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Age-Structured Population Dynamics with Nonlocal Diffusion

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Abstract

Random diffusive age-structured population models have been studied by many researchers. Though nonlocal diffusion processes are more applicable to many biological and physical problems compared with random diffusion processes, there are very few theoretical results on age-structured population models with nonlocal diffusion. In this paper our objective is to develop basic theory for age-structured population dynamics with nonlocal diffusion. In particular, we study the semigroup of linear operators associated to an age-structured model with nonlocal diffusion and use the spectral properties of its infinitesimal generator to determine the stability of the zero steady state. It is shown that (i) the structure of the semigroup for the age-structured model with nonlocal diffusion is essentially determined by that of the semigroups for the age-structured model without diffusion and the nonlocal operator when both birth and death rates are independent of spatial variables; (ii) the asymptotic behavior can be determined by the sign of spectral bound of the infinitesimal generator when both birth and death rates are dependent on spatial variables; (iii) the weak solution and comparison principle can be established when both birth and death rates are dependent on spatial variables and time; and (iv) the above results can be generalized to an age-size structured model. In addition, we compare our results with the age-structured model with Laplacian diffusion in the first two cases (i) and (ii).

Keywords Age structure · Nonlocal diffusion · Semigroup theory · Infinitesimal generator · Spectrum theory · Stability

Mathematics Subject Classification 35K55 · 35K90 · 92D25

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1 Introduction

Let $u(t, a, x)$ denote the density of a population at time t with age a at location $x \in \Omega$, where Ω is a bounded region in $\mathbb{R}^N$, $N \geq 1$ is an integer, and $a^+ < \infty$ is the maximum age. Consider the following age-structured model with nonlocal dispersal:

$$\begin{cases} \frac{\partial u(t,a,x)}{\partial t} + \frac{\partial u(t,a,x)}{\partial a} = d(J * u - u)(t, a, x) - \mu(a, x)u(t, a, x), & t, a > 0, x \in \Omega, \\ u(0, a, x) = u_0(a, x), \quad a > 0, x \in \Omega, \\ u(t, 0, x) = \int_0^{a^+} \beta(a, x)u(t, a, x)da, \quad t > 0, x \in \Omega, \\ u(t, a, x) = 0, \quad t, a > 0, x \in \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.1}$$

where d is the diffusion rate, J is a diffusion kernel which is a C^0, compactly supported, nonnegative radial function with unit integral representing the spatial dispersal, i.e.,

$$\int_{\mathbb{R}^N} J(x)dx = 1, \quad J(x) \geq 0, \quad \forall x \in \mathbb{R}^N, \quad J(0) > 0$$

and

$$(J * u - u)(t, a, x) = \int_{\mathbb{R}^N} J(x - y)u(t, a, y)dy - u(t, a, x).$$

The convolution $\int_{\mathbb{R}^N} J(x - y)u(t, a, y)dy$ is the rate at which individuals are arriving at position x from other places and $\int_{\mathbb{R}^N} J(y - x)u(t, a, x)dy$ is the rate at which they are leaving location x to travel to other sites. The mortality rate $\mu(a, x)$ is a positive and bounded measurable function in any district $[0, a_c] \times \Omega$, $a_c < a^+ < \infty$, and satisfies

$$\sup_{a\in[0,a_c]} \int_\Omega \mu^2(a, x)dx \text{ is continuous with respect to } a_c \in [0, a^+)$$

and

$$\int_0^a \overline{\mu}(\rho)d\rho < \infty \text{ for } a \in [0, a^+) \text{ with } \int_0^{a^+} \underline{\mu}(\rho)d\rho = \infty,$$

in which $\underline{\mu}(a) = \inf_{x\in\Omega} \mu(a, x)$ and $\overline{\mu}(a) = \sup_{x\in\Omega} \mu(a, x)$. While the fertility rate $\beta(a, x)$ is a bounded nonnegative measurable function on $[0, a^+]$ and satisfies

$$\text{mes}\Big\{a \Big| a \in [0, a^+), \beta(a) = \inf_{x\in\Omega} \beta(a, x) > 0\Big\} > 0.$$

$u_0(a, x)$ is the initial data with $u_0(a, x) \geq 0$.

In the following, we list some special cases of Eq. (1.1) or related models which have been studied in the literature.

1.1 Age-Structured Models

When $u(t, a, x) = u(t, a)$ does not depend on the spatial variable x, Eq. (1.1) reduces to the well-known age-structured model (McKendrick [39], von Foerster [48], Gurtin and MacCamy [20])):

$$\begin{cases} \frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -\mu(a)u(t, a), \quad t, a > 0, \\ u(0, a) = u_0(a), \quad a > 0, \\ u(t, 0) = \int_0^{a^+} \beta(a)u(t, a)da, \quad t > 0. \end{cases} \tag{1.2}$$

Properties of solutions to the age-structured model (1.2) have been extensively studied and well-understood, we refer to the monographs by Anita [2], Iannelli [27], Inaba [30], Magal and Ruan [37], and Webb [53] for basic theories, results and references.

1.2 Nonlocal Diffusion Models

If $u(t, a, x) = u(t, x)$ does not depend on the age variable a, then Eq. (1.1) becomes the nonlocal diffusion model:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d(J * u - u)(t, x) - \mu(x)u(t, x), \quad t > 0,\ x \in \Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega, \\ u(t, x) = 0, \quad t > 0,\ x \in \mathbb{R}^N\backslash\Omega. \end{cases} \tag{1.3}$$

Great attention has also been paid to the dynamics of model (1.3), we refer to the monograph by Andreu-Vaillo et al. [1] for fundamental theories and results and surveys by Bates [3] and Ruan [42] for applications in materials science and epidemiology, respectively. As pointed out in Bates et al. [3], $J(x - y)$ is viewed as the probability distribution of jumping from location y to location x, namely the convolution $\int_\Omega J(x - y)u(t, y)dy$ is the rate at which individuals are arriving to position x from other places and $\int_\Omega J(y - x)u(t, x)dy$ is the rate at which they are leaving location x to travel to other sites.

1.3 Age-Structured Models with Laplace Diffusion

If the population moves randomly and a Laplace operator is used to described such movement, then we have the following age-structured reaction-diffusion equation

$$\begin{cases} \frac{\partial u(t,a,x)}{\partial t} + \frac{\partial u(t,a,x)}{\partial a} = d\Delta u(t, a, x) - \mu(a, x)u(t, a, x), \quad t, a > 0,\ x \in \Omega, \\ u(0, a, x) = u_0(a, x), \quad a > 0,\ x \in \Omega, \\ u(t, 0, x) = \int_0^{a^+} \beta(a, x)u(t, a, x)da, \quad t > 0,\ x \in \Omega, \\ Bu(t, a, x) = 0, \quad t, a > 0,\ x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $Bu = 0$ represents one of the regular boundary conditions (Dirichlet, Nuemann or Robin), Δ is a Laplace operator. Such age-structured models with Laplace diffusion were first proposed by Gurtin [19] and were consequently investigated by Chan and Guo [5], Di Blasio [14], Ducrot et al. [15], Guo and Chan [18], Gurtin and MacCamy [21], Hastings [23], Huyer [26], Langlais [34], MacCamy [36], Walker [49,51], Webb [52], and so on. We refer to a survey by Webb [55] for detailed results and references.

Compared with random Laplace diffusion problems, nonlocal diffusion problems on one hand are more applicable to many biological and physical phenomena (Bates [3], Ruan [42], Zhao and Ruan [57]) and, on the other hand, post more technical challenges in analysis since the semigroup generated by the associated operator is not compact for any $t \geq 0$ (Andreu-Vaillo et al. [1]). Though age-structured models with Laplace diffusion have been studied extensively in the literature, in most existing references the age-structured diffusion equations in $u(t, a, x)$ were changed into reaction-diffusion equations in $u(t, x)$ by transforming the age structure a into a delay structure, for example, see Lin and Weng [35] and the references cited therein. To the best of our knowledge there are very few studies on the original age-structured models in $u(t, a, x)$ with nonlocal diffusion. The purpose of this paper is to provide a systematical and theoretical treatment of the age-structured problem (1.1) with nonlocal diffusion. More specifically, we study the semigroup of linear operators associated to the problem and use spectral properties of its infinitesimal generator to determine the stability of the zero steady state. It is shown that (i) the structure of the semigroup for the age-structured model with nonlocal diffusion is essentially determined by those of the semigroup for the age-structured model without diffusion and the nonlocal operator when both birth rate and death rates are independent of spatial variable; (ii) the asymptotic behavior can be determined by the sign of spectral bound of the infinitesimal generator when both birth and death rates are dependent on spatial variables; (iii) the weak solution and comparison principles can be established when both birth and death rates depend on not only time t but also spatial variables; and (iv) such results can be extended to an age-size structured model. In addition, we compare our results with the age-structured model with Laplacian diffusion in the first two cases (i) and (ii).

2 A Special Case When Both Birth and Death Rates are Independent of Spatial Variables

In this section, we first consider a special case where both birth and death rate functions are spatial homogeneous; that is,

$$\begin{cases} \frac{\partial u(t,a,x)}{\partial t} + \frac{\partial u(t,a,x)}{\partial a} = d(J * u - u)(t, a, x) - \mu(a)u(t, a, x), \quad t, a > 0, \ x \in \Omega, \\ u(0, a, x) = u_0(a, x), \quad a > 0, \ x \in \Omega, \\ u(t, 0, x) = \int_0^{a^+} \beta(a)u(t, a, x)da, \quad t > 0, \ x \in \Omega, \\ u(t, a, x) = 0, \quad t, a > 0, \ x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.1}$$

where $\mu(a)$ satisfies

$$\int_0^a \mu(\rho)d\rho < \infty \text{ for } a \in [0, a^+) \text{ with } \int_0^{a^+} \mu(\rho)d\rho = \infty,$$

and β is bounded and nonnegative on $[0, a^+)$.

2.1 The Linear Operator

Introducing the state space $E = L^2((0, a^+) \times \Omega)$ with the usual norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and defining an operator $A : E \to E$ by

$$A\phi(a,x) = d(J * \phi - \phi)(a,x) - \frac{\partial \phi(a,x)}{\partial a} - \mu(a)\phi(a,x), \quad \forall \phi \in D(A),$$

$$D(A) = \left\{\phi(a,x) \middle| \phi, A\phi \in E, \phi|_{\mathbb{R}^N \setminus \Omega} = 0, \phi(0,x) = \int_0^{a^+} \beta(a)\phi(a,x)da\right\}, \tag{2.2}$$

we can write Eq. (2.1) as an abstract Cauchy problem on the state space E:

$$\begin{cases} \frac{du(t,a,x)}{dt} = Au(t,a,x), \quad t > 0, \\ u(0,a,x) = u_0(a,x). \end{cases} \tag{2.3}$$

In what follows, we are only interested in the existence and uniqueness of principal eigenvalue of A and leave the dependence of the principal eigenvalue on the diffusion rate d in the future. Thus without loss of generality we assume that $d = 1$ in the following context. Now we introduce the eigenvalues and eigenfunctions of the nonlocal problem with Dirichlet boundary condition, which are denoted by $(\lambda_i, \varphi_i)_{i\geq 0}$, in the domain $\Omega \subset \mathbb{R}^N$; that is,

$$\begin{cases} -(J * \varphi_i - \varphi_i)(x) = \lambda_i \varphi_i(x), \quad x \in \Omega \\ \varphi_i(x) = 0, \quad x \in \mathbb{R}^N \setminus \Omega \end{cases} \tag{2.4}$$

with

$$\int_\Omega \varphi_i^2(x)dx = 1, \quad i \geq 0. \tag{2.5}$$

Note that the eigenfunctions φ_i of (2.4) satisfy $\varphi_i = 0$ in $\mathbb{R}^N \setminus \Omega$, the integral in the convolution term can indeed be considered in Ω. Therefore, we define the operator

$$L_0 u(x) = \int_\Omega J(x-y)u(y)dy.$$

Now observe that λ is an eigenvalue of (2.4)–(2.5) if and only if $\hat{\lambda} = 1 - \lambda$ is an eigenvalue of L_0 in $L^2(\Omega)$. It is easy to see that L_0 is compact and self-adjoint in $L^2(\Omega)$. Hence, by the classical spectral theorem, there exists an orthonormal basis consisting of eigenvectors of L_0 with corresponding eigenvalues $\{\hat{\lambda}_n\} \subset \mathbb{R}$ and $\hat{\lambda}_n \to 0$. Furthermore, we are interested in the existence of a principal eigenvalue, that is an eigenvalue associated to a nonnegative eigenfunction. We state a result related to the principal eigenvalue (See Coville et al. [7], García-Melián and Rossi [17], and Hutson et al. [25]).

Theorem 2.1 (García-Melián and Rossi [17]) *Problem* (2.4)–(2.5) *admits an eigenvalue* λ_0 *associated to a positive eigenfunction* $\varphi_0 \in L^2(\Omega)$. *Moreover, it is simple and unique and satisfies* $0 < \lambda_0 < 1$. *Furthermore,* λ_0 *can be variationally characterized as*

$$\lambda_0 = 1 - \left(\sup_{u \in L^2(\Omega), \|u\|_{L^2(\Omega)}=1} \int_\Omega \left(\int_\Omega J(x-y)u(y)dy\right)^2 dx\right)^{1/2}. \tag{2.6}$$

For other eigenvalues we can arrange them as $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to 1$. Next, we denote the usual population operator by B without diffusion defined in $L^2(0, a^+)$:

$$B\eta(a) = -\frac{\partial \eta(a)}{\partial a} - \mu(a)\eta(a), \quad \forall \eta \in D(B), \tag{2.7}$$

$$D(B) = \left\{\eta(a) | \eta, B\eta \in L^2(0, a^+), \quad \eta(0) = \int_0^{a^+} \beta(a)\eta(a)da\right\} \tag{2.8}$$

and $\{\gamma_j\}_{j\geq 0}$ be the eigenvalues of B, i.e., the solution of the following equation

$$\int_0^{a^+} \beta(a)e^{-\gamma a}\pi(a)da = 1,$$

where $\pi(a) = e^{-\int_0^a \mu(\rho)d\rho}$. Arrange γ in the following way:

$$\gamma_0 > \mathrm{Re}\gamma_1 \geq \mathrm{Re}\gamma_2 \geq \cdots$$

Next let us solve the resolvent equation

$$(\sigma I - A)\phi = \psi, \quad \forall \psi \in E.$$

If for any $i, j \geq 0, \sigma + \lambda_i \neq \gamma_j$, then define

$$\phi_\psi(a, x) = \sum_{i=0}^{\infty} \big((\sigma + \lambda_i)I - B\big)^{-1} \langle \psi(a, \cdot), \varphi_i \rangle_{L^2(\Omega)} \varphi_i(x),$$

where $\langle \psi(a, \cdot), \varphi_i \rangle_{L^2(\Omega)} = \int_\Omega \psi(a, x)\varphi_i(x)dx$. Since B is the infinitesimal generator of a bounded strongly continuous semigroup, there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|(\sigma I - B)^{-1}\| \leq \frac{M}{\mathrm{Re}\sigma - \omega}, \quad \forall\, \mathrm{Re}\sigma > \omega.$$

Recall that $\lambda_i > 0$ for all i, then $\mathrm{Re}(\sigma + \lambda_i) > \omega$ for all $i > 0$ provided $\mathrm{Re}\sigma > \omega$,

$$\begin{aligned}
&\sum_{i=0}^{\infty} \|\big((\sigma + \lambda_i)I - B\big)^{-1} \langle \psi(a, \cdot), \varphi_i \rangle_{L^2(\Omega)}\|^2 \\
&\quad \leq \left[\frac{M}{\mathrm{Re}(\sigma + \lambda_0) - \omega}\right]^2 \sum_{i=0}^{\infty} \|\langle \psi(a, \cdot), \varphi_i \rangle_{L^2(\Omega)}\|^2 \\
&\quad \leq \left[\frac{M}{\mathrm{Re}(\sigma + \lambda_0) - \omega}\right]^2 \|\psi\|^2 < \infty.
\end{aligned} \tag{2.9}$$

