(g(\omega)\big), F_{\mu_2}(\theta^n\omega)\big)$ is measurable, which can be achieved using Assumption 1.1 with a proof similar to Lemma 1.4. □

Lemma 1.6 *For every $i \ge 0$, there is a constant $M_i > 0$ such that*

$$\|[\psi_\omega^1]_{\mu_{i+1}}\| < M_i \|\psi_\omega^1\|$$

for every $\omega \in \tilde{\Omega}$.

Proof Since $\dim[\frac{F_{\mu_i}(\omega)}{F_{\mu_{i+1}}(\omega)}] = m_i$, we can choose $H_\omega \subset F_{\mu_i}(\omega)$ such that

$$H_\omega \oplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega) \quad \text{and} \quad \|\Pi_{H_\omega || F_{\mu_{i+1}}(\omega)}\| \le \sqrt{m_i} + 2 =: M_i, \tag{1.4}$$

cf. [2, Lemma 2.3]. Let $\xi_\omega \in F_{\mu_i}(\omega) \backslash F_{\mu_{i+1}}(\omega)$ with corresponding decomposition $\xi_\omega = h_\omega + f_\omega \in H_\omega \oplus F_{\mu_{i+1}}(\omega)$. From (1.4), we know that $\frac{\|h_\omega\|}{\|[\xi_\omega]_{\mu_{i+1}}\|} \le M_i$ and consequently

$$\frac{\|[\psi_\omega^1(\xi_\omega)]_{\mu_{i+1}}\|}{\|[\xi_\omega]_{\mu_{i+1}}\|} \le M_i \frac{\|[\psi_\omega^1(h_\omega)]_{\mu_{i+1}}\|}{\|h_\omega\|} \le M_i \frac{\|\psi_\omega^1(h_\omega)\|}{\|h_\omega\|} \le M_i \|\psi_\omega^1\|.$$

The claim follows. □

Lemma 1.7 *Assume that $\{f_n(\omega)\}_{n\ge1}$ is a subadditive sequence with respect to θ and set $g_n(\omega) := f_n(\sigma^n\omega)$. Assume $f_1^+(\omega) \in L^1(\Omega)$. Then there is a θ-invariant set $\tilde{\Omega} \in \mathcal{F}$ with full measure such that for every $\omega \in \tilde{\Omega}$,*

$$\lim_{n\to\infty} \frac{1}{n} f_n(\omega) = \lim_{n\to\infty} \frac{1}{n} g_n(\omega) \in [-\infty, \infty)$$

where the limit does not depend on ω.

Proof We can easily check that $\{g_n(\omega)\}_{n\geq 1}$ is a subadditive sequence with respect to σ. Since $f_n(\omega)$ and $g_n(\omega)$ have same law, the result follows from Kingman's Subadditive Ergodic Theorem. □

As a consequence, we obtain the following:

Lemma 1.8 *There is a θ-invariant set of full measure $\tilde{\Omega} \in \mathcal{F}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log D_k\big(\psi^n_\omega\big) = \lim_{n\to\infty} \frac{1}{n} \log D_k\big(\psi^n_{\sigma^n\omega}\big) = \lim_{n\to\infty} \frac{1}{n} \log D_k\big((\psi^n_{\sigma^n\omega})^*\big) = \Lambda_k \tag{1.5}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log D_k\big(\psi^n_\omega\ |_{F_{\mu_2}(\omega)}\ \big) = \lim_{n\to\infty} \frac{1}{n} \log D_k\big(\psi^n_{\sigma^n\omega}\ |_{F_{\mu_2}(\sigma^n\omega)}\ \big)$$
$$= \lim_{n\to\infty} \frac{1}{n} \log D_k\big((\psi^n_{\sigma^n\omega})^*\ |_{\left(F_{\mu_2}(\sigma^n\omega)\right)^*}\ \big] = \Lambda_{k+m_1} - \Lambda_{m_1} \tag{1.6}$$

Proof We already noted that $\lim_{n\to\infty} \frac{1}{n} \log D_k\big(\psi^n_\omega\big) = \Lambda_k$. The equality

$$\lim_{n\to\infty} \frac{1}{n} \log D_k\big(\psi^n_\omega\ |_{F_{\mu_2}(\omega)}\ \big) = \Lambda_{k+m_1} - \Lambda_{m_1} \tag{1.7}$$

was a partial result in the proof of Theorem [12, Theorem4.17]. The remaining inequalities follow by a combination of all Lemmas 1.2–1.7. □

From now on, we will assume that $\tilde{\Omega}$ is the set provided in Lemma 1.8.

Lemma 1.9 *Fix $\omega \in \tilde{\Omega}$ and let $(\xi_{\sigma^n\omega})_n$ be a sequence such that $\xi_{\sigma^n\omega} \in F_{\mu_1}(\sigma^n\omega)\backslash F_{\mu_2}(\sigma^n\omega)$ and $\|[\xi_{\sigma^n\omega}]_{\mu_2}\| = 1$ for every $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \|[\psi^n_{\sigma^n\omega}(\xi_{\sigma^n\omega})]_{\mu_2}\| = \mu_1. \tag{1.8}$$

Proof By applying Lemmas 1.5, 1.6 and 1.7, Kingman's Subadditive Ergodic Theorem shows that

$$\lim_{n\to\infty} \frac{1}{n} \log D_k\big(\big[\psi^n_\omega\big]_{\mu_2}\big) = \lim_{n\to\infty} \frac{1}{n} \log D_k\big(\big[\psi^n_{\sigma^n\omega}\big]_{\mu_2}\big)$$

exist for every $k \geq 1$. Let H_ω be a complement subspace for $F_{\mu_2}(\omega)$ in $F_{\mu_1}(\omega)$. Using a slight generalization of [12, Lemma 4.4], we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{\psi^n_\omega(H_\omega)||F_{\mu_2}(\theta^n\omega)}\| = 0.$$

For $\xi_\omega \in F_{\mu_1}(\omega)\backslash F_{\mu_2}(\omega)$, since

$$\frac{\|\psi^n_\omega(\Pi_{H_\omega||F_{\mu_2}(\omega)}(\xi_\omega))\|}{\|[\psi^n_\omega(\xi_\omega)]_{\mu_2}\|} \leq \|\Pi_{\psi^n_\omega(H_\omega)||F_{\mu_2}(\theta^n\omega)}\|$$

it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log \|[\psi^n_\omega(\xi_\omega)]_{\mu_2}\| = \mu_1. \tag{1.9}$$

Let

$$k := \max\big\{m : \lim_{n\to\infty} \frac{1}{n} \log D_m\big(\big[\psi^n_\omega\big]_{\mu_2}\big) = m\mu_1\big\}.$$

We claim $k = m_1$. Indeed, otherwise from [12, Proposition4.15], there exists a subspace $F_\omega \subset \frac{F_{\mu_1}(\omega)}{F_{\mu_2}(\omega)}$ with codimension k such that for every $\xi_\omega \in F_\omega$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|[\psi_\omega^n(\xi_\omega)]_{\mu_2}\| < \mu_1.$$

Since $\dim[\frac{F_{\mu_1}(\omega)}{F_{\mu_2}(\omega)}] = m_1$, we can find a non-zero element in F_ω which contradicts (1.9). Hence we have shown that

$$\lim_{n\to\infty} \frac{1}{n} \log D_{m_1}\big(\big[\psi_\omega^n\big]_{\mu_2}\big) = m_1\mu_1.$$

Therefore, for every $n \in \mathbb{N}$, we can find $\{\xi^j_{\sigma^n\omega}\}_{1\leq j\leq m_1} \subset F_{\mu_1}(\sigma^n\omega)$ such that $\|[\xi^j_\omega]_{\mu_2}\| = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Vol}\big([\psi^n_{\sigma^n\omega}(\xi^1_{\sigma^n\omega})]_{\mu_2}, \ldots, [\psi^n_{\sigma^n\omega}(\xi^{m_1}_{\sigma^n\omega})]_{\mu_2})\big] = m_1\mu_1. \tag{1.10}$$

Using the definition of Vol, it follows that for every $2 \leq t \leq m_1$,

$$\lim_{n\to\infty} \frac{1}{n} \log d\big([\psi^n_{\sigma^n\omega}(\xi^t_\omega)]_{\mu_2}, \langle[\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})]_{\mu_2}\rangle_{1\leq j\leq t-1}\big) = \mu_1. \tag{1.11}$$

We have $\xi_{\sigma^n\omega} = \sum_{1\leq j\leq m_1} \alpha_j \xi^j_{\sigma^n\omega} \mod F_{\mu_2}(\sigma^n\omega)$. In the proof of [12, Lemma 4.7], we already saw that the the Vol-function is symmetric up to a constant. By our assumption on $\xi_{\sigma^n\omega}$, we can therefore assume that $\alpha_{m_1} \geq \frac{1}{m_1}$. Finally from (1.11)

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log \|[\psi^n_{\sigma^n\omega}(\xi_{\sigma^n\omega})]_{\mu_2}\| \\
&\quad= \lim_{n\to\infty} \frac{1}{n}\big[d\big([\psi(\xi^{m_i}_{\sigma^n\omega})]_{\mu_2}, \langle[\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})]_{\mu_2}\rangle_{1\leq j\leq m_1-1}\big) = \mu_1.
\end{aligned}$$

□

Definition 1.10 Let X be a Banach space. We define $G(X)$ to be the Grassmanian of closed subspaces of X equipped with the Hausdorff distance

$$d_H(A, B) := \max\{\sup_{a\in S_A} d(a, S_B), \sup_{b\in S_B} d(b, S_A)\}$$

where $S_A = \{a \in A \ : \ \|a\| = 1\}$. Set

$$G_k(X) = \{A \in G(X) \ : \ \dim[A] = k\} \quad \text{and} \quad G^k(X) = \{A \in G(X) \ : \ \dim[X/A] = k\}.$$

It can be shown that $(G(X), d_H)$ is a complete metric space and that $G_k(X)$ and $G^k(X)$ are closed subsets [13, Chapter IV]. The following lemma will be useful.

Lemma 1.11 *For $A, B \in G(X)$ set*

$$\delta(A, B) := \sup_{a\in S_A} d(a, B).$$

Then the following holds:

(i) $d_H(A, B) \leq 2\max\{\delta(A, B), \delta(B, A)\}$.
(ii) If $A, B \in G_k(X)$ with $d(A, B) < \frac{1}{k}$ for some $k \in \mathbb{N}$, we have

$$\delta(B, A) \leq \frac{k\delta(A, B)}{1 - k\delta(A, B)}.$$

Proof [2, Lemma 2.6]. □

Proposition 1.12 *Assume $\mu_1 > -\infty$. Fix $\omega \in \tilde{\Omega}$. For every $n \in \mathbb{Z}$, let $H^n_{\sigma^n\omega} \subset F_{\mu_1}(\sigma^n\omega)$ be a complementary subspace for $F_{\mu_2}(\omega)$ satisfying (1.4). Set $\tilde{H}^n_\omega := \psi^n_{\sigma^n\omega}(H^n_{\sigma^n\omega})$. Then the sequence $\{\tilde{H}^n_\omega\}_{n\geq 1}$ is Cauchy in $\big(G_{m_1}(F_{\mu_1}(\omega)), d_H\big)$.*

Proof From (1.4), we can deduce that for every $n \in \mathbb{N}$ and $\xi_{\sigma^n\omega} \in S_{H^n_{\sigma^n\omega}}$,

$$\frac{1}{M_1} < \|[\xi_{\sigma^n\omega}]_{\mu_2}\| \leq 1. \tag{1.12}$$

Note that $\psi^k_{\sigma^n\omega}|_{H^n_{\sigma^n\omega}}$ is injective for any $k \geq 1$, therefore $\dim(\tilde{H}^n_\omega) = \dim(H^n_{\sigma^n\omega}) = m_1$. Since $\mu_2 < \mu_1$, we know that $\tilde{H}^n_\omega \cap F_{\mu_2}(\omega) = \{0\}$ and since $\dim[\frac{F_{\mu_1}(\omega)}{F_{\mu_2}(\omega)}] = m_1$, we obtain that

$$\tilde{H}^n_\omega \oplus F_{\mu_2}(\omega) = F_{\mu_1}(\omega)$$

for any $n \in \mathbb{N}$. Let $\{\xi^j_{\sigma^n\omega}\}_{1\leq j\leq m_1} \subset S_{F_{\mu_1}(\sigma^n\omega)}$ be a base for $H^n_{\sigma^n\omega}$. Then for $\xi_{\sigma^{n+1}\omega} \in S_{F_{\mu_1}(\sigma^{n+1}\omega)} \cap H^{n+1}_{\sigma^{n+1}\omega}$, there exist $\{\beta_j\}_{1\leq j\leq m_1} \subset \mathbb{R}$ such that

$$Z^n_\omega := \frac{\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|} - \sum_{1\leq j\leq m_1} \beta_j \frac{\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})}{\|\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})\|} \in F_{\mu_2}(\omega).$$

