Minimal Non-invertible Maps on the Pseudo-Circle

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Abstract

In this article we show that R.H. Bing's pseudo-circle admits a minimal non-invertible map. This resolves a conjecture raised by Bruin, Kolyada and Snoha in the negative. The main tool is a variant of the Denjoy–Rees technique, further developed by Béguin–Crovisier–Le Roux, combined with detailed study of the structure of the pseudo-circle. This is the first example of a planar 1-dimensional space that admits both minimal homeomorphisms and minimal noninvertible maps.

Keywords Pseudo-circle · Pseudoarc · Minimal · Noninvertible

Mathematics Subject Classification 37B05 · 37B20 · 37B45

1 Introduction

In the late 1960s J. Auslander raised a question concerning the existence of minimal noninvertible maps. Since then examples of such maps have been given, but the question as for what spaces such maps exist remains particularly interesting for compact and connected spaces

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(continua). In the present paper we are going to give the first example of a 1-dimensional space in the plane that admits both minimal homeomorphisms and minimal noninvertible maps. In general, for compact spaces the existence of an invertible minimal map is independent of the existence of a noninvertible one, and there exist spaces that admit both types of maps, either one of them, or none [26]. In [1] Auslander and Yorke observed that the Cantor set admits minimal maps of both kinds. In Kolyada et al. [26] showed how to obtain minimal noninvertible maps on the 2-torus from a class of minimal homeomorphisms. An analogous result was obtained in [39] by Sotola and Trofimchuk for the Klein bottle. In [9] Clark, the first and the last mentioned authors showed the existence of both types of minimal maps for any compact finite-dimensional metric space that admits an aperiodic minimal flow. The first author showed in [10] the existence of minimal noninvertible maps for all compact manifolds of dimension at least 2, that admit minimal homeomorphisms. Auslander and Katznelson showed that the circle admits no minimal noninvertible maps, despite supporting minimal homeomorphisms [3]. In this context Bruin, Kolyada and Snoha asked the following question:

Question 1 [12] Is the circle the only infinite continuum that admits a minimal homeomorphism but no noninvertible minimal map?

In the same paper Bruin, Kolyada and Snoha conjectured that R.H. Bing's pseudocircle might provide a counterexample (see also [38]).

Conjecture 1 [12] The pseudo-circle is another continuum that admits minimal homeomorphisms, but no noninvertible minimal maps.

Quite recently, in [16] a construction of a family of continua that admit minimal homeomorphisms, but no minimal noninvertible maps was given, resolving Question 1 (see also [11]). However, Conjecture 1 has remained open until now. The pseudocircle seems indeed a natural candidate for another counterexample to Question 1, as it shares several properties with the circle. Topologically it is a circle-like planar cofrontier, whose homeomorphism group contains periodic rotations [22], and it appears in various contexts in dynamical systems. Handel showed that it appears as both a minimal subset of a volume preserving, C^∞ -smooth planar diffeomorphism [19], and a minimal attractor of a C^∞ -smooth planar diffeomorphism. Herman [21] extended Handel's construction to show that the pseudo-circle can bound an open disk in the complex plane, on which a smooth planar diffeomorphism is complex analytic and complex analytically conjugate to an irrational rotation, and on the complement of which the diffeomorphism is smoothly conjugate to such a rotation. This result was further improved by Chéritat who showed that the pseudo-circle can appear as the boundary of a Siegel disk [13]. In [8] the first and last author showed that it also appears as a Birkhoff-like attractor [8]. Other related constructions were given by Yorke and the second author in [23–25]. Most recently, a continuous decomposition of the 2-torus into pseudo-circles was given by Béguin, Crovisier, and Jäger [4], that is invariant under a homeomorphism of the torus semi-conjugate to an irrational circle rotation. In the present paper, relying on Handel's construction, we adapt and modify the results of Béguin, Crovisier and Le Roux [5] to the very fine and complex structure of the pseudo-circle, to provide a counterexample to the aforementioned conjecture.

Theorem 1.1 *The pseudo-circle admits minimal noninvertible maps.*

The results of [5], that are discussed in more detail in Sect. 3, are a far-reaching generalization of the Denjoy–Rees technique [15,34] that allowed Rees to construct the first examples

of minimal torus homeomorphisms with positive entropy. These results are crucial to our construction. However, it should be noted that our proof is not a mere application of very general results of [5] for manifolds to a special case of a particular 1-dimensional pathological fractal. Quite a lot of care and effort is needed to make sure that the construction will not change the underlying space, that is the pseudo-circle, into a different one-dimensional space. The general approach that we take is as follows. First, blow up a forward orbit of Handel's minimal homeomorphism h contained in the pseudo-circle P , to a null sequence of pseudo-arcs $\{P_n\}_{n=0}^\infty$, with $h(P_n) = P_{n+1}$ for each $n \in \mathbb{N}$ to obtain a "new" pseudo-circle Q (Theorem 3.1). Second, define an equivalence relation ' $\equiv$ ' between points in Q that identifies any two points in each pseudo-arc in the sequence. This relation is monotone, and closed since the sequence of pseudo-arcs in null. Now consider the quotient space $P' = Q/\equiv$. This new space P' is a pseudo-circle (Proposition 4.1). The homeomorphism h induces a minimal map on P' , that is not invertible on $h^{-1}(P_0)/\equiv = h^{-1}(P_0)$ (Theorem 4.2), and the goal of the construction is achieved. Therefore the success of the construction relies on the ability to guarantee that Q is indeed the pseudo-circle. To this end, one could think of a straightforward application of the following proposition from [5].

Proposition 1.2 (Béguin et al. [5]) *Let R be a homeomorphism on a compact manifold M , and x a point of M which is not periodic under R . Consider a compact subset D of M which can be written as the intersection of a strictly decreasing sequence $\{D_n : n \in \mathbb{N}\}$ of tamely embedded topological closed balls. Then there exist a homeomorphism $f : M \rightarrow M$ and a continuous onto map $\phi : M \rightarrow M$ such that $\phi \circ f = R \circ \phi$, and such that*

- $\phi^{-1}(x) = D$;
- $\phi^{-1}(y)$ is a single point if y does not belong to the R -orbit of x .

However, this is not sufficient to obtain our result. The problem, again, is that 'inserting' the pseudo-arcs *ad hoc* does not guarantee that the resulting space will not change, as there is even no guarantee that such an insertion would allow these pseudo-arcs to have empty interior in the resulting space, as is required for the pseudo-circle. Moreover, even if one can ensure that the special nontrivial fibres indeed have empty interior in the new space, one still has to ensure that the insertion is such that the resulting space remains circle-like (i.e. it can be covered by circular chains of arbitrarily small mesh; see Section. 2 for precise definitions). Consequently, we were lead to an adjustment of the method in [5] (see the comments preceding Lemma 3.4 for more details) and combine it with a careful analysis of the hereditary indecomposable structure of the pseudo-circle. To that end, the notion of an internal composant introduced by Krasinkiewicz in [27] came in handy (see Sect. 4). Nevertheless, our construction made us appreciate the Denjoy–Rees technique, as presented in [5], even more and there are clearly other novel applications of it to come.

2 Preliminaries and Some Previous Constructions

In this section we define the notation and terms we use as well as recall some results from the literature that are used in the arguments.

2.1 Basic Notation in Topology and Dynamics

For any planar set E , we denote by $\text{Int}(E)$ the interior of the set E . We work with the closed annulus $\mathbb{A} = S^1 \times [0, 1]$, and its lift to $\tilde{\mathbb{A}} = \mathbb{R} \times [0, 1]$, with π denoting the lift function from $\tilde{\mathbb{A}}$ to $\mathbb{A}$.

Given a family $\mathcal{E}$ of connected subsets of $\mathbb{A}$, define

$$\text{mesh}(\mathcal{E}) = \max\{\text{diam}(X) \mid X \in \mathcal{E}\}. \quad (2.1)$$

For any family $\mathcal{E}$ of subsets of $\mathbb{A}$, denote by $\mathfrak{s}(\mathcal{E})$ the union of all the elements of $\mathcal{E}$, called the *realization* of $\mathcal{E}$. A *non-degenerate continuum* is a compact, connected metric space, which contains at least two elements. We call a continuum *planar* if it can be embedded into the plane. A continuum X is called *indecomposable* if X is not the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if every subcontinuum of it is also indecomposable. For any $x \in X$, the *composant* of x , denoted $C(x)$, is the union of all proper subcontinua in X containing x . Moreover, for any indecomposable continuum the following properties hold:

- If x, y are points in X , then either $C(x) = C(y)$ or $C(x) \cap C(y) = \emptyset$ ([33], Theorem 11.17),
- There exists uncountably many composants ([33], Theorem 11.15),
- Each composant is a dense first-category connected set in X .

Note that the first two properties do not hold for decomposable continua. Such continua have either one or three composants, and they are not pairwise disjoint ([33], Theorem 11.13). As for indecomposable continua, let us consider some examples well known from the theory of dynamical systems. The first class of such examples involves solenoids, which are defined as inverse limits of circles, with finite-to-one covering maps as bonding maps. Equivalently, solenoids can be defined as suspensions (mapping tori) of odometers. Smale used solenoids to provide one of the first examples of hyperbolic attractors [37]. Any solenoid S is an indecomposable continuum and composants of S coincide with arc-components of S ; i.e. the arc-component of any point is equal to the composant of that point. This follows directly from the fact that each arc-component is dense, and there are uncountably many such components, whose union is the entire solenoid. The same is true about the composants of derived from Anosov diffeomorphism attractors, Plykin attractors, and Denjoy minimal sets, with all three classes of continua (unlike the solenoids [7]) being embeddable in surfaces (see e.g. [20], ([36], Appendix) and [31] for definitions). However, there are indecomposable continua that do not share this property, and in particular the pseudo-circle that we shall consider in the present paper contains no arcs at all(!), since it is hereditarily indecomposable. In particular, all arc-components of the pseudo-circle are degenerate, but its composants are unions of pseudo-arcs; see Remark 2.9.