Thus, $\phi_\psi(a, x)$ is well defined. Moreover, for any $n > 0$,

$$\begin{aligned}
&(\sigma I - A) \sum_{i=0}^{n} \big((\sigma + \lambda_i)I - B\big)^{-1} \langle \psi(a, \cdot), \varphi_i \rangle_{L^2(\Omega)} \varphi_i(x) \\
&= \sum_{i=0}^{n} \langle \psi(a, \cdot), \varphi_i \rangle_{L^2(\Omega)} \varphi_i(x) \to \psi(a, x) \quad \text{in} \quad E \quad \text{as} \quad n \to \infty
\end{aligned}$$

Since B and $L := J * -I$ are both closed operators on E, so is A. Hence $(\sigma I - A)\phi_\psi = \psi$, i.e. $\phi_\psi(a, x)$ is a solution of the resolvent equation. Now choose $\phi \in D(A)$, we have

$$\begin{aligned}
\langle A\phi, \phi \rangle &= \int_{(0,a^+)\times\Omega} -\frac{\partial \phi(a, x)}{\partial a} \phi(a, x)dadx - \int_{(0,a^+)\times\Omega} \mu(a)|\phi(a, x)|^2 dadx \\
&\quad + \int_{(0,a^+)\times\Omega} (J * \phi(a, x) - \phi(a, x))\phi(a, x)dadx \\
&\leq \frac{1}{2} \int_\Omega |\phi(0, x)|^2 dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_\Omega \left[\int_0^{a^+} \beta(a)\phi(a,x)da\right]^2 dx \\
&\le \frac{1}{2}\int_\Omega \left[\int_0^{a^+} \beta^2(a)da\right]\left[\int_0^{a^+} \phi^2(a,x)da\right] dx \\
&= \frac{1}{2}\|\beta\|^2_{L^2(0,a^+)}\|\phi\|^2_E,
\end{aligned}
\tag{2.10}
$$

where we used the symmetry of J

$$
\begin{aligned}
&\int_{(0,a^+)\times\Omega} (J*\phi(a,x)-\phi(a,x))\phi(a,x)dadx \\
&= \int_{(0,a^+)}\int_\Omega\int_\Omega J(x-y)(\phi(a,y)-\phi(a,x))\phi(a,x)dydxda \\
&= -\frac{1}{2}\int_{(0,a^+)}\int_\Omega\int_\Omega J(x-y)(\phi(a,y)-\phi(a,x))^2dydxda \le 0.
\end{aligned}
\tag{2.11}
$$

It follows that for all sufficiently large σ, $A-\sigma I$ is a dissipative operator on E.

On the other hand, it can be shown that ϕ is the unique solution of the resolvent equation by the uniqueness resolvent solution of age-structured models with orthonormal basis in $L^2(\Omega)$, and thus $\sigma \in \rho(A)$, the resolvent set of A, and

$$
(\sigma I - A)^{-1}\psi = \sum_{i=0}^{\infty}\big((\sigma+\lambda_i)I - B\big)^{-1}\langle\psi(a,\cdot),\varphi_i\rangle_{L^2(\Omega)}\varphi_i(x). \tag{2.12}
$$

It follows that $\mathcal{R}(\sigma I - A)$, the range of $\sigma I - A$ is equal to the whole space E, and by (2.10), $A-\sigma I$ is dissipative when σ is sufficiently large, it implies from [41, Chapter I, Theorem 4.6] that $D(A-\sigma I)$ is dense and $\overline{D(A-\sigma I)} = E$, so does $D(A)$ and $\overline{D(A)} = E$. Moreover, from (2.9) we have

$$
\|(\sigma I - A)^{-1}\| \le \frac{M}{\mathrm{Re}(\sigma+\lambda_0)-\omega}.
$$

It follows from Hille–Yosida theorem that A is an infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\ge 0}$. (In fact, one can conclude the same result by using Lumer–Phillips theorem in [41].)

If there are some i, j such that $\sigma+\lambda_i = \gamma_j$, then

$$
\phi_i(a,x) = e^{-(\sigma+\lambda_i)a}\pi(a)\varphi_i(x)
$$

satisfies $(\sigma I - A)\phi_i = 0$, i.e. $\sigma \in \sigma_p(A)$, the point spectrum of A. Furthermore, if $(\sigma I - A)\phi = 0$, expanding the known initial function $\phi(0,x)$ as

$$
\phi(0,x) = \sum_{i=0}^{\infty}\alpha_i\varphi_i(x) \quad \text{in } L^2(\Omega),
$$

then we have

$$
\phi(a,x) = \sum_{i=0}^{\infty}\alpha_i e^{-(\sigma+\lambda_i)a}\pi(a)\varphi_i(x).
$$

In view of the initial condition

$$\phi(0,x)=\int_0^{a^+}\beta(a)\phi(a,x)da,$$

we get for each i that either $\alpha_i=0$ or $\int_0^{a^+}\beta(a)e^{-(\sigma+\lambda_i)a}\pi(a)da=1$. In particular, for $\sigma_0=\gamma_0-\lambda_0$, which is the dominant eigenvalue of A, $(\sigma I-A)\phi=0$ has only one independent linear solution, which is

$$\phi_{\sigma_0}(a,x)=e^{-\gamma_0 a}\pi(a)\varphi_0(x), \tag{2.13}$$

so σ_0 is of geometric multiplicity one.

Define an operator

$$\mathscr{H}_\sigma=\int_0^{a^+}\beta(a)e^{-\sigma a}\pi(a)e^{La}da.$$

It is easy to see that $\mathscr{H}_\sigma$ is a positive and self-adjoint operator in $L^2(\Omega)$ since L is self-adjoint and that $\varphi_0(x)$ is the eigenfunction of the eigenvalue 1 of $\mathscr{H}_{\sigma_0}$. Thus, $r(\mathscr{H}_{\sigma_0})\geq 1$.

In addition, note that $\{\varphi_i\}_{i\geq 0}$ are indeed in $C(\Omega)$ due to the fact that J is continuous and $e^{La}: C_b(\Omega)\to C_b(\Omega)$ is a contraction mapping with contraction coefficient e^{-a} (see [16]), where $C_b(\Omega)$ represents the space of continuous bounded functions in Ω. It follows that

$$r_e(\mathscr{H}_{\sigma_0})\leq\|\mathscr{H}_{\sigma_0}\|_e\leq\int_0^{a^+}\beta(a)e^{-\sigma_0 a}\pi(a)\|e^{La}\|_e da\leq\int_0^{a^+}\beta(a)e^{-\gamma_0 a}\pi(a)e^{-(1-\lambda_0)a}da<1,$$

where $r_e(A)$ and $\|A\|_e$ represent the essential spectral radius and essential norm of operator A, respectively. Now suppose that $r(\mathscr{H}_{\sigma_0})>1$, for the sake of contraction, we then see from the generalized Krein–Rutman theorem (see Nussbaum [40] or Zhang [56]) that $r(\mathscr{H}_{\sigma_0})$ is an eigenvalue of $\mathscr{H}_{\sigma_0}$ corresponding to a positive eigenvector $\psi\in L^2(\Omega)$. It follows that

$$r(\mathscr{H}_{\sigma_0})\langle\psi,\varphi_0\rangle_{L^2(\Omega)}=\langle\mathscr{H}_{\sigma_0}\psi,\varphi_0\rangle_{L^2(\Omega)}=\langle\psi,\mathscr{H}_{\sigma_0}\varphi_0\rangle_{L^2(\Omega)}=\langle\psi,\varphi_0\rangle_{L^2(\Omega)},$$

which implies that $r(\mathscr{H}_{\sigma_0})=1$ since $\langle\psi,\varphi_0\rangle>0$. This is a contradiction. Thus $r(\mathscr{H}_{\sigma_0})=1$.

In summary, we have the following theorem.

Theorem 2.2 *The following statements are valid.*

(i) *The operator A defined in (2.2) generates a strongly continuous semigroup* $\{S(t)\}_{t\geq 0}$ *on E;*
(ii) $\sigma(A)=\sigma_P(A)=\{\gamma_i-\lambda_j\}_{i,j=0}^{\infty}$;
(iii) *The operator A has a real dominant eigenvalue* σ_0 *corresponding to the eigenfunction* ϕ_{σ_0} *defined in* (2.13)*, that is* σ_0 *is greater than any real part of eigenvalues of A;*
(iv) σ_0 *is a simple eigenvalue of A;*
(v) *For the operator* $\mathscr{H}_{\sigma_0}$, 1 *is an eigenvalue with an eigenfunction* $\varphi_0(x)$. *Furthermore,* $r(\mathscr{H}_{\sigma_0})=1$.

The proofs of (i)–(iv) are similar to those in [5, Theorem 1]. We omit them here. The proof of (v) is shown in the above argument.

2.2 The Semigroup

In this section, we discuss the C_0-semigroup $\{S(t)\}_{t\geq 0}$ generated by the operator A by using the idea from Chan and Guo [5]. For every $\phi\in E$, define a family of operators $\{\hat{S}(t)\}_{t\geq 0}$ as follows:

$$\hat{S}(t)\phi(a,x)=\sum_{i=0}^{\infty}e^{Bt}e^{-\lambda_i t}\langle\phi(a,\cdot),\varphi_i\rangle_{L^2(\Omega)}\varphi_i(x), \tag{2.14}$$

where e^{Bt} is the semigroup generated by B.

It is obvious that $\hat{S}(t)$ is a well-defined bounded linear operator on E for every $t\geq 0$ and for all $\phi_{n,q}(a,x)=\sum_{j=0}^{n}q_j(a)\varphi_j(x)$, $q_j(a)\in L^2(0,a^+)$, $j=0,1,\ldots,n$, $n>0$, we can directly verify that

$$\hat{S}(t+s)\phi_{n,q}(a,x)=\hat{S}(t)\hat{S}(s)\phi_{n,q}(a,x),\quad \forall t,s\geq 0.$$

Since $\{\phi_{n,q}(a,x)\}$ is dense in E, so $\hat{S}(t+s)=\hat{S}(t)\hat{S}(s)$ for all $t,s\geq 0$. Moreover,

$$\lim_{t\to 0}\hat{S}(t)\phi_{n,q}=\phi_{n,q}$$

and since $\|\hat{S}(t)\|\leq Me^{(\omega-\lambda_0)t}$, it follows that

$$\lim_{t\to 0}\hat{S}(t)\phi=\phi,\quad \forall\phi\in E.$$

This shows that $\{\hat{S}(t)\}_{t\geq 0}$ is also a C_0-semigroup on E. A simple calculation shows that

$$\lim_{t\to 0}\frac{\hat{S}(t)-I}{t}\phi_{n,q}=A\phi_{n,q}$$

for all $\phi_{n,q}$. Hence, $\hat{S}(t)=S(t)$ for all $t\geq 0$. We have the following result.

Theorem 2.3 *The operator A defined in (2.2) is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on the state space E. Thus, if $u_0\in E$, then there exists a unique mild solution to (2.3) such that*

$$u(t,\cdot,\cdot)=S(t)u_0\in C([0,\infty),E),$$

and if $u_0\in D(A)$, then there exists a classical solution to (2.3) such that

$$u(t,\cdot,\cdot)=S(t)u_0\in C^1([0,\infty),E).$$

2.3 Asymptotic Behavior

First, we recall the asymptotic behavior of solutions to the following age-structured model without diffusion:

$$\begin{cases}\frac{\partial u(t,a)}{\partial t}+\frac{\partial u(t,a)}{\partial a}+\mu(a)u(t,a)=0, & t,a>0,\\ u(t,0)=\int_0^{a^+}\beta(a)u(t,a)da, & t>0,\\ u(0,a)=u_0\in L^1(0,a^+), & a>0.\end{cases} \tag{2.15}$$

Denote

$$R=\int_0^{a^+}\beta(a)e^{-\int_0^a\mu(s)ds}da.$$

In the case that $u_0\not\equiv 0$ ($c>0$ where c is defined in the following (2.16)), we can infer that (see, e.g., [2, Remark 2.3.2])

Proposition 2.4 *The following results hold,*

(i) $\lim_{t\to\infty} \|u(t)\|_{L^\infty(0,a^+)} = 0$ *if* $R < 1$*;*
(ii) $\lim_{t\to\infty} \|u(t)\|_{L^1(0,a^+)} = \infty$ *if* $R > 1$*;*
(iii) $\lim_{t\to\infty} \|u(t) - \tilde{u}\|_{L^\infty(0,a^+)} = 0$ *if* $R = 1$,

where $\tilde{u}(a) = ce^{-\int_0^a \mu(s)ds}$, $\forall a \in (0, a^+)$ *is a nontrivial steady state of* (2.15) *and*

$$c = \frac{\int_0^{a^+} \beta(a)\left[\int_0^a u_0(s)e^{\int_0^s \mu(\xi)d\xi}ds\right]e^{-\int_0^a \mu(\xi)d\xi}da}{\int_0^{a^+} a\beta(a)e^{-\int_0^a \mu(\xi)d\xi}da}. \tag{2.16}$$

Now we establish the asymptotic behavior of the age-structured model with nonlocal diffusion under Dirichlet boundary conditions.

Theorem 2.5 *Assume that* $a^+ < \infty$.

(i) *If* $R < 1$, *then* $\lim_{t\to\infty} \|u(t)\|_{L^2((0,a^+)\times\Omega)} = 0$*;*
(ii) *If* $R = 1$, *then* $\lim_{t\to\infty} \|u(t)\|_{L^2((0,a^+)\times\Omega)} = 0$*;*
(iii) *If* $R > 1$, $\gamma_0 < \lambda_0$, *then* $\lim_{t\to\infty} \|u(t)\|_{L^2((0,a^+)\times\Omega)} = 0$*; while if* $\gamma_0 > \lambda_0$ *and* u_0 *is a nontrivial datum, then* $\lim_{t\to\infty} \|u(t)\|_{L^2((0,a^+)\times\Omega)} = \infty$.

Proof Based on Theorem 2.2, we see that if $R < 1$, $\gamma_0 < 0$, then the principal eigenvalue $\sigma_0 = \gamma_0 - \lambda_0 < 0$, which implies (i). If $R = 1$, $\gamma_0 = 0$, then $\sigma_0 = -\lambda_0 < 0$, which implies (ii). When $R > 1$, $\gamma_0 > 0$, if $\gamma_0 < \lambda_0$, then $\sigma_0 < 0$ which implies $\lim_{t\to\infty} \|u(t)\|_{L^2((0,a^+)\times\Omega)} = 0$; while if $\gamma_0 > \lambda_0$, then $\sigma_0 > 0$, hence the solution u will blow up in E, which implies (iii). □

Remark 2.6 There are some differences between Laplace diffusion and nonlocal diffusion. For the Laplace diffusion problem, from Chan and Guo [5] we have the following asymptotic expression for the solution $u(t, a, x)$:

$$u(t, a, x) = C_{u_0}(x)e^{-\mu_0 a - \int_0^a \mu(\rho)d\rho}e^{\sigma_0 t}\varphi_0(x) + o(e^{(\sigma_0 - \epsilon)t}),$$

where $\sigma_0 = \gamma_0 - \lambda_0$ is the dominant eigenvalue of A, ϵ is a small positive number such that $\sigma(A) \cap \{\sigma | \sigma_0 - \epsilon \le \mathrm{Re}\sigma < \sigma_0\} = \emptyset$, and

$$C_{u_0}(x) = \frac{\int_0^{a^+} \beta(a)\left[\int_0^a e^{-\gamma_0(a-s) - \int_s^a \mu(\rho)d\rho}\langle u_0(a, x), \varphi_0(x)\rangle ds\right]da}{-\int_0^{a^+} a\beta(a)e^{-\gamma_0 a - \int_0^a \mu(\rho)d\rho}da},$$

since the semigroup $\{\hat{S}(t)\}_{t\ge0}$ generated by the operator $\hat{A}$, where the nonlocal diffusion is replaced by Laplace diffusion in A, is compact when $t \ge a^+$. However, for the nonlocal diffusion problem, the semigroup $\{S(t)\}_{t\ge0}$ generated by the operator A is not compact for any $t \ge 0$, so such an asymptotic expression for solutions of the nonlocal diffusion problem does not hold. In fact, such asymptotic expression comes from *asynchronous exponential growth* which was introduced in details by Webb [54] and is based on the spectral bound and essential growth rate of A.