It follows that

$$Y^n_{\sigma^n\omega} := \frac{\psi^1_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|} - \sum_{1\leq j\leq m_1} \beta_j \frac{\xi^j_{\sigma^n\omega}}{\|\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})\|} \in F_{\mu_2}(\sigma^n\omega),$$

thus

$$\begin{aligned}\Big\| \sum_{1\leq j\leq m_1} \beta_j \frac{\xi^j_{\sigma^n\omega}}{\|\psi^n_{\sigma^n\omega}(\xi^j_{\sigma^n\omega})\|}\Big\| &\leq \|\Pi_{H^n_{\sigma^n\omega}||F_{\mu_2}(\sigma^n\omega)}\| \frac{\|\psi^1_{\sigma^{n+1}\omega}\|}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|}\\ &\leq M_1 \frac{\|\psi^1_{\sigma^{n+1}}\|}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|}\end{aligned}$$

and so

$$\begin{aligned} d\bigg(\frac{\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|}, \tilde{H}^n_\omega\bigg) &\leq \|Z^n_\omega\| = \|\psi^n_{\sigma^n\omega}(Y^n_{\sigma^n\omega})\|\\ &\leq (M_1+1)\frac{\|\psi^n_{\sigma^n\omega}|_{F_{\mu_2}(\sigma^n\omega)}\|\|\psi^1_{\sigma^{n+1}\omega}\|}{\|\psi^{n+1}_{\sigma^{n+1}\omega}(\xi_{\sigma^{n+1}\omega})\|}. \end{aligned} \tag{1.13}$$

Note that $\lim_{n\to\infty}\frac{1}{n}\log\|\psi^1_{\sigma^n\omega}\| = 0$ from Birkhoff's Ergodic Theorem. Using Lemma 1.7 and (1.7) for $k = 1$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\|\psi^n_{\sigma^n\omega}|_{F_{\mu_2}(\sigma^n\omega)}\| \leq \mu_2.$$

From Lemma 1.9 the estimate 1.12 and Lemma 1.11, (1.13) implies that for $\epsilon > 0$ small and large n,

$$d_H\big(\tilde{H}^n_\omega, \tilde{H}^{n+1}_\omega\big) < M\exp\big(n(\mu_2 - \mu_1 + \epsilon)\big)$$

for a constant $M > 0$. The claim is proved. □

Next, we collect some facts about the limit of the sequence above.

Lemma 1.13 *Assume $\tilde{H}^n_\omega \xrightarrow{d_H} \tilde{H}_\omega$. Then the following holds:*

(i) $\tilde{H}_\omega$ is invariant, i.e. $\psi^k_\omega(\tilde{H}_\omega) = \tilde{H}_{\theta^k\omega}$ for any $k \geq 0$.
(ii) $\tilde{H}_\omega \cap F_{\mu_2}(\omega) = \{0\}$.
(iii) $\tilde{H}_\omega$ only depends on ω. In particular, it does not depend on the choice of the sequence $\{\tilde{H}^n_\omega\}_{n\geq 1}$.

Proof By construction, $\tilde{H}_\omega$ is invariant. We proceed with (ii). Consider the dual map

$$\left(\psi^n_{\sigma^n\omega}\right)^*_{\mu_1} : \left(F_{\mu_1}(\omega)\right)^* \to \left(F_{\mu_1}(\sigma^n\omega)\right)^*.$$

It is straightforward to see that $\left(\psi^n_{\sigma^n\omega}\right)^*_{\mu_1}$ enjoys the cocycle property. From (1.5) and [12, Proposition 4.15], we can find a closed subspace $G^*_{\mu_2}(\omega) \subset \left(F_{\mu_1}(\omega)\right)^*$ such that $\dim[(F_{\mu_1}(\omega))^*/G^*_{\mu_2}(\omega)] = m_1$ and for $\xi^*_\omega \in G^*_{\mu_2}(\omega)$, $\limsup_{n\to\infty}\frac{1}{n}\log\left\|\left(\psi^n_{\sigma^n\omega}\right)^*_{\mu_1}(\xi^*_\omega)\right\| \leq \mu_2$. Set

$$\left(F_{\mu_2}(\omega)\right)^\perp_{\mu_1} = \left\{\xi^*_\omega \in \left(F_{\mu_1}(\omega)\right)^* \;:\; \xi^*_\omega|_{F_{\mu_2}(\omega)} = 0\right\}.$$

By Hahn–Banach separation theorem,

$$\dim\left[\left(F_{\mu_2}(\omega)\right)^\perp_{\mu_1}\right] = \dim\left[F_{\mu_1}(\omega)/F_{\mu_2}(\omega)\right] = m_1.$$

Let $\xi^*_\omega \in \left(F_{\mu_2}(\omega)\right)^\perp_{\mu_1} \cap G^*_{\mu_2}(\omega)$ and assume that $\xi^*_\omega \neq 0$. Then for some $\xi_\omega \notin F_{\mu_1}(\omega)\backslash F_{\mu_2}(\omega)$, $\langle \xi^*_\omega, \xi_\omega\rangle = 1$. Using surjectivity of $[\psi^n_{\sigma^n\omega}]_{\mu_2}$, for every $n \in \mathbb{N}$, we can find $\xi_{\sigma^n\omega} \in H^n_{\sigma^n\omega}$ such that

$$\psi^n_{\sigma^n\omega}(\xi_{\sigma^n\omega}) = \xi_\omega \mod F_{\mu_2}(\omega).$$

Consequently, $\langle(\psi^n_{\sigma^n\omega})^*_{\mu_1}(\xi^*_\omega), \xi_{\sigma^n\omega}\rangle = 1$. From Lemma 1.9 ,

$$\lim_{n\to\infty}\frac{1}{n}\log\left\|\left[\psi^n_{\sigma^n\omega}\left(\frac{\xi_{\sigma^n\omega}}{\|[\xi_{\sigma^n\omega}]_{\mu_2}\|}\right)\right]_{\mu_2}\right\| = \lim_{n\to\infty}\frac{1}{n}\log\left\|\frac{\|[\xi_\omega]_{\mu_2}\|}{\|[\xi_{\sigma^n\omega}]_{\mu_2}\|}\right\| = \mu_1. \tag{1.14}$$

Hence for $\epsilon > 0$ and large n,

$$\|[\xi_{\sigma^n\omega}]_{\mu_2}\| < \exp(-n\big(\mu_1 - \epsilon)\big)$$

which is a contradiction since $\|(\psi^n_{\sigma^n\omega})^*_{\mu_1}(\xi^*_\omega)\| \leq \exp\big(n(\mu_2 + \epsilon)\big)$. Thus we have shown that

$$\left(F_{\mu_1}(\omega)\right)^* = \left(F_{\mu_2}(\omega)\right)^\perp_{\mu_1} \oplus G^*_{\mu_2}(\omega). \tag{1.15}$$

Now let $\xi_\omega \in \tilde{H}_\omega \cap F_{\mu_2}(\omega)$ and assume that $\|\xi_\omega\| = 1$. From 1.15, we can find $\xi^*_\omega \in G^*_{\mu_2}(\omega)$ such that $\langle\xi^*_\omega, \xi_\omega\rangle = 1$. By definition of $\tilde{H}_\omega$, there exist $\xi^n_{\sigma^n\omega} \in S_{H^n_{\sigma^n\omega}}$ such that $\frac{\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})}{\|\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})\|} \to \xi_\omega$ as $n \to \infty$, and consequently

$$\langle\xi^*_\omega, \frac{\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})}{\|\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})\|}\rangle = \langle(\psi^n_{\sigma^n\omega})^*(\xi^*_\omega), \frac{\xi^n_{\sigma^n\omega}}{\|\psi^n_{\sigma^n\omega}(\xi^n_{\sigma^n\omega})\|}\rangle \to 1$$

as $n \to \infty$. With Lemma 1.9 and a similar argument as above, this is again a contradiction and we have shown (ii). It remains to prove (iii). For $\xi_\omega \in \tilde{H}_\omega \subset (F_{\mu_1}(\omega))^{**}$, $\xi_\omega^* \in G_{\mu_2}^*(\omega)$ and a sequence $\xi_{\sigma^n\omega}^n$ chosen as above,

$$\langle \frac{\psi_{\sigma^n\omega}^n(\xi_{\sigma^n\omega}^n)}{\|\psi_{\sigma^n\omega}^n(\xi_{\sigma^n\omega}^n)\|}, \xi_\omega^* \rangle \to 0$$

as $n \to \infty$. Therefore, $\tilde{H}_\omega \subset \left(G_{\mu_2}^*(\omega)\right)_{\mu_1}^{\perp} = \left\{\xi_\omega^{**} \in \left(F_{\mu_1}(\omega)\right)^{**} : \xi_\omega^{**}|_{G_{\mu_2}^*(\omega)} = 0\right\}$ and since $\dim\left[\left(G_{\mu_2}^*(\omega)\right)_{\mu_1}^{\perp}\right] = m_1$, we obtain $\tilde{H}_\omega = \left(G_{\mu_2}^*(\omega)\right)_{\mu_1}^{\perp}$ which proves (iii). □

Combining Proposition 1.12 and Lemma 1.13, we see that if $\mu_1 > -\infty$, there is a θ-invariant set $\tilde{\Omega} \subset \Omega$ of full measure such that for every $\omega \in \tilde{\Omega}$, there is an m_1-dimensional subspace H_ω^1 with the properties

- $\psi_\omega^k(H_\omega^1) = H_{\theta^k\omega}^1$ for every $k \geq 0$ and
- $H_\omega^1 \oplus F_{\mu_2}(\omega) = F_{\mu_1}(\omega)$.

Thanks to the following lemma, we can invoke an induction argument to deduce the existence of a sequence of invariant spaces H_ω^i, $i \geq 1$.

Lemma 1.14 *The family of Banach spaces $\{F_{\mu_2}(\omega)\}_{\omega\in\tilde{\Omega}}$ is a measurable field of Banach spaces with*

$$\tilde{\Delta} = \{\tilde{g} := \Pi_{F_{\mu_2}||H^1} \circ g,\ g \in \Delta\} \quad \textit{and} \quad \tilde{\Delta}_0 = \{\tilde{g} := \Pi_{F_{\mu_2}||H^1} \circ g,\ g \in \Delta_0\}.$$

In addition, $\psi_\omega|_{F_{\mu_2}(\omega)} : F_{\mu_2}(\omega) \to F_{\mu_2}(\theta\omega)$ is a linear compact cocycle satisfying Assumption 1.1 with Δ replaced by $\tilde{\Delta}$. Moreover, the maps

$$f_1(\omega) := \|\Pi_{H_\omega^1||F_{\mu_2}(\omega)}\| \quad \textit{and} \quad f_2(\omega) := \|\Pi_{F_{\mu_2}(\omega)||H_\omega^1}\|$$

are measurable.

Proof The only non-trivial part in proving that $\{F_{\mu_2}(\omega)\}_{\omega\in\tilde{\Omega}}$ is a measurable field of Banach spaces is to show that

$$\omega \mapsto \|\Pi_{F_{\mu_2}(\omega)||H_\omega^1}(g(\omega))\| \tag{1.16}$$

is measurable for every $g \in \Delta$. Let

$$\{g_i : i \in \mathbb{N}\} = \Delta_0 \quad \text{and} \quad \{(g_{k_1}, \ldots, g_{k_{m_1}}) : k \in \mathbb{N}\} = \Delta_0^{m_1}.$$

Fix $n \in \mathbb{N}$ and $\omega \in \tilde{\Omega}$. We define $\{U_{\sigma^n\omega}^k\}_{k\geq 1}$ to be the family of subspaces of $E_{\sigma^n\omega}$ given by $U_{\sigma^n\omega}^k = \langle g_{k_i}(\sigma^n\omega)\rangle_{1\leq i\leq m_1, g_{k_i}\in\Delta_0}$. Using the same technique as in Lemma 1.5, one can show that the map

$$\omega \mapsto G_k(\sigma^n\omega) = \begin{cases} \|\Pi_{U_{\sigma^n\omega}^k||F_{\mu_2}(\sigma^n\omega)}\| & U_{\sigma^n\omega}^k \oplus F_{\mu_2}(\sigma^n\omega) = F_{\mu_1}(\sigma^n\omega) \\ \infty & \text{otherwise} \end{cases}$$

is measurable. Set $\psi_n(\omega) := \inf\{k : G_k(\sigma^n\omega) < M_1\}$ with M_1 as in Lemma 1.6. This map is clearly measurable. By Proposition 1.12, $\tilde{H}_\omega^n := \psi_{\sigma^n\omega}^n\left(U_{\sigma^n\omega}^{\psi_n(\omega)}\right) \xrightarrow{d_H} H_\omega^1$ and consequently

$$\Pi_{\tilde{H}_\omega^n||F_{\mu_2}(\omega)} \to \Pi_{H_\omega^1||F_{\mu_2}(\omega)} \quad \text{as } n \to \infty. \tag{1.17}$$