Definition 2.1 Let X be a planar continuum. A composant C of X is called *internal* if every continuum L intersecting both C and the complement of X intersects all composants of X .

We will need the following theorem proved by Krasinkiewicz (see Main Theorems of [27,28]).

Lemma 2.2 For any indecomposable planar continuum X , the union of all the internal composants is a G_δ subset of X . In particular, every indecomposable planar continuum contains an internal composant.

Let $f: X \rightarrow X$, $g: Y \rightarrow Y$ be two homeomorphisms on compact metric spaces X and Y , respectively. Let $\psi: X \rightarrow Y$ be a continuous surjective map satisfying $\psi \circ f = g \circ \psi$. In this case, we say ψ is a *semi-conjugacy* between (X, f) and (Y, g) . The map $\psi: X \rightarrow Y$ is called *monotone* if for any $y \in Y$, $\psi^{-1}(y)$ is connected. We say the semi-conjugacy ψ is *almost 1-1* if there exists a residual subset $Y_1 \subset Y$, such that for each $y \in Y_1$, $\phi^{-1}(y)$ is a singleton.

Given a monotone map $\psi: X \rightarrow Y$, the partition space $\{\psi^{-1}(y) | y \in Y\}$ is known to be *upper semi-continuous*, i.e., each set $\psi^{-1}(y)$ is closed in X , and for any open set U in X , the union of all $\psi^{-1}(y)$ contained in U is open. We need the following well known properties for upper semi-continuous decompositions. (See for example Proposition 1 and Proposition 2 in chapter 1 of [14].)

Lemma 2.3 *Let $\phi: X \rightarrow Y$ be such that, $\{\phi^{-1}(y) | y \in Y\}$ forms an upper semi-continuous decomposition of X . Then the following hold:*

- (1) *For any open set U containing some pre-image $\phi^{-1}(y_0)$, the set $W = \bigcup_{y \in \phi(U)} \phi^{-1}(y)$ is open.*
- (2) *Suppose a sequence of closed sets $\phi^{-1}(y_n) \subset X$ converges to $A \subset X$ in the Hausdorff topology. Then for any $x \in A$, $A \subset \phi^{-1} \circ \phi(x)$.*

A *Moore decomposition* of the closed annulus $\mathbb{A}$ is an upper semi-continuous decomposition $\mathcal{M}$ such that any $M \in \mathcal{M}$ is contained in a topological disk. The following is the classic Moore's theorem.

Lemma 2.4 (Section 25, Theorem 1 in [14]) *For any Moore decomposition $\mathcal{M}$ on $\mathbb{A}$, the partition space is again $\mathbb{A}$. Moreover, there exists a continuous monotone surjection $\phi: \mathbb{A} \rightarrow \mathbb{A}$, such that the preimage of each point is precisely one element $M \in \mathcal{M}$.*

2.2 Circular Chains and Pseudo-Circles

Let $\mathbb{Z}_m$ denote the finite additive Abelian group $\mathbb{Z}_m = \{0, \dots, m-1\}$. For any two elements $\bar{i}, \bar{j} \in \mathbb{Z}_m$, we use the convenient notation ρ for their distance. More precisely, denote $\rho(\bar{i}, \bar{j}) = \min(|i - j|, |m - |i - j||)$.

Definition 2.5 For $m \geq 3$, a *circular chain* is a finite family of open subset $\mathcal{D} = \{D_{\bar{\ell}}\}_{\bar{\ell} \in \mathbb{Z}_m}$, not necessarily connected, such that

$$D_{\bar{\ell}} \cap D_{\bar{k}} \neq \emptyset \text{ if and only if } \rho(\bar{k}, \bar{\ell}) \leq 1. \quad (2.2)$$

Each element $D_{\bar{\ell}}$ is called a link.

Definition 2.6 Let $\mathcal{D} = \{D_{\bar{\ell}} : \bar{\ell} \in \mathbb{Z}_m\}$ and $\mathcal{C} = \{C_{\bar{\ell}} : \bar{\ell} \in \mathbb{Z}_s\}$ be two circular chains. We say $\mathcal{D}$ is *crooked inside* $\mathcal{C}$ if there exists a map $\pi: \mathbb{Z}_m \rightarrow \mathbb{Z}_s$ with the following properties.

- (i) For any $\bar{i} \in \mathbb{Z}_m$, the closure of $D_{\bar{i}}$ is contained in $C_{\pi(\bar{i})}$.
- (ii) With properly chosen circular orders, let $\bar{i} < \bar{j}$ be two elements in $\mathbb{Z}_m$, such that $\pi(\bar{i}) \leq \pi(\bar{j})$. Suppose for any $\bar{i} < \bar{k} < \bar{j}$, $\pi(\bar{i}) \leq \pi(\bar{k}) \leq \pi(\bar{j})$. Then there exist $\bar{i} < \bar{u} < \bar{v} < \bar{j}$, such that,

$$\rho(\pi(\bar{i}), \pi(\bar{v})) \leq 1. \quad (2.3)$$

$$\rho(\pi(\bar{j}), \pi(\bar{u})) \leq 1. \quad (2.4)$$

We call a continuum X *circle-like* if for any $\varepsilon > 0$, X can be covered by a circular chain $\mathcal{C}$ with $\text{mesh}(\mathcal{C}) < \varepsilon$.

Definition 2.7 (See [6,17] for the uniqueness part.) The *pseudo-circle* is the unique circle-like plane separating continuum that can be covered by a decreasing family of circular chains $\{\mathcal{C}_n\}_{n \geq 1}$, such that $\mathcal{C}_{n+1}$ is crooked inside $\mathcal{C}_n$ for each $n \geq 0$, and $\text{mesh}(\mathcal{C}_n) \rightarrow 0$ as n tends to ∞ .

Remark 2.8 Alternatively, the pseudo-circle can also be characterised as circle-like, hereditarily indecomposable plane separating continuum which can be embedded in any two dimensional manifold (see [7]).

Remark 2.9 A pseudo-arc is similarly defined replacing the objects “circular chain” and “circle-like continua” by “linear chain” and “arc-like continua”, respectively. For precise definitions, see for example the book [33]. Here we want to mention several important facts. First, the pseudo-arc is also unique up to homeomorphism (see [6]). Then it follows from both definitions that any non-degenerate proper subcontinuum of a pseudo-circle is a pseudo-arc. Another useful fact is that, any non-degenerate subcontinuum of a pseudo-arc is also a pseudo-arc (this is the main theorem of [32]).

We also need the following Theorem from [22].

Lemma 2.10 (Theorem 3 of [22]) *Let f be a homeomorphism of the pseudo-circle P . Let C be a composant of P and suppose $f(C) = C$. Then f admits a fixed point in C .*

2.3 The Denjoy–Rees Technique: Settings

Denote by $\mathbb{A} = S^1 \times [0, 1]$ for the closed annulus. Let $f : \mathbb{A} \rightarrow \mathbb{A}$ be a homeomorphism which preserves both the orientation and the boundaries. The following definitions are adopted from [5].

Definition 2.11 For an integer $p \geq 1$, a finite family $\mathcal{E}$ of closed disks, contained in $S^1 \times (0, 1)$, is called *p-iterable* if for any $X, Y \in \mathcal{E}$ and integers $-p \leq k, s \leq p$, either $f^k(X) = f^s(Y)$ or $f^k(X) \cap f^s(Y) = \emptyset$.

For any *p-iterable* family of closed disks $\mathcal{E}$ and any $0 \leq n \leq p$, we denote

$$\mathcal{E}^n = \bigcup_{|k| \leq n} f^k(\mathcal{E}), \quad (2.5)$$

where $f(\mathcal{E}) = \{f(X) \mid X \in \mathcal{E}\}$. For any $0 \leq n < p$, define an oriented graph $G(\mathcal{E}^n)$, where the vertices are elements of $\mathcal{E}^n$, and there is an edge from X to Y if and only if $f(X) = Y$. For a map h such that $h(X) \subset X$ and $h(Y) \subset Y$, we say that it commutes with f along edge $X \rightarrow Y$ if $f \circ h(x) = h \circ f(x)$.

Definition 2.12 Let $\mathcal{E}, \mathcal{F}$ be two finite families of closed disks, all contained in the open set $S^1 \times (0, 1)$. By $\mathfrak{s}(\mathcal{E})$ we denote the union of the elements of $\mathcal{E}$. We say $\mathcal{F}$ refines $\mathcal{E}$ if the following conditions hold.

- (1) Every element of $\mathcal{E}$ contains at least one element of $\mathcal{F}$.
- (2) For any $X \in \mathcal{E}, Y \in \mathcal{F}$, either $X \cap Y = \emptyset$ or $Y \subset \text{int}(X)$.

Definition 2.13 Let $\mathcal{E}, \mathcal{F}$ be two families of closed disks. For an integer $p \geq 0$, we say $\mathcal{F}$ is compatible with $\mathcal{E}$ for p iterates, if the following conditions hold.