3 The General Case When Both Birth and Death Rates are Dependent on Spatial Variables

Now we consider the general case where both birth and death rate functions depend on the spatial variable. In this section, one will see that since the semigroup or evolution family generated by nonlocal diffusion loses compactness, we need to deal with it via different arguments compared with the case of Laplace diffusion. Introducing the state space $E = L^2((0, a^+) \times \Omega)$ with the usual norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and defining the operator $A' : E \to E$ by

$$A'\phi(a, x) = d(J * \phi - \phi)(a, x) - \frac{\partial \phi(a, x)}{\partial a} - \mu(a, x)\phi(a, x), \quad \forall \phi \in D(A'),$$
$$D(A') = \big\{\phi(a, x)\big|\phi, A'\phi \in E, \phi|_{\mathbb{R}^N \setminus \Omega} = 0, \phi(0, x) = \int_0^{a^+} \beta(a, x)\phi(a, x)da\big\}, \tag{3.1}$$

we can write the Eq. (1.1) as an abstract Cauchy problem on the state space E:

$$\begin{cases} \frac{du(t,a,x)}{dt} = A'u(t, a, x) \\ u(0, a, x) = u_0(a, x). \end{cases} \tag{3.2}$$

3.1 The Semigroup

In the following without loss of generality we assume that $d = 1$. In addition, we need a lemma similar to Lemma 1 in Guo and Chan [18].

Lemma 3.1 *For any $0 \le a_0 < a^+$, there exists a unique mild solution $u(a, x)$, $0 \le \tau \le a \le a^+ - a_0$, to the evolution equation on E for any initial function $\phi \in L^2(\Omega)$:*

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = [L - \mu(a_0 + a, x)]u(a, x) \\ u(\tau, x) = \phi(x), \end{cases} \tag{3.3}$$

where $L := J * -I$ is the part of a nonlocal operator with Dirichlet boundary condition. Define the solution operator of the initial value problem (3.3) by

$$\mathscr{F}(a_0, \tau, a)\phi(x) = u(a, x), \quad \forall \phi \in L^2(\Omega), \tag{3.4}$$

then $\{\mathscr{F}(a_0, \tau, a)\}_{0 \le \tau \le a \le a^+ - a_0}$ is a family of uniformly linear bounded positive operators on E and is strong continuous in τ and a.

Proof We first consider (3.3) in the interval $[0, \bar{a}]$, where $\bar{a} < a^+ - a_0$ is any given number. Define a mapping $\mathscr{F}(a_0)$ from $C([\tau, \bar{a}], L^2(\Omega))$ into itself by

$$\mathscr{F}(a_0)u(a, x) = e^{La}\phi(x) - \int_\tau^a e^{L(a-\sigma)}\mu(a_0 + \sigma, x)u(\sigma, x)d\sigma. \tag{3.5}$$

Denoting $\|u\|_\infty = \max_{\tau \le a \le \bar{a}} \|u(a, \cdot)\|$ and $M = \sup_{\tau \le a \le \bar{a}, x \in \Omega} \mu(a_0 + a, x)$, it is easy to check that

$$\|\mathscr{F}^n(a_0)u - \mathscr{F}^n(a_0)v\|_\infty \le \frac{M^n(\bar{a} - \tau)^n}{n!}\|u - v\|_\infty, \quad \forall u, v \in C([\tau, \bar{a}], L^2(\Omega)),\ n \ge 1.$$

For n large enough, $M^n(\bar{a} - s)^n/n! < 1$, and by the well known Banach contraction mapping theorem, $\mathscr{F}(a_0)$ has a unique fixed point u in $C([\tau, \bar{a}], L^2(\Omega))$ for which

$$u(a, x) = e^{La}\phi(x) - \int_\tau^a e^{L(a-\sigma)}\mu(a_0 + \sigma, x)u(\sigma, x)d\sigma.$$

By assumption on μ and the property of e^{Lt}, the right hand side of the above expression is continuous. Thus $\{\mathscr{F}(a_0, \tau, a)\}_{0 \le \tau \le a \le a^+ - a_0}$ is a family of uniformly linear continuous positive operators on E and is strongly continuous in τ, a. And it is easy to note that when $\mu(a, x) = \mu(a)$,

$$\mathscr{F}(a_0, \tau, a) = e^{-\int_\tau^a \mu(a_0+\sigma)d\sigma} e^{L(a-\tau)}.$$

This completes the proof. □

Furthermore, we have

$$e^{-\int_\tau^a \overline{\mu}(a_0+\rho)d\rho} e^{L(a-\tau)}\phi \leq \mathscr{F}(a_0, \tau, a)\phi \leq e^{-\int_\tau^a \underline{\mu}(a_0+\rho)d\rho} e^{L(a-\tau)}\phi, \quad \forall \phi \in L_+^2(\Omega). \tag{3.6}$$

Now, we consider the resolvent equation

$$(\lambda I - A')\phi = \varphi, \quad \forall \varphi \in E,$$

i.e.

$$\begin{cases} \frac{\partial \phi(a,x)}{\partial a} = (J * -I)\phi(a, x) - (\lambda + \mu(a, x))\phi(a, x) + \varphi(a, x), \\ \phi(0, x) = \int_0^{a^+} \beta(a, x)\phi(a, x)da, \\ \phi(a, x) = 0, \quad x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.7}$$

Letting $\mathscr{F}(0, \tau, a) = \mathscr{F}(\tau, a)$, we have

$$\phi(a, x) = e^{-\lambda a}\mathscr{F}(0, a)\phi(0, x) + \int_0^a e^{-\lambda(a-\sigma)}\mathscr{F}(\sigma, a)\varphi(\sigma, x)d\sigma,$$

and accordingly

$$\begin{aligned} &\phi(0, x) - \int_0^{a^+} \beta(a, x)e^{-\lambda a}\mathscr{F}(0, a)\phi(0, x)da \\ &= \int_0^{a^+} \beta(a, x) \int_0^a e^{-\lambda(a-\sigma)}\mathscr{F}(\sigma, a)\varphi(\sigma, x)d\sigma da. \end{aligned}$$

Define an operator $\mathscr{G}_\lambda : L^2(\Omega) \to L^2(\Omega)$ for $\lambda \in \mathbb{R}$ by

$$\mathscr{G}_\lambda\phi(x) = \int_0^{a^+} \beta(a, x)e^{-\lambda a}\mathscr{F}(0, a)\phi(x)da, \quad \forall \phi \in L^2(\Omega). \tag{3.8}$$

Then

$$\lambda \in \rho(A') \Leftrightarrow 1 \in \rho(\mathscr{G}_\lambda),$$

and for $\lambda \in \rho(A')$,

$$\begin{aligned} (\lambda I - A')^{-1}\varphi(a, x) = e^{-\lambda a}\mathscr{F}(0, a)(I - \mathscr{G}_\lambda)^{-1} \int_0^{a^+} \beta(a, x)da \int_0^a e^{-\lambda(a-\sigma)}\mathscr{F}(\sigma, a)\varphi(\sigma, x)d\sigma \\ + \int_0^a e^{-\lambda(a-\sigma)}\mathscr{F}(\sigma, a)\varphi(\sigma, x)d\sigma. \end{aligned} \tag{3.9}$$

It is easy to see from (3.8) that

$$\lim_{\lambda\to\infty} \|\mathscr{G}_\lambda\| = 0.$$

Hence, for all sufficiently large $\lambda > 0$, $(I - \mathscr{G}_\lambda)^{-1}$ exists and so does the operator $(\lambda I - A')^{-1}$. It follows that for sufficient large $\lambda > 0$, $\mathcal{R}(I - (A' - \lambda I))$, the range of the operator $I - (A' - \lambda I)$ is equal to the whole space E.

Next, by similar computations in (2.10), we have

$$\langle A'\phi, \phi\rangle \le \frac{1}{2}\sup_{x\in\Omega}\int_0^{a^+}\beta^2(a,x)da\|\phi\|^2, \quad \forall\phi \in D(A'),$$

which implies that A' is dissipative for sufficient large $\lambda > 0$. Together with $\mathcal{R}(I-(A'-\lambda I)) = E$ we can conclude that $D(A'-\lambda I)$ is dense in E and so is $D(A')$. It follows from Lumer–Phillips Theorem in Pazy [41] that $A'-\lambda I$ is the infinitesimal generator of a C_0-semigroup of contractions on E. Hence, A' is also the infinitesimal generator of a C_0-semigroup $\{S'(t)\}_{t\ge 0}$ (but of not contractions) on E.

In summary, we have a theorem similar to Theorem 2.3 in the following.

Theorem 3.2 *The operator A' defined in (3.1) is the infinitesimal generator of a C_0-semigroup $\{S'(t)\}_{t\ge0}$ on the state space E. Thus, if $u_0 \in E$, then there exists a unique mild solution to (2.3) such that*

$$u(t,\cdot,\cdot) = S'(t)u_0 \in C([0,\infty), E),$$

and if $u_0 \in D(A')$, then there exists a classical solution

$$u(t,\cdot,\cdot) = S'(t)u_0 \in C^1([0,\infty), E).$$

Moreover, when $\lambda \in \sigma_p(A')$, its corresponding eigenfunction $\phi(a,x)$ can be expressed as

$$\phi(a,x) = e^{-\lambda a}\mathscr{F}(0,a)\phi_0(x),$$

where $\phi_0(x)$ is the nonzero solution of

$$\phi(x) - \int_0^{a^+}\beta(a,x)e^{-\lambda a}\mathscr{F}(0,a)\phi(x)da = 0.$$

3.2 The Infinitesimal Generator

In this section we study the spectral bound of the operator A' and therefore determine the stability of the zero steady state.

Define

$$\mathscr{C}_\lambda = \int_0^{a^+}\underline{\beta}(a)e^{-\lambda a}\pi'(a)e^{La}da,$$

where $\underline{\beta}(a) = \inf_{x\in\Omega}\beta(a,x)$ and $\pi'(a) = e^{-\int_0^a\overline{\mu}(\sigma)}d\sigma$ with $\overline{\mu}(a) = \sup_{x\in\Omega}\mu(a,x)$. It is easy to see that $\mathscr{G}_\lambda \ge \mathscr{C}_\lambda$ in the positive operator sense.

Now we claim that $r(\mathscr{G}_\lambda)$ is continuous with respect to the parameter $\lambda \in \mathbb{R}$.

Lemma 3.3 *$r(\mathscr{G}_\lambda)$ is continuous with respect to the parameter $\lambda \in \mathbb{R}$. Moreover, it is decreasing with respect to λ.*

Proof Note that $\mathscr{G}_\lambda$ is a positive operator in $L^2(\Omega)$ with a positive cone $L^2_+(\Omega)$ which consists of nonnegative functions in $L^2(\Omega)$, since $\mathscr{F}(0,a)$ is a positive operator in $L^2(\Omega)$ with the same positive cone $L^2_+(\Omega)$. We claim that $\forall\lambda_1 \ge \lambda_2$, one can obtain

$$e^{(\lambda_2-\lambda_1)a^+}r(\mathscr{G}_{\lambda_2}) \le r(\mathscr{G}_{\lambda_1}) \le r(\mathscr{G}_{\lambda_2}), \tag{3.10}$$

which implies that $r(\mathscr{G}_\lambda)$ is decreasing in λ. In fact, it is sufficient to show that

$$e^{(\lambda_2-\lambda_1)a^+}\mathscr{G}_{\lambda_2}\phi \le \mathscr{G}_{\lambda_1}\phi \le \mathscr{G}_{\lambda_2}\phi, \quad \forall \phi \in L^2_+(\Omega),$$

where the second inequality is obvious. Thus, we only need to show the first inequality. It is easy to see that

$$\begin{aligned}\mathscr{G}_{\lambda_1}\phi &= \int_0^{a^+} \beta(a,x)e^{-\lambda_1 a}\mathscr{F}(0,a)\phi(x)da\\ &= \int_0^{a^+} \beta(a,x)e^{-(\lambda_1-\lambda_2)a}e^{-\lambda_2 a}\mathscr{F}(0,a)\phi(x)da\\ &\ge e^{-(\lambda_1-\lambda_2)a^+}\mathscr{G}_{\lambda_2}\phi, \quad \forall \phi \in L^2_+(\Omega).\end{aligned} \tag{3.11}$$

It follows that $e^{(\lambda_2-\lambda_1)a^+}\mathscr{G}_{\lambda_2}\phi \le \mathscr{G}_{\lambda_1}\phi$, $\forall \phi \in L^2_+(\Omega)$, which is our desired result. □

Next, we claim that $r(\mathscr{G}_\lambda)$ is log convex with respect to $\lambda \in \mathbb{R}$.

Lemma 3.4 *$r(\mathscr{G}_\lambda)$ is log convex with respect to $\lambda \in \mathbb{R}$.*

Proof We use the generalized Kingman's theorem from Kato [32] to show it. First claim that $\lambda \to \mathscr{G}_\lambda$ is completely monotonic. Then, $\lambda \to r(\mathscr{G}_\lambda)$ is superconvex by Thieme [45, Theorem 2.5] and hence log convex. By the definition from Thieme [45], an infinitely often differentiable function $f : (a, \infty) \to X_+$ is called *completely monotonic* if

$$(-1)^n f^{(n)}(\lambda) \in X_+, \quad \forall \lambda > a, n \in \mathbb{N},$$

where X_+ is a normal and generating cone of an ordered Banach space X. A family $\{F(\lambda)\}_{\lambda>a}$ of positive operators on X is called *completely monotonic* if $f(\lambda) = F(\lambda)x$ is completely monotonic for every $x \in X_+$. For our case, $\mathscr{G}_\lambda$ is indeed infinitely often differentiable with respect to $\lambda \in \mathbb{R}$ and

$$(-1)^n\mathscr{G}^{(n)}_\lambda\phi = \int_0^{a^+} \beta(a,x)a^n e^{-\lambda a}\mathscr{F}(0,a)\phi(x)da \in L^2_+(\Omega), \quad \lambda \in \mathbb{R}, n \in \mathbb{N}, \phi \in L^2_+(\Omega).$$

Thus, our result follows. □

Remark 3.5 Here we need to emphasize that, the continuity and strict monotonicity of $\lambda \to r(\mathscr{G}_\lambda)$ was established by Delgado et al. [13] and Walker [50] based on the fact that $r(\mathscr{G}_\lambda)$ is an eigenvalue of $\mathscr{G}_\lambda$ via showing $\mathscr{G}_\lambda$ is a compact and strongly positive operator and then using Krein–Rutman theorem. However, in our case, $\mathscr{G}_\lambda$ loses compactness since the evolution family $\mathscr{F}(0, a)$ generated by a nonlocal operator does not have regularity, which is a key difference and a main difficulty compared with the case of Laplace diffusion.