Let $g \in \Delta$. Then we have a decomposition of the form

$$\Pi_{\tilde{H}^n_\omega || F_{\mu_2}(\omega)} g(\omega) = \sum_{1 \le t \le m_1} \alpha_t(\omega) \psi^n_{\sigma^n \omega}(g_{\iota_t(\omega)}(\sigma^n \omega))$$

where $\iota_1, \ldots, \iota_{m_1} : \Omega \to \mathbb{N}$ are measurable. We assume $m_1 = 1$ first. To ease notation, set $\iota := \iota_1$. Since $g(\omega) - \alpha_1(\omega)\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega)) \in F_{\mu_2}(\omega)$, we have $\|[g(\omega)]_{\mu_2}\| = |\alpha_1(\omega)|\|[\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega))]\|$ and therefore

$$|\alpha_1(\omega)| = \frac{d\big(g(\omega), F_{\mu_2}(\omega)\big)}{d\big(\psi_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega)), F_{\mu_2}(\omega)\big)}.$$

Set

$$d_0(\omega) := d\big(g(\omega), F_{\mu_2}(\omega)\big) \quad \text{and} \quad d_1(\omega) := d\big(\psi_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega)), F_{\mu_2}(\omega)\big).$$

From Lemma 1.4, we know that d_0 is measurable. Furthermore, a slight adaptation of the proof yields the measurability of $\omega \mapsto d\big(\psi_{\sigma^n\omega}(g_k(\sigma^n\omega)), F_{\mu_2}(\omega)\big)$ for any fixed $k \in \mathbb{N}$. Since ι is measurable, this implies the measurability of d_1, too. We have

$$\Pi_{\tilde{H}^n_\omega || F_{\mu_2}(\omega)} g(\omega) = G(\omega) \frac{d_0(\omega)}{d_1(\omega)} \psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega))$$

where $G(\omega)$ takes values in $\{-1, 0, 1\}$. Set $h_0(\omega) := g(\omega) - \frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega))$ and $h_1(\omega) := g(\omega) + \frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota(\omega)}(\sigma^n\omega))$ and define

$$J_0(\omega) := \lim_{m\to\infty} \frac{1}{m} \log \big\|\psi^m_\omega\big(h_0(\omega)\big)\big\|, \qquad J_1(\omega) := \lim_{m\to\infty} \frac{1}{m} \log \big\|\psi^m_\omega\big(h_1(\omega)\big)\big\|.$$

It follows that J_0 and J_1 are measurable and that

$$\Pi_{\tilde{H}^n_\omega || F_{\mu_2}(\omega)} g(\omega) = (1 - \chi_{\{g(\omega)\in F_{\mu_2}(\omega)\}}) \big[g(\omega) - \chi_{\mu_2}\big(J_0(\omega)\big)h_0(\omega) - \chi_{\mu_2}\big(J_1(\omega)\big)h_1(\omega)\big]. \tag{1.18}$$

Then (1.18) and (1.17) prove the measurability of (1.16) for every $g \in \Delta$ in the case $m_1 = 1$. Furthermore, measurability of f_1 and f_2 and Assumption 1.1 for $\tilde{\Delta}$ can also be deduced. It remains to consider the case $m_1 > 1$ for which we invoke the same technique: Let

$$d_0(\omega) = d\big(g(\omega), F_{\mu_2}(\omega) \oplus \langle\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))\rangle_{2\le t\le m_1}\big),$$
$$d_1(\omega) = d\big(\psi^n_{\sigma^n\omega}(g_{\iota_1(\omega)}(\sigma^n\omega)), F_{\mu_2}(\omega) \oplus \langle\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))\rangle_{2\le t\le m_1}\big).$$

For $h_0(\omega) = g(\omega) - \frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota_1(\omega)}(\sigma^n\omega))$ and $h_1(\omega) = g(\omega) + \frac{d_0(\omega)}{d_1(\omega)}\psi^n_{\sigma^n\omega}(g_{\iota_1(\omega)}(\sigma^n\omega))$ let

$$d_{i0}(\omega) := d\big(h_i(\omega), F_{\mu_2}(\omega) \oplus \langle\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))\rangle_{3\le t\le m_1}\big), \; i \in \{0, 1\}$$
$$d_{01}(\omega) = d_{11}(\omega) = d\big(\psi^n_{\sigma^n\omega}(g_{\iota_2(\omega)}(\sigma^n\omega)), F_{\mu_2}(\omega) \oplus \langle\psi^n_{\sigma^n\omega}(g_{\iota_t(\omega)}(\sigma^n\omega))\rangle_{3\le t\le m_1}\big).$$

For $i \in \{0, 1\}$ define

$$h_{0,i} = h_0(\omega) + (-1)^{i+1} \frac{d_{00}(\omega)}{d_{01}(\omega)} \psi^n_{\sigma^n\omega}(g_{\iota_2(\omega)}(\sigma^n\omega))$$
$$h_{1,i} = h_1(\omega) + (-1)^{i+1} \frac{d_{10}(\omega)}{d_{11}(\omega)} \psi^n_{\sigma^n\omega}(g_{\iota_2(\omega)}(\sigma^n\omega)).$$

We repeat the same procedure with our four new functions. Iterating this, we end up with 2^{m_1} functions $\{I_t(\omega)\}_{1\leq t\leq 2^{m_1}}$ for which we define $J_t(\omega) := \lim_{m\to\infty}\frac{1}{m}\log\left\|\psi_\omega^m(I_t(\omega))\right\|$. Since

$$\Pi_{\tilde{H}_\omega^n||F_{\mu_2}(\omega)}g(\omega) = (1-\chi_{\{g(\omega)\in F_{\mu_2}(\omega)\}})\left[g(\omega) - \sum_{0\leq t\leq 2^{m_1}} \chi_{\mu_2}\big(J_t(\omega)\big)I_t(\omega)\right],$$

our claim follows for arbitrary m_1. □

Proposition 1.15 *Let $i \in \mathbb{N}$ and assume $\mu_i > \infty$. Then there is a θ-invariant set of full measure $\tilde{\Omega}$ such that for every $\omega \in \tilde{\Omega}$, there is an m_i-dimensional space H_ω^i with the properties*

(1) $\psi_\omega^k(H_\omega^i) = H_{\theta^k\omega}^i$ for every $k \geq 0$ and
(2) $H_\omega^i \oplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega)$.

Proof For $i = 1$, the statement follows from Proposition 1.12 and Lemma 1.13. For $i = 2$, we consider the restricted cocycle $\psi_\omega^k|_{F_{\mu_2}(\omega)}$. From Lemma 1.14, we know that this cocycle acts on the measurable field of Banach spaces $\{F_{\mu_2}(\omega)\}_{\omega\in\Omega}$ and we can thus apply Proposition 1.12 and Lemma 1.13 to this cocycle again. It remains to make sure that the top Lyapunov exponent of the restricted cocycle coincides with μ_2. This, however, was deduced in Lemma 1.8. We can now repeat the argument until we reach i. □

From now on, H_ω^i will always denote the spaces deduced in Proposition 1.15.

Remark 1.16 Using identities of the form

$$\Pi_{F_{\mu_j}(\omega)||\oplus_{l\leq i<j}H_\omega^i} = \Pi_{F_{\mu_j}(\omega)||H_\omega^{j-1}} \circ \Pi_{F_{\mu_{j-1}}(\omega)||H_\omega^{j-2}} \circ \cdots \circ \Pi_{F_{\mu_{l+1}}(\omega)||H_\omega^l},$$

we can use the same strategy as in Lemma 1.14 to see that for each $1 \leq l \leq j$ and $k \geq 0$,

$$f_1(\omega) := \left\|\Pi_{\oplus_{l\leq i<j}H_\omega^i\oplus F_{\mu_j}(\omega)}\right\|,\ f_2(\omega) := \left\|\Pi_{F_{\mu_j}(\omega)||\oplus_{l\leq i<j}H_\omega^i}\right\| \text{ and } f_3(\omega) := \|\psi_\omega^k|_{\oplus_{l\leq i<j}H_\omega^i}\|$$

are measurable.

Lemma 1.17 *For a measurable and non-negative function $f:\Omega\to\mathbb{R}$*

$$\lim_{n\to\infty}\frac{1}{n}f(\theta^n\omega) = 0 \textit{ a.s.} \quad \textit{if and only if} \quad \lim_{n\to\infty}\frac{1}{n}f(\sigma^n\omega) = 0 \textit{ a.s.}$$

Proof The main idea is due to Jack Feldman, cf. [16, Lemma 7.2]. Assume that $\lim_{n\to\infty}\frac{1}{n}f(\theta^n\omega) = 0$ on a set of full measure Ω^0. Let $\epsilon > 0$ and set

$$\Omega_n := \{\omega \in \Omega^0 \ :\ \forall i \geq n \ \ \frac{f(\theta^i\omega)}{i} \leq \epsilon\}.$$

Fom our assumptions, for some $n_0 \in \mathbb{N}$,

$$\mathbb{P}(\Omega_{n_0}) > \frac{9}{10}.$$

From Birkhoff's ergodic theorem, there is a set of full measure Ω^1 such that for every $\omega \in \Omega^1$, we can find $m_0 = m_\omega$ such that for $m \geq m_0$,

$$\frac{1}{m} \sum_{0 \leq j \leq m} \chi_{\Omega_{n_0}}(\sigma^j \omega) > \frac{9}{10}. \tag{1.19}$$

W.l.o.g., we may assume that $\Omega^0 = \Omega^1$. Now for $k \geq \max\{3n_0, m_0\}$, set $m = \lfloor \frac{5}{3} k \rfloor + 1$. Then from (1.19)

$$\frac{1}{m}\Big[\sum_{0 \leq j \leq \frac{4m}{5}} \chi_{\Omega_{n_0}}(\sigma^j \omega) + \sum_{\frac{4m}{5} < j \leq m} \chi_{\Omega_{n_0}}(\sigma^j \omega)\Big] > \frac{9}{10}.$$

Consequently, there exists $\frac{4m}{5} < j \leq m$ such that $\sigma^j \omega \in \Omega_{n_0}$. Set $i := j - k > n_0$. Then by the definition of Ω_{n_0},

$$\frac{f(\theta^i \sigma^j \omega)}{i} = \frac{f(\sigma^k \omega)}{j - k} \leq \epsilon.$$

Since $j - k \leq \frac{2}{3} k + 1$ and ϵ is arbitrary, our claim is shown. The other direction can be proved similarly. □

As a consequence, we obtain the following:

Lemma 1.18 *For each $1 \leq l \leq j$ and $\omega \in \tilde{\Omega}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{\oplus_{l \leq i < j} H^i_{\theta^n \omega} || F_{\mu_j}(\theta^n \omega)}\| = \lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{\oplus_{l \leq i < j} H^i_{\sigma^n \omega} || F_{\mu_j}(\sigma^n \omega)}\| = 0. \tag{1.20}$$

Proof Follows from a straightforward generalization of [12, Lemma 4.4] and Lemma 1.17. □

The following lemma characterizes the spaces H^i_ω as 'fast-growing' subspaces.

Proposition 1.19 *For $\omega \in \tilde{\Omega}$, every $i \geq N$ and $\xi_\omega \in H^i_\omega \backslash \{0\}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(\xi_\omega)\| = \lim_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega|_{H^i_\omega}\| = \mu_i \tag{1.21}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)\| = \lim_{n\to\infty} \frac{1}{n} \log \|(\psi^n_{\sigma^n \omega}|_{H^i_\omega})^{-1}\| = -\mu_i. \tag{1.22}$$

Proof The equalities (1.21) follow by applying the Multiplicative Ergodic Theorem [12, Theorem 4.17] to the map $\psi^n_\omega|_{H^i_\omega} : H^i_\omega \to H^i_{\theta^n \omega}$. It remains to prove (1.22). By definition, for every $\xi_\omega \in H^i_\omega$,

$$\begin{aligned} \frac{\|(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)\|}{\|[\xi_\omega]_{\mu_{i+1}}\|} \times \frac{\big\|\big[\psi^n_{\sigma^n \omega}\big((\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)\big)\big]_{\mu_{i+1}}\big\|}{\|[(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)]_{\mu_{i+1}}\|} &= \frac{\|(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)\|}{\|[(\psi^n_{\sigma^n \omega})^{-1}(\xi_\omega)]_{\mu_{i+1}}\|} \\ &\leq \|\Pi_{H^i_{\sigma^n \omega} || F_{\mu_{i+1}}(\sigma^n \omega)}\|. \end{aligned}$$

From Lemma 1.9,

$$\lim_{n\to\infty} \frac{1}{n} \inf_{\bar{\xi}_{\sigma^n \omega} \in H^i_{\sigma^n \omega}} \frac{\|[\psi^n_{\sigma^n \omega}(\bar{\xi}_{\sigma^n \omega})]_{\mu_{i+1}}\|}{\|[\bar{\xi}_{\sigma^n \omega}]_{\mu_{i+1}}\|} = \lim_{n\to\infty} \frac{1}{n} \frac{\|[\psi^n_{\sigma^n \omega}(\hat{\xi}_{\sigma^n \omega})]_{\mu_{i+1}}\|}{\|[\hat{\xi}_{\sigma^n \omega}]_{\mu_{i+1}}\|} = \mu_i$$

where $\hat{\xi}_{\sigma^n\omega} \in H^i_{\sigma^n\omega}$ is chosen such that

$$\frac{\|[\psi^n_{\sigma^n\omega}(\hat{\xi}_{\sigma^n\omega})]_{\mu_{i+1}}\|}{\|[\hat{\xi}_{\sigma^n\omega}]_{\mu_{i+1}}\|} = \min_{\bar{\xi}_{\sigma^n\omega}\in H^i_{\sigma^n\omega}} \frac{\|[\psi^n_{\sigma^n\omega}(\bar{\xi}_{\sigma^n\omega})]_{\mu_{i+1}}\|}{\|[\bar{\xi}_{\sigma^n\omega}]_{\mu_{i+1}}\|}.$$