- (1) $\mathcal{E}$ is p -iterable.
- (2) $\mathcal{F}$ is $(p + 1)$ -iterable.
- (3) $\mathfrak{s}(\mathcal{F}) \subset \mathfrak{s}(\mathcal{E})$, and $\mathcal{F}^{p+1}$ refines $\mathcal{E}^p$.
- (4) For every k with $|k| \leq p$,

$$\mathfrak{s}(\mathcal{F}^{p+1}) \cap f^k(\mathfrak{s}(\mathcal{E})) = f^k(\mathfrak{s}(\mathcal{F})). \quad (2.6)$$

Let $\{\mathcal{E}_{(n)}\}_{n \geq 0}$ denote a sequence of families of closed disks. The following list of hypotheses was formulated in Section 2 of [5].

A₁. For every $n \geq 0$,

- (1) $\mathcal{E}_{(n)}$ is $(n + 1)$ -iterable and the graph $G(\mathcal{E}_{(n)}^n)$ has no cycle.
- (2) $\mathcal{E}_{(n)}^{n+1}$ refines $\mathcal{E}_{(m)}^{m+1}$ for any $0 \leq m < n$;
- (3) $\mathcal{E}_{(n)}$ is compatible with $\mathcal{E}_{(n-1)}$ for n iterates.

A₂. The following holds.

$$\lim_{n \rightarrow \infty} \text{mesh}(\mathcal{E}_{(n)}^n) = 0. \quad (2.7)$$

Next, we want to choose a sequence of homeomorphisms $\{h_n\}_{n \geq 1}$, such that the following hold for every $n \geq 1$.

B₁. The closure $\overline{\{x : h_n(x) \neq x\}}$ is contained in the set $\mathfrak{s}(\mathcal{E}_{(n-1)}^{n-1})$.

B₂. h_n and f commute along edges of the graph $G(\mathcal{E}_{(n-1)}^{n-1})$.

B₃. The following holds.

$$\lim_{n \rightarrow \infty} \text{mesh}(h_1^{-1} \circ \dots \circ h_{n-1}^{-1}(\mathcal{E}_{(n)}^{n+1} \setminus \mathcal{E}_{(n)}^{n-1})) = 0. \quad (2.8)$$

Whenever we have such a sequence of homeomorphisms h_n , for any $n \geq 1$, we define the homeomorphisms ψ_n, g_n as follows.

$$\psi_n = h_n \circ \dots \circ h_1. \quad (2.9)$$

$$g_n = \psi_n^{-1} \circ f \circ \psi_n. \quad (2.10)$$

Finally we set $\psi_0 = \text{id}$ and $g_0 = f$.

2.4 The Denjoy–Rees Technique: Results

The following results proved in [5] form the starting point of our proof. They ensure proper convergence of the sequence of conjugated homeomorphisms, while the conjugacies converge to a continuous map, which provides a semi-conjugacy. Later we will have to adjust this approach, when defining maps h_n to ensure sufficiently fast convergence. This will be possible due to different approach to sets X_n in (2.11). For purposes of [5] sets X_n were specified as the beginning, and then the sequence was refined sufficiently fast. Unfortunately, this approach is likely to lead to a situation as on Fig. 1, destroying chainability or other properties. To deal with this problem, sets X_n will be defined dynamically as the construction progresses (and so the continuum in the fiber is unknown to the very end of the construction).

Lemma 2.14 (Proposition 3.1 in [5]) *Assume that hypotheses **A₁**, **A₂**, **B₁**, **B₂**, **B₃** are satisfied. Then:*

Fig. 1 Fiber nowhere dense in $\mathbb{T}^2$ but large in “blown up” continuum

- (1) The sequence $\{\psi_n\}_{n \geq 0}$ converges uniformly to a continuous surjective map $\psi: \mathbb{A} \rightarrow \mathbb{A}$.
- (2) The sequences $\{g_n\}_{n \geq 1}$ and $\{g_n^{-1}\}_{n \geq 1}$ converge uniformly to a homeomorphism $g: \mathbb{A} \rightarrow \mathbb{A}$ and its inverse g^{-1} , respectively.
- (3) The homeomorphism f is semi-conjugate to g via the map ψ .

Lemma 2.15 (Proposition 3.4 in [5]) Assume that hypotheses **A**₁, **A**₂, **B**₁, **B**₂, **B**₃ are satisfied. Let $K = \bigcap_{n \geq 0} \mathfrak{s}(\mathcal{E}_n)$.

- (1) Suppose $x \in \mathbb{A}$ and there exists $m \in \mathbb{Z}$ such that $x \in f^m(K)$. Let $\{X_n\}_{n \geq |m|} \subset \mathcal{E}_{(n)}^{|m|}$ denote the decreasing sequence of closed disks containing x . Then

$$\psi^{-1}(x) = \bigcap_{n \geq |m|} \psi_n^{-1}(X_n). \quad (2.11)$$

- (2) For every x which does not belong to the orbit of K , $\psi^{-1}(x)$ is a singleton.

3 Minimal Non-invertible Map on the Pseudo-Circle

The main result of this section is the following Theorem.

Theorem 3.1 There exists a homeomorphism $f: \mathbb{A} \rightarrow \mathbb{A}$ with an invariant pseudo-circle $Q \subset \mathbb{A}$ such that (Q, f) is minimal and there exists a pseudo-arc $A \subset Q$ such that

$$\lim_{|n| \rightarrow \infty} \text{diam}(f^n(A)) = 0. \quad (3.1)$$

The proof of Theorem 3.1 goes through the whole section. In fact we will prove more than the statement here. We will provide a semi-conjugacy between two annulus homeomorphisms. The conclusion of the theorem is based on the existence of such a semi-conjugacy.

Let us give an outline before getting into details. In Sect. 3.1, we recall Handel's construction of an annulus homeomorphism f , which admits an invariant pseudo-circle $P \subset \mathbb{A}$. Then, we construct the semi-conjugacy in Sect. 3.2 between $(\mathbb{A}, f)$ and $(\mathbb{A}, g)$ for some other homeomorphism g on $\mathbb{A}$. This is the most delicate part of the construction. In Sect. 3.3, we show the pre-image of P under the semi-conjugacy is still a pseudo-circle, and, finally, the rest of the proof of Theorem 3.1 is given in Sect. 3.4.

Based on Theorem 3.1, the proof of Theorem 1.1 is given in Sect. 4.

3.1 Minimal Homeomorphism on the Pseudo-Circle

Here we define a homeomorphism f on the closed annulus $\mathbb{A}$ which preserves a minimal invariant pseudo-circle P . The construction follows directly from [19], so we refer to this

paper for more details (and some pictures for the construction). Roughly speaking, the pseudo-circle in construction in [19] is obtained as a nested sequence of annuli. Annulus in the n -th step of the construction is identified with $S^1 \times [0, 1]$ and then circle S^1 is split into exactly m_n sub-intervals of equal lengths, say, $I_1^{(0)}, \dots, I_{m_n}^{(0)}$. This way we define circular chain C_n , taking consecutive link of the form $(I_k^{(0)} \cup I_{k+1}^{(0)}) \times (0, 1)$, for $k = 1, \dots, m_n - 1$, and putting as the last $(I_{m_n}^{(0)} \cup I_1^{(0)}) \times (0, 1)$. This allows us to express the main result of [19] in the following form.

Lemma 3.2 *Let $\mathbb{A} = S^1 \times [0, 1]$. There exists a sequence of circular chains $\{C_n\}_{n \geq 0}$ with links whose closures are closed discs, and a homeomorphism $f : \mathbb{A} \rightarrow \mathbb{A}$, with the following properties.*

(1) *The following conditions are satisfied.*

$$\overline{\mathfrak{s}(C_0)} \subset S^1 \times (0, 1). \quad (3.2)$$

$$\overline{\mathfrak{s}(C_{n+1})} \subset \mathfrak{s}(C_n), \text{ for any } n \geq 0. \quad (3.3)$$

(2) *There is an f -invariant minimal pseudo-circle P such that*

$$P = \bigcap_{n \geq 0} \mathfrak{s}(C_n) = \bigcap_{n \geq 0} \overline{\mathfrak{s}(C_n)}. \quad (3.4)$$

3.2 Construction of the Semi-Conjugacy

In this subsection, we construct the semi-conjugacy. The main tool from this subsection is Lemma 3.4.