On the other hand, from Theorem 2.2-(v) there exists a unique simple real value λ_0 such that $r(\mathscr{C}_{\lambda_0}) = 1$. Therefore, by the theory of positive operators,

$$r(\mathscr{G}_{\lambda_0}) \ge r(\mathscr{C}_{\lambda_0}) = 1.$$

Moreover, $\lim_{\lambda\to\infty} r(\mathscr{G}_\lambda) = 0$. Now since $r(\mathscr{G}_\lambda)$ is continuous and decreasing with respect to λ by Lemma 3.3, there exists a real $\hat{\lambda}_0$ such that $r(\mathscr{G}_{\hat{\lambda}_0}) = 1$. Since $\mathscr{G}_\lambda$ is positive, $1 = r(\mathscr{G}_{\hat{\lambda}_0}) \in \sigma(\mathscr{G}_{\hat{\lambda}_0}) \ne \emptyset$, which implies that $\hat{\lambda}_0 \in \sigma(A')$ thus $\sigma(A') \ne \emptyset$. Assume that there is $\lambda_1 < \lambda_2$ such that $r(\mathscr{G}_{\lambda_1}) = r(\mathscr{G}_{\lambda_2}) = 1$. Since $\lambda \to r(\mathscr{G}_\lambda)$ is decreasing and log convex, it follows that $r(\mathscr{G}_\lambda) = 1$ for all $\lambda \ge \lambda_1$. This contradicts the fact that $r(\mathscr{G}_\lambda) \to 0$

as $\lambda \to \infty$. So there is a unique $\hat{\lambda}_0 \in \mathbb{R}$ such that $r(\mathscr{G}_{\hat{\lambda}_0}) = 1$. This is equivalent to the uniqueness of $\hat{\lambda}_0 \in \sigma(A')$. Moreover, we have shown that the mapping $\lambda \to r(\mathscr{G}_\lambda)$ is either strictly decreasing on the interval $(-\infty, \infty)$ or strictly decreasing on some interval $(-\infty, \lambda_0)$ with $r(\mathscr{G}_\lambda) = 0$ for all $\lambda \geq \lambda_0$.

Furthermore, for any $\lambda \in \mathbb{R}$, when $\lambda > \hat{\lambda}_0$ we have $r(\mathscr{G}_\lambda) < r(\mathscr{G}_{\hat{\lambda}_0}) = 1$, $(I - \mathscr{G}_\lambda)^{-1}$ exists and is positive. Moreover, $1 \in \rho(\mathscr{G}_\lambda) \Rightarrow \lambda \in \rho(A')$. Therefore, the semigroup $\{S'(t)\}_{t\geq 0}$ generated by A' is positive and further $\hat{\lambda}_0$ is larger than any other real spectral values in $\sigma(A')$. It follows that $\hat{\lambda}_0 = s_{\mathbb{R}}(A') := \sup\{\lambda \in \mathbb{R} : \lambda \in \sigma(A')\}$. Next we claim that A' is a *resolvent positive operator*. In fact, it is easy to see the resolvent set of A' contains an infinite ray $(\hat{\lambda}_0, \infty)$ and $(\lambda I - A')^{-1}$ is a positive operator for $\lambda > \hat{\lambda}_0$ by (3.9) since $\mathscr{F}(\tau, a)$ is positive. But since $E = L^2((0, a^+) \times \Omega)$ is a Banach lattice with normal and generating cone $E_+ = L^2_+((0, a^+) \times \Omega)$ and $s(A') \geq \hat{\lambda}_0 > -\infty$ due to $\hat{\lambda}_0 \in \sigma(A')$, we can conclude from Theorem 3.5 in Thieme [45] that $s(A') = s_{\mathbb{R}}(A') = \hat{\lambda}_0$. Thus we have the following result.

Theorem 3.6 *For the operator A' defined in* (3.1)*, there is only one real value $\hat{\lambda}_0 \in \sigma(A')$ satisfying $r(\mathscr{G}_{\hat{\lambda}_0}) = 1$ such that $s(A') = \hat{\lambda}_0$.*

Now we expect to establish the criterion of stability of the zero steady state via $s(A')$. Fortunately, for a Banach lattice $E = L^2((0, a^+) \times \Omega)$ we have $s(A') = \omega_0(A')$, where $\omega_0(A) := \lim_{t\to\infty} \frac{\log \|S(t)\|}{t}$ is the *growth bound* of $\{S(t)\}_{t\geq 0}$ with infinitesimal generator A (see Thieme [46, Theorem 6.4] or Clément et al. [6, Theorem 9.7]). Thus, we have the following result.

Theorem 3.7 *If $\mathscr{R}_0 := r(\mathscr{G}_0) < 1$, then the zero steady state of system* (3.2) *is globally exponentially stable. Otherwise, if $\mathscr{R}_0 > 1$, then the zero steady state is unstable.*

Proof If $\mathscr{R}_0 = r(\mathscr{G}_0) < 1$, there exists a unique $\lambda_0 < 0$ such that $r(\mathscr{G}_{\lambda_0}) = 1$, then by Theorem 3.6, $\omega_0(A') = s(A') = \lambda_0 < 0$, thus the zero steady state is globally exponentially stable. While, if $\mathscr{R}_0 = r(\mathscr{G}_0) > 1$, there exists a unique $\lambda_0 > 0$ such that $r(\mathscr{G}_{\lambda_0}) = 1$, then by Theorem 3.6, $\omega_0(A') = s(A') = \lambda_0 > 0$. Thus the zero steady state is unstable by Webb [53, Proposition 4.12].

Remark 3.8 Note that Thieme [46] discussed age-structured models with an additional structure and gave a theorem which states that $s(A')$ has the same sign as $r(\mathscr{G}_0) - 1$, provided $\omega(\mathscr{F}) < 0$. In fact, in our present paper, we obtain the same result. Moreover, we find $s(A')$ explicitly in Theorem 3.6, after setting the concrete function space $E = L^2((0, a^+) \times \Omega)$ and spatial diffusion defined in (3.1) when one notes that

$$\sup_{\tau\geq 0} \int_\tau^{a^+} \|\mathscr{F}(\tau, a)x\|^2_{L^2(\Omega)} da < \infty, \quad x \in L^2(\Omega)$$

indeed holds in our case due to (3.6), which is equivalent to $\omega(\mathscr{F}) < 0$.

Remark 3.9 As was seen above, we analyzed the spectrum of A' partially and established the stability criterion via the sign of $s(A')$. But, it is noted that we do not know whether such $s(A')$ is an eigenvalue of A' due to the lost of compactness of $\mathscr{G}_\lambda$. In fact, it is interesting and worth to analyze the essential spectrum of $\mathscr{G}_\lambda$ via the fact that the semigroup generated by nonlocal operators is an α-contraction and obtain an estimate of its essential spectral radius

$$r_e(\mathscr{G}_\lambda) \leq \|\mathscr{G}_\lambda\|_e \leq \int_0^{a^+} \overline{\beta}(a) e^{-\lambda a} e^{-\int_0^a \underline{\mu}(\rho) d\rho} e^{-a} da,$$

where $r_e(A)$ and $\|A\|_e$ represent the essential spectral radius and essential norm of operator A, respectively. However, we do not know whether $r_e(\mathscr{G}_\lambda) < r(\mathscr{G}_\lambda)$ holds and then use the generalized Krein–Rutman theorem to conclude that $s(A')$ is indeed an eigenvalue.

4 Weak Solutions and Comparison Principle: Nonautonomous Case

In the previous sections, we considered the autonomous cases where β and μ depend on age a or/and spatial variable x and analyzed the spectra of the infinitesimal generators and determined the asymptotic behavior of solutions. In this section we will establish the existence and uniqueness of the weak solutions (see the definition in the following) of the following nonautonomous and nonhomogeneous age-structured model with nonlocal diffusion where β and μ are both dependent on t, a, x:

$$\begin{cases} \frac{\partial u(t,a,x)}{\partial t} + \frac{\partial u(t,a,x)}{\partial a} = d(J * u - u)(t,a,x) - \mu(t,a,x)u(t,a,x) + f(t,a,x) \quad \text{a.e. in } Q_T \\ u(0,a,x) = u_0(a,x) \quad \text{in } L^2(\Omega) \quad \text{a.e.}\, a \in (0,a^+) \\ u(t,0,x) = \int_0^{a^+} \beta(t,a,x)u(t,a,x)da \quad \text{in } L^2(\Omega) \quad \text{a.e.}\, t \in (0,T) \\ u(t,a,x) = 0 \quad \text{a.e. } (0,T) \times (0,a^+) \times \mathbb{R}^N \backslash \Omega, \end{cases} \tag{4.1}$$

where the mortality rate $\mu(t,a,x)$ satisfies

$$\mu(t,a,x) \in L^\infty_{\text{loc}}(Q_T),\ \mu(t,a,x) \geq \mu_0(t,a) \geq 0\, a.e. \text{ in } Q_T, \quad \mu_0 \in L^\infty_{\text{loc}}((0,T) \times (0,a^+)), \tag{4.2}$$

and in addition, it satisfies that $\int_0^{a^+} \mu_0(t-a^+ + a, a)da = \infty$ a.e. $t \in (0,T)$, where $\mu_0(t,a)$ is extended by zero on $(-\infty, 0) \times (0,a^+)$. Then the solution u is zero, for $a = a^+$; i.e.

$$\lim_{\epsilon \to 0^+} u(t-\epsilon, a^+ - \epsilon, x) = 0, \quad \text{a.e. } (t,x) \in (0,T) \times \Omega$$

while the fertility rate $\beta(t,a,x)$ is a bounded nonnegative measurable function satisfying

$$\beta(t,a,x) \in L^\infty(Q_T),\ \beta(t,a,x) \geq 0\ a.e. \text{ in } Q_T, \tag{4.3}$$

in which $Q_T := (0,T) \times (0,a^+) \times \Omega$. $f(t,a,x)$ is the recruitment term which satisfies $f(t,a,x) \in L^2(Q_T)$ and $f(t,a,x) \geq 0$ a.e. in Q_T. $u_0(a,x) \in L^2((0,a^+) \times \Omega)$ is the initial data with $u_0(a,x) \geq 0$ a.e.

Since the system is nonautonomous, it is not easy to study it by the method of semigroups as above, especially the spectrum analysis. We would like to use energy estimates to deal with it. Moreover, we will also establish the comparison principle. In the following we will prove lemmas and theorems under the conditions $\mu \in L^\infty(Q_T)$ and $\mu(t,a,x) \geq 0$ a.e. in Q_T. For $\mu \in L^\infty_{\text{loc}}(Q_T)$, one can use the truncation technique in Anita [2, Theorem 4.1.3], so we omit it.

Lemma 4.1 *For any $u_0 \in L^2(\Omega)$, $g \in L^2((0,T) \times \Omega)$, there exists a unique solution*

$$u \in C([0,T], L^2(\Omega)) \cap AC((0,T), L^2(\Omega)) \cap L^2((0,T) \times \Omega)$$

of the following system

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - d(J * u - u)(t,x) = g(t,x), \quad t > 0, x \in \Omega, \\ u(t,x) = 0, \quad t > 0, x \in \mathbb{R}^N \backslash \Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{4.4}$$

Moreover, if $0 \le u_0^1 \le u_0^2$, $g_1 \le g_2$ *a.e., then* $0 \le u_1(t,x) \le u_2(t,x)$ *a.e. in* $(0,T) \times \Omega$.

Proof The proof is in the "Appendix". □

Using the notation of Anita [2], we introduce the definition of a solution of (4.1). By a *solution* to system (4.1) we mean a function $u \in L^2(Q_T)$, which belongs to $C(\overline{L}, L^2(\Omega)) \cap AC(L, L^2(\Omega)) \cap L^2(L, L^2(\Omega))$ for almost any characteristic line L of equation

$$a - t = a_0 - t_0 \quad (a,t) \in (0,a^+) \times (0,T),$$

where $(a_0, t_0) \in \{0\} \times (0,T) \cup (0,a^+) \times \{0\}$, and satisfies

$$\begin{cases} \frac{\partial u(t,a,x)}{\partial t} + \frac{\partial u(t,a,x)}{\partial a} = d(J*u-u)(t,a,x) - \mu(t,a,x)u(t,a,x) + f(t,a,x) \quad \text{a.e. in } Q_T, \\ \lim\limits_{\epsilon \to 0^+} u(\epsilon, a+\epsilon, \cdot) = u_0(a,\cdot) \quad \text{in } L^2(\Omega) \quad \text{a.e. } a \in (0,a^+), \\ \lim\limits_{\epsilon \to 0^+} u(t+\epsilon, \epsilon, \cdot) = \int_0^{a^+} \beta(t,a,\cdot)u(t,a,\cdot)da \quad \text{in } L^2(\Omega) \quad \text{a.e. } t \in (0,T), \\ u(t,a,x) = 0 \quad \text{a.e. } (0,T) \times (0,a^+) \times \mathbb{R}^N \backslash \Omega. \end{cases} \tag{4.5}$$

We can write the characteristic line as

$$L = \{(a,t) \in (0,a^+) \times (0,T) : a - t = a_0 - t_0\} = \{(a_0 + s, t_0 + s : s \in (0,\alpha)\},$$

here $(a_0+\alpha, t_0+\alpha) \in a^+ \times (0,T) \cup (0,a^+) \times \{T\}$. We give the following lemmas motivated by Anita [2].

Lemma 4.2 *For any* $f \in L^2(Q_T)$, $b \in L^2((0,T) \times \Omega)$*, there exists a unique solution* $u_b \in L^2(Q_T)$ *of the following system*

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - d(J*u-u) + \mu(t,a,x)u = f(t,a,x) \quad \textit{a.e. in } Q_T, \\ u(0,a,x) = u_0(a,x) \quad \textit{in } L^2(\Omega) \quad \textit{a.e. } a \in (0,a^+), \\ u(t,0,x) = b(t,x) \quad \textit{in } L^2(\Omega), \quad \textit{a.e. } t \in (0,T), \\ u(t,a,x) = 0, \quad (0,T) \times (0,a^+) \times \mathbb{R}^N \backslash \Omega. \end{cases} \tag{4.6}$$

Proof Fix any $w \in L^2(Q_T)$, we first prove that the following system has a unique solution $u_{b,w}(t,a,x)$:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - d(J*u-u) + \mu(t,a,x)w = f(t,a,x) \quad \text{a.e. in } Q_T, \\ u(0,a,x) = u_0(a,x) \quad \text{in } L^2(\Omega) \quad \text{a.e. } a \in (0,a^+), \\ u(t,0,x) = b(t,x) \quad \text{in } L^2(\Omega) \quad \text{a.e. } t \in (0,T), , \\ u(t,a,x) = 0, \quad x \in \mathbb{R}^N \backslash \Omega. \end{cases} \tag{4.7}$$

We can view (4.7) as a collection of linear nonlocal equations on the characteristic line L. Define

$$\begin{cases} \tilde{u}(s,x) = u(t_0+s, a_0+s, x), \quad (s,x) \in (0,\alpha) \times \Omega, \\ \tilde{w}(s,x) = w(t_0+s, a_0+s, x), \quad (s,x) \in (0,\alpha) \times \Omega, \\ \tilde{f}(s,x) = f(t_0+s, a_0+s, x), \quad (s,x) \in (0,\alpha) \times \Omega, \\ \tilde{\mu}(s,x) = u_0(t_0+s, a_0+s, x), \quad (s,x) \in (0,\alpha) \times \Omega. \end{cases}$$