Consequently, from (1.20),

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(\psi^n_{\sigma^n\omega}|_{H^i_\omega})^{-1}\| \le -\mu_i$$

Finally, from inequality $\|\xi_\omega\| \le \|\psi^n_{\sigma^n\omega}|_{H^i_{\sigma^n\omega}}\|\|(\psi^n_{\sigma^n\omega})^{-1}(\xi_\omega)\|$, Lemma 1.7 and (1.21), the equalities (1.22) can be deduced. □

Lemma 1.20 *Let $\omega \in \tilde{\Omega}$ and $i < k$. For every $i \le j < k$, let $\{\xi^t_\omega\}_{t\in I_j}$ be a basis of H^j_ω. Set $I := \cup_{i\le j<k} I_j$ and assume $\xi^t_\omega \in H^j_\omega$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log d(\psi^n_\omega(\xi^t_\omega), \langle\psi^n_\omega(\xi^{t'}_\omega)\rangle_{t'\in I\setminus\{t\}}) = \mu_j \tag{1.23}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log d((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega), \langle(\psi^n_{\sigma^n\omega})^{-1}(\xi^{t'}_\omega)\rangle_{t'\in I\setminus\{t\}}) = -\mu_j. \tag{1.24}$$

Proof We will prove (1.24) only, the proof for (1.23) is completely analogous. First, we claim that the statement is true for $j = i$ and $k = i+1$. Indeed, in this case we have the inequalities

$$\frac{1}{\|\psi^n_{\sigma^n\omega}|_{H^i_{\sigma^n\omega}}\|} \le \frac{d\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega), \langle(\psi^n_{\sigma^n\omega})^{-1}(\xi^{t'}_\omega)\rangle_{t'\in I\setminus\{t\}}\big)}{d\big(\xi^t_\omega, \langle\xi^{t'}_\omega\rangle_{t'\in I\setminus\{t\}}\big)} \le \|(\psi^n_{\sigma^n\omega})^{-1}|_{H^i_\omega}\|$$

and we can conclude with Proposition 1.19. For arbitrary k and $j = i$, we can use the inequalities

$$1 \le \frac{d\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega), \langle(\psi^n_{\sigma^n\omega})^{-1}(\xi^{t'}_\omega)\rangle_{t'\in I_i\setminus\{t\}}\big)}{d\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega), \langle(\psi^n_{\sigma^n\omega})^{-1}(\xi^{t'}_\omega)\rangle_{t'\in I\setminus\{t\}}\big)} \le \|\Pi_{H^i_{\sigma^n\omega}\|F_{\mu_{i+1}}(\sigma^n\omega)}\|,$$

Lemma 1.18 and our previous result above. The definition of Vol allows to deduce that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}\left(\big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega)\big)_{t\in I_{k-1}}, \dots, \big((\psi^n_{\sigma^n\omega})^{-1}(\xi^t_\omega)\big)_{t\in I_i}\right) = \sum_{i\le j<k} -\mu_j|I_j|. \tag{1.25}$$

Since Vol is symmetric up to a constant, the claim (1.24) follows for arbitrary j. □

The following theorem is the announced semi-invertible Oseledets theorem on fields of Banach spaces. It summarizes the main result of this section.

Theorem 1.21 *There is a θ-invariant set of full measure $\tilde{\Omega}$ such that for every $i \ge 1$ with $\mu_i > \mu_{i+1}$ and $\omega \in \tilde{\Omega}$, there is an m_i-dimensional subspace H^i_ω with the following properties:*

(i) (Invariance) $\psi^k_\omega(H^i_\omega) = H^i_{\theta^k\omega}$ for every $k \ge 0$.
(ii) (Splitting) $H^i_\omega \oplus F_{\mu_{i+1}}(\omega) = F_{\mu_i}(\omega)$. In particular,

$$E_\omega = H^1_\omega \oplus \cdots \oplus H^i_\omega \oplus F_{\mu_{i+1}}(\omega).$$

(iii) ('Fast-growing' subspace I) For each $h_\omega \in H^i_\omega \backslash \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\psi^n_\omega(h_\omega)\| = \mu_i.$$

(iv) ('Fast-growing' subspace II) For each $h_\omega \in H^i_\omega \backslash \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\psi^n_{\sigma^n\omega})^{-1}(h_\omega)\| = -\mu_i.$$

(v) If $\{\xi^t_\omega\}_{1\leq t\leq m}$ *is a basis of* $\oplus_{1\leq i\leq j} H^i_\omega$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}\left(\psi^n_\omega(\xi^1_\omega), \ldots, \psi^n_\omega(\xi^m_\omega)\right) = \sum_{1\leq i\leq j} m_i\mu_i \quad \textit{and}$$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}\left((\psi^n_{\sigma^n\omega})^{-1}(\xi^1_\omega), \ldots, (\psi^n_{\sigma^n\omega})^{-1}(\xi^m_\omega)\right) = \sum_{1\leq i\leq j} -m_i\mu_i. \quad (1.26)$$

Moreover, the properties (i)–(iv) uniquely determine the spaces H^i_ω.

Proof Properties (i) and (ii) are proven in Proposition 1.15. (iii) and (iv) are shown in Proposition 1.19 and (v) can be deduced from Lemma 1.20, using the definition of Vol and symmetry modulo a constant of this function. It remains to prove the uniqueness statement. Fix $i \geq 1$ and assume $\mu_i > \mu_{i+1}$. We define $G^*_{\mu_{i+1}}(\omega)$ and $\left(G^*_{\mu_{i+1}}(\omega)\right)^\perp_{\mu_i}$ as in Lemma 1.13 and claim that

$$H^i_\omega = \left(G^*_{\mu_{i+1}}(\omega)\right)^\perp_{\mu_i}. \quad (1.27)$$

Let $h_\omega \in H^i_\omega$, $h^*_\omega \in G^*_{\mu_{i+1}}(\omega)$ and set $h_{\sigma^n\omega} := (\psi^n_{\sigma^n\omega})^{-1}(h_\omega)$. Property (iv) implies that there is an $\epsilon > 0$ sufficiently small such that

$$\langle h_\omega, h^*_\omega\rangle = \langle \psi^n_{\sigma^n\omega}(h_{\sigma^n\omega}), h^*_\omega\rangle = \langle h_{\sigma^n\omega}, (\psi^n_{\sigma^n\omega})^*(h^*_\omega)\rangle \leq \exp\left(-n(\mu_i - \mu_{i+1} - \epsilon)\right) \to 0$$

as $n \to \infty$ which reveals $H^i_\omega \subset \left(G^*_{\mu_{i+1}}(\omega)\right)^\perp_{\mu_i}$. Finally, since these spaces have the same dimension, (1.27) follows. □

Remark 1.22 Property (iv) seems to be new in the context of Banach spaces. As seen in the proof, it is crucial for the uniqueness statement

2 Invariant Manifolds

Let $\{E_\omega\}_{\omega\in\Omega}$ be a measurable field of Banach spaces and φ^n_ω a nonlinear cocycle on acting on it, i.e.

$$\varphi^n_\omega \colon E_\omega \to E_{\theta^n\omega}$$
$$\varphi^{n+m}_\omega(.) = \varphi^n_{\theta^m\omega}\left(\varphi^m_\omega(.)\right).$$

Definition 2.1 We say that φ^n_ω admits a *stationary solution* if there exists a map $Y : \Omega \longrightarrow \prod_{\omega\in\Omega} E_\omega$ such that

(i) $Y_\omega \in E_\omega$,
(ii) $\varphi^n_\omega(Y_\omega) = Y_{\theta^n\omega}$ and
(iii) $\omega \to \|Y_\omega\|$ is measurable.

Stationary solutions should be thought of random analogues to fixed points in (deterministic) dynamical systems. If φ_ω^n is Fréchet differentiable, one can easily check that the derivative around a stationary solution also enjoys the cocycle property, i.e for $\psi_\omega^n(.) = D_{Y_\omega}\varphi_\omega^n(.)$, one has

$$\psi_\omega^{n+m}(.) = \psi_{\theta^m\omega}^n\big(\psi_\omega^m(.)\big).$$

In the following, we will assume that φ is Fréchet differentiable, that there exists a stationary solution Y and that the linearized cocycle ψ around Y is compact and satisfies Assumption 1.1. Furthermore, we will assume that

$$\log^+ \|\psi_\omega\| \in L^1(\Omega).$$

Therefore, we can apply the MET to ψ. In the following, we will use the same notation as in the previous section.

2.1 Stable Manifolds

Definition 2.2 Let Y be a stationary solution, let $\{\cdots < \mu_j < \mu_{j-1} < \cdots < \mu_1\} \in [-\infty, \infty)$ be the corresponding Lyapunov spectrum and $\tilde{\Omega}$ the θ-invariant set on which the MET holds. Set $\mu_{j_0} = \max\{\mu_j : \mu_j < 0\}$ and $\mu_{j_0} = -\infty$ if all finite μ_j are nonnegative. We define the *stable subspace*

$$S_\omega := F_{\mu_{j_0}}(\omega).$$

By the *unstable subspace* we mean

$$U_\omega := \oplus_{1\le i<j_0} H_\omega^i.$$

Note that $\dim[E_\omega/S_\omega] = \dim[U_\omega] =: k < \infty$ for every $\omega \in \tilde{\Omega}$.

Lemma 2.3 *For $\omega \in \tilde{\Omega}$ and $\epsilon \in (0, -\mu_{j_0})$, set*

$$F(\omega) := \sup_{p\ge 0} \exp[-p(\mu_{j_0} + \epsilon)] \|\psi_\omega^p|_{S_\omega}\|.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log^+ \big[F(\theta^n\omega)\big] = 0. \tag{2.1}$$

Proof Follows from (1.7). □

Lemma 2.4 *Let $\omega \in \tilde{\Omega}$, $U_\omega = \langle \xi_\omega^t \rangle_{1\le t\le k}$ and $n, p \ge 0$. Then*

$$\|[\psi_{\theta^p\omega}^n]^{-1}\|_{L[U_{\theta^{n+p}\omega}, U_{\theta^p\omega}]} \le \sum_{1\le t\le k} \frac{\|\psi_\omega^p(\xi_\omega^t)\|}{\|\psi_\omega^{n+p}(\xi_\omega^t)\|} \times \frac{\|\psi_\omega^{n+p}(\xi_\omega^t)\|}{d\big(\psi_\omega^{n+p}(\xi_\omega^t), \langle \psi_\omega^{n+p}(\xi_\omega^{t'})\rangle_{t'\neq t}\big)} \tag{2.2}$$

and

$$\|[\psi_{\sigma^n\omega}^p]^{-1}\|_{L[U_{\sigma^{n-p}\omega}, U_{\sigma^n\omega}]} \le \sum_{1\le t\le k} \frac{\|(\psi_{\sigma^n\omega}^n)^{-1}(\xi_\omega^t)\|}{\|(\psi_{\sigma^{n-p}\omega}^{n-p})^{-1}(\xi_\omega^t)\|} \times \frac{\|(\psi_{\sigma^{n-p}\omega}^{n-p})^{-1}(\xi_\omega^t)\|}{d\big((\psi_{\sigma^{n-p}\omega}^{n-p})^{-1}(\xi_\omega^t), \langle (\psi_{\sigma^{n-p}(\omega)}^{n-p})^{-1}(\xi_\omega^{t'})\rangle_{t'\neq t}\big)}. \tag{2.3}$$

Proof Choose $u \in U_{\theta^p\omega}$ and assume that $u = \sum_{1\le t\le k} u^t \frac{\psi_\omega^p(\xi_\omega^t)}{\|\psi_\omega^p(\xi_\omega^t)\|}$. Then

$$\frac{|u^t|}{\|u\|} \le \frac{\|\psi_\omega^p(\xi_\omega^t)\|}{d\big(\psi_\omega^p(\xi_\omega^t), \langle\psi_\omega^p(\xi_\omega^{t'})\rangle_{t'\neq t}\big)}. \tag{2.4}$$

From $\psi_{\theta^p\omega}^n u = \sum_{1\le t\le k} u^t \frac{\|\psi_\omega^{n+p}(\xi_\omega^t)\|}{\|\psi_\omega^p(\xi_\omega^t)\|} \frac{\psi_\omega^{n+p}(\xi_\omega^t)}{\|\psi_\omega^{n+p}(\xi_\omega^t)\|}$ and (2.4),

$$\frac{|u^t|}{\|\psi_{\theta^p\omega}^n u\|} \le \frac{\|\psi_\omega^p(\xi_\omega^t)\|}{\|\psi_\omega^{n+p}(\xi_\omega^t)\|} \times \frac{\|\psi_\omega^{n+p}(\xi_\omega^t)\|}{d\big(\psi_\omega^{n+p}(\xi_\omega^t), \langle\psi_\omega^{n+p}(\xi_\omega^{t'})\rangle_{t'\neq t}\big)}$$

and (2.2) follows. The estimate (2.3) is proven similarly. □

Definition 2.5 For $\omega \in \Omega$ set $\Sigma_\omega := \prod_{j\ge 0} E_{\theta^j\omega}$. For $\upsilon > 0$ we define

$$\Sigma_\omega^\upsilon := \left\{\Gamma \in \Sigma_\omega : \|\Gamma\| = \sup_{j\ge 0}\left[\|\Pi_\omega^j\Gamma\| \exp(\upsilon j)\right] < \infty\right\}$$

where $\Pi_\omega^j : \prod_{i\ge 0} E_{\theta^i\omega} \to E_{\theta^j\omega}$ denotes the projection map.