Lemma 3.3 *For the homeomorphism $f : \mathbb{A} \rightarrow \mathbb{A}$ constructed in Lemma 3.2, there exists a sequence of pseudo-arcs $A_0 \supset A_1 \supset \dots$ contained in P , such that,*

$$\lim_{n \rightarrow \infty} \text{diam}(A_n) = 0. \quad (3.5)$$

Moreover, for all $n \geq 0$, each A_{n+1} is contained in an internal composant of A_n . In particular, $\bigcap_{n \geq 0} A_n$ is a singleton, containing a unique point, called p_ .*

Proof By Lemma 2.2, there exists an internal composant in P . So, we can choose $A_0 \subset P$ to be a pseudo-arc in P , with diameter smaller than 2^{-1} , which is contained in an internal composant of P . Then, for the same reason, we can choose A_1 to be a pseudo-arc contained in some internal composant of A_0 , of diameter less than 2^{-2} . Inductively, we can choose a sequence of pseudo-arcs, namely $\{A_n\}_{n \geq 1}$, such that, for any $n \geq 0$, the diameter $\text{diam}(A_n)$ is smaller than 2^{-n} , and each A_{n+1} is contained in an internal composant of A_n . The proof is completed. $\square$

Now we will perform a construction inspired by [5]. For our proofs it is essential to have both conditions (3.10) and (3.11) below satisfied and have a pseudoarc in one of the fibres. To do it following a scheme from [5] we would start with a continuum $A \subset P$ that we want to have in a fibre and then we use a decreasing family of its neighborhoods U_n such that $\bigcap_n U_n = A$. However, if one checks the details carefully, then it becomes evident that in this approach always one of the conditions (3.10) or (3.11) will be broken and it is impossible to satisfy them simultaneously. Therefore we need an adjustment in the construction. The idea here is that we do not fix the sequence U_n apriori, but construct it inductively (U_{n+1} can be

defined after $h_1, \dots, h_n$ are known). In the process, we have to ensure that the sequence of connected open set U_n is nested (i.e. $\overline{U_{n+1}} \subset U_n$) and continuum obtained at the end $A = \bigcap_n U_n$ is within the family of continua that we are interested in (in our case, the pseudo-arc). These are main issues where we have to be especially careful.

Lemma 3.4 *Denote by f , P and p_* the objects obtained in Lemmas 3.2 and 3.3. Then there exist homeomorphisms h_n on $\mathbb{A}$ such that if we denote*

$$\psi_n := h_n \circ \dots \circ h_1. \quad (3.6)$$

$$g_n = \psi_n^{-1} \circ f \circ \psi_n. \quad (3.7)$$

the the following limits exist (in the topology of uniform convergence)

$$g := \lim_{n \rightarrow \infty} g_n. \quad (3.8)$$

$$\psi := \lim_{n \rightarrow \infty} \psi_n \quad (3.9)$$

and dynamical system $(\mathbb{A}, f)$ is a factor of $(\mathbb{A}, g)$ by the factor map ψ . For every $x \in \mathbb{A}$ not in the orbit of p_ (i.e. $f^n(x) \neq p_*$ for every $n \in \mathbb{Z}$) the set $\psi^{-1}(x)$ is a singleton, $\psi^{-1}(p_*)$ is a pseudoarc and additionally there is an increasing sequence k_ℓ such that:*

$$h_{\ell+1}^{-1}(\overline{\mathfrak{s}(C_{k_\ell+1})}) \subset \mathfrak{s}(C_{k_\ell}). \quad (3.10)$$

$$\text{mesh}((h_\ell \circ \dots \circ h_1)^{-1}(C_{k_\ell})) < 2^{-\ell}. \quad (3.11)$$

for every $\ell \geq 1$

Proof Let $p_* \in \bigcap_n A_n \subset P$ be the pseudo-arc provided by Lemma 3.3. Recall that in the construction of f in Lemma 3.2, we have chosen a sequence of circular chains $\{C_n\}_{n \geq 0}$, such that $P = \bigcap_{n \geq 0} \mathfrak{s}(C_n)$. Then, define V_n to be an element contained in C_n which contains p_* . We also require (refining sequence C_n when necessary), that the elements of $\{f^i(\overline{V_n})\}_{i=-(n+2)}^{n+2}$ are pairwise disjoint and the sequence V_n is nested.

As U_0 we take a neighborhood of the pseudo-arc A_0 such that $U_0 \subset \mathfrak{s}(C_0)$ and $\{f^i(\overline{U_0})\}_{i=-2}^2$ has pairwise disjoint elements. Further sets U_n will be defined inductively.

Regarding Lemmas 2.14 and 2.15, we define inductively the following objects.

- (1) The sequence of families of closed disks, denoted by $\{\mathcal{E}_{(n)}\}_{n \geq 0}$, satisfying conditions **A₁** and **A₂** in Sect. 2.3.
- (2) The sequence of homeomorphisms $\{h_n\}_{n \geq 1}$ on the annulus $\mathbb{A}$ satisfying conditions **B₁**, **B₂**, **B₃** in Sect. 2.3.

We start by setting $\mathcal{E}_{(0)} = \{\overline{U_0}\}$, and $k_0 = 0$. It is immediate from definition that $\mathcal{E}_{(0)}$ satisfies conditions of **A₁** (only item (a) of **A₁** needs to be checked for $n = 0$). Note that

$$\mathcal{E}_{(0)}^1 = \{f^{-1}(\overline{U_0}), \overline{U_0}, f(\overline{U_0})\}. \quad (3.12)$$

Let $U \subset \overline{U} \subset U_0$ be another neighborhood of A_0 , such that $\overline{U}$ is homeomorphic to a closed disc, and is contained in the union of links of a chain cover of A_0 by discs of diameter at most 2^{-1} . In other words, we may treat U as an open set and as a chain cover. Take any integers $s_1 > 0$ such that $V_{s_1} \subset U$.

Then we define the homeomorphism $h_1: \mathbb{A} \rightarrow \mathbb{A}$, with the following properties.

$$h_1(U) = V_{s_1}. \quad (3.13)$$

$$h_1(p_*) = p_*. \quad (3.14)$$

$$h_1 \text{ restricted to } \mathbb{A} \setminus U_0 \text{ is the identity.} \quad (3.15)$$

Note that the condition $\mathbf{B}_1$ for $n = 1$ follows from (3.15). Condition $\mathbf{B}_2$ is empty for $n = 1$. Denote $\gamma = \text{diam } A_0/2$. Let $k_1 > 0$ be an integer large enough, so that

$$\text{mesh}(h_1^{-1}(\mathcal{C}_{k_1})) < 2^{-1}$$

and put $U_{k_1} = U$. Define now

$$\mathcal{E}_{(1)} := \{\overline{V_{s_1}}\}. \quad (3.16)$$

By definition $\mathcal{E}_{(1)}$ is $(s_1 + 1)$ -iterable, and $s_1 \geq 1$. It also follows that $\mathcal{E}_{(1)}^2$ refines $\mathcal{E}_{(0)}^1$, and $\mathcal{E}_{(1)}$ is compatible with $\mathcal{E}_{(0)}$ for 1 iterate. In other words, the three conditions in $\mathbf{A}_1$ are all satisfied for $n = 1$.

Now we will proceed with main construction, which will be repeated inductively.

Take any $m > k_1 + s_1$ sufficiently large so that, if $R \ni p_*$ is a link of $\mathcal{C}_m$ then the sets $\{f^i(\overline{R}) : i = -3s_1, \dots, 3s_1\}$ are pairwise disjoint and $\overline{R} \subset V_{s_1}$. Let s be any integer such that $A_s \subset R$.

Since in the definition of h_1 we only require (3.13) and (3.14), we can freely modify it on points of $U_{k_1} \supset A_0$, that is we may assume that $\text{diam } h_1^{-1}(A_s)$ is arbitrarily close to $\text{diam } U_{k_1}$, in particular exceeds γ .

Note that $A_s \subset P \subset \mathfrak{s}(\mathcal{C}_m)$. Let $m' > m$ be such that

$$f^i(\mathfrak{s}(\mathcal{C}_{m'})) \subset \mathfrak{s}(\mathcal{C}_m) \quad \text{for } |i| \leq 1,$$

and

$$\text{mesh } h_1^{-1}(\mathcal{C}_{m'}) < 2^{-2}.$$

We also assume that $h_1^{-1}(\mathcal{C}_{m'-1})$ defines a chain cover of $h_1^{-1}(A_s)$, which is a refinement of U , treated as a chain. Recall that $\mathcal{C}_{m'}$ is crooked inside $\mathcal{C}_{m'-1}$ by definition. Let $p_* \in W \subset \overline{W} \subset U_{k_1}$ be a neighborhood of $h_1^{-1}(A_s)$ obtained as the union of elements of a 2^{-2} -chain cover of the pseudoarc $h_1^{-1}(A_s)$ consisting of elements of $h_1^{-1}(\mathcal{C}_{m'})$. Since links of $\mathcal{C}_{m'}$ are connected, W is connected as well.

If we view U_{k_1} and W as chains, then we may assume that W is $1/4$ -chain crooked inside U_{k_1} (in particular, it intersects all links that U_{k_1} is union of). Let W' be an open neighborhood of A_s such that $\overline{W'}$ is homeomorphic to a closed disc and

$$A_s \subset W' \subset \overline{W'} \subset h_1(W) \subset V_{s_1}$$

and take any $s_2 > m' + k_1 + s_1$ such that $\overline{V_{s_2}} \subset W'$.

Recall that the elements of $\{f^i(\overline{V_{s_2}}) : |i| \leq 2\}$ are pairwise disjoint sets. Then we can define the homeomorphism h_2 satisfying the following properties.

$$h_2(W') = V_{s_2}, \quad (3.17)$$

$$h_2(p_*) = p_*, \quad (3.18)$$

$$h_2 \text{ commutes with } f \text{ on } \bigcup_{|i| \leq 1} f^i(\overline{h_1(W)}), \quad (3.19)$$

$$h_2 \text{ restricted to } \mathbb{A} \setminus \bigcup_{i=-1}^1 f^i(h_1(W)) \text{ is the identity.} \quad (3.20)$$

For (3.19), note that the sets $f^{-1}(V_{s_1})$, V_{s_1} , and $f(V_{s_1})$ are pairwise disjoint, so first we define the restriction of h_2 to $\overline{W'}$ and then extend it to V_{s_1} so that $h_2|_{(V_{s_1} \setminus h_1(W))}$ is the identity. Next

put $f^i \circ h_2|_{V_{s_1}} \circ f^{-i}$ for $|i| \leq 1$, and finally put $h_2(x) = x$ for all other points $x \in \mathbb{A}$. This way a homeomorphism h_2 on $\mathbb{A}$ is defined, which satisfies (3.19) and (3.20).

