Discretisation of Global Attractors for Lattice Dynamical Systems

Xiaoying Han¹ · Peter E. Kloeden² · Stefanie Sonner³

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Abstract

The existence of numerical attractors for lattice dynamical systems is established, where the implicit Euler scheme is used for time discretisation. Infinite dimensional discrete lattice systems as well as their finite dimensional truncations are considered. It is shown that the finite dimensional numerical attractors converge upper semicontinuously to the global attractor of the original lattice model as the discretisation step size tends to zero.

Keywords Lattice dynamical system · Global attractor · Implicit Euler scheme · Numerical attractor · Upper semicontinuous convergence

Mathematics Subject Classification 34D45 · 37L60 · 65L20

1 Introduction

Using a finite difference quotient to discretise the Laplace operator in a reaction–diffusion equation defined on the real line $\mathbb{R}$,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + f(u) + g(x), \quad \nu > 0,$$

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✉ Stefanie Sonner
s.sonner@ru.nl

Xiaoying Han
xzh0003@auburn.edu

Peter E. Kloeden
kloeden@math.uni-frankfurt.de

¹ Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA

² School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China

³ IMAPP - Mathematics, Radboud University Nijmegen, PO Box 9010, 6500 GL Nijmegen, The Netherlands

leads to a lattice dynamical system (LDS). In particular, setting the step size equal to 1 we obtain the infinite dimensional system of ordinary differential equations (ODEs)

$$\frac{du_i}{dt} = v(u_{i-1} - 2u_i + u_{i+1}) + f(u_i) + g_i, \quad i \in \mathbb{Z}, \quad (1)$$

where $u_i(t) = u(i, t)$ and $g_i = g(i)$. On the other hand, lattice dynamical systems also arise naturally from a large variety of applications with discrete spatial structures (see, e.g., [2–4, 9, 10] and references therein).

Attractors for various types of lattice dynamical systems have been extensively studied during the past years (see, e.g., [1, 2, 6, 13, 14] and references therein). In particular, the existence and upper semicontinuity of global attractors for the LDS (1) and its finite dimensional truncations were studied by Bates et al. [2] in the Hilbert space ℓ^2 of square summable bi-infinite real-valued sequences.

The goal of this work is to investigate the existence of numerical attractors for the LDS (1), where the implicit Euler scheme (IES) is used for time-discretisation. We show that the numerical attractors converge upper semicontinuously to the global attractor for the continuous time lattice model as the discretisation step size tends to zero. Subsequently, we consider m -dimensional truncations of the implicit Euler scheme. We prove the existence of finite dimensional numerical attractors and their convergence to the attractor for the infinite dimensional IES as m tends to infinity. Hence, the upper semicontinuous convergence of the finite dimensional numerical attractors to the global attractor of the original lattice system (1) follows.

The paper is organised as follows. In Sect. 2 we formulate our hypotheses and recall the existence result for the global attractor for the lattice model (1) in [2]. In Sect. 3 we apply the implicit Euler scheme to discretise the LDS in time, and investigate the existence of the corresponding numerical attractors. In addition, an estimate for the global discretisation error is established that is later needed to prove the convergence of the numerical attractors. In Sect. 4 we state and prove our main results. We consider finite dimensional truncations of the discretised lattice system and show that their numerical attractors converge upper semicontinuously to the global attractor of the original lattice model (1). Some closing remarks will be given in Sect. 5.

2 Global Attractor for the LDS (1)

Introduce the Hilbert space

$$\ell^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}} : \sum_{i \in \mathbb{Z}} u_i^2 < \infty \right\}$$

with the norm and inner product

$$\|u\| = \sqrt{\sum_{i \in \mathbb{Z}} u_i^2}, \quad \langle u, v \rangle = \sum_{i \in \mathbb{Z}} u_i v_i, \quad u, v \in \ell^2.$$

For any $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, define the bounded linear operators A , B and B^* from ℓ^2 to ℓ^2 by

$$(Au)_i = v(u_{i-1} - 2u_i + u_{i+1}),$$

and

$$(Bu)_i = v^{\frac{1}{2}}(u_{i+1} - u_i), \quad (B^*u)_i = v^{\frac{1}{2}}(u_{i-1} - u_i).$$

We observe that

$$\langle B^*u, v \rangle = \langle u, Bv \rangle \quad \forall u, v \in \ell^2,$$

and $-A = BB^* = B^*B$, which implies that

$$\langle Au, u \rangle = -\|Bu\|^2 \leq 0 \quad \forall u \in \ell^2.$$

Moreover, since A is a bounded linear operator in ℓ^2 , it generates a uniformly continuous semigroup in ℓ^2 .