One can check that Σ_ω^υ is a Banach space.

Lemma 2.6 *Let $\omega \in \Omega$ and $0 < \upsilon < -\mu_{j_0}$. Define*

$$\begin{aligned} P_\omega &: E_\omega \to E_{\theta\omega} \\ &\quad \xi_\omega \mapsto \varphi_\omega^1(Y_\omega + \xi_\omega) - \varphi_\omega^1(Y_\omega) - \psi_\omega^1(\xi_\omega). \end{aligned}$$

Let $\rho\colon \Omega \to \mathbb{R}^+$ be a random variable with the property that

$$\liminf_{n\to\infty} \frac{1}{n} \log \rho(\theta^n\omega) \ge 0$$

almost surely. Assume that for $\|\xi_\omega\|, \|\tilde\xi_\omega\| < \rho(\omega)$,

$$\|P_\omega(\xi_\omega) - P_\omega(\tilde\xi_\omega)\| \le \|\xi_\omega - \tilde\xi_\omega\| f(\omega) h(\|\xi_\omega\| + \|\tilde\xi_\omega\|) \tag{2.5}$$

almost surely where $f : \Omega \to \mathbb{R}^+$ is a measurable function such that $\lim_{n\to\infty} \frac{1}{n}\log^+ f(\theta^n\omega) = 0$ almost surely and $h(x) = x^r g(x)$ for some $r > 0$ where $g : \mathbb{R} \to \mathbb{R}^+$ is an increasing C^1 function. Set

$$\tilde\rho(\omega) := \inf_{n\ge 0} \exp(n\upsilon)\rho(\theta^n\omega). \tag{2.6}$$

Then the map

$$I_\omega\colon S_\omega \times \Sigma_\omega^\upsilon \cap B(0, \tilde\rho(\omega)) \to \Sigma_\omega^\upsilon,$$
$$\Pi_\omega^n\big[I_\omega(v_\omega, \Gamma)\big]$$
$$= \begin{cases} \psi_\omega^n(v_\omega) + \sum_{0\le j\le n-1}\left[\psi_{\theta^{1+j}\omega}^{n-1-j} \circ \Pi_{S_{\theta^{1+j}\omega}\|U_{\theta^{1+j}\omega}}\right] P_{\theta^j\omega}\big(\Pi_\omega^j[\Gamma]\big) & \\ \quad - \sum_{j\ge n}\left[[\psi_{\theta^n\omega}^{j-n+1}]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}\right] P_{\theta^j\omega}\big(\Pi_\omega^j[\Gamma]\big) & \textit{for } n \ge 1, \\ v_\omega - \sum_{j\ge 0}\left[[\psi_\omega^{j+1}]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}\right] P_{\theta^j\omega}\big(\Pi_\omega^j[\Gamma]\big) & \textit{for } n = 0. \end{cases}$$

is well-defined on a θ-invariant set of full measure $\tilde\Omega$.

Proof We collect some estimates first. Let $\epsilon \in (0, -\mu_{j_0})$. From (1.20), we can find a random variable $R(\omega) > 1$ such that for $j \geq 0$,

$$\|\Pi_{U_{\theta^j\omega}\|S_{\theta^j\omega}}\| \leq R(\omega)\exp(\epsilon j)\ , \qquad \|\Pi_{S_{\theta^j\omega}\|U_{\theta^j\omega}}\| \leq R(\omega)\exp(\epsilon j). \tag{2.7}$$

Also from (2.1), for $n, p \geq 0$,

$$\|\psi^p_{\theta^n\omega}|_{S_{\theta^n\omega}}\| \leq R(\omega)\exp\big(p\mu_{j_0} + \epsilon(n+p)\big). \tag{2.8}$$

In addition, from (1.23) and (2.2) for $n, p \geq 0$,

$$\|[\psi^n_{\theta^p\omega}]^{-1}\|_{L[U_{\theta^{n+p}\omega}, U_{\theta^p\omega}]} \leq R(\omega)\exp\big(\epsilon(n+p)\big)\exp(-n\mu_{j_0-1}). \tag{2.9}$$

From our assumptions,

$$\big\|P_{\theta^j\omega}\big(\Pi^j_\omega[\Gamma]\big)\big\| \leq \big\|\Pi^j_\omega[\Gamma]\big\|^{1+r}\big[f(\theta^j\omega)g(\|\Pi^j_\omega[\Gamma]\|)\big].$$

So for $j \geq 0$ and a random variable $\tilde{R}(\omega) > 1$,

$$\big\|P_{\theta^j\omega}\big(\Pi^j_\omega[\Gamma]\big)\big\| \leq \tilde{R}(\omega)\big\|\Pi^j_\omega[\Gamma]\big\|^{1+r}g(\|\Pi^j_\omega[\Gamma]\|)\exp(\epsilon j). \tag{2.10}$$

Now from (2.7), (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned}
&\big\|\Pi^n_\omega\big[I_\omega(v_\omega,\Gamma)\big]\big\| \leq R(\omega)\Bigg[\exp((\mu_{j_0}+\epsilon)n)\|v_\omega\| + \\
&\sum_{0\leq j\leq n-1} R(\omega)\tilde{R}(\omega)\exp\big(\epsilon n + 2\epsilon(1+j) + (n-1-j)\mu_{j_0}\big)\|\Pi^j_\omega(\Gamma)\|^{1+r}g(\|\Pi^j_\omega[\Gamma]\|) + \\
&\sum_{j\geq n} R(\omega)\tilde{R}(\omega)\exp\big(3\epsilon(1+j) - (j-n+1)\mu_{j_0-1}\big)\|\Pi^j_\omega(\Gamma)\|^{1+r}g(\|\Pi^j_\omega[\Gamma]\|)\Bigg].
\end{aligned}$$

Since g is increasing,

$$\begin{aligned}
&\big\|\Pi^n_\omega\big[I_\omega(v_\omega,\Gamma)\big]\big\| \leq R(\omega)\Bigg[\exp\big((\mu_{j_0}+\epsilon)n\big).\|v_\omega\| + \\
&R(\omega)\tilde{R}(\omega)\|\Gamma\|^{1+r}_{\Sigma^\upsilon_\omega}g(\|\Gamma\|_{\Sigma^\upsilon_\omega})\exp\big(\epsilon n + 2\epsilon + (n-1)\mu_{j_0}\big)\sum_{0\leq j\leq n-1}\exp\big(j(2\epsilon - \mu_{j_0} - (1+r)\upsilon)\big) + \\
&R(\omega)\tilde{R}(\omega)\|\Gamma\|^{1+r}_{\Sigma^\upsilon_\omega}g(\|\Gamma\|_{\Sigma^\upsilon_\omega})\exp\big(3\epsilon + (n-1)\mu_{j_0-1}\big)\sum_{j\geq n}\exp\big(j(3\epsilon - \mu_{j_0-1} - (1+r)\upsilon)\big)\Bigg].
\end{aligned}$$

Since $\mu_{j_0-1} \geq 0$ and $0 < \upsilon < -\mu_{j_0}$, we can choose $\epsilon > 0$ smaller if necessary to see that

$$\sup_{n\geq 0}\Big[\big\|\Pi^n_\omega\big[I_\omega(v_\omega,\Gamma)\big]\big\|\exp(\upsilon n)\Big] < \infty.$$

As a result, I_ω is well-defined . □

Lemma 2.7 *With the same setting as in Lemma 2.6, for* $\Gamma \in \Sigma^\upsilon_\omega \cap B(0, \tilde{\rho}(\omega))$,

$$I_\omega[v_\omega,\Gamma] = \Gamma \iff \forall j \geq 0 : \Pi^j_\omega[\Gamma] = \varphi^j_\omega(Y_\omega + \xi_\omega) - \varphi^j_\omega(Y_\omega) \tag{2.11}$$

where

$$\xi_\omega = v_\omega - \sum_{j\geq 0}\big[[\psi^{j+1}_\omega]^{-1}\circ\Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}\big]P_{\theta^j\omega}\big(\Pi^j_\omega[\Gamma]\big). \tag{2.12}$$

Proof The strategy of the proof is similar to [17, Lemma VI.5]. Let $I_\omega[v_\omega, \Gamma] = \Gamma$. Then $\xi_\omega = \Pi^0_\omega[\Gamma]$ and the claim is shown for $j = 0$. We proceed by induction. Assume that $\Pi^n_\omega[\Gamma] = \varphi^n_\omega(Y_\omega + \xi_\omega) - \varphi^n_\omega(Y_\omega)$. By definition,

$$\begin{aligned}&\varphi^{n+1}_\omega(Y_\omega + \xi_\omega) - \varphi^{n+1}_\omega(Y_\omega) = \varphi^1_{\theta^n\omega}\big(\varphi^n_\omega(Y_\omega + \xi_\omega)\big) - \varphi^1_{\theta^n\omega}(Y_{\theta^n\omega}) = \\ &P_{\theta^n\omega}\big(\varphi^n_\omega(Y_\omega + \xi_\omega) - Y_{\theta^n\omega}\big) \\ &+ \psi^1_{\theta^n\omega}\big(\varphi^n_\omega(Y_\omega + \xi_\omega) - Y_{\theta^n\omega}\big) = P_{\theta^n\omega}(\Pi^n_\omega[\Gamma]) + \psi^1_{\theta^n\omega}\big(\Pi^n_\omega\big[I_\omega(v_\omega, \Gamma)\big]\big).\end{aligned}$$

Note that for $j \geq n$,

$$\psi^1_{\theta^n\omega} \circ [\psi^{j-n+1}_{\theta^n\omega}]^{-1} = [\psi^{j-n}_{\theta^{n+1}\omega}]^{-1} : U_{\theta^{1+j}\omega} \to U_{\theta^{1+n}\omega}.$$

By definition

$$\psi^1_{\theta^n\omega}\big(\Pi^n_\omega[I_\omega(v_\omega, \Gamma)]\big) = \psi^{n+1}_\omega(v_\omega) + \sum_{0\leq j\leq n-1}\big[\psi^{n-j}_{\theta^{1+j}\omega} \circ \Pi_{S_{\theta^{1+j}\omega}\|U_{\theta^{1+j}\omega}}\big]P_{\theta^j\omega}\big(\Pi^j_\omega[\Gamma]\big) -$$
$$\sum_{j\geq n}\big[[\psi^{j-n}_{\theta^n\omega}]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}\big]P_{\theta^j\omega}\big(\Pi^j_\omega[\Gamma]\big).$$

Consequently, $\Pi^{n+1}_\omega[\Gamma] = \varphi^{n+1}_\omega(Y_\omega + \xi_\omega) - \varphi^{n+1}_\omega(Y_\omega)$ which finishes the induction step.