According to Lemma 4.1, the following system admits a unique solution $\tilde{u} \in C([0,\alpha], L^2(\Omega)) \cap AC((0,\alpha), L^2(\Omega)) \cap L^2((0,\alpha)\times\Omega)$:

$$\begin{cases} \frac{\partial \tilde{u}}{\partial s} - d(J * \tilde{u} - \tilde{u}) = \tilde{f} - \tilde{\mu}\tilde{w} \quad (s,x) \in (0,\alpha)\times\Omega, \\ \tilde{u}(0,x) = \begin{cases} b(t_0,x), & a_0 = 0, x \in \Omega, \\ u_0(a_0,x), & t_0 = 0, x \in \Omega, \end{cases} \\ \tilde{u}(s,x) = 0, \quad (s,x) \in (0,\alpha)\times\mathbb{R}^N\setminus\Omega. \end{cases} \tag{4.8}$$

In fact, multiplying the first equation of system (4.8) by $\tilde{u}$ and integrating on $(0,s)\times\Omega$, we have

$$\|\tilde{u}(s)\|^2_{L^2(\Omega)} \le \|\tilde{u}(0)\|^2_{L^2(\Omega)} + \|\tilde{f} - \tilde{\mu}\tilde{w}\|^2_{L^2((0,\alpha)\times\Omega)} + \int_0^s \|\tilde{u}(\tau)\|^2_{L^2(\Omega)} d\tau,$$

where we used inequality (2.11). Then by Gronwall's inequality we obtain

$$\|\tilde{u}(s)\|^2_{L^2(\Omega)} \le C\left(\|\tilde{u}(0)\|^2_{L^2(\Omega)} + \|\tilde{f} - \tilde{\mu}\tilde{u}\|^2_{L^2((0,\alpha)\times\Omega)}\right)e^{\alpha}, \quad \forall s \in [0,\alpha]. \tag{4.9}$$

Now if we denote

$$u_{b,w}(t_0+s, a_0+s, x) = \tilde{u}(s,x), \quad (s,x) \in (0,\alpha)\times\Omega$$

for any characteristic line L. It follows from Lemma 4.1 and (4.9) that

$$u_{b,w} \in C(\overline{L}, L^2(\Omega)) \cap AC(L, L^2(\Omega)) \cap L^2(L, L^2(\Omega))$$

for almost any characteristic line L, and $u_{b,w}$ satisfies

$$\begin{cases} \frac{\partial u_{b,w}}{\partial t} + \frac{\partial u_{b,w}}{\partial a} - d(J * u_{b,w} - u_{b,w}) + \mu(t,a,x)w = f(t,a,x) \quad \text{a.e. in } Q_T, \\ u_{b,w}(0,a,x) = u_0(a,x) \quad \text{a.e. in } (0,a^+)\times\Omega, \\ u_{b,w}(t,0,x) = b(t,x) \quad \text{a.e. in } (0,T)\times\Omega, \\ u_{b,w}(t,a,x) = 0 \quad \text{a.e. in } (0,T)\times(0,a^+)\times\mathbb{R}^N\setminus\Omega, \end{cases} \tag{4.10}$$

where

$$u_{b,w}(t,0,\cdot) = \lim_{\epsilon\to 0^+} u_{b,w}(t+\epsilon,\epsilon,\cdot), \quad \text{in } L^2(\Omega) \text{ a.e. } t \in (0,T)$$

and

$$u_{b,w}(0,a,\cdot) = \lim_{\epsilon\to 0^+} u_{b,w}(\epsilon,a+\epsilon,\cdot), \quad \text{in } L^2(\Omega) \text{ a.e. } a \in (0,a^+).$$

Now we need to show that $u_{b,w} \in L^2(Q_T)$ to make $u_{b,w}$ a solution of (4.10). Recall that there exists an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}} \subset L^2(\Omega)$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$, $\lambda_j \to 1$ as $j\to\infty$, such that

$$\begin{cases} -(J * \varphi_j - \varphi_j) = \lambda_j\varphi_j, \\ \varphi_j = 0, \quad x \in \mathbb{R}^N\setminus\Omega. \end{cases}$$

Then we have

$$f(t,a,x) - \mu(t,a,x)w = \sum_{j=0}^{\infty} v^j(t,a)\varphi_j(x) \quad \text{in } L^2(\Omega) \text{ a.e. } (t,a) \in (0,T)\times(0,a^+),$$

$$b(t,x) = \sum_{j=0}^{\infty} b^j(t)\varphi_j(x) \quad \text{in } L^2(\Omega) \text{ a.e. } t \in (0,T),$$

$$u_0(a,x) = \sum_{j=0}^{\infty} u_0^j(a)\varphi_j(x) \quad \text{in } L^2(\Omega) \text{ a.e. } a \in (0,a^+),$$

and

$$u_{b,w}(t,a,x) = \sum_{j=0}^{\infty} u_{b,w}^j(t,a)\varphi_j(x), \quad \text{in } L^2(\Omega) \text{ a.e. } (t,a) \in (0,T)\times(0,a^+).$$

Plugging it into (4.10), we have

$$\begin{cases} \frac{\partial u_{b,w}^j}{\partial t} + \frac{\partial u_{b,w}^j}{\partial a} + d\lambda_j u_{b,w}^j = v^j(t,a), \quad (t,a) \in (0,T)\times(0,a^+), \\ u_{b,w}^j(0,a) = u_0^j(a), \quad a \in (0,a^+), \\ u_{b,w}^j(t,0) = b^j(t), \quad t \in (0,T). \end{cases} \tag{4.11}$$

Multiplying the first equation of (4.11) by $u_{b,w}^j$ we obtain that

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}|u_{b,w}^j(t,a)|^2 + \frac{1}{2}\frac{\partial}{\partial a}|u_{b,w}^j(t,a)|^2 &\le |v^j(t,a)||u_{b,w}^j(t,a)| \\ &\le \frac{1}{2}|u_{b,w}^j(t,a)|^2 + \frac{1}{2}|v^j(t,a)|^2 \end{aligned}$$

and integrating on $(0,t)\times(0,a^+)$ we get

$$\begin{aligned} \int_0^{a^+} |u_{b,w}^j(t,a)|^2 da \ &\le \int_0^{a^+} |u_0^j(a)|^2 da + \int_0^t |b^j(s)|^2 ds + \int_0^t \int_0^{a^+} |u_{b,w}^j(s,a)|^2 da ds \\ &+ \int_0^t \int_0^{a^+} |v^j(s,a)|^2 da ds. \end{aligned}$$

By Gronwall's inequality we have

$$\|u_{b,w}^j(t)\|_{L^2(0,a^+)}^2 \le e^t \left(\|u_0^j\|_{L^2(0,a^+)}^2 + \|b^j\|_{L^2(0,T)}^2 + \|v^j\|_{L^2((0,T)\times(0,a^+))}^2 \right), \quad \forall t \in (0,T)$$

and it follows that

$$\begin{aligned} &\|u_{b,w}^j\|_{L^2((0,T)\times(0,a^+))}^2 \\ &\le (e^T - 1)\left(\|u_0^j\|_{L^2(0,a^+)}^2 + \|b^j\|_{L^2(0,T)}^2 + \|v^j\|_{L^2((0,T)\times(0,a^+))}^2 \right), \quad \forall j \in \mathbb{N}. \end{aligned}$$

In conclusion, $u_{b,w} \in L^2(Q_T)$ and

$$\begin{aligned} \|u_{b,w}\|_{L^2(Q_T)}^2 &= \sum_{j=0}^{\infty} \|u_{b,w}^j\|_{L^2((0,T)\times(0,a^+))}^2 \\ &\le (e^T - 1)\left(\|u_0\|_{L^2((0,a^+)\times\Omega)}^2 + \|b\|_{L^2((0,T)\times\Omega)}^2 + \|f - \mu w\|_{L^2(Q_T)}^2 \right). \end{aligned} \tag{4.12}$$

Now for any $w(t,a,x) \in L^2(Q_T)$, we prove that system (4.7) has a solution $u_{b,w} \in L^2(Q_T)$. Define a mapping $\mathscr{F} : L^2(Q_T) \to L^2(Q_T)$ by $\mathscr{F}(w_i(t,a,x)) = u_{b,w_i}(t,a,x), i = 1,2$. Take any two functions $w_1, w_2 \in L^2(Q_T)$, then $u_{b,w_1} - u_{b,w_2}$ satisfies

$$\begin{cases} \frac{\partial(u_{b,w_1}-u_{b,w_2})}{\partial t} + \frac{\partial(u_{b,w_1}-u_{b,w_2})}{\partial a} - d(J*(u_{b,w_1}-u_{b,w_2}) - (u_{b,w_1}-u_{b,w_2})) + \mu(w_1 - w_2) = 0 \quad \text{a.e. in } Q_T, \\ (u_{b,w_1}-u_{b,w_2})(0,a,x) = 0 \quad \text{a.e. in } (0,a^+)\times\Omega, \\ (u_{b,w_1}-u_{b,w_2})(t,0,x) = 0 \quad \text{a.e. in } (0,T)\times\Omega, \\ (u_{b,w_1}-u_{b,w_2})(t,a,x) = 0 \quad \text{a.e. in } (0,T)\times(0,a^+)\times\mathbb{R}^N\backslash\Omega. \end{cases} \tag{4.13}$$

By the result of (4.12), we have

$$\|u_{b,w_1} - u_{b,w_2}\|^2_{L^2(Q_T)} \le (e^T - 1)(\|\mu(w_1 - w_2)\|^2_{L^2(Q_T)}) \quad \text{in } L^2(Q_T).$$

It is easy to see that when T is small enough, $u_{b,w}(t, a, x)$ is a contraction mapping with respect to $w(t, a, x)$. Consequently, there exists a unique solution u_b of system (4.6) for sufficient small T. Furthermore, we can extend T by following previous steps for $t \in (T, 2T)$ since this is a linear equation. Hence, system (4.6) has a unique solution $u_b \in L^2(Q_T)$ with $u_b \in C(\overline{L}, L^2(\Omega)) \cap AC(L, L^2(\Omega)) \cap L^2(L, L^2(\Omega))$ for almost any characteristic line L. □

Next we need to establish the following auxiliary result.

Lemma 4.3 *Under the hypotheses of Lemma 4.2, for any $b_1, b_2 \in L^2((0, T) \times \Omega)$, $0 \le b_1(t, x) \le b_2(t, x)$ a.e. in $(0, T) \times \Omega$, we have that*

$$0 \le u_{b_1}(t, a, x) \le u_{b_2}(t, a, x) \quad \text{a.e. in } Q_T.$$

Proof Consider $v = u_{b_1} - u_{b_2}$, which is a solution of

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} - d(J * v - v) + \mu(t, a, x)v = 0 \quad \text{a.e. in } Q_T, \\ v(0, a, x) = 0 \quad \text{a.e. in } (0, a^+) \times \Omega, \\ v(t, 0, x) = b_1(t, x) - b_2(t, x) \quad \text{a.e. in } (0, T) \times \Omega, \\ v(t, a, x) = 0 \quad \text{a.e. in } (0, T) \times (0, a^+) \times \mathbb{R}^N \backslash \Omega. \end{cases} \tag{4.14}$$

Multiplying the first equation by v^+ and integrating over Q_t, we obtain

$$\frac{1}{2}\int_0^{a^+}\int_\Omega |v^+(t, a, x)|^2 dxda \le \int_0^t \int_\Omega (b_1 - b_2)(s, x)v^+(s, 0, x)dxds, \quad \forall t \in [0, T].$$

Consequently $v^+ = 0$ a.e. in Q_T. So we get the conclusion of the lemma. □

We now show that system (4.1) has a unique solution $u(t, a, x) \in L^2(Q_T)$.

Lemma 4.4 *There exists a unique solution $u \in L^2(Q_T)$ of system (4.1).*

Proof Define an operator $\mathscr{G} : L^2((0, T) \times \Omega) \to L^2((0, T) \times \Omega)$ by

$$(\mathscr{G}b)(t, x) = \int_0^{a^+} \beta(t, a, x)u_b(t, a, x)da \quad \text{a.e. in} (0, T) \times \Omega.$$

For any fixed $b_i \in L^2((0, T) \times \Omega)$, $i = 1, 2$, let u_{b_1} and u_{b_2} be the solutions of system (4.6) with $b_1(t, x)$ and $b_2(t, x)$, respectively. Let $v(t, a, x) = u_{b_1}(t, a, x) - u_{b_2}(t, a, x)$. Then $v(t, a, x)$ satisfies

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} - d(J * v - v) + \mu(t, a, x)v = 0 \quad \text{a.e. in } Q_T, \\ v(0, a, x) = 0 \quad \text{a.e. in } (0, a^+) \times \Omega, \\ v(t, 0, x) = b_1(t, x) - b_2(t, x) \quad \text{a.e. in } (0, T) \times \Omega, \\ v(t, a, x) = 0 \quad \text{a.e. in } (0, T) \times (0, a^+) \times \mathbb{R}^N \backslash \Omega. \end{cases} \tag{4.15}$$

Multiplying the first equation by v and integrating over Q_t, we obtain

$$\|v(t)\|^2_{L^2((0,a^+)\times\Omega)} \le \int_0^t \|b_1(s) - b_2(s)\|^2_{L^2(\Omega)} ds.$$

Consequently,

$$\int_0^T e^{-\lambda t}\|v(t)\|^2_{L^2((0,a^+)\times\Omega)}dt = \int_0^T\int_0^t e^{-\lambda t}\|b_1(s)-b_2(s)\|^2_{L^2(\Omega)}dsdt$$
$$= \int_0^T\int_s^T e^{-\lambda t}\|b_1(s)-b_2(s)\|^2_{L^2(\Omega)}dtds.$$

This implies that

$$\int_0^T e^{-\lambda t}\|w(t)\|^2_{L^2((0,a^+)\times\Omega)}dt \le \frac{1}{\lambda}\int_0^T e^{-\lambda t}\|b_1(t)-b_2(t)\|^2_{L^2(\Omega)}dt \tag{4.16}$$

for any $\lambda \in (0,\infty)$. Consider $L^2((0,T)\times\Omega)$ with the norm

$$\|g\| = \left(\int_0^T e^{-\lambda t}\|b(t)\|^2_{L^2(\Omega)}dt\right)^{1/2}, \quad \forall g \in L^2((0,T)\times\Omega),$$

which is equivalent to the usual norm (the constant λ will be given later). Then we have

$$\|\mathscr{G}b_1 - \mathscr{G}b_2\|^2 = \int_0^T e^{-\lambda t}\|\int_0^{a^+}\beta(t,a,x)(u_{b_1}(t,a,x)-u_{b_2}(t,a,x)da\|^2_{L^2(\Omega)}dt$$
$$\le a^+\|\beta\|^2_{L^\infty(Q_T)}\int_0^T e^{-\lambda t}\|v(t)\|^2_{L^2((0,a^+)\times\Omega)}dt$$
$$\le \frac{a^+}{\lambda}\|\beta\|^2_{L^\infty(Q_T)}\|b_1-b_2\|^2. \tag{4.17}$$

It is now obvious that for any $\lambda > a^+\|\beta\|^2_{L^\infty(Q_T)}$, $\mathscr{G}$ is a contraction mapping on $(L^2((0,T)\times\Omega), d)$, where d is the metric defined by

$$d(b_1,b_2) = \|b_1-b_2\|, \quad \forall b_1,b_2 \in L^2((0,T)\times\Omega).$$

Now Banach contraction mapping theorem allows us to conclude that there exists a unique $b \in L^2((0,T)\times\Omega)$ such that $b = \mathscr{G}b$.

Finally we present the comparison principle for system (4.1).