Now take k_2 sufficiently large, so that

$$h_2^{-1}(\overline{\mathfrak{s}(C_{k_2})}) \subset \mathfrak{s}(C_{k_1}), \quad (3.21)$$

$$\text{mesh}((h_2 \circ h_1)^{-1}(C_{k_2})) < 2^{-2}, \quad (3.22)$$

and put $U_{k_2} = h_1^{-1}(W')$.

As long as (3.22) is fairly easy to satisfy, (3.21) is possible since

$$\{x : h_2(x) \neq x\} \subset \cup_{|i| \leq 1} f^i(h_1(W)) \subset \cup_{|i| \leq 1} f^i(\mathfrak{s}(C_{m'})) \subset \mathfrak{s}(C_m) \subset \mathfrak{s}(C_{k_1}),$$

in particular $h_2^{-1}(P) \subset \mathfrak{s}(C_{k_1})$. Note that U_{k_2} is connected, so we also may view it as a chain with links induced by W . To finish the construction, put

$$\mathcal{E}_{(2)} := \{\overline{V_{s_2}}\}. \quad (3.23)$$

Directly from the definition of h_2 , conditions **B**₁ and **B**₂ are satisfied for $n = 2$. Sets $f^i(V_{s_2})$ are pairwise disjoint for $|i| \leq s_2$ and $s_2 \geq 4$, so we obtain that $\mathcal{E}_{(2)}$ is at least 4-iterable. Since V_{s_2} is a neighborhood of p_* , it is easy to satisfy the requirement that $\mathcal{E}_{(2)}^3$ refines $\mathcal{E}_{(1)}^2$, and clearly also $\mathcal{E}_{(2)}$ is compatible with $\mathcal{E}_{(1)}$ for 2 iterates. Summing up, the three conditions in **A**₁ are easily satisfied for $n = 2$.

Assume now that for $i = 1, 2, \dots, n$ we have already defined increasing sequences s_i, k_i , open sets U_{k_i} and maps $h_i : \mathbb{A} \rightarrow \mathbb{A}$ such that $h_i(p_*) = p_*$ and:

- (1) $U_{k_i} \subset \overline{U_{k_i}} \subset U_{k_{i-1}}$, $h_1 \circ \dots \circ h_i(U_{k_i}) = V_{s_i}$ and for some $s > s_n$ we have $A_s \subset V_{s_n}$ and $\text{diam}(h_1 \circ \dots \circ h_n)^{-1}(A_s) > \gamma$.
- (2) $\mathcal{E}_{(n)} = \{\overline{V_{s_n}}\}$ and conditions **A**₁, **B**₁, **B**₂ and **B**₃ are satisfied for each $i \leq n$.
- (3) Each $\overline{U_{k_i}}$ is homeomorphic to a closed disc, we can present each set U_{k_i} as a union of elements of a $1/2^i$ -chain and under this representation U_{k_i} is crooked in $U_{k_{i-1}}$.
- (4) The following conditions are satisfied:

$$h_{i-1}^{-1}(\overline{\mathfrak{s}(C_{k_i})}) \subset \mathfrak{s}(C_{k_{i-1}}). \quad (3.24)$$

$$\text{mesh}((h_i \circ \dots \circ h_1)^{-1}(C_{k_i})) < 2^{-i}. \quad (3.25)$$

Take any $m > k_n + s_n$ sufficiently large so that, if $R \ni p_*$ is a link of $\mathcal{C}_m$ then sets $\{f^i(\overline{R}) : i = -3s_n, \dots, 3s_n\}$ are pairwise disjoint and $\overline{R} \subset V_{s_n}$. Let r be any integer such that $A_r \subset R$. The only requirements on h_n are that $h_n(p_*) = p_*$ and $h_n(h_{n-1} \circ \dots \circ h_1(U_{k_n})) = V_{s_n}$. Furthermore $A_r \subset A_s \subset V_{s_n}$ and $\text{diam}(h_n \circ \dots \circ h_1)^{-1}(A_s) > \gamma$. Therefore, we may adjust definition of h_n on the set $h_{n-1} \circ \dots \circ h_1(U_{k_n})$ in such a way that also $\text{diam}(h_n \circ \dots \circ h_1)^{-1}(A_r) > \gamma$ and all other assumed properties hold (in particular, we have to adjust it also on some images of U_{k_n} so that commutativity in **B**₂ remains satisfied).

Note that $A_r \subset P \subset \mathfrak{s}(C_m)$. Let $m' > m$ be such that

$$f^i(\mathfrak{s}(C_{m'})) \subset \mathfrak{s}(C_m) \quad \text{for } |i| \leq n,$$

and

$$\text{mesh}((h_n \circ \dots \circ h_1)^{-1}(C_{m'})) < 2^{-n-1}.$$

We also assume that $(h_n \circ \dots \circ h_1)^{-1}(C_{m'-1})$ defines a chain cover of $(h_n \circ \dots \circ h_1)^{-1}(A_r)$, which is a refinement of U_{k_n} , treated as a chain. Let $p_* \in W \subset \overline{W} \subset U_{k_n}$ be a neighborhood of $(h_n \circ \dots \circ h_1)^{-1}(A_r)$ obtained as the union of elements of a 2^{-n-1} -chain cover of the

pseudoarc $(h_n \circ \dots \circ h_1)^{-1}(A_r)$ consisting of elements of $(h_n \circ \dots \circ h_1)^{-1}(C_{m'})$. Since links of $C_{m'}$ are connected, W is connected as well.

If we view U_{k_n} and W as chains, then we may assume that W is 2^{-n-1} -chain crooked inside U_{k_n} (in particular intersects all links that U_{k_n} is union of). Let W' be an open neighborhood of A_r such that W' is homeomorphic to a closed disc and

$$A_r \subset W' \subset \overline{W'} \subset h_n \circ \dots \circ h_1(W) \subset V_{s_n}$$

and take any $s_{n+1} > m' + k_n + s_n$ such that $\overline{V_{s_{n+1}}} \subset W'$.

Recall that the elements of $\{f^i(\overline{V_{s_{n+1}}}) : |i| \leq n+1\}$ are pairwise disjoint sets. Then we can define the homeomorphism h_{n+1} satisfying the following properties.

$$h_{n+1}(W') = V_{s_n}, \quad (3.26)$$

$$h_{n+1}(p_*) = p_*, \quad (3.27)$$

$$h_{n+1} \text{ commutes with } f \text{ on } \bigcup_{|i| \leq n} f^i(\overline{h_n(W)}), \quad (3.28)$$

$$h_{n+1} \text{ restricted to } \mathbb{A} \setminus \bigcup_{i=-n}^n f^i(h_n(W)) \text{ is the identity.} \quad (3.29)$$

For (3.28), note that the sets $f^i(V_{s_n})$, $|i| \leq n$ are pairwise disjoint, so first we define the restriction of h_{n+1} to $h_n(W) \subset V_{s_n}$, extend it by identity to V_{s_n} , put $f^i \circ h_{n+1}|_{V_{s_n}} \circ f^{-i}$ for $i = -n, \dots, n$, and finally put $h_{n+1}(x) = x$ for all other points $x \in \mathbb{A}$. In this way a homeomorphism h_{n+1} on $\mathbb{A}$, which satisfies (3.28) is defined.

Now take k_{n+1} sufficiently large, so that

$$h_{n+1}^{-1}(\overline{\mathfrak{s}(C_{k_{n+1}})}) \subset \mathfrak{s}(C_{k_n}), \quad (3.30)$$

$$\text{mesh}((h_{n+1} \circ \dots \circ h_1)^{-1}(C_{k_{n+1}})) < 2^{-n-1} \quad (3.31)$$

and put $U_{k_{n+1}} = (h_{n+1} \circ \dots \circ h_1)^{-1}(W')$. Note that $U_{k_{n+1}}$ contains $(h_n \circ \dots \circ h_1)^{-1}(A_r)$, in particular $\text{diam}(U_{k_{n+1}}) > \gamma$.

As before (3.31) is easy to satisfy and (3.30) is possible, because

$$\begin{aligned} \{x : h_{n+1}(x) \neq x\} &\subset \bigcup_{|i| \leq n} f^i(h_n \circ \dots \circ h_1(W)) \\ &\subset \bigcup_{|i| \leq n} f^i(\mathfrak{s}(C_{m'})) \subset \mathfrak{s}(C_m) \subset \mathfrak{s}(C_{k_n}), \end{aligned}$$

in particular $h_{n+1}^{-1}(P) \subset \mathfrak{s}(C_{k_n})$. Note that $U_{k_{n+1}}$ is connected, so we also may view it as a chain with links induced by W . Additionally, modifying h_{n+1} accordingly on $U_{k_{n+1}}$, we may assume that there is $\hat{s} > 0$ such that $A_{\hat{s}} \subset V_{s_{n+1}}$ such that $\text{diam}((h_{n+1} \circ \dots \circ h_1)^{-1}(A_{\hat{s}})) > \gamma$.