Conversely, for $\xi_\omega \in E_\omega$ and $\Gamma \in \Sigma^\upsilon_\omega \cap B(0, \tilde{\rho}(\omega))$, assume that for every $j \geq 0$, $\Pi^j_\omega[\Gamma] = \varphi^j_\omega(Y_\omega + \xi_\omega) - \varphi^j_\omega(Y_\omega)$. Set

$$v_\omega := \xi_\omega + \sum_{j\geq 0}\big[[\psi^{j+1}_\omega]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega}\|S_{\theta^{1+j}\omega}}\big]P_{\theta^j\omega}\big(\Pi^j_\omega[\Gamma]\big).$$

Similar to Lemma 2.6, we can see that v_ω is well-defined. Morever,

$$\begin{aligned}\Pi^n_\omega\big[I_\omega(v_\omega, \Gamma)\big] &= \psi^n_\omega(\xi_\omega) + \sum_{0\leq j\leq n-1}\psi^{n-1-j}_{\theta^{1+j}\omega}P_{\theta^j\omega}\big(\Pi^j_\omega[\Gamma]\big) \\ &= \varphi^j_\omega(Y_\omega + \xi_\omega) - \varphi^j_\omega(Y_\omega) = \Pi^j_\omega[\Gamma]\end{aligned}$$

which proves the claim. □

Lemma 2.8 *Under the same assumptions as in Lemma 2.7, set*

$$h^\upsilon_1(\omega) := \sup_{n\geq 0}\big[\exp(n\upsilon)\|\psi^n_\omega|_{S_\omega}\|\big] \quad \textit{and}$$

$$\begin{aligned}h^\upsilon_2(\omega) := \sup_{n\geq 0}\Big[&\exp(n\upsilon)\sum_{0\leq j\leq n-1}\exp(-j\upsilon(1+r))f(\theta^j\omega)\|\psi^{n-j}_{\theta^{j+1}\omega}|_{S_{\theta^{j+1}\omega}}\|\|\Pi_{S_{\theta^{j+1}\omega}\|U_{\theta^{j+1}\omega}}\| \\ &+ \exp(n\upsilon)\sum_{j\geq n}\exp(-j\upsilon(1+r))f(\theta^j\omega)\|(\psi^{j-n+1}_{\theta^n\omega}|_{U_{\theta^{j+1}}})^{-1}\|\|\Pi_{U_{\theta^{j+1}\omega}\|S_{\theta^{j+1}\omega}}\|\Big].\end{aligned}$$

Then h^υ_1 and h^υ_2 are measurable and finite on a θ-invariant set of full measure $\tilde{\Omega}$. In addition,

$$\lim_{n\to\infty}\frac{1}{n}\log^+ h^\upsilon_1(\theta^n\omega) = \lim_{n\to\infty}\frac{1}{n}\log^+ h^\upsilon_2(\theta^n\omega) = 0$$

for every $\omega \in \tilde{\Omega}$. Furthermore, the estimates

$$\begin{aligned}\|I_\omega(v_\omega, \Gamma)\| &\leq h^\upsilon_1(\omega)\|v_\omega\| + h^\upsilon_2(\omega)\|\Gamma\|^{1+r}g(\|\Gamma\|) \quad \textit{and} \\ \|I_\omega(v_\omega, \Gamma) - I_\omega(v_\omega, \tilde{\Gamma})\| &\leq h^\upsilon_2(\omega)h(\|\Gamma\| + \|\tilde{\Gamma}\|)\,\|\Gamma - \tilde{\Gamma}\|\end{aligned}$$

hold for every $\omega \in \tilde{\Omega}$, $\Gamma, \tilde{\Gamma} \in \Sigma^\upsilon_\omega \cap B(0, \tilde{\rho}(\omega))$ and $v_\omega \in S_\omega$.

Proof The statements about h_1^υ and h_2^υ follow from our assumption on f, (1.7), Lemma 1.8 and Proposition 1.19. The claimed estimates follow by definition of I_ω. □

Recall that $h(x) = x^r g(x)$. In particular, h is invertible and h and h^{-1} are strictly increasing.

Lemma 2.9 *Assume that for $v_\omega \in S_\omega$,*

$$\|v_\omega\| \leq \frac{1}{2h_1^\upsilon(\omega)} \min\left\{\frac{1}{2}h^{-1}\left(\frac{1}{2h_2^\upsilon(\omega)}\right), \tilde{\rho}(\omega)\right\}.$$

Then the equation

$$I_\omega(v_\omega, \Gamma) = \Gamma$$

admits a uniques solution $\Gamma = \Gamma(v_\omega)$ and the bound

$$\|\Gamma(v_\omega)\| \leq \min\left\{\frac{1}{2}h^{-1}\left(\frac{1}{2h_2^\upsilon(\omega)}\right), \tilde{\rho}(\omega)\right\} =: H_1^\upsilon(\omega) \tag{2.13}$$

holds true.

Proof We can use the estimates provided in Lemma 2.8 to conclude that $I(v_\omega, \cdot)$ is a contraction on the closed ball with radius $\min\left\{\frac{1}{2}h^{-1}\left(\frac{1}{2h_2^\upsilon(\omega)}\right), \tilde{\rho}(\omega)\right\}$. □

Now we can formulate the main theorem about the existence of local stable manifolds.

Theorem 2.10 *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic measure-preserving dynamical systems and φ a Fréchet-differentiable cocycle acting on a measurable field of Banach spaces $\{E_\omega\}_{\omega\in\Omega}$. Assume that φ admits a stationary solution Y and that the linearized cocycle ψ around Y is compact, satisfies Assumption 1.1 and the integrability condition*

$$\log^+ \|\psi_\omega\| \in L^1(\omega).$$

Moreover, assume that (2.5) holds for φ and ψ. Let $\mu_{j_0} < 0$ and S_ω be defined as in Definition 2.2. For $0 < \upsilon < -\mu_{j_0}$, $\omega \in \Omega$ and $R^\upsilon(\omega) := \frac{1}{2h_1^\upsilon(\omega)} \min\left\{\frac{1}{2}h^{-1}\left(\frac{1}{2h_2^\upsilon(\omega)}\right), \tilde{\rho}(\omega)\right\}$ with $\tilde{\rho}$ defined as in (2.6), let

$$S_{loc}^\upsilon(\omega) := \left\{Y_\omega + \Pi_\omega^0[\Gamma(v_\omega)], \quad \|v_\omega\| < R^\upsilon(\omega)\right\}. \tag{2.14}$$

Then there is a θ-invariant set of full measure $\tilde{\Omega}$ on which the following properties are satisfied for every $\omega \in \tilde{\Omega}$:

(i) There are random variables $\rho_1^\upsilon(\omega)$, $\rho_2^\upsilon(\omega)$, positive and finite on $\tilde{\Omega}$, for which

$$\liminf_{p\to\infty} \frac{1}{p} \log \rho_i^\upsilon(\theta^p\omega) \geq 0, \quad i = 1, 2 \tag{2.15}$$

and such that

$$\left\{Z_\omega \in E_\omega \,:\, \sup_{n\geq 0} \exp(n\upsilon)\|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| < \rho_1^\upsilon(\omega)\right\} \subseteq S_{loc}^\upsilon(\omega)$$
$$\subseteq \left\{Z_\omega \in E_\omega \,:\, \sup_{n\geq 0} \exp(n\upsilon)\|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| < \rho_2^\upsilon(\omega)\right\}.$$

(ii) $S_{loc}^\upsilon(\omega)$ of E_ω and

$$T_{Y_\omega} S_{loc}^\upsilon(\omega) = S_\omega.$$

(iii) *For* $n \geq N(\omega)$,

$$\varphi_\omega^n(S_{loc}^\upsilon(\omega)) \subseteq S_{loc}^\upsilon(\theta^n\omega).$$

(iv) *For* $0 < \upsilon_1 \leq \upsilon_2 < -\mu_{j_0}$,

$$S_{loc}^{\upsilon_2}(\omega) \subseteq S_{loc}^{\upsilon_1}(\omega).$$

Also for $n \geq N(\omega)$,

$$\varphi_\omega^n(S_{loc}^{\upsilon_1}(\omega)) \subseteq S_{loc}^{\upsilon_2}(\theta^n(\omega))$$

and consequently for $Z_\omega \in S_{loc}^\upsilon(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| \leq \mu_{j_0}. \tag{2.16}$$

(v)

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\sup \left\{ \frac{\|\varphi_\omega^n(Z_\omega) - \varphi_\omega^n(\tilde{Z}_\omega)\|}{\|Z_\omega - \tilde{Z}_\omega\|}, \quad Z_\omega \neq \tilde{Z}_\omega,\ Z_\omega, \tilde{Z}_\omega \in S_{loc}^\upsilon(\omega) \right\} \right] \leq \mu_{j_0}.$$

Proof We start with (i). For the first inclusion, note that we can find a random variable $\rho_1^\upsilon(\omega)$ satisfying

$$\liminf_{p\to\infty} \frac{1}{p} \log \rho_1^\upsilon(\theta^p\omega) \geq 0 \tag{2.17}$$

and such that whenever $\|\Gamma\| \leq \rho_1^\upsilon(\omega)$,

$$\|\Gamma\| + h_2^\upsilon(\omega)\|\Gamma\|^{r+1} g(\|\Gamma\|) \leq \frac{1}{2h_1^\upsilon(\omega)} \min \left\{ \frac{1}{2} h^{-1}\left(\frac{1}{2h_2^\upsilon(\omega)}\right), \tilde{\rho}(\omega) \right\} =: H_2^\upsilon(\omega).$$

For example, we can define

$$\rho_1^\upsilon(\omega) := \min \left\{ h^{-1}\left(\frac{1}{h_2^\upsilon(\omega)}\right), H_2^\upsilon(\omega)/2, H_1^\upsilon(\omega) \right\}$$

with H_1^υ defined as in (2.13). Assume that $Z_\omega \in E_\omega$ has the property that

$$\sup_{n\geq 0} \exp(n\upsilon)\|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| < \rho_1^\upsilon(\omega).$$

Setting

$$\tilde{\upsilon}_\omega := Z_\omega - Y_\omega + \sum_{j\geq 0} \left[[\psi_\omega^{j+1}]^{-1} \circ \Pi_{U_{\theta^{1+j}\omega} \| S_{\theta^{1+j}\omega}} \right] P_{\theta^j\omega}\left(\Pi_\omega^j[\tilde{\Gamma}]\right),$$

it follows that $\|\tilde{\upsilon}_\omega\| < R^\upsilon(\omega)$. From Lemma 2.7, we conclude that $I_\omega[\tilde{\upsilon}_\omega, \tilde{\Gamma}] = \tilde{\Gamma}$. By uniqueness of the fixed point map, we have $\tilde{\Gamma} = \Gamma(\tilde{\upsilon}_\omega)$, therefore $Z_\omega = Y_\omega + \Pi_\omega^0(\Gamma(\tilde{\upsilon}_\omega)) \in S_{loc}^\upsilon(\omega)$. Next, let $Z_\omega \in S_{loc}^\upsilon(\omega)$, i.e. $Z_\omega = Y_\omega + \Pi_\omega^0(\Gamma(\upsilon_\omega))$ for some $\|\upsilon_\omega\| < R^\upsilon(\omega)$. From Lemmas 2.7 and 2.9,

$$\|\Gamma(\upsilon_\omega)\| = \sup_{n\geq 0} \exp(n\upsilon)\|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| \leq R^\upsilon(\omega).$$

We can therefore choose $\rho_2^\upsilon(\omega) = R^\upsilon(\omega)$ and the second inclusion is shown.

The second item immediately follows from our definition for $S_{loc}^\upsilon(\omega)$.

For item (iii), by (2.15), we can find $N(\omega)$ such that for $n \geq N(\omega)$,

$$\exp(-n\upsilon)\rho_2^{\upsilon}(\omega) \leq \rho_1^{\upsilon}(\theta^n\omega).$$

Now the claim follows from item (i).

For item (iv), note first that $R^{\upsilon_2}(\omega) \leq R^{\upsilon_1}(\omega)$. By definition of $\Gamma_\omega^{\upsilon}(v_\omega)$, it immediately follows that

$$S_{loc}^{\upsilon_2}(\omega) \subseteq S_{loc}^{\upsilon_1}(\omega).$$

Now take $Z_\omega \in S_{loc}^{\upsilon_1}(\omega)$. From Lemma 1.18 and (i), we can find $N(\omega)$ such that for $n \geq N(\omega)$,

$$\|\Pi_{S_{\theta^n\omega}\|U_{\theta^n\omega}}\big(\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\big)\| < R^{\upsilon_2}(\theta^n\omega).$$

We may also assume that $\exp(-n\upsilon_1)\rho_2^{\upsilon_1}(\omega) \leq \rho_1^{\upsilon_1}(\theta^n\omega)$ for $n \geq N(\omega)$. For

$$v_{\theta^n\omega} := \Pi_{S_{\theta^n\omega}\|U_{\theta^n\omega}}\big(\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\big)$$

let

$$Z_{\theta^n\omega} := \Pi_{\theta^n\omega}^0(\Gamma(v_{\theta^n\omega})) + Y_{\theta^n\omega} \in S_{loc}^{\upsilon_2}(\theta^n\omega) \subset S_{loc}^{\upsilon_1}(\theta^n\omega).$$

We claim that $Z_{\theta^n\omega} = \varphi_\omega^n(Z_\omega)$. Since $Z_\omega \in S_{loc}^{\upsilon_1}(\omega)$,

$$\sup_{j\geq 0}\exp(j\upsilon_1)\|\varphi_{\theta^n\omega}^j(\varphi_\omega^n(Z_\omega)) - Y_{\theta^j\theta^n\omega}\| \leq \exp(-n\upsilon_1)\rho_2^{\upsilon_1}(\omega) \leq \rho_1^{\upsilon_1}(\theta^n\omega).$$

So from item (i), $\varphi_\omega^n(Z_\omega) \in S_{loc}^{\upsilon_1}(\theta^n\omega)$. Remember $Z_{\theta^n\omega} \in S_{loc}^{\upsilon_1}(\theta^n\omega) \cap S_{loc}^{\upsilon_2}(\theta^n\omega)$ and

$$\Pi_{S^{\theta^n\omega}||U^{\theta^n\omega}}(Z_{\theta^n\omega} - Y_{\theta^n\omega}) = \Pi_{S^{\theta^n\omega}||U^{\theta^n\omega}}(\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}).$$

So by uniqueness of the fixed point, we indeed have

$$\varphi_\omega^n(Z_\omega) = Z_{\theta^n\omega} \in S_{loc}^{\upsilon_2}(\theta^n\omega).$$

To prove (2.16), let $\upsilon \leq \upsilon_2 < -\mu_0$ and take $Z_\omega \in S_{loc}^{\upsilon}(\omega)$. Then we know that for large enough N, $\varphi_\omega^N(Z_\omega) \in S_{loc}^{\upsilon_2}(\theta^N\omega)$, therefore

$$\sup_{j\geq 0}\exp(j\upsilon_2)\|\varphi_\omega^{j+N}(Z_\omega) - Y_{\theta^{j+N}\omega}\| < \infty$$

and it follows that

$$\limsup_{n\to\infty}\frac{1}{n}\log\|\varphi_\omega^n(Z_\omega) - Y_{\theta^n\omega}\| \leq -\upsilon_2.$$

We can choose υ_2 arbitrarily close to $-\mu_0$, therefore the claim follows and item (iv) is proved.