Theorem 4.5 *There is a unique solution $u \in L^2(Q_T)$ of system (4.1) with μ satisfying (4.2). If u_1 and u_2 are the solutions of system (4.1) with $\mu_1, f_1, \beta_1, u_{01}$ and $\mu_2, f_2, \beta_2, u_{02}$, respectively, and $\mu_1 \ge \mu_2$ satisfy (4.2), $f_1 \le f_2, \beta_1 \le \beta_2, u_{01} \le u_{02}$, then*

$$0 \le u_1(t,a,x) \le u_2(t,a,x) \quad \textit{a.e.} \quad \textit{in } Q_T.$$

Proof Define operators $\mathscr{G}_i : L^2((0,T)\times\Omega) \to L^2((0,T)\times\Omega)$, $i = 1,2$, by

$$(\mathscr{G}_i b)(t,x) = \int_0^{a^+}\beta(t,a,x)u_i(t,a,x)da \quad \text{a.e. } (t,x) \in (0,T)\times\Omega,$$

where $u_i (i = 1,2)$ are solutions of (4.6) corresponding to $(\beta_i, \mu_i, f_i, u_{0i})$ respectively. By Lemma 4.1 we have for any $b \in L^2((0,T)\times\Omega)$ that

$$u_{1,b}(s,x) \le u_{2,b}(s,x) \quad \text{for almost every characteristic line } L,$$

which implies that

$$u_{1,b}(t,a,x) \le u_{2,b}(t,a,x) \quad \text{a.e. in } Q_T,$$

It follows that

$$(\mathscr{G}_1 b)(t,x) \le (\mathscr{G}_2 b)(t,x) \quad \text{a.e. in } (0,T)\times\Omega.$$

We have already proved that there exists a unique fixed point b_2 of $\mathscr{G}_2$ in $L^2((0,T)\times\Omega)$. Since the set

$$\mathscr{C} = \{b \in L^2((0,T)\times\Omega);\, 0 \le b(t,x) \le b_2(t,x) \text{ a.e. in } (0,T)\times\Omega\}$$

is closed, it follows that $\mathscr{G}$ has a unique fixed point $b_1 \in \mathscr{C}$. Thus,

$$0 \le b_1(t,x) \le b_2(t,x) \text{ a.e. in } (0,T)\times\Omega$$

and the conclusion follows from Lemma 4.3. □

Remark 4.6 For the semigroup $\{S(t)\}_{t\ge 0}$ generated by A, we have determined its dominant eigenvalue σ_0 in Theorem 2.2. The stability of the zero steady state is also determined by the sign of σ_0 via the comparison principle. In fact, let $u(t,\phi_{\sigma_0}) = e^{\sigma_0 t}\phi_{\sigma_0}$, where $\phi_{\sigma_0} \in E$ is the eigenfunction corresponding to σ_0, then $u(t,\phi_{\sigma_0})$ is obvious a solution of (2.3). Now for any $\Phi \in E$ with $\Phi \ge 0$, there exists $M > 0$, such that $\Phi \le M\phi_{\sigma_0}$, then by comparison principle, we have

$$0 \le u(t,\Phi) \le u(t, M\phi_{\sigma_0}) = Me^{\sigma_0 t}\phi_{\sigma_0}, \quad \forall t \ge 0.$$

Hence, when $\sigma_0 < 0$, $u(t,\Phi) \to 0$ as $t \to \infty$, which implies stability of the zero steady state.

5 An Age- and Size-Structured Model with Nonlocal Diffusion

In population dynamics, besides the age structure, there are some other physiological structures that need to be taken into account, for example the size of individuals which is used to distinguish cohorts (Tucker and Zimmerman [47], Webb [55]), infection age which is the time elapsed since infection, and recovery age which is the time elapsed since the last infection (Inaba [30]). In fact, various size-structured models have been studied in the literature, see Cushing [8–11], Calsina [4] and Gwiazda, Lorenz and Marciniak [22]. Kooijman and Metz [33] considered a nonlinear age-size structured model with experiments and later Thieme [44] analyzed the model by formulating it as an integral equation.

In this section, we consider the following age- and size-structured model with nonlocal diffusion:

$$\begin{cases} \frac{\partial u(t,a,s,x)}{\partial t} + \frac{\partial u(t,a,s,x)}{\partial a} + \frac{\partial u(t,a,s,x)}{\partial s} = d(J * u - u)(t,a,s,x) - \mu(a,s)u(t,a,s,x), & t,a,s > 0, x \in \Omega,\\ u(0,a,s,x) = u_0(a,s,x), \quad a,s > 0, x \in \Omega, \\ u(t,0,s,x) = \int_0^{s+}\int_0^{a^+} \beta(a,p,s)u(t,a,p,x)dadp, \quad t,s > 0, x \in \Omega, \\ u(t,a,0,x) = \int_0^{a^+}\int_0^{s^+} \chi(a,p,s)u(t,p,s,x)dsdp, \quad t,a > 0,, x \in \Omega, \\ u(t,a,s,x) = 0, \quad t,a,s > 0, x \in \mathbb{R}^N\backslash\Omega, \end{cases} \tag{5.1}$$

where $s \in [0, s^+]$ is the size of the population. We assume that $a^+, s^+ < \infty$.

Assumption 5.1 (i) The rate $\beta \in L^2([0,a^+)\times[0,s^+)\times[0,s^+))$ is Lipschitz continuous and zero outside the domain,

$$\sup_{(a,p)\in[0,a^+)\times[0,s^+)} \beta(a,p,s) \le \overline{\beta}(s) \quad \text{and} \quad \overline{\beta} \in L^2(0,s^+),$$

while $\chi \in L^2([0, a^+) \times [0, a^+) \times [0, s^+))$ is Lipschitz continuous and zero outside the domain,

$$\sup_{(p,s)\in[0,a^+)\times[0,s^+)} \chi(a, p, s) \le \overline{\chi}(a) \quad \text{and} \quad \overline{\chi}(a) \in L^2(0, a^+);$$

(ii) The following limits

$$\lim_{h\to 0} \int_0^{s^+} |\beta(a, p, s+h) - \beta(a, p, s)|^2 ds = 0$$

and

$$\lim_{h\to 0} \int_0^{a^+} |\chi(a+h, p, s) - \chi(a, p, s)|^2 da = 0$$

hold uniformly for $(a, p) \in (0, a^+) \times (0, s^+)$ and $(p, s) \in (0, a^+) \times (0, s^+)$, respectively;

(iii) There exists nonnegative functions $\epsilon_1(p)$ and $\epsilon_2(p)$ such that $\beta(a, p, s) \ge \epsilon_1(p) > 0$ and $\chi(a, p, s) \ge \epsilon_2(p) > 0$, respectively, for all $a, s > 0$ $(a, s > 0)$;

(iv) $\int_0^{a^+} \mu(a, s)da = \infty$ uniformly in $s \in (0, s^+)$ and $\int_0^{s^+} \mu(a, s)ds = \infty$ uniformly in $a \in (0, a^+)$, where $\mu(a, s)$ is extended by zero outside its domain.

This assumption implies that $u(t, a^+, s) = 0$ for any $t, s > 0$ and $u(t, a, s^+) = 0$ for any $t, a > 0$ respectively.

We consider Eq. (5.1) in the state space $\tilde{E} := L^2((0, a^+) \times (0, s^+) \times \Omega)$ with the usual norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Define the operator $\tilde{A} : \tilde{E} \to \tilde{E}$ as follows:

$$\begin{aligned} \tilde{A}\phi(a, s, x) &= d(J * \phi - \phi)(a, s, x) - \frac{\partial \phi(a, s, x)}{\partial a} \\ &\quad - \frac{\partial \phi(a, s, x)}{\partial s} - \mu(a, s)\phi(a, s, x), \quad \forall \phi \in D(\tilde{A}), \\ D(\tilde{A}) &= \Big\{ \phi(a, s, x) | \phi, \tilde{A}\phi \in \tilde{E}, \phi|_{\mathbb{R}^N\setminus\Omega} = 0, \phi(0, s, x) \\ &= \int_0^{s^+} \int_0^{a^+} \beta(a, p, s)\phi(a, p, x)dadp, \\ \phi(a, 0, x) &= \int_0^{a^+} \int_0^{s^+} \chi(a, p, s)\phi(p, s, x)dsdp \Big\}. \end{aligned} \tag{5.2}$$

Thus, Eq. (5.1) can be written as an evolutionary equation on $\tilde{E}$:

$$\begin{cases} \frac{du(t,a,s,x)}{dt} = \tilde{A}u(t, a, s, x) \\ u(0, a, s, x) = u_0(a, s, x). \end{cases} \tag{5.3}$$

Now define an operator H on $L^2((0, a^+) \times (0, a^+))$ by

$$\begin{aligned} H\phi(a, s) &= -\frac{\partial \phi(a, s)}{\partial a} - \frac{\partial \phi(a, s)}{\partial s} - \mu(a, s)\phi(a, s), \quad \phi \in D(H), \\ D(H) &= \Big\{ \phi(a, s) | \phi, H\phi \in L^2((0, a^+) \times (0, s^+)), \phi(0, s) \\ &= \int_0^{s^+} \int_0^{a^+} \beta(a, p, s)\phi(a, p)dadp, \end{aligned}$$

$$\phi(a,0)=\int_0^{a^+}\int_0^{s^+}\chi(a,p,s)\phi(p,s)dsdp\Big\}. \tag{5.4}$$

Solving the resolvent equation

$$(\lambda I-H)\phi(a,s)=\psi(a,s),\quad \forall\psi\in L^2((0,a^+)\times(0,s^+)),\lambda\in\mathbb{C},$$

i.e.

$$\begin{cases}\frac{\partial\phi}{\partial a}+\frac{\partial\phi}{\partial s}+(\lambda+\mu(a,s))\phi(a,s)=\psi(a,s),\\ \phi(0,s)=\int_0^{s^+}\int_0^{a^+}\beta(a,p,s)\phi(a,p)dadp,\\ \phi(a,0)=\int_0^{a^+}\int_0^{s^+}\chi(a,p,s)\phi(p,s)dsdp\end{cases} \tag{5.5}$$

by using the method of characteristic lines, we obtain

$$\phi(a,s)=\begin{cases}\psi(a-s,0)e^{-\lambda s}\Pi(a,s,s)+\int_0^s e^{-\lambda\sigma}\Pi(a,s,\sigma)\psi(a-\sigma,s-\sigma)d\sigma, & a-s\geq 0,\\ \psi(0,s-a)e^{-\lambda a}\Pi(a,s,a)+\int_0^a e^{-\lambda\sigma}\Pi(a,s,\sigma)\psi(a-\sigma,s-\sigma)d\sigma, & a-s<0,\end{cases} \tag{5.6}$$

where $\Pi(a,s,\sigma)=e^{-\int_0^\sigma\mu(a-\tau,s-\tau)d\tau}$, $\alpha(s)=\phi(0,s)$ and $\eta(a)=\phi(a,0)$. Plugging them into the boundary conditions in (5.5), we obtain that

$$\begin{aligned}\alpha(t)=&\int_0^{a^+}\int_0^a f_1(a,p,t)\eta(a-p)e^{-\lambda p}dpda+\int_0^{s^+}\int_0^p f_2(a,p,t)\alpha(p-a)e^{-\lambda a}dadp\\ &+\int_0^{a^+}\int_0^a K_1(t,a,p)\psi(a,p)dpda+\int_0^{s^+}\int_0^p K_2(t,a,p)\psi(a,p)dadp,\\ \eta(t)=&\int_0^{a^+}\int_0^p f_3(t,p,s)\eta(p-s)e^{-\lambda s}dsdp+\int_0^{s^+}\int_0^s f_4(t,p,s)\alpha(s-p)e^{-\lambda p}dpds\\ &+\int_0^{a^+}\int_0^p K_3(t,p,s)\psi(p,s)dsdp+\int_0^{s^+}\int_0^s K_4(t,p,s)\psi(p,s)dpds,\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}&f_1(a,p,t)=\beta(a,p,t)\Pi(a,p,p),\\ &f_2(a,p,t)=\beta(a,p,t)\Pi(a,p,a),\\ &f_3(t,p,s)=\chi(t,p,s)\Pi(p,s,s),\\ &f_4(t,p,s)=\chi(t,p,s)\Pi(p,s,p),\\ &K_1(t,a,p)\psi(a,p)=\beta(a,p,t)\int_0^p e^{-\lambda\sigma}\Pi(a,p,\sigma)\psi(a-\sigma,p-\sigma)d\sigma,\\ &K_2(t,a,p)\psi(a,p)=\beta(a,p,t)\int_0^a e^{-\lambda\sigma}\Pi(a,p,\sigma)\psi(a-\sigma,p-\sigma)d\sigma,\\ &K_3(t,p,s)\psi(p,s)=\chi(t,p,s)\int_0^s e^{-\lambda\sigma}\Pi(p,s,\sigma)\psi(p-\sigma,s-\sigma)d\sigma,\\ &K_4(t,p,s)\psi(p,s)=\chi(t,p,s)\int_0^p e^{-\lambda\sigma}\Pi(p,s,\sigma)\psi(p-\sigma,s-\sigma)d\sigma.\end{aligned} \tag{5.8}$$

One can rewrite (5.7) as the following functional equations:

$$\begin{pmatrix}\alpha\\ \eta\end{pmatrix}=F_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}+\begin{pmatrix}G_\lambda^1\psi\\ G_\lambda^2\psi\end{pmatrix}, \tag{5.9}$$

where

$$
\begin{aligned}
&F_\lambda = (F_{1\lambda}, F_{2\lambda}) : L^2(0, s^+) \times L^2(0, a^+) \to L^2(0, s^+) \times L^2(0, a^+),\\
&F_{1\lambda}(\alpha, \eta) = \int_0^{a^+}\int_0^{a} f_1(a, p, t)\eta(a-p)e^{-\lambda p}dpda + \int_0^{s^+}\int_0^{p} f_2(a, p, t)\alpha(p-a)e^{-\lambda a}dadp,\\
&F_{2\lambda}(\alpha, \eta) = \int_0^{a^+}\int_0^{p} f_3(t, p, s)\eta(p-s)e^{-\lambda s}dsdp + \int_0^{s^+}\int_0^{s} f_4(t, p, s)\alpha(s-p)e^{-\lambda p}dpds,\\
&G^1_\lambda\psi = \int_0^{a^+}\int_0^{a} K_1(t, a, s)\psi(a, s)dsda + \int_0^{s^+}\int_0^{s} K_2(t, a, s)\psi(a, s)dads,\\
&G^2_\lambda\psi = \int_0^{a^+}\int_0^{a} K_3(t, a, s)\psi(a, s)dsda + \int_0^{s^+}\int_0^{s} K_4(t, a, s)\psi(a, s)dads.
\end{aligned}
\tag{5.10}
$$