To finish the construction, put

$$\mathcal{E}_{(n+1)} := \{\overline{V_{s_{n+1}}}\}. \quad (3.32)$$

Directly from the definition of h_{n+1} , conditions **B**₁ and **B**₂ are satisfied for $n+1$. Sets $f^i(V_{s_{n+1}})$ are pairwise disjoint for $|i| \leq s_{n+1}$ and $s_{n+1} \geq 2^{n+1}$, we obtain that $\mathcal{E}_{(n+1)}$ is at least 2^{n+1} -iterable. Since $V_{s_{n+1}}$ is a neighborhood of p_* , it is easy to satisfy the requirement that $\mathcal{E}_{(n+1)}^{n+2}$ refines $\mathcal{E}_{(n)}^{n+1}$, and clearly also $\mathcal{E}_{(n+1)}$ is compatible with $\mathcal{E}_{(n)}$ for $n+1$ iterates. Summing up, the three conditions in **A**₁ are easily satisfied for $n+1$. This completes the induction.

Since family of sets V_n intersects in a single point, condition **A**₂ is satisfied, proving all the properties claimed for the construction.

Define for all $n \geq 1$,

$$\psi_n := h_n \circ \cdots \circ h_1. \quad (3.33)$$

$$g_n = \psi_n^{-1} \circ f \circ \psi_n. \quad (3.34)$$

Then, we apply Lemma 2.14, to obtain a homeomorphism g and a semi-conjugacy ψ , as the limits

$$g := \lim_{n \rightarrow \infty} g_n. \quad (3.35)$$

$$\psi := \lim_{n \rightarrow \infty} \psi_n. \quad (3.36)$$

Observe that the set $A = \bigcap_n U_{k_n}$ is continuum, as intersection of decreasing family of continua, all of diameter at least γ . As we announced in the construction, all sets U_{k_n} can be viewed as 2^{-n} -chains, so A is chainable, and each $U_{k_{n+1}}$ is crooked inside U_{k_n} , so A is hereditarily indecomposable, therefore a pseudoarc (see [6]).

Now we apply Lemma 2.15, obtaining that

$$\psi^{-1}(y) \text{ is a singleton for any } y \notin \overline{\{f^n(p_*)\}_{n \in \mathbb{Z}}}. \quad (3.37)$$

$$\psi^{-1}(p_*) = \bigcap_{n \geq 0} \psi_n^{-1}(\overline{V_{s_n}}) = \bigcap_{n \geq 0} U_{k_n} = A. \quad (3.38)$$

The proof of Lemma 3.4 is completed. $\square$

3.3 Properties of “Blow Up” of P

In this subsection, we prove the following.

Theorem 3.5 *Let ψ denote the semi-conjugacy map $\mathbb{A}$ obtained by construction in Lemma 3.4, and denote $Q = \psi^{-1}(P)$. Then Q is a pseudo-circle.*

Before we can prove the above result, we again need some preparation.

Recall that by Lemma 3.4 all preimages $\psi^{-1}(x)$ for $x \in \mathbb{A}$ are connected, so ψ is a monotone map. It follows immediately that Q is a continuum.

Lemma 3.6 *The pseudo-arc $A = \psi^{-1}(p_*) \subset Q$ is nowhere dense in Q .*

Proof Suppose to the contrary that A contains interior in Q . Equivalently, there exists an open subset $W \subset S^1 \times (0, 1)$ such that $W \cap Q \neq \emptyset$ and $W \cap Q = W \cap A$. Now we consider some connected component U of $W \setminus Q$ such that $\overline{U} \cap Q \neq \emptyset$. Since Q is closed, U is open. We can choose an arc $\alpha : [0, 1] \rightarrow W$ connecting some point in U to $\partial U \cap Q$. More precisely, we can define $\alpha : [0, 1] \rightarrow W$, such that, $\alpha(0) \in A$ and $\alpha((0, 1]) \subset U$. In particular, $\alpha((0, 1]) \cap Q = \emptyset$. Then $\psi|_{\alpha([0, 1])}$ is a homeomorphism, and by the construction of ψ (see Lemma 3.4), $\psi \circ \alpha(0) = p_*$, while $\psi \circ \alpha((0, 1]) \cap P = \emptyset$. However, $\psi \circ \alpha : [0, 1] \rightarrow \psi(W)$ is an arc containing both p_* and some point in the complement of P . Since p_* is contained in an internal composant of P , $\psi \circ \alpha([0, 1])$ intersects all composants of P . In particular it contains many points of P , which is a contradiction. The proof of the lemma is completed. $\square$

Lemma 3.7 *There exists a dense G_δ subset $I \subset Q$, such that, for every point $x \in I$,*

$$\psi^{-1} \circ \psi(x) = \{x\}. \quad (3.39)$$

Proof Note that $f|_Q : Q \rightarrow Q$ is a homeomorphism, and by Lemma 3.6, A is nowhere dense in Q . Thus, $f^n(A)$ is nowhere dense for all $n \in \mathbb{Z}$. Consider the set $\bigcup_{n \in \mathbb{Z}} f^n(A)$, which is a countable union of nowhere dense closed sets. Denote the complement by $I = Q \setminus \bigcup_{n \in \mathbb{Z}} f^n(A)$. Then by the Baire category theorem, I as an intersection of a sequence of open and dense sets is a dense G_δ set. Moreover, it follows from (3.37) and (3.38) that, for any $x \in I$,

$$\psi^{-1} \circ \psi(x) = \{x\}. \quad (3.40)$$

The proof is completed. $\square$

Lemma 3.8 *Q is an indecomposable continuum.*

Proof Suppose to the contrary that Q is decomposable. Then we can write $Q = K \cup L$, where K and L are proper subcontinua of Q . It follows that $\psi(K)$ and $\psi(L)$ are two subcontinua of P as well as $P = \psi(K) \cup \psi(L)$. Since P is indecomposable, it follows either $\psi(K)$ or $\psi(L)$ is equal to P . Suppose without loss of generality that $\psi(L) = P$. Now recall that $K \setminus L = Q \setminus L \neq \emptyset$ is open in Q . So by Lemma 3.7, there exists $x \in (K \setminus L) \cap I$, where $I = (Q \setminus \bigcup_{n \in \mathbb{Z}} f^n(A))$. In particular, $\psi^{-1} \circ \psi(x) = \{x\}$. On the other hand, by assumption that $\psi(L) = P$, there always exists $y \in L$ such that $\psi(y) = x$. This provides a contradiction. Thus, Q is indecomposable. $\square$

Lemma 3.9 *The continuum Q is hereditarily indecomposable.*

Proof By comparing definitions, let us first point out the fact that, the composants of Q are precisely the preimages of the composants of P under the semi-conjugacy ψ .

Suppose to the contrary that a non-degenerate subcontinuum $M \subset Q$ is decomposable. By Lemma 3.8 M is proper, hence $Q \setminus M$ is open in Q , and by Lemma 3.7, $(Q \setminus M) \cap I \neq \emptyset$, where $I = Q \setminus \bigcup_{n \in \mathbb{Z}} f^n(A)$, restricted to which, the semi-conjugacy ψ is 1 to 1. In particular $\psi(M) \neq P$, so $\psi(M)$ is a proper subcontinuum of P . It follows that, $\psi(M)$ is either a singleton or a pseudo-arc. It is not hard to see that $\psi(M)$ can not be a singleton, because otherwise, by the construction of ψ , $M \subset f^n(A)$ for some $n \in \mathbb{Z}$. Then, as a non-degenerate subcontinuum of the pseudo-arc $f^n(A)$, by Remark 2.9, M itself has to be a pseudo-arc, which is indecomposable.

Now we suppose $\psi(M)$ is a pseudo-arc. By assumption, there are proper subcontinua K and L of M , such that $K \cup L = M$. It follows that $\psi(K)$ and $\psi(L)$ are two subcontinua of $\psi(M)$, which in turn is a subcontinuum of P . They satisfy

$$\psi(K) \cup \psi(L) = \psi(M). \quad (3.41)$$

Since $\psi(M)$ is hereditarily indecomposable, either $\psi(L)$ or $\psi(K)$ is equal to $\psi(M)$. Let us assume $\psi(L) = \psi(M)$ for definiteness. Then, for any $x \in K \setminus L$, there exists some $y \in L$, such that $\psi(x) = \psi(y)$. It follows that $x \notin I$, i.e., there exists some $n \in \mathbb{Z}$ such that $x \in f^n(A)$.

Lemma 3.8 tells us Q is indecomposable. Now by Lemma 2.10, $f|_Q$ is fixed point free, which implies that no two members of the sequence $\{f^n(A)\}_{n \in \mathbb{Z}}$ of pseudo-arcs belong to the same component of Q . Thus, $K \setminus L$ is contained in $f^n(A)$ for some $n \in \mathbb{Z}$. Up to composing by f^{-n} if necessary, we simply assume that $K \setminus L \subset A$. Since $K \setminus L = M \setminus L$ is open in M , there is some nonempty open set $W \subset S^1 \times (0, 1)$ such that,

$$W \cap (K \setminus L) = W \cap M \neq \emptyset. \quad (3.42)$$

$$W \cap L = \emptyset. \quad (3.43)$$

Since $\psi(K \setminus L) = p_*$, as in the proof of Lemma 3.6, we can find some arc $\beta : [0, 1] \rightarrow W$ such that

$$\beta(0) \in A. \quad (3.44)$$

$$\beta((0, 1]) \cap M = \emptyset. \quad (3.45)$$

Since $\psi(M)$ is a pseudo-arc in P containing the point p_* , for each $i \in \mathbb{Z}$, it follows $A_i \cap \psi(M) \neq \emptyset$. Since the sets A_i are pseudo-arcs, it follows either $A_i \subset \psi(M)$ or $\psi(M) \subset A_i$. Since by Corollary 3.11, $\lim_{n \rightarrow \infty} \text{diam}(A_n) = 0$, there is $k > 0$ such that $A_k \subset \psi(M)$. Now, by Lemma 3.3, $A_{k+1} \subset A_k$ contains p_* , and A_{k+1} is in an internal composant of A_k .