For item (v), first by definition,

$$\begin{aligned}\|\Gamma(v_\omega) - \Gamma(\tilde{v}_\omega)\| &= \|I_\omega(v_\omega, \Gamma(v_\omega)) - I_\omega(\tilde{v}_\omega, \Gamma(\tilde{v}_\omega))\| \\ &\leq \|I_\omega(v_\omega, \Gamma(v_\omega)) - I_\omega(\tilde{v}_\omega, \Gamma(v_\omega))\| + \|I_\omega(\tilde{v}_\omega, \Gamma(v_\omega)) - I_\omega(\tilde{v}_\omega, \Gamma(\tilde{v}_\omega))\| \\ &\leq h_1^{\upsilon}(\omega)\|v_\omega - \tilde{v}_\omega\| + \frac{1}{2}\|\Gamma(v_\omega) - \Gamma(\tilde{v}_\omega)\|\end{aligned}$$

for every $v_\omega, \tilde{v}_\omega \in S_\omega$ with $\|v_\omega\|, \|\tilde{v}_\omega\| \leq R^{\upsilon}(\omega)$. Consequently,

$$\|\Gamma(v_\omega) - \Gamma(\tilde{v}_\omega)\| \leq 2h_1^{\upsilon}(\omega)\|v_\omega - \tilde{v}_\omega\|. \tag{2.18}$$

Also by definition, cf. (2.12),

$$\begin{aligned}&\|\Pi^0_\omega(\Gamma(v_\omega)) - \Pi^0_\omega(\Gamma(\tilde v_\omega))\|\\ &\quad\geq \|v_\omega - \tilde v_\omega\| - h^\upsilon_2(\omega)\,\|\Gamma(v_\omega) - \Gamma_\omega(\tilde v_\omega)\|\,h(\|\Gamma(v_\omega)\| + \|\Gamma_\omega(\tilde v_\omega)\|).\end{aligned}$$

So from (2.18)

$$\|\Pi^0_\omega(\Gamma(v_\omega)) - \Pi^0_\omega(\Gamma(\tilde v_\omega))\| \geq \|v_\omega - \tilde v_\omega\|\big[1 - 2h^\upsilon_1(\omega)h^\upsilon_2(\omega)h(\|\Gamma(v_\omega)\| + \|\Gamma_\omega(\tilde v_\omega)\|)\big]. \tag{2.19}$$

First assume that

$$\max\{\|\Gamma(v_\omega), \Gamma(\tilde v_\omega)\|\} \leq \frac{1}{2}h^{-1}(\frac{1}{4h^\upsilon_1(\omega)h^\upsilon_2(\omega)}).$$

Then from (2.18) and (2.19),

$$\frac{\|\Gamma(v_\omega) - \Gamma(\tilde v_\omega)\|}{\|\Pi^0_\omega(\Gamma(v_\omega)) - \Pi^0_\omega(\Gamma(\tilde v_\omega))\|} \leq 4h^\upsilon_1(\omega). \tag{2.20}$$

Thus if $Z_\omega = Y_\omega + \Pi^0_\omega[\Gamma(v_\omega)]$ and $\tilde Z_\omega = Y_\omega + \Pi^0_\omega[\Gamma(v_\omega)]$, it follows that

$$\frac{\|\varphi^n_\omega(Z_\omega) - \varphi^n_\omega(\tilde Z_\omega)\|}{\|Z_\omega - \tilde Z_\omega\|} \leq 4\exp(-n\upsilon)h^\upsilon_1(\omega)$$

for every $n \geq 1$. In the general case, we can use item (i) and that $h^{-1}(\frac{1}{4h^\upsilon_1(\omega)h^\upsilon_2(\omega)})$ satisfies (2.15) to see that for some $N = N(\omega)$,

$$\begin{aligned}&\sup_{j\geq 0}\exp(j\upsilon)\|\varphi^j_{\theta^N\omega}(\varphi^N_\omega(Z_\omega)) - Y_{\theta^j\theta^N\omega}\| \leq \exp(-N\upsilon)\rho^\upsilon_2(\omega)\\ &\quad\leq \frac{1}{2}h^{-1}(\frac{1}{4h^\upsilon_1(\theta^N\omega)h^\upsilon_2(\theta^N\omega)}).\end{aligned}$$

Consequently, from (2.20),

$$\sup_{j\geq 0}\frac{\exp(j\upsilon)\|\varphi^{j+N}_\omega(Z_\omega) - \varphi^{j+N}_\omega(\tilde Z_\omega)\|}{\|\varphi^N_\omega(Z_\omega) - \varphi^N_\omega(\tilde Z_\omega)\|} \leq 4h^\upsilon_1(\theta^N\omega)$$

and hence for every $n \geq N$,

$$\frac{\|\varphi^n_\omega(Z_\omega) - \varphi^n_\omega(\tilde Z_\omega)\|}{\|Z_\omega - \tilde Z_\omega\|} \leq 4\exp((-n - N)\upsilon)h^\upsilon_1(\theta^N\omega)H^\upsilon_N(\omega) \tag{2.21}$$

where

$$H^\upsilon_N(\omega) = \sup\left\{\frac{\|\varphi^N_\omega(Z_\omega) - \varphi^N_\omega(\tilde Z_\omega)\|}{\|Z_\omega - \tilde Z_\omega\|},\quad Z_\omega \neq \tilde Z_\omega,\; Z_\omega, \tilde Z_\omega \in S^\upsilon_{loc}(\omega)\right\}.$$

We claim that $H^\upsilon_N(\omega)$ is finite. Indeed, by assumption (2.5),

$$\begin{aligned}\|\varphi^N_\omega(Z_\omega) - \varphi^N_\omega(\tilde Z_\omega)\| &\leq \|\psi^1_{\theta^{N-1}\omega}\|\;\|\varphi^{N-1}_\omega(Z_\omega) - \varphi^{N-1}_\omega(\tilde Z_\omega)\|\\ &\quad + f(\theta^N\omega)\;\|\varphi^{N-1}_\omega(Z_\omega) - \varphi^{N-1}_\omega(\tilde Z_\omega)\|h\\ &\quad \times\big(\|\varphi^{N-1}_\omega(Z_\omega) - Y_{\theta^{N-1}\omega}\| + \|\varphi^{N-1}_\omega(\tilde Z_\omega) - Y_{\theta^{N-1}\omega}\|\big)\end{aligned}$$

and we can proceed by induction to conclude. Finally, from (2.21) and item (iv), our claim is proved. □

Remark 2.11 Assume that for $\omega \in \tilde{\Omega}$ the function φ_ω is C^m. Then, since

$$I_\omega(0,0) = \frac{\partial}{\partial \Gamma} I_\omega(0,0) = 0,$$

we can deduce from the Implicit function theorem that $S^{\upsilon}_{loc}(\omega)$ is locally C^{m-1}.

2.2 Unstable Manifolds

We invoke same strategy for proving the existence of unstable manifolds. Since the arguments are very similar, we will only sketch them briefly. In this section, we will assume that the largest Lyapunov exponent is strictly positive, i.e. that $\mu_1 > 0$.

Definition 2.12 Set $k_0 := \min\{k : \mu_k > 0\}$, $\tilde{S}_\omega := F_{\mu_{k_0+1}}(\omega)$ and $\tilde{U}_\omega = \oplus_{1\leq i\leq k_0} H^i_\omega$ for $\omega \in \tilde{\Omega}$. For $\tilde{\Sigma}_\omega := \prod_{j\geq 0} E_{\sigma^j\omega}$ and $\upsilon > 0$, we define the Banach space

$$\tilde{\Sigma}^{\upsilon}_\omega := \left\{ \Gamma \in \tilde{\Sigma}_\omega \ : \ \|\Gamma\| = \sup_{k\geq 0} \left[\|\tilde{\Pi}^k_\omega \Gamma\| \exp(k\upsilon) \right] < \infty \right\}$$

where $\tilde{\Pi}^k_\omega : \prod_{i\geq 0} E_{\sigma^i\omega} \to E_{\sigma^k\omega}$ is the projection map. Similar to last section, we also set

$$\begin{aligned}
\tilde{h}^{\upsilon}_1(\omega) &:= \sup_{n\geq 0} \left[\exp(n\upsilon) \|(\psi^n_{\sigma^n\omega}|_{\tilde{U}_\omega})^{-1}\| \right] \quad \text{and} \\
\tilde{h}^{\upsilon}_2(\omega) &:= \sup_{n\geq 0} \Big[\exp(n\upsilon) \sum_{0\leq k\leq n-1} \exp\big(-\upsilon(n-k)(1+r)\big) f(\sigma^{n-k}\omega) \|(\psi^{k+1}_{\sigma^n\omega}|_{\tilde{U}_{\sigma^{n-1-k}\omega}})^{-1}\| \\
&\qquad \times \|\Pi_{\tilde{U}_{\sigma^{n-1-k}\omega}\|\tilde{S}_{\sigma^{n-1-k}\omega}}\| \\
&\qquad + \exp(n\upsilon) \sum_{k\geq n} \exp(-\upsilon(k+1)(1+r)) f(\sigma^{k+1}\omega) \|\psi^{k-n}_{\sigma^k\omega}|_{\tilde{S}_{\sigma^k\omega}}\| \|\Pi_{\tilde{S}_{\sigma^k\omega}\|\tilde{U}_{\sigma^k\omega}}\| \Big].
\end{aligned}$$

Lemma 2.13 *Let $\omega \in \Omega$, $0 < \upsilon < \mu_{k_0}$ and assume that $\rho\colon \Omega \to \mathbb{R}^+$ satisfies*

$$\liminf_{n\to\infty} \frac{1}{n} \log \rho(\sigma^n\omega) \geq 0 \tag{2.22}$$

almost surely. Define P as in Lemma 2.6 and assume that (2.5) holds for a random variable $f\colon \Omega \to \mathbb{R}^+$ which satisfies $\lim_{n\to\infty} f(\sigma^n\omega) = 0$ almost surely. Set

$$\tilde{\rho}(\omega) := \inf_{n\geq 0} \exp(n\upsilon)\rho(\sigma^n\omega). \tag{2.23}$$

Then the map

$$\begin{aligned}
&\tilde{I}_\omega : \tilde{U}_\omega \times \tilde{\Sigma}^{\upsilon}_\omega \cap B(0,\tilde{\rho}(\omega)) \to \tilde{\Sigma}^{\upsilon}_\omega, \\
&\tilde{\Pi}^n_\omega\big[\tilde{I}_\omega(u_\omega,\Gamma)\big] \\
&\quad = \begin{cases}
[\psi^n_{\sigma^n\omega}]^{-1}(u_\omega) \\
\quad - \sum_{0\leq k\leq n-1} \big[[\psi^{k+1}_{\sigma^n\omega}]^{-1} \circ \Pi_{\tilde{U}_{\sigma^{n-1-k}\omega}\|\tilde{S}_{\sigma^{n-1-k}\omega}}\big] P_{\sigma^{n-k}\omega}\big(\tilde{\Pi}^{n-k}_\omega[\Gamma]\big) \\
\quad + \sum_{k\geq n} \big[\psi^{k-n}_{\sigma^k\omega} \circ \Pi_{\tilde{S}_{\sigma^k\omega}\|\tilde{U}_{\sigma^k\omega}}\big] P_{\sigma^{k+1}\omega}\big(\tilde{\Pi}^{k+1}_\omega[\Gamma]\big) & \text{for } n \geq 1, \\
u_\omega + \sum_{k\geq 0} \big[\psi^k_{\sigma^k\omega} \circ \Pi_{\tilde{S}_{\sigma^k\omega}\|\tilde{U}_{\sigma^k\omega}}\big] P_{\sigma^{k+1}\omega}\big(\tilde{\Pi}^{k+1}_\omega[\Gamma]\big) & \text{for } n = 0.
\end{cases}
\end{aligned}$$

is well-defined on a θ-invariant set of full measure $\tilde{\Omega}$.