In fact, α, η are respectively in the projection space of $L^2((0, s^+) \times (0, a^+))$, that is, α is obtained from $\phi(a, s) \in L^2((0, s^+) \times (0, a^+))$ by fixing $a = 0$, while η is obtained by fixing $s = 0$. However, $L^2((0, s^+) \times (0, a^+)) \subset L^2(0, s^+) \times L^2(0, a^+)$. Thus, we consider $(\alpha, \eta) \in L^2(0, a^+) \times L^2(0, a^+)$. It is easy to check that F_λ maps $L^2(0, s^+) \times L^2(0, a^+)$ into itself since by Holder inequality

$$
\begin{aligned}
&\|F_{1\lambda}(\alpha, \eta)\|^2\\
&= \int_0^{s^+}\left[\int_0^{a^+}\int_0^{a} f_1(a, p, t)\eta(a-p)e^{-\lambda p}dpda + \int_0^{s^+}\int_0^{p} f_2(a, p, t)\alpha(p-a)e^{-\lambda a}dadp\right]^2 dt\\
&\le 2\int_0^{s^+}\left[\int_0^{a^+}\int_0^{a} f_1(a, p, t)\eta(a-p)e^{-\lambda p}dpda\right]^2 dt + 2\int_0^{s^+}\left[\int_0^{s^+}\int_0^{p} f_2(a, p, t)\alpha(p-a)e^{-\lambda a}dadp\right]^2 dt\\
&\le 2\int_0^{s^+}\overline{\beta}^2(t)dt\left[\int_0^{a^+}\eta(a-p)da\int_0^{a} e^{-(\lambda+\underline{\mu})p}dp\right]^2 + 2\int_0^{s^+}\overline{\beta}^2(t)dt\left[\int_0^{s^+}\alpha(p-a)dp\int_0^{p} e^{-(\lambda+\underline{\mu})a}da\right]^2\\
&\le 2\|\overline{\beta}\|^2_{L^2(0,s^+)}\frac{1}{(\lambda+\underline{\mu})^2}\left[\left(\int_0^{a^+}\eta(a-p)da\right)^2 + \left(\int_0^{s^+}\alpha(p-a)dp\right)^2\right]\\
&\le \frac{2\|\overline{\beta}\|^2_{L^2(0,s^+)}}{(\lambda+\underline{\mu})^2}\max\{a^+, s^+\}\|(\alpha, \eta)\|^2,
\end{aligned}
\tag{5.11}
$$

where $\|\cdot\|$ denotes the norm in the space $L^2_+(0, s^+) \times L^2_+(0, a^+)$ in this section. Similarly, for $F_{2\lambda}$, we also have an estimate as follows:

$$
\|F_{2\lambda}(\alpha, \eta)\|^2 \le \frac{2\|\chi\|^2_{L^2(0,a^+)}}{(\lambda+\underline{\mu})^2}\max\{a^+, s^+\}\|(\alpha, \eta)\|^2. \tag{5.12}
$$

Lemma 5.2 *Let Assumption 5.1 hold. Then the operator F_λ is compact and non-supporting for all $\lambda \in \mathbb{R}$.*

Proof For the compactness of F_λ, it is equivalent to show that for a bounded set K of $L^2((0, s^+)) \times L^2((0, a^+))$,

$$
\lim_{h\to 0}\int_0^{s^+} |F_{1\lambda}(\alpha, \eta)(t+h) - F_{1\lambda}(\alpha, \eta)(t)|^2 dt = 0
$$
$$
\text{uniformly for}\quad (\alpha, \eta) \in L^2((0, s^+)) \times L^2((0, a^+))
$$

and

$$
\lim_{h\to 0}\int_0^{a^+} |F_{2\lambda}(\alpha, \eta)(t+h) - F_{2\lambda}(\alpha, \eta)(t)|^2 dt = 0
$$

$$\text{uniformly for} \quad (\alpha, \eta) \in L^2((0, s^+)) \times L^2((0, a^+)).$$

Now consider $F_{1\lambda}$, that is,

$$\begin{aligned}
&\int_0^{s^+} \Big[\int_0^{a^+} \int_0^a f_1(a, p, t+h) e^{-\lambda p} \eta(a-p) dp da + \int_0^{s^+} \int_0^p f_2(a, p, t+h) e^{-\lambda a} \alpha(p-a) da dp \\
&\quad - \int_0^{a^+} \int_0^a f_1(a, p, t) e^{-\lambda p} \eta(a-p) dp da - \int_0^{s^+} \int_0^p f_2(a, p, t) e^{-\lambda a} \alpha(p-a) da dp \Big]^2 dt \\
&\leq \int_0^{s^+} \Big[\int_0^{a^+} \int_0^a |f_1(a, p, t+h) - f_1(a, p, t)| |e^{-\lambda p}| |\eta(a-p)| dp da \\
&\quad + \int_0^{s^+} \int_0^p |f_2(a, p, t+h) - f_2(a, p, t)| |e^{-\lambda a}| |\alpha(p-a)| da dp \Big]^2 dt \\
&\leq 2 \int_0^{s^+} \Big[\int_0^{a^+} \int_0^a |f_1(a, p, t+h) - f_1(a, p, t)| |e^{-\lambda p}| |\eta(a-p)| dp da \Big]^2 dt \\
&\quad + 2 \int_0^{s^+} \Big[\int_0^{s^+} \int_0^p |f_2(a, p, t+h) - f_2(a, p, t)| |e^{-\lambda a}| |\alpha(p-a)| da dp \Big]^2 dt \\
&\to 0 \quad \text{as} \quad h \to 0
\end{aligned} \tag{5.13}$$

by Assumption 5.1-(ii) on β, χ and μ. Similarly, we can show the convergence for $F_{2\lambda}$, which implies that F_λ is a compact operator for any $\lambda \in \mathbb{R}$.

Next, for $\lambda \in \mathbb{R}$ define a positive functional $\mathscr{F}_\lambda = (\mathscr{F}_{1\lambda}, \mathscr{F}_{2\lambda})$ by

$$\begin{aligned}
\langle \mathscr{F}_{1\lambda}, (\alpha, \eta) \rangle &:= \int_0^{a^+} \int_0^a \epsilon_1(p) \Pi(a, p, p) e^{-\lambda p} \eta(a-p) dp da \\
&\quad + \int_0^{s^+} \int_0^p \epsilon_1(p) \Pi(a, p, a) e^{-\lambda a} \alpha(p-a) da dp \\
\langle \mathscr{F}_{2\lambda}, (\alpha, \eta) \rangle &:= \int_0^{a^+} \int_0^p \epsilon_2(p) \Pi(p, s, s) e^{-\lambda s} \eta(p-s) ds dp \\
&\quad + \int_0^{s^+} \int_0^s \epsilon_2(p) \Pi(p, s, p) e^{-\lambda p} \alpha(s-p) dp ds,
\end{aligned}$$

where $\Pi(a, p, t) := \exp[-\int_0^t \mu(a-\tau, p-\tau) d\tau]$. From Assumption 5.1-(iii), $\mathscr{F}_\lambda$ is a strictly positive functional and we have

$$\begin{aligned}
&F_\lambda(\alpha, \eta) = (F_{1\lambda}(\alpha, \eta), F_{2\lambda}(\alpha, \eta)) \geq (\langle \mathscr{F}_{1\lambda}, (\alpha, \eta) \rangle e_1, \langle \mathscr{F}_{2\lambda}, (\alpha, \eta) \rangle e_2), \\
&\lim_{\lambda \to -\infty} (\langle \mathscr{F}_{1\lambda}, (e_1, e_2) \rangle, \langle \mathscr{F}_{2\lambda}, (e_1, e_2) \rangle) = (+\infty, +\infty),
\end{aligned} \tag{5.14}$$

where $(e_1, e_2) \equiv 1$ is a quasi-interior point in $L^2(0, s^+) \times L^2(0, a^+)$. Moreover, for any integer n, we have

$$F_\lambda^2(\alpha, \eta) = F_\lambda(F_{1\lambda}(\alpha, \eta), F_{2\lambda}(\alpha, \eta)) = (F_{1\lambda}(F_{1\lambda}(\alpha, \eta), F_{2\lambda}(\alpha, \eta)), F_{2\lambda}(F_{1\lambda}(\alpha, \eta), F_{2\lambda}(\alpha, \eta))),$$

where

$$\begin{aligned}
F_{i\lambda}(F_{1\lambda}(\alpha, \eta), F_{2\lambda}(\alpha, \eta)) &\geq \langle \mathscr{F}_{i\lambda}, (F_{1\lambda}(\alpha, \eta), F_{2\lambda}(\alpha, \eta)) \rangle e_i \\
&\geq \langle \mathscr{F}_{i\lambda}, (\langle \mathscr{F}_{1\lambda}, (\alpha, \eta) \rangle e_1, \langle \mathscr{F}_{2\lambda}, (\alpha, \eta) \rangle e_2) \rangle e_i \\
&\geq \min\{\langle \mathscr{F}_{1\lambda}, (\alpha, \eta) \rangle, \langle \mathscr{F}_{2\lambda}, (\alpha, \eta) \rangle\} \langle \mathscr{F}_{i\lambda}, (e_1, e_2) \rangle e_i \\
&:= \min \langle \mathscr{F}_\lambda, (\alpha, \eta) \rangle \langle \mathscr{F}_{i\lambda}, (e_1, e_2) \rangle e_i, \quad i = 1, 2.
\end{aligned}$$

It follows that

$$F_\lambda^2(\alpha,\eta) \geq \min\langle \mathscr{F}_\lambda, (\alpha,\eta)\rangle(\langle \mathscr{F}_{1\lambda}, (e_1,e_2)\rangle e_1, \langle \mathscr{F}_{2\lambda}, (e_1,e_2)\rangle e_2)$$
$$\geq \min\langle \mathscr{F}_\lambda, (\alpha,\eta)\rangle \min\langle \mathscr{F}_\lambda, (e_1,e_2)\rangle (e_1,e_2).$$

By induction we have

$$F_\lambda^{n+1}(\alpha,\eta) \geq \min\langle \mathscr{F}_\lambda, (\alpha,\eta)\rangle \, [\min\langle \mathscr{F}_\lambda, (e_1,e_2)\rangle]^n \, (e_1,e_2).$$

Then we obtain $\langle \mathscr{F}, F_\lambda^n(\alpha,\eta)\rangle > 0, n \geq 1$, for every pair $(\alpha,\eta) \in L_+^2(0,s^+) \times L_+^2(0,a^+)\backslash\{(0,0)\}$, $\mathscr{F} \in (L_+^2(0,s^+))^* \times (L_+^2(0,a^+))^*\backslash\{(0,0)\}$; that is, F_λ is a non-supporting operator. In summary, F_λ is a compact and non-supporting operator.

Remark 5.3 More results on non-supporting operators are given in "Appendix A.1".

Now we give a proposition on the spectrum of H with its dominant eigenvalue.

Proposition 5.4 *We have the following statements:*

(i) $\Gamma := \{\lambda \in \mathbb{C} : 1 \in \sigma(F_\lambda)\} = \{\lambda \in \mathbb{C} : 1 \in \sigma_P(F_\lambda)\}$, *where* $\sigma(A)$ *and* $\sigma_P(A)$ *are the spectrum and point spectrum of the operator* A, *respectively;*
(ii) *There exists a unique real number* $\vartheta_0 \in \Gamma$ *such that* $r(F_{\vartheta_0}) = 1$ *and* $\vartheta_0 > 0$ *if* $r(F_0) > 1$; $\vartheta_0 = 0$ *if* $r(F_0) = 1$; *and* $\vartheta_0 < 0$ *if* $r(F_0) < 1$;
(iii) $\vartheta_0 > \sup\{\mathrm{Re}\theta : \lambda \in \Gamma\backslash\{\vartheta_0\}\}$;
(iv) $\{\lambda \in \mathbb{C} : \lambda \in \rho(H)\} = \{\lambda \in \mathbb{C} : 1 \in \rho(F_\theta)\}$, *where* $\rho(A)$ *is the resolvent set of* A;
(v) ϑ_0 *is the dominant eigenvalue of* H; *i.e.* ϑ_0 *is greater than all real parts of eigenvalues of* H. *Moreover, it is a simple eigenvalue of* H;
(vi) $\vartheta_0 = s(H) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(H)\}$.

We refer to Kang et al. [31] for the proof and for more results on population models with two physiological structures in $L_+^1((0,s^+) \times (0,a^+))$.

Note that by Proposition 5.4 the eigenvalues of H are countable. Thus, we can denote the eigenvalues of H by $\{\vartheta_0, \vartheta_1, \vartheta_2, \ldots\}$ to distinguish those with a nonlocal operator. As before, we can rearrange $\{\vartheta_j\}$ in the following way:

$$\vartheta_0 > \mathrm{Re}\vartheta_1 \geq \mathrm{Re}\vartheta_2 \geq \cdots$$

For any $i, j \geq 0, \sigma + \lambda_i \neq \vartheta_j$, define

$$\phi_\psi(a,s,x) = \sum_{i=0}^{\infty}((\sigma+\lambda_i)I - H)^{-1}\langle \psi(a,s,\cdot), \varphi_i\rangle \varphi_i(x),$$

where $\langle \psi(a,s,\cdot), \varphi_i\rangle = \int_\Omega \psi(a,s,x)\varphi_i(x)dx$. Now following the same steps as the age-structured model with the nonlocal diffusion in the Sect. 2.1, we can obtain similar results as in Theorems 2.2, 2.3 and 2.5.

Remark 5.5 There is only a slight difference between these two cases once one verifies that $\tilde{A}$ is a dissipative operator. We point it out here.

$$\langle \tilde{A}\phi, \phi\rangle = \int_{(0,a^+)\times(0,s^+)\times\Omega} -\frac{\partial \phi(a,s,x)}{\partial a}\phi(a,s,x)dadsdx$$
$$-\int_{(0,a^+)\times(0,s^+)\times\Omega} \frac{\partial \phi(a,s,x)}{\partial s}\phi(a,s,x)dadsdx$$

$$
\begin{aligned}
&-\int_{(0,a^+)\times(0,s^+)\times\Omega}\mu(a,s)|\phi(a,s,x)|^2 dadsdx \\
&+\int_{(0,a^+)\times(0,s^+)\times\Omega}(J*\phi(a,s,x)-\phi(a,s,x))\phi(a,s,x)dadsdx \\
\le\, &\frac{1}{2}\int_{(0,a^+)\times\Omega}|\phi(a,0,x)|^2 dadx+\frac{1}{2}\int_{(0,s^+)\times\Omega}|\phi(0,s,x)|^2 dsdx \\
=\, &\frac{1}{2}\int_{(0,a^+)\times\Omega}\left[\int_0^{a^+}\int_0^{s^+}\chi(a,p,s)\phi(p,s,x)dsdp\right]^2 dadx \\
&+\frac{1}{2}\int_{(0,s^+)\times\Omega}\left[\int_0^{s^+}\int_0^{a^+}\beta(a,p,s)\phi(a,p,x)dadp\right]^2 dsdx \\
\le\, &\frac{1}{2}\int_0^{a^+}\int_0^{a^+}\int_0^{s^+}\chi^2(a,p,s)dsdpda\int_\Omega\int_0^{a^+}\int_0^{s^+}\phi^2(p,s,x)dsdpdx \\
&+\frac{1}{2}\int_0^{s^+}\int_0^{s^+}\int_0^{a^+}\beta^2(a,p,s)dadpds\int_\Omega\int_0^{s^+}\int_0^{a^+}\phi^2(a,p,x)dadpdx \\
\le\, &\frac{1}{2}\|\beta\|^2\|\phi\|^2_{\tilde{E}}+\frac{1}{2}\|\chi\|^2\|\phi\|^2_{\tilde{E}}.
\end{aligned} \tag{5.15}
$$

6 Discussion

In modeling real world problems (such as transmission dynamics of infectious diseases, population dynamics, cancer therapy, etc.), one usually needs to consider multiple internal variables such as time, age, size, stage, location and so on. For example, for the transmission of coronavirus disease (COVID-19) it is now known that the mortality rate in seniors is significantly higher than in juniors, so age structure of the host population plays a crucial role; when an individual who was infected in Asia or Europe is traveling to the U.S., it is very likely to import the virus to there, thus the location and spatial movement of infected individuals are also very important in modeling the spatio-temporal transmission dynamics of COVID-19. Such models would be described by partial differential equations with time, age, and spatial variable.

Great attention has been paid to the study of age-structured population dynamics with random (Laplace) diffusion (see Chan and Guo [5], Di Blasio [14], Guo and Chan [18], Gurtin [19], Gurtin and MacCamy [21], Hastings [23], Huyer [26], Langlais [34], MacCamy [36], Walker [49,51], Webb [52], and a survey by Webb [55]). Notice that these models are constructed under the assumption that species or individuals disperse in the connected spatial domain randomly. As far as the geographical spread of infectious diseases such as the spatial spread of COVID-19 by long distance traveling is concerned, the random Laplace diffusion is no longer valid and nonlocal convolution diffusion seems to be more appropriate. In fact nonlocal convolution diffusion processes are more applicable to many biological populations and physical materials (Andreu-Vaillo et al. [1], Bates [3], Ruan [42]) compared with random diffusion processes. However, there are very few theoretical studies on age-structured population models with nonlocal diffusion due to the complexity of these equations and the lack of methods and techniques.