Thus, denote $\beta' := \psi \circ \beta([0, 1])$ which is an arc in $\mathbb{A}$ containing the point p_* . Since β' is in an internal composant of A_k , it must intersect all the composants of A_k . Then there exists $y \in A_k \cap \beta'$, with $y \neq p_*$. But $y \in \psi(M)$, and there is some $x \in M$ such that $\psi(x) = y$. Then $x \in L$, because otherwise $\psi(x) = p_*$. But then $x \in W \cap L$ which is a contradiction to (3.43). So Q is hereditarily indecomposable, and the proof is completed. $\square$

Lemma 3.10 *The continuum Q is circle-like and separates $\mathbb{A}$.*

Proof By the construction, denote by $\{C_k\}_{k \geq 0}$ for the defining sequence of circular chains for P , and let $\{h_n\}_{n \geq 0}$ and $\{\psi_n\}_{n \geq 0}$ be the sequence of homeomorphisms constructed in Lemma 3.4. Since the sequence $\{\text{mesh}(C_k)\}_{k \geq 0}$ converges to 0, we can choose an increasing integer subsequence $\{k_j\}_{j \geq 0}$, such that $k_0 = 0$ and for any $j \geq 0$ we have

$$h_{j+1}^{-1}(\overline{\mathfrak{s}(C_{k_{j+1}})}) \subset \mathfrak{s}(C_{k_j}). \quad (3.46)$$

$$\text{mesh}(\psi_{j+1}^{-1}(C_{k_{j+1}})) < 2^{-j}. \quad (3.47)$$

Now we denote $\mathcal{D}_j = C_{k_j}$ for all $j \geq 0$. Clearly, $\{\psi_n^{-1}(\mathcal{D}_n)\}_{n \geq 0}$ is a sequence of circular chains, and (3.46) show inductively this sequence is in fact decreasing. By (3.47), it follows that the continuum $\bigcap_{n \geq 0} \mathfrak{s}(\mathcal{D}_n)$ is circle-like. So we are left to show the following.

$$Q = \bigcap_{n=0}^{\infty} \psi_n^{-1}(\mathfrak{s}(\mathcal{D}_n)). \quad (3.48)$$

Choose any $x \in \bigcap_{n=0}^{\infty} \psi_n^{-1}(\mathfrak{s}(\mathcal{D}_n))$. By the choice, $\psi_n(x) \in \mathfrak{s}(\mathcal{D}_n)$ for all $n \geq 0$. Noting (3.36) and (3.3), it follows that $\psi(x) \in \mathfrak{s}(\mathcal{D}_n)$ for each $n \geq 0$. Thus,

$$\psi(x) \in \bigcap_{n=0}^{\infty} \mathfrak{s}(\mathcal{D}_n) = P. \quad (3.49)$$

Then $x \in \psi^{-1} \circ \psi(x) \subset \psi^{-1}(P) = Q$ and so $\bigcap_{n=0}^{\infty} \psi_n^{-1}(\mathfrak{s}(\mathcal{D}_n)) \subseteq Q$.

Next, suppose $Q' = \bigcap_{n=0}^{\infty} \psi_n^{-1}(\mathfrak{s}(\mathcal{D}_n))$ is a proper subcontinuum of Q . By definition $\psi_n(p_*) = p_*$ for all $n \geq 1$ and therefore $\psi_n^{-1}(p_*) = p_* \in \psi_n^{-1}(\mathfrak{s}(\mathcal{D}_n))$ for each $n \geq 0$, which yields that $p_* \in Q'$. Then Q' as a proper subcontinuum of Q , must be contained in the composant $C(p_*)$ of Q that contains p_* . Then $\psi(Q')$ is a proper subcontinuum of P . So, $\psi(Q')$ is either a pseudo-arc or a singleton. In particular, $\psi(Q')$ does not separate $\mathbb{A}$.

On the other hand, we know $\mathfrak{s}(\mathcal{D}_0)$ separates $\mathbb{A}$. More precisely, the set $\mathbb{A} \setminus \mathfrak{s}(\mathcal{D}_0)$ has two connected components, and we call them S and T , respectively. Since $\psi(Q')$ is not separating, there exists an arc $\beta : [0, 1] \rightarrow \mathbb{A}$ such that $\beta(0) \in \psi(S)$ and $\beta(1) \in \psi(T)$, and $\beta([0, 1]) \cap \psi(Q') = \emptyset$. Let $L := \psi^{-1}(\beta([0, 1]))$. Then L is a continuum contained in $S^1 \times (0, 1)$. Moreover,

$$L \cap Q' = \emptyset. \quad (3.50)$$

However, $L \cap \psi_n^{-1}(\overline{\mathfrak{s}(\mathcal{D}_n)}) \neq \emptyset$ for any $n \geq 0$, since it intersects both S and T . Then for any $n \geq 0$, we choose $y_n \in L \cap \psi_n^{-1}(\overline{\mathfrak{s}(\mathcal{D}_n)})$. By extracting a subsequence if necessary, we can assume the sequence $\{y_n\}_{n \geq 0}$ converges to a point $y_* \in L$.

We can also deduce from (3.46) the following for all $n > k \geq 1$.

$$\begin{aligned}\psi_n^{-1}(\overline{\mathfrak{s}(\mathcal{D}_n)}) &= \psi_{n-1}^{-1} h_n^{-1}(\overline{\mathfrak{s}(\mathcal{D}_n)}) \\ &\subset \psi_{n-1}^{-1}(\overline{\mathfrak{s}(\mathcal{D}_{n-1})}) \subset \dots \\ &\subset \psi_k^{-1}(\overline{\mathfrak{s}(\mathcal{D}_k)}).\end{aligned}\quad (3.51)$$

Therefore $y_n \in \psi_k^{-1}(\overline{\mathfrak{s}(\mathcal{D}_k)})$ for any $n \geq k \geq 0$. It follows that $y \in \psi_k^{-1}(\overline{\mathfrak{s}(\mathcal{D}_k)})$ for all $k \geq 0$. Thus $y \in L \cap Q'$, which is a contradiction to (3.50). This contradiction shows $Q' = Q$. We have completed the proof of the lemma. $\square$

Proof (End of Proof of Proposition 3.5) It follows directly from Lemma 3.9, Lemma 3.10 and the characterization stated in Remark 2.8, that Q is a pseudo-circle. $\square$

Corollary 3.11 *Let $f: \mathbb{A} \rightarrow \mathbb{A}$ be the homeomorphism given in Lemma 3.2 with semiconjugacy $\psi: (\mathbb{A}, g) \rightarrow (\mathbb{A}, f)$, mapping the pseudo-circle Q onto the pseudo-circle P provided by Lemma 3.4. Then, for the pseudo-arc $A = \psi^{-1}(p_*)$ we have*

$$\lim_{|n| \rightarrow \infty} \text{diam}(g^n(A)) = 0. \quad (3.52)$$

Proof Note that the semi-conjugacy ψ on $\mathbb{A}$ can be regarded as a continuous map from Q to P . Suppose to the contrary that for some integer subsequence $\{n_k\}_{k \geq 0}$ and some $\delta > 0$, every pseudo-arc $f^{n_k}(A)$ has diameter at least δ . Let us fix some point $x \in Q$ such that

$$\psi^{-1} \circ \psi(x) = \{x\}. \quad (3.53)$$

By minimality of f on P , we can extract a further subsequence if necessary, still called $\{n_k\}_{k \geq 0}$, such that $g^{n_k}(A)$ converges to some set K containing x . By definition $g^n(A) = \psi^{-1}(f^n(p_*))$ and so by Lemma 2.3, $\psi^{-1} \circ \psi(x) \supset K$. However, K has diameter at least δ , which is a contradiction with (3.53). $\square$

3.4 Proof of Theorem 3.1

Proof of Theorem 3.1 We have already proved in Theorem 3.5 that Q is a pseudo-circle. By Lemma 3.6, any iterate $g^n(A)$ has empty interior in Q for any $n \in \mathbb{Z}$. Thus, the set $Q \setminus \bigcap_{n \in \mathbb{Z}} g^n(A)$ is a dense G_δ subset of Q . This means that the set of points $y \in Q$ such that $\psi^{-1} \circ \psi(y) = \{y\}$ is a dense G_δ -set. In particular, ψ is an almost 1-to-1 semi-conjugacy between (Q, g) and (P, f) . Therefore, it follows immediately that (Q, g) is minimal since (P, f) is minimal. (see for example [18]). Equation (3.1) was established in Corollary 3.11. The proof is completed. $\square$

4 Proof of the Main Theorem

In this section we finish the proof of our main result. It seems that it is known that a monotone image of the pseudo-circle is homeomorphic to it (see e.g. p. 91, [30]), but for completeness we provide a short justification below.