Throughout the paper we make the following assumptions:

(A) $g = (g_i)_{i \in \mathbb{Z}} \in \ell^2$ and the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function satisfying

$$f(s)s \leq -\alpha s^2 \quad \forall s \in \mathbb{R}, \quad (2)$$

for some $\alpha > 0$.

Remark 1 Since f is smooth, the dissipativity assumption (2) implies that $f(0) = 0$.

Under hypothesis (A), the lattice system (1) can be written as the infinite dimensional ordinary differential equation (ODE) in ℓ^2

$$\frac{du}{dt} = Au + F(u) + g, \quad (3)$$

where $g = (g_i)_{i \in \mathbb{Z}}$ and the Nemytskii operator $F : \ell^2 \rightarrow \ell^2$ is defined componentwise by

$$F(u) = (F_i(u))_{i \in \mathbb{Z}} := (f(u_i))_{i \in \mathbb{Z}}.$$

2.1 Properties of the Nemytskii Operator F

Assumption (2) ensures that $F : \ell^2 \rightarrow \ell^2$ is Lipschitz continuous on bounded sets. Indeed, let $B \subset \ell^2$ be a bounded set contained in a ball $\mathbb{B}_{r_B}(0)$ with radius $r_B > 0$ and center 0 in ℓ^2 . Then, since f is smooth, for every $u, v \in B$ we have

$$\|F(u) - F(v)\|^2 = \sum_{i \in \mathbb{Z}} |f(u_i) - f(v_i)|^2 = \sum_{i \in \mathbb{Z}} |f'(\xi_i)|^2 |u_i - v_i|^2,$$

for some ξ_i with $|\xi_i| \leq |u_i| + |v_i| \leq \|u\| + \|v\| \leq 2r_B$, $i \in \mathbb{Z}$. Hence,

$$\|F(u) - F(v)\| \leq L_{r_B} \|u - v\| \quad (4)$$

for some constant $L_{r_B} > 0$ depending on r_B . In particular, since $f(0) = 0$ implies that $F(0) = 0$, estimate (4) with $v = 0$ yields

$$\|F(u)\| \leq L_{r_B} \|u\|,$$

which shows that $F(u) \in \ell^2$.

For given initial data $u(0) = u_o \in \ell^2$, the existence and uniqueness of a global solution $u(\cdot; u_o) \in C([0; \infty), \ell^2)$ of the ODE system (3) follows by standard arguments (e.g., see [2]). Moreover, the lattice model (3) generates a semi dynamical system $\varphi(t) : \ell^2 \rightarrow \ell^2$, $t \geq 0$, defined by

$$\varphi(t)u_o := u(t; u_o), \quad u_o \in \ell^2.$$

2.2 Existence of the Global Attractor

Taking the inner product of Eq. (3) with u and using assumption (2) we obtain

$$\begin{aligned}\frac{d}{dt}\|u\|^2 &= 2\langle Au, u \rangle + 2\langle F(u), u \rangle + 2\langle g, u \rangle \\ &\leq 2 \sum_{i \in \mathbb{Z}} u_i f(u_i) + 2 \sum_{i \in \mathbb{Z}} g_i u_i \\ &\leq -\alpha \|u\|^2 + \frac{\|g\|^2}{\alpha},\end{aligned}$$

where the last step follows from Young's inequality. Hence, Gronwall's lemma implies that

$$\|u(t)\|^2 \leq \|u_o\|^2 e^{-\alpha t} + \frac{\|g\|^2}{\alpha^2} (1 - e^{-\alpha t}), \quad t \geq 0. \quad (5)$$