In this paper we provided a systematical and theoretical treatment of the age-structured problem (1.1) with nonlocal diffusion. More specifically, first we studied a special case when both birth and death rate functions are independent of the spatial variable. Then we considered the general case when both birth and death rate functions are dependent of the spatial variable. In each case we studied the semigroup of linear operators associated to the nonlocal diffusion problem and used the spectral properties of its infinitesimal generator to determine the stability of the zero steady state. In addition, we compared our results with that for the age-structured model with random Laplace diffusion. It is shown that the structure of the semigroup for the age-structured model with nonlocal diffusion is essentially determined by those of the semigroup for the age-structured model without diffusion and the nonlocal operator when birth and death rates are only dependent on age. Next we studied the weak solution and the comparison principle for a nonautonomous and nonhomogeneous age-structured model where birth and death rate functions depend on all variables t, a, x with nonlocal diffusion. Then we generalized our techniques and results to a nonlocal diffusion model with age and size structures.

We expect that age-structured models with nonlocal diffusion exhibit more interesting dynamics such as the existence of bifurcations and traveling wave solutions and leave this for future consideration. consideration. Also, it will be very interesting to propose an age-structured susceptible-exposed-infectious-recovered (SEIR) model with nonlocal diffusion to study the spatio-temporal transmission dynamics of COVID-19 via long-distance traveling.

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A Appendix

A.1 Positive Operators

In this section, we recall some definitions and results of positive operator theory on ordered Banach spaces from Inaba [28,29]. For more complete exposition, we refer to Daners et al. [12], Heijmans [24], Marek [38], and Sawashima [43].

Let E be a real or complex Banach space and E^* be its dual (the space of all linear functionals on E). Write the value of $f \in E^*$ at $\psi \in E$ as $\langle f, \psi \rangle$. A nonempty closed subset E_+ is called a *cone* if the following hold: (i) $E_+ + E_+ \subset E_+$, (ii) $\lambda E_+ \subset E_+$ for $\lambda \geq 0$, (iii) $E_+ \cap (-E_+) = \{0\}$. Define the *order* in E such that $x \leq y$ if and only if $y - x \in E_+$ and $x < y$ if and only if $y - x \in E_+ \backslash \{0\}$. The cone E_+ is called *total* if the set $\{\psi - \phi : \psi, \phi \in E_+\}$ is dense in E. The *dual cone* E_+^* is the subset of E^* consisting of all positive linear functionals on E; that is, $f \in E_+^*$ if and only if $\langle f, \psi \rangle \geq 0$ for all $\psi \in E_+$. $\psi \in E_+$ is called a *quasi-interior point* if $\langle f, \psi \rangle > 0$ for all $f \in E_+^* \backslash \{0\}$. $f \in E_+^*$ is said to be *strictly positive* if $\langle f, \psi \rangle > 0$ for all $\psi \in E_+ \backslash \{0\}$. The cone E_+ is called *generating* if $E = E_+ - E_+$ and is called *normal* if $E^* = E_+^* - E_+^*$.

An ordered Banach space $(E, \leq)$ is called a *Banach lattice* if (i) any two elements $x, y \in E$ have a supremum $x \vee y = \sup\{x, y\}$ and an infimum $x \wedge y = \inf\{x, y\}$ in E; and (ii) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in E$, where the modulus of x is defined by $|x| = x \vee (-x)$.

Let $B(E)$ be the set of bounded linear operators from E to E. $T \in B(E)$ is said to be *positive* if $T(E_+) \subset E_+$. For $T, S \in B(E)$, we say $T \geq S$ if $(T - S)(E_+) \subset E_+$. A positive operator $T \in B(E)$ is called *non-supporting* if for every pair $\psi \in E_+ \backslash \{0\}$, $f \in E_+^* \backslash \{0\}$, there exists a positive integer $p = p(\psi, f)$ such that $\langle f, T^n \psi \rangle > 0$ for all $n \geq p$. The *spectral radius* of $T \in B(E)$ is denoted by $r(T)$. $\sigma(T)$ denotes the *spectrum* of T and $\sigma_P(T)$ denotes the *point spectrum* of T.

From results in Sawashima [43] and Inaba [29], we state the following proposition.

Proposition A.1 *Let E be a Banach lattice and let $T \in B(E)$ be compact and non-supporting. Then the following statements holds:*

(i) *$r(T) \in \sigma_P(T)\setminus\{0\}$ and $r(T)$ is a simple pole of the resolvent; that is, $r(T)$ is an algebraically simple eigenvalue of T;*
(ii) *The eigenspace of T corresponding to $r(T)$ is one-dimensional and the corresponding eigenvector $\psi \in E_+$ is a quasi-interior point. The relation $T\phi = \mu\phi$ with $\phi \in E_+$ implies that $\phi = c\psi$ for some constant c;*
(iii) *The eigenspace of T^* corresponding to $r(T)$ is also a one-dimensional subspace of E^* spanned by a strictly positive functional $f \in E^*_+$.*

The following comparison theorem is due to Marek [38]:

Proposition A.2 *Suppose that E is a Banach lattice. Let S and T be positive operators in $B(E)$.*

(i) *If $S \le T$, then $r(S) \le r(T)$;*
(ii) *If S and T are semi-nonsupporting and $r(S)$, $r(T)$ are respectively eigenvalues of S, T, then $S \le T$, $S \ne T$ and $r(T) \ne 0$ imply that $r(S) < r(T)$.*

A.2 Linear Nonlocal Problems

We prove the following lemma which is actually Lemma 4.1 and the proof is motivated by Anita [2].

Lemma A.3 *If $y_0 \in L^2(\Omega)$ and $g \in L^2((0,T)\times\Omega)$, then there exists a unique $y \in C([0,T], L^2(\Omega)) \cap AC((0,T), L^2(\Omega) \cap L^2((0,T)\times\Omega)$ such that*

$$\begin{cases} \frac{\partial y}{\partial t} - k(J * u - u)(t,x) = g(t,x) \quad a.e.\ in\ (0,T)\times\Omega, \\ y(t,x) = 0, \quad t > 0, x \in \mathbb{R}^N\setminus\Omega, \\ y(0,x) = y_0(x) \quad a.e.\ in\ \Omega. \end{cases} \tag{A.1}$$

If $y_0 \ge 0$ a.e. in Ω and $g(t,x) \ge 0$ a.e. in $(0,T)\times\Omega$, then $y(t,x) \ge 0$ a.e. $(t,x) \in (0,T)\times\Omega$.

Proof We are particularly interested in the constructive proof. Since we know that there is an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}} \subset L^2(\Omega)$ and $\{\lambda_j\} \subset \mathbb{R}^+$, $0 < \lambda_0 < 1$, $\lambda_j \to 1$ as $j \to \infty$, such that

$$\begin{cases} -(J * \varphi_j(x) - \varphi_j(x)) = \lambda_j \varphi(x), \quad x \in \Omega, \\ \varphi_j(x) = 0, \quad x \in \mathbb{R}^N\setminus\Omega. \end{cases} \tag{A.2}$$

We are looking for a solution y of the form

$$y(t,x) = \sum_{j=0}^{\infty} y^j(t)\varphi(x), \quad (t,x) \in (0,T)\times\Omega, \tag{A.3}$$

where y^j are unknown functions. Formally substituting (A.3) into (A.1), we obtain for y^j the differential equations

$$\begin{cases} (y^j)'(t) + \lambda_j k y^j(t) = g_j(t), \quad t \in (0, T), \\ y^j(0) = y_0^j, \quad j \in \mathbb{N}, \end{cases} \tag{A.4}$$

where

$$y_0^j = (y_0, \varphi_j) = \int_\Omega y_0(x)\varphi_j(x)dx$$

and

$$g_j(t) = (g(t), \varphi_j) = \int_\Omega g(t, x)\varphi_j(x)dx.$$

Here $(\cdot, \cdot)$ denotes the usual product in $L^2(\Omega)$. Solving the problem (A.4), we obtain

$$y^j(t) = e^{-\lambda_j kt} y_0^j + \int_0^t e^{-\lambda_j k(t-s)} g_j(s)ds, \quad t \in [0, T],$$

and by (A.3) we have

$$y(t, x) = \sum_{j=0}^{\infty} \left(e^{-\lambda_j kt} y_0^j + \int_0^t e^{-\lambda_j k(t-s)} g_j(s)ds \right) \varphi_j(x). \tag{A.5}$$

Let us prove that $y \in C([0, T], L^2(\Omega))$. It is sufficient to show that the series

$$\sum_{j=0}^{\infty} \left(e^{-\lambda_j kt} y_0^j + \int_0^t e^{-\lambda_j k(t-s)} g_j(s)ds \right)^2 \tag{A.6}$$

is uniformly convergent in $[0, T]$. Indeed, we have

$$\begin{aligned} (y^j)^2(t) &= \left(e^{-\lambda_j kt} y_0^j + \int_0^t e^{-\lambda_j k(t-s)} g_j(s)ds \right)^2 \\ &\le 2e^{-2\lambda_j kt}(y_0^j)^2 + 2\left(\int_0^t e^{-\lambda_j k(t-s)} g_j(s)ds \right)^2 \\ &\le 2(y_0^j)^2 + 2\int_0^T |g_j(s)|^2 ds, \quad \forall j \in \mathbb{N}, t \in [0, T]. \end{aligned}$$

Now the Parseval's formula allows us to conclude that

$$\|y_0\|^2_{L^2(\Omega)} = \sum_{j=0}^{\infty} (y_0^j)^2, \quad \|g(t)\|^2_{L^2(\Omega)} = \sum_{j=0}^{\infty} |g_j(t)|^2, \quad t \in [0, T].$$

Then by Beppo–Levi Theorem, we have

$$\sum_{j=0}^{\infty} \int_0^T |g_j(s)|^2 ds = \int_0^T \left(\sum_{j=0}^{\infty} |g_j(s)|^2 \right) ds = \int_{(0,T)\times\Omega} |g(t, x)|^2 dxdt.$$

This implies that the series (A.6) is uniformly convergent in $[0, T]$. Next we shall prove that $y \in L^2((0, T) \times \Omega)$. By (A.5) we may rewrite y as

$$y(t, x) = \sum_{j=0}^{\infty} \sqrt{1 + \lambda_j} y^j(t)\phi_j(x), \quad (t, x) \in (0, T) \times \Omega,$$

where $\phi_j(x) = (\sqrt{1+\lambda_j})^{-1}\varphi_j$. Since $\{\phi_j\}_{j\in\mathbb{N}}$ is an orthonormal complete system in $L^2(\Omega)$, it suffices to show that the series

$$\sum_{j=0}^{\infty}(1+\lambda_j)\|y^j\|^2_{L^2(0,T)}$$

is convergent, or equivalently that the series

$$\sum_{j=0}^{\infty}\lambda_j\|y^j\|^2_{L^2(0,T)} \tag{A.7}$$

is convergent. This yields by (A.6) that

$$\lambda_j(y^j)^2(t) \le 2\lambda_j e^{-2\lambda_j kt}(y_0^j)^2 + \frac{1}{k}\int_0^T |g_j(s)|^2 ds, \quad \forall j \in \mathbb{N}$$

and by Parseval's formula we have

$$\sum_{j=0}^{\infty}\lambda_j\|y^j\|^2_{L^2(0,T)} \le \frac{1}{k}\left(\|y_0\|^2_{L^2(\Omega)} + \|g\|_{L^2((0,T)\times\Omega)}\right).$$

It follows that (A.7) holds and so $y \in L^2((0,T)\times\Omega)$.

In order to prove that $y_t \in L^2((\delta, T-\delta), L^2(\Omega))$, it suffices to show that the series $\sum_{j=0}^{\infty}\|y_j'\|^2_{L^2(\delta,T-\delta)}$ is convergent for any $\delta > 0$ small enough. Since $(y_j)'(t) = -\lambda_j k y^j(t) + g_j(t)$, and since

$$\sum_{j=0}^{\infty}\|g_j\|^2_{L^2(0,T)} = \|g\|^2_{L^2((0,T)\times\Omega)},$$

it suffices to prove the convergence of the series

$$\sum_{j=0}^{\infty}\lambda_j^2 k^2\|y^j\|^2_{L^2(\delta,T-\delta)}$$

for any $\delta > 0$ small enough. By the solution of y^j, we have

$$\begin{aligned}\lambda_j^2k^2(y^j)^2(t) &\le 2\lambda_j^2k^2e^{-2\lambda_j kt}(y_0^j)^2 + 2\lambda_j^2k^2e^{-2\lambda_j kt}\left(\int_0^t e^{\lambda_j ks}ds\right)\left(\int_0^t e^{\lambda_j ks}|g_j(s)|^2ds\right)\\ &\le \frac{2}{t^2}(y_0^j)^2 + 2\lambda_j k e^{-\lambda_j kt}\int_0^t e^{\lambda_j ks}|g_j(s)|^2 ds.\end{aligned}$$

Since the series

$$\sum_{j=0}^{\infty}\int_\delta^{T-\delta}\frac{2}{t^2}(y_0^j)^2dt$$

is convergent and

$$\begin{aligned}\sum_{j=0}^{\infty}\int_\delta^{T-\delta}\lambda_j k e^{-\lambda_j kt}\int_0^t e^{\lambda_j ks}|g_j(s)|^2dsdt \le 2\sum_{j=0}^{\infty}\int_0^T e^{\lambda_j ks}|g_j(s)|^2\\ \int_s^T(-e^{-\lambda_j kt})'dtds \le 2\|g\|^2_{L^2((0,T)\times\Omega)},\end{aligned}$$

we conclude that

$$\sum_{j=0}^{\infty} \|(y^j)'\|^2_{L^2(\delta, T-\delta)}$$

is convergent. It follows that $y_t \in L^2((\delta, T-\delta), L^2(\Omega))$ and that

$$y \in AC((0, T), L^2(\Omega)).$$

For now we mollify the initial data y_0 and the force term g by a positive smooth function $\rho_\epsilon = \rho_\epsilon(x)$, denote them by $y_{0\epsilon} := \rho_\epsilon * y_0$ and $g_\epsilon := \rho_\epsilon * g$, respectively, where $*$ represents the convolution. It follows that the solution y_ϵ will be smooth. Let us now consider the mollified solution y_ϵ and assume by contradiction that $y_\epsilon(t, x)$ is negative somewhere. Let $v_\epsilon(t, x) = y_\epsilon(t, x) + \delta t$ with δ so small that v_ϵ is still negative somewhere. Then, if (t_0, x_0) is a point where v_ϵ attains its negative minimum, we have $t_0 > 0$ and

$$\begin{aligned} v_{\epsilon t}(t_0, x_0) = y(t_0, x_0) + \delta &> \int_{\mathbb{R}^N} J(x-z) y_\epsilon((t_0, z) - y_\epsilon(t_0, x_0)) dz \\ &= \int_{\mathbb{R}^N} J(x-z)(v_\epsilon(t_0, y) - v_\epsilon(t_0, y_0)) dy \geq 0, \end{aligned} \tag{A.8}$$

which is a contradiction. Thus, $y_\epsilon \geq 0$. Since $y_\epsilon(t, \cdot) \to y(t, \cdot)$ in $L^2(\Omega)$ for any $t \in (0, T)$ as $\epsilon \to 0$ and ρ_ϵ is positive, it implies that $y \geq 0$ a.e. □

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