Proposition 4.1 *Suppose $\phi : \mathbb{A} \rightarrow \mathbb{A}$ is a monotone continuous surjective map. Assume there is a non-degenerate continuum $Y \subset \mathbb{A}$, such that $\phi^{-1}(Y) = P$. Then Y is a pseudo-circle.*

Proof We need to show that Y is circle-like, plane separating and hereditarily indecomposable.

It is well known that a monotone image of a circle-like continuum is circle-like (see e.g. Lemma 9, [29]). Moreover Y is non-degenerate and $\phi^{-1}(Y) = P$ is plane separating. By the continuity of ϕ , it is immediate that Y is also plane separating. It remains to show that Y is hereditarily indecomposable. But this is also immediate, since if a subcontinuum K of Y were decomposable, then $\phi^{-1}(K)$ would also be a decomposable continuum, resulting in a contradiction. This shows that Y is a pseudo-circle. $\square$

Theorem 1.1 follows directly from the following statement.

Theorem 4.2 *There exists a continuous surjection $g' : \mathbb{A} \rightarrow \mathbb{A}$ with an invariant pseudo-circle $P \subset \mathbb{A}$ such that (g', P) is minimal but is not one-to-one.*

Proof Let $g : \mathbb{A} \rightarrow \mathbb{A}$ denote the homeomorphism provided by Proposition 3.1 minimal on pseudo-circle Q and let $A \subset Q$ be the associated pseudo-arc. Define the equivalence relation $\sim$ in $\mathbb{A}$, such that, $x \sim y$ if and only if either $x = y$, or $x, y \in g^n(A)$ for some $n \geq 0$. Since by Proposition 3.1,

$$\lim_{|n| \rightarrow \infty} \text{diam}(g^n(A)) = 0, \quad (4.1)$$

it follows that $\sim$ is a closed equivalence relation with connected equivalence classes. Moreover, it is not hard to see the family of equivalence classes form an upper semi-continuous family. Thus, by Moore's theorem (see Lemma 2.4), there exists a semi-conjugacy $\pi : \mathbb{A} \rightarrow \mathbb{A}$, which collapses each equivalence class into one point in $\mathbb{A}$. Then it induces a map $\hat{g} : \mathbb{A} \rightarrow \mathbb{A}$, such that $\pi \circ g = \hat{g} \circ \pi$. Note that $\hat{g}$ is a continuous non-invertible surjection. Indeed, $\pi(g^{-1}(A))$ is nondegenerate but $\hat{g}(\pi(g^{-1}(A))) = \pi(A)$ is a point in $\pi(Q)$, so $\hat{g}$ is non-invertible on $\pi(g^{-1}(A))$. In addition, $\pi : (\mathbb{A}, g) \rightarrow (\mathbb{A}, \hat{g})$ is a monotone surjective continuous map (a factor map). It follows that $\hat{g}$ must be minimal. Then, by Proposition 4.1, the proof is completed. $\square$

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References

1. Auslander, J., Yorke, J.A.: Interval maps, factors of maps, and chaos. *Tôhoku Math. J.* **32**(2), 177–188 (1980)
2. Anosov, D., Katok, A.: New examples in smooth ergodic theory and ergodic diffeomorphisms. *Trudy Moskovskogo Matematicheskogo Obshchestva* **23**, 3–36 (1970)
3. Auslander, J., Katznelson, Y.: Continuous maps of the circle without periodic points. *Isr. J. Math.* **32**(4), 375–381 (1979)
4. Béguin, F., Crovisier, S., Jäger, T.: A dynamical decomposition of the torus into pseudo-circles. *Modern theory of dynamical systems*, 39–50, *Contemp. Math.*, 692, Amer. Math. Soc., Providence, RI (2017)
5. Béguin, F., Crovisier, S., Le Roux, F.: Construction of curious minimal uniquely ergodic homeomorphisms on manifolds: the denjoy-rees technique. *Annales Scientifiques de l'Ecole Normale Supérieure* **40**, 251–308 (2007)
6. Bing, R.H.: Concerning hereditarily indecomposable continua. *Pac. J. Math.* **1**(1), 43–51 (1951)
7. Bing, R.H.: Embedding circle-like continua in the plane. *Can. J. Math.* **14**, 113–128 (1962)
8. Boroński, J.P., Oprocha, P.: Rotational chaos and strange attractors on the 2-torus. *Math. Z.* **279**(3–4), 689–702 (2015)
9. Boroński, J.P., Alex Clark, P.: Oprocha: a compact minimal space Y such that its square $Y \times Y$ is not minimal. *Adv. Math.* **335**, 261–275 (2018)
10. Boroński, J.P., Kozłowski, G.: On Minimal Manifolds. *Proceedings of the American Mathematical Society*. <https://doi.org/10.1090/proc/15186>
11. Boroński, J.P., Činč, J., Foryś-Krawiec, M.: On rigid minimal spaces. *J. Dyn. Differ. Equ.* (2020). <https://doi.org/10.1007/s10884-020-09845-4>
12. Bruin, H., Kolyada, S., Snoha, L.: Minimal nonhomogeneous continua. *Colloquium Mathematicum* **95**, 123–132 (2003)
13. Chéritat, A.: Relatively compact siegel disks with non-locally connected boundaries. *Math. Ann.* **349**(3), 529–542 (2011)
14. Daverman, R.J.: *Decompositions of Manifolds*, vol. 124. Academic Press, Cambridge (1986)
15. Denjoy, A.: Sur les courbes définies par les équations différentielles à la surface du tore. *J. Math. Pures Appl. Ser. IX* **11**, 333–375 (1932)
16. Downarowicz, T., Snoha, L., Tywoniuk, D.: Minimal spaces with cyclic group of homeomorphisms. *J. Dyn. Differ. Equ.* **29**(1), 243–257 (2017)
17. Fearnley, L.: The pseudo-circle is unique. *Bull. Am. Math. Soc.* **75**(2), 398–401 (1969)
18. Glasner, E.: *Ergodic Theory Via Joinings*, vol. 101. American Mathematical Society, New York (2003)
19. Handel, M.: A pathological area preserving C^∞ diffeomorphism of the plane. *Proc. Am. Math. Soc.* **86**(1), 163–168 (1982)
20. Hasselblatt, B., Katok, A.: *A First Course in Dynamics: With a Panorama of Recent Developments*. Cambridge University Press, Cambridge (2003)
21. Herman, M.: Construction of some curious diffeomorphisms of the Riemann sphere. *J. Lond. Math. Soc.* **2**(34), 375–384 (1986)
22. Kennedy, J., Rogers Jr., J.T.: Orbits of the pseudocircle. *Trans. Am. Math. Soc.* **296**, 327–340 (1986)
23. Kennedy, J., Yorke, J.A.: Pseudocircles in dynamical systems. *Trans. Am. Math. Soc.* **343**, 349–366 (1994)
24. Kennedy, J., Yorke, J.A.: Bizarre topology is natural in dynamical systems. *Bull. Am. Math. Soc.* **32**, 309–316 (1995)
25. Kennedy, J., Yorke, J.A.: Pseudocircles, diffeomorphisms and perturbable dynamical systems. *Ergodic Theory Dyn. Syst.* **16**, 1031–1057 (1996)
26. Kolyada, S., Snoha, L., Trofimchuk, S.: Noninvertible minimal maps. *Fund. Math.* **168**, 141–163 (2001)
27. Krasinkiewicz, J.: Composants of indecomposable plane continua. *Bulletin de l'Académie Polonaise des Sciences Mathématiques Astronomiques et Physiques* **20**(11), 935–940 (1972)
28. Krasinkiewicz, J.: On internal composants of indecomposable plane continua. *Fundamenta Mathematicae* **84**(3), 255–263 (1974)
29. Krupski, P.: Solenoids and inverse limits of sequences of arcs with open bonding maps. *Fundamenta Mathematicae* **120**(1), 41–52 (1984)
30. Lewis, W.: The classification of homogeneous continua. *Soochow J. Math.* **18**(1), 85–121 (1992)
31. McCann, S.M.: A classification of toral and planar attractors and substitution tiling spaces. Thesis Ph.D. University of Leicester (United Kingdom) (2013)
32. Moise, Edwin: An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua. *Trans. Am. Math. Soc.* **63**(3), 581–594 (1948)

33. Nadler, S.: *Continuum Theory. An Introduction*. Monographs and Textbooks in Pure and Applied Mathematics, vol. 158. Marcel Dekker, Inc., New York (1992)
34. Rees, M.: A minimal positive entropy homeomorphism of the 2-torus. *J. Lond. Math. Soc.* **2**(3), 537–550 (1981)
35. Rosenholtz, I.: Open maps of chainable continua. *Proc. Am. Math. Soc.* **42**(1), 258–264 (1974)
36. Schweitzer, P.A.: Counterexamples to the Seifert conjecture and opening closed leaves of foliations. *Ann. Math. (2)* **100**, 386–400 (1974)
37. Smale, S.: Differentiable dynamical systems. *Bull. Am. Math. Soc.* **73**, 747–817 (1967)
38. Snoha, L.: On some problems in discrete dynamics (cycles, chaos, topological entropy, minimality). In: *D.Sc. Thesis. Academy of Sciences of Czech Republic, Prague* (2005)
39. Sotola, J., Trofimchuk, S.: Construction of minimal non-invertible skew-product maps on 2-manifolds. *Proc. Am. Math. Soc.* **144**, 723–732 (2016)

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