Define the closed ball $\mathcal{B}$ in ℓ^2 by

$$\mathcal{B} := \left\{ u \in \ell^2 : \|u\|^2 \leq (r^*)^2 := \frac{2\|g\|^2}{\alpha^2} + 1 \right\} = \mathbb{B}_{r^*}(0).$$

Estimate (5) then implies that $\mathcal{B}$ is an absorbing set for the semi dynamical system $\{\varphi(t)\}_{t \geq 0}$. In fact, for any bounded set $B \subset \ell^2$ contained in a ball $\mathbb{B}_{r_B}(0)$ in ℓ^2 , it is straightforward to check that

$$\varphi(t)B \subset \mathcal{B} \quad \forall t \geq \frac{1}{\alpha} \ln \left(\frac{\alpha^2 r_B^2}{\|g\|^2 + \alpha^2} \right).$$

Moreover, for every $u_o \in \mathcal{B}$ we have

$$\begin{aligned}\|\varphi(t)u_o\|^2 &\leq \left(\frac{2\|g\|^2}{\alpha^2} + 1 \right) e^{-\alpha t} + \frac{\|g\|^2}{\alpha^2} (1 - e^{-\alpha t}) \\ &\leq \frac{2\|g\|^2}{\alpha^2} + 1 = (r^*)^2,\end{aligned}$$

i.e., $\mathcal{B}$ is positive invariant.

Using uniform estimates on the “tail ends” (i large) of solutions one can, furthermore, prove that $\{\varphi(t)\}_{t \geq 0}$ is asymptotically compact in ℓ^2 and hence, the existence of the global attractor follows (e.g., see [2]).

Theorem 1 *The semi dynamical system $\{\varphi(t)\}_{t \geq 0}$ generated by the ODE (3) possesses a global attractor $\mathcal{A}$ in ℓ^2 .*

The aim of this work is to construct numerical approximations for the global attractor $\mathcal{A}$ and to show that the numerical attractors converge to the analytical attractor $\mathcal{A}$ as the time step tends to zero. For time discretisation the implicit Euler scheme is used.

3 Numerical Attractors for the Implicit Euler Scheme

Applying the implicit Euler scheme (IES) with constant time step $h > 0$ to discretise the ODE (3) in time results in

$$\begin{aligned} u_{n+1}^{(h)} &= u_n^{(h)} + hAu_{n+1}^{(h)} + hF\left(u_{n+1}^{(h)}\right) + hg, \quad n \in \mathbb{N}_0, \\ u_o &\in \ell^2. \end{aligned} \quad (6)$$

Remark 2 The subscript n denotes the time step, while the subscript i corresponds to the point in the lattice. More precisely, we have

$$u_n^{(h)} = \left(u_{n,i}^{(h)}\right)_{i \in \mathbb{Z}} \quad \forall n \in \mathbb{N}_0.$$

The IES (6) is well defined (all terms belong to ℓ^2) and, restricted to bounded sets in ℓ^2 , uniquely solvable for sufficiently small step sizes h (see [7,12] and Lemma 2 below). Note that the numerical scheme does not need to possess a globally attracting set even if the original attractor is; see [7, Example 2.13]. Since we aim to construct approximations for the global attractor $\mathcal{A}$, and $\mathcal{A}$ is contained in the bounded absorbing set $\mathcal{B}$, we will restrict the IES to $\mathcal{B}$, and show that it generates a discrete semi dynamical system in $\mathcal{B}$.

Lemma 1 *Let Assumption (A) hold. Then, for any $h > 0$ and $n \in \mathbb{N}_0$, the solutions of (6) satisfy*

$$\left\|u_{n+1}^{(h)}\right\|^2 \leq \frac{1}{1+h\alpha} \left\|u_n^{(h)}\right\|^2 + \frac{h}{1+h\alpha} \frac{\|g\|^2}{\alpha}.$$

In particular, if $u_o \in \mathcal{B}$, then $u_n \in \mathcal{B}$ for all $n \in \mathbb{N}_0$, i.e., the bounded positive invariant absorbing set $\mathcal{B}$ for the lattice system (3) is also positive invariant for the IES (6).