Proof We can use Lemma 1.17 to obtain a version of Lemma 2.3 where we replace θ by σ. The rest of the proof is similar to Lemma 2.6. □

Lemma 2.14 *For* $0 < \upsilon < \mu_{k_0}$, $\omega \in \tilde{\Omega}$ *and* $\Gamma \in \Sigma_\omega^\upsilon \cap B(0, \tilde{\rho}(\omega))$,

$$\tilde{I}_\omega(u_\omega, \Gamma) = \Gamma \iff \forall\, 0 \le k \le n: \quad \tilde{\Pi}_\omega^{n-k}\Gamma = \varphi^k_{\sigma^n\omega}(\tilde{\Pi}^n_\omega \Gamma + Y_{\sigma^n\omega}) - Y_{\sigma^{n-k}\omega}. \tag{2.24}$$

Proof Similar to Lemma 2.7. □

Lemma 2.15 *For* $0 < \upsilon < \mu_{k_0}$, $\tilde{h}_1^\upsilon$ *and* $\tilde{h}_2^\upsilon$ *are measurable and finite on a* θ*-invariant set of full measure* $\tilde{\Omega}$*. Moreover,*

$$\lim_{p\to\infty} \frac{1}{p} \log^+ \tilde{h}_1^\upsilon(\sigma^p\omega) = \lim_{p\to\infty} \frac{1}{p} \log^+ \tilde{h}_2^\upsilon(\sigma^p\omega) = 0 \tag{2.25}$$

and

$$\begin{aligned} \|\tilde{I}_\omega(u_\omega, \Gamma)\| &\le \tilde{h}_1^\upsilon(\omega)\|u_\omega\| + \tilde{h}_2^\upsilon(\omega)\|\Gamma\|^{r+1} g(\|\Gamma\|) \\ \|\tilde{I}_\omega(u_\omega, \Gamma) - \tilde{I}_\omega(u_\omega, \tilde{\Gamma})\| &\le \tilde{h}_2^\upsilon(\omega) h(\|\Gamma\| + \|\tilde{\Gamma}\|)\, \|\Gamma - \tilde{\Gamma}\| \end{aligned}$$

hold for every $\omega \in \tilde{\Omega}$, $\Gamma, \tilde{\Gamma} \in \tilde{\Sigma}_\omega^\upsilon \cap B(0, \tilde{\rho}(\omega))$ *and* $u_\omega \in \tilde{U}_\omega$.

Proof As in Lemma 2.8. □

Lemma 2.16 *Assume that for* $u_\omega \in \tilde{U}_\omega$,

$$\|u_\omega\| \le \frac{1}{2\tilde{h}_1^\upsilon(\omega)} \min\Big\{\frac{1}{2} h^{-1}\Big(\frac{1}{2\tilde{h}_2^\upsilon(\omega)}\Big), \tilde{\rho}(\omega)\Big\}.$$

Then the equation

$$\tilde{I}_\omega(u_\omega, \Gamma) = \Gamma$$

admits a uniques solution $\Gamma = \Gamma(u_\omega)$ *and the bound*

$$\|\Gamma(u_\omega)\| \le \min\Big\{\frac{1}{2} h^{-1}\Big(\frac{1}{2\tilde{h}_2^\upsilon(\omega)}\Big), \tilde{\rho}(\omega)\Big\}$$

holds true.

Proof We can show that $\tilde{I}(u_\omega, \cdot)$ is a contraction using Lemma 2.15. □

Finally we can formulate our main results about the existence of local unstable manifolds.

Theorem 2.17 *Let* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ *be an ergodic measure-preserving dynamical systems,* $\sigma := \theta^{-1}$ *and* φ *a Fréchet-differentiable cocycle acting on a measurable field of Banach spaces* $\{E_\omega\}_{\omega\in\Omega}$*. Assume that* φ *admits a stationary solution* Y *and that the linearized cocycle* ψ *around* Y *is compact, satisfies Assumption 1.1 and the integrability condition*

$$\log^+ \|\psi_\omega\| \in L^1(\omega).$$

Moreover, assume that (2.5) *holds for* φ *and* ψ *and a random variable* $\rho\colon \Omega \to \mathbb{R}^+$ *satisfying* (2.22)*. Assume that* $\mu_1 > 0$ *and let* $\mu_{k_0} > 0$ *and* $\tilde{U}_\omega$ *be defined as in Definition 2.12. For*

$0 < \upsilon < \mu_{k_0}$, $\omega \in \Omega$ *and* $R^\upsilon(\omega) := \frac{1}{2\tilde{h}_1^\upsilon(\omega)} \min\left\{\frac{1}{2}h^{-1}(\frac{1}{2\tilde{h}_2^\upsilon(\omega)}), \tilde{\rho}(\omega)\right\}$ *with* $\tilde{\rho}$ *defined as in* (2.23)*, let*

$$U_{loc}^\upsilon(\omega) := \left\{Y_\omega + \tilde{\Pi}_\omega^0[\Gamma(u_\omega)], \ \ \|u_\omega\| < \tilde{R}^\upsilon(\omega)\right\}. \tag{2.26}$$

Then there is a θ*-invariant set of full measure* $\tilde{\Omega}$ *on which the following properties are satisfied for every* $\omega \in \tilde{\Omega}$*:*

(i) There are random variables $\tilde{\rho}_1^\upsilon(\omega)$, $\tilde{\rho}_2^\upsilon(\omega)$*, positive and finite on* $\tilde{\Omega}$*, for which*

$$\liminf_{p\to\infty} \frac{1}{p} \log \tilde{\rho}_i^\upsilon(\sigma^p\omega) \geq 0, \quad i = 1, 2$$

and such that

$$\Big\{Z_\omega \in E_\omega \ : \ \exists\{Z_{\sigma^n\omega}\}_{n\geq 1} \text{ s.t. } \varphi_{\sigma^n\omega}^m(Z_{\sigma^n\omega}) = Z_{\sigma^{n-m}\omega} \text{ for all } 0 \leq m \leq n \text{ and}$$
$$\sup_{n\geq 0} \exp(n\upsilon)\|Z_{\sigma^n\omega} - Y_{\sigma^n\omega}\| < \tilde{\rho}_1^\upsilon(\omega)\Big\} \subseteq U_{loc}^\upsilon(\omega) \subseteq \Big\{Z_\omega \in E_\omega \ : \ \exists\{Z_{\sigma^n\omega}\}_{n\geq 1} \text{ s.t.}$$
$$\varphi_{\sigma^n\omega}^m(Z_{\sigma^n\omega}) = Z_{\sigma^{n-m}\omega} \text{ for all } 0 \leq m \leq n \text{ and } \sup_{n\geq 0} \exp(n\upsilon)\|Z_{\sigma^n\omega} - Y_{\sigma^n\omega}\| < \tilde{\rho}_2^\upsilon(\omega)\Big\}.$$

(ii) $U_{loc}^\upsilon(\omega)$ *is an immersed submanifold of* E_ω *and*

$$T_{Y_\omega} U_{loc}^\upsilon(\omega) = \tilde{U}_\omega.$$

(iii) For $n \geq N(\omega)$*,*

$$U_{loc}^\upsilon(\omega) \subseteq \varphi_{\sigma^n\omega}^n(U_{loc}^\upsilon(\sigma^n\omega)).$$

(iv) For $0 < \upsilon_1 \leq \upsilon_2 < \mu_{k_0}$*,*

$$U_{loc}^{\upsilon_2}(\omega) \subseteq U_{loc}^{\upsilon_1}(\omega).$$

Also for $n \geq N(\omega)$*,*

$$U_{loc}^{\upsilon_1}(\omega) \subseteq \varphi_{\sigma^n\omega}^n(U_{loc}^{\upsilon_2}(\sigma^n(\omega))$$

and consequently for $Z_\omega \in U_{loc}^\upsilon(\omega)$*,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Z_{\sigma^n\omega} - Y_{\sigma^n\omega}\| \leq -\mu_{k_0}.$$

(v)

$$\limsup_{n\to\infty} \frac{1}{n} \log\left[\sup\left\{\frac{\|Z_{\sigma^n\omega} - \tilde{Z}_{\sigma^n\omega}\|}{\|Z_\omega - \tilde{Z}_\omega\|}, \ \ Z_\omega \neq \tilde{Z}_\omega, \ Z_\omega, \tilde{Z}_\omega \in U_{loc}^\upsilon(\omega)\right\}\right] \leq -\mu_{k_0}.$$

Proof One uses the same arguments as in the proof of Theorem 2.10. □

Remark 2.18 (i) As in the stable case, if φ_ω is C^m for every $\omega \in \tilde{\Omega}$, one can deduce that $U_{loc}^\upsilon(\omega)$ is locally C^{m-1}.

(ii) In the *hyperbolic* case, i.e. if all Lyapunov exponents are non-zero, if the assumptions of Theorem 2.10 and 2.17 are satisfied, we have $S_\omega = \tilde{S}_\omega$ and $U_\omega = \tilde{U}_\omega$. In particular, the submanifolds $S_{loc}^\upsilon(\omega)$ and $U_{loc}^\upsilon(\omega)$ are *transversal*, i.e.

$$E_\omega = T_{Y_\omega} U_{loc}^\upsilon(\omega) \oplus T_{Y_\omega} S_{loc}^\upsilon(\omega).$$

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References

1. Arnold, L.: Random Dynamical Systems. Springer Monographs in Mathematics, Springer, Berlin (1998)
2. Blumenthal, A.: A volume-based approach to the multiplicative ergodic theorem on Banach spaces. Discrete Contin. Dyn. Syst. **36**(5), 2377–2403 (2016)
3. Caraballo, T., Garrido-Atienza, M.J., Schmalfuss, B.: Existence of exponentially attracting stationary solutions for delay evolution equations. Discrete Contin. Dyn. Syst. **18**(2–3), 271–293 (2007)
4. Caraballo, T., Kloeden, P.E., Schmalfuß, B.: Exponentially stable stationary solutions for stochastic evolution equations and their perturbation. Appl. Math. Optim. **50**(3), 183–207 (2004)
5. Conti, M., Pata, V., Temam, R.: Attractors for processes on time-dependent spaces. Applications to wave equations. J. Differ. Equ. **255**(6), 1254–1277 (2013)
6. Doan, T.S.: Lyapunov Exponents for Random Dynamical Systems. Ph.D. thesis, Technische Universität Dresden (2009)
7. Di Plinio, F., Duane, G.S., Temam, R.: Time-dependent attractor for the oscillon equation. Discrete Contin. Dyn. Syst. **29**(1), 141–167 (2011)
8. Friz, P.K., Hairer, M.: A Course on Rough Paths with an Introduction to Regularity Structures. Universitext, vol. XIV. Springer, Berlin (2014)
9. Flandoli, F., Schmalfuss, B.: Random attractors for the 3D stochastic Navier–Stokes equation with multiplicative white noise. Stoch. Stoch. Rep. **59**(1–2), 21–45 (1996)
10. González-Tokman, C., Quas, A.: A semi-invertible operator Oseledets theorem. Ergodic Theory Dyn. Syst. **34**(4), 1230–1272 (2014)
11. González-Tokman, C., Quas, A.: A concise proof of the multiplicative ergodic theorem on Banach spaces. J. Mod. Dyn. **9**, 237–255 (2015)
12. Varzaneh, M.G., Riedel, S., Scheutzow, M.: A dynamical theory for singular stochastic delay differential equations I: linear equations and a multiplicative ergodic theorem on fields of Banach spaces. arXiv:1903.01172v3 (2019)
13. Kato, T.: Perturbation Theory for Linear Operators. Classics in Mathematics. Springer, Berlin (1995). (**Reprint of the 1980 edition**)
14. Lions, J.-L.: Équations différentielles opérationnelles et problèmes aux limites. Die Grundlehren der mathematischen Wissenschaften, Bd. 111. Springer, Berlin (1961)
15. Lian, Z., Lu, K.: Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space. Mem. Am. Math. Soc. **206**(967), vi+106 (2010)
16. Lyons, R., Pemantle, R., Peres, Y.: Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. Ann. Probab. **23**(3), 1125–1138 (1995)
17. Mañé, R.: Lyapounov exponents and stable manifolds for compact transformations. In Geometric Dynamics (Rio de Janeiro, 1981), volume 1007 of Lecture Notes in Mathematics, pp. 522–577. Springer, Berlin (1983)
18. Oseledec, V.I.: A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems. Trudy Mosk. Mat. Obšč. **19**, 179–210 (1968)
19. Raghunathan, M.S.: A proof of Oseledec's multiplicative ergodic theorem. Isr. J. Math. **32**(4), 356–362 (1979)

20. Ruelle, D.: Ergodic theory of differentiable dynamical systems. Inst. Hautes Études Sci. Publ. Math. **50**, 27–58 (1979)
21. Ruelle, D.: Characteristic exponents and invariant manifolds in Hilbert space. Ann. Math. (2) **115**(2), 243–290 (1982)
22. Thieullen, P.: Fibrés dynamiques asymptotiquement compacts. Exposants de Lyapounov. Entropie. Dimension. Ann. Inst. H. Poincaré Anal. Non Linéaire **4**(1), 49–97 (1987)
23. Walters, P.: A dynamical proof of the multiplicative ergodic theorem. Trans. Am. Math. Soc. **335**(1), 245–257 (1993)