Proof Taking the inner product of (6) with $u_{n+1}^{(h)}$ and using (A) we obtain

$$\begin{aligned} \left\|u_{n+1}^{(h)}\right\|^2 &= \left\langle u_n^{(h)}, u_{n+1}^{(h)} \right\rangle + h \left\langle Au_{n+1}^{(h)}, u_{n+1}^{(h)} \right\rangle + h \left\langle F\left(u_{n+1}^{(h)}\right) + g, u_{n+1}^{(h)} \right\rangle \\ &\leq \frac{1}{2} \left\|u_{n+1}^{(h)}\right\|^2 + \frac{1}{2} \left\|u_n^{(h)}\right\|^2 + h \sum_{i \in \mathbb{Z}} u_{n+1,i}^{(h)} f(u_{n+1,i}^{(h)}) + \frac{h}{2} \left(\alpha \left\|u_{n+1}^{(h)}\right\|^2 + \frac{1}{\alpha} \|g\|^2 \right) \\ &\leq \left(\frac{1-h\alpha}{2} \right) \left\|u_{n+1}^{(h)}\right\|^2 + \frac{1}{2} \left\|u_n^{(h)}\right\|^2 + \frac{h}{2\alpha} \|g\|^2, \end{aligned}$$

which implies the desired estimate.

Moreover, if $u_n^{(h)} \in \mathcal{B}$, then $\left\|u_n^{(h)}\right\|^2 \leq (r^*)^2 = \frac{2\|g\|^2}{\alpha^2} + 1$. Hence, it follows that

$$\begin{aligned} \left\|u_{n+1}^{(h)}\right\|^2 &\leq \frac{1}{1+h\alpha} \left(\frac{2\|g\|^2}{\alpha^2} + 1 \right) + \frac{h}{1+h\alpha} \frac{\|g\|^2}{\alpha} \\ &\leq \frac{1}{1+h\alpha} \left(\frac{(2+2h\alpha)\|g\|^2}{\alpha^2} + 1 \right) \leq (r^*)^2, \end{aligned}$$

which proves the positive invariance of $\mathcal{B}$. $\square$

Lemma 2 *Let Assumption (A) hold. Then, there exists $h^* > 0$ such that the IES (6) is uniquely solvable for every $u_o \in \mathcal{B}$ and $h \in (0, h^*)$.*

Proof For $h > 0$ and given $u_n \in \mathcal{B}$ we define the mapping $\Phi : \mathcal{B} \rightarrow \mathcal{B}$ by

$$\Phi_h(w) := u_n + hAw + hF(w) + hg, \quad w \in \mathcal{B}.$$

The positive invariance of $\mathcal{B}$ was shown in Lemma 1 and implies that $\Phi_h(w) \in \mathcal{B}$. For every $w, v \in \mathcal{B}$ the difference between $\Phi_h(w)$ and $\Phi_h(v)$ satisfies

$$\|\Phi_h(w) - \Phi_h(v)\| \leq h\|A(w - v)\| + h\|F(w) - F(v)\|. \quad (7)$$

The definition of A implies that $\|Au\| \leq 4v\|u\|$ for every $u \in \ell^2$, and hence

$$\|A(w - v)\| \leq 4v\|w - v\|.$$

Moreover, since f is smooth, we have

$$\|F(w) - F(v)\| \leq \sup_{s \in [-2r^*, 2r^*]} |f'(s)| \cdot \|w - v\| =: L_{r^*} \|w - v\|.$$

Inserting the above two inequalities in (7) yields the estimate

$$\|\Phi_h(w) - \Phi_h(v)\| \leq h(4v + L_{r^*})\|w - v\|.$$

Consequently, for all $h < h^* = \frac{1}{4v + L_{r^*}}$, by the contraction mapping principle it follows that Φ_h possesses a unique fixed point in $\mathcal{B}$ which implies that the IES (6) is uniquely solvable for all $n \in \mathbb{N}_0$. $\square$

By Lemma 2, for all $h \in (0, h^*)$ the IES (6) can be written in an explicit form $u_{n+1}^{(h)} = \phi^{(h)}(u_n^{(h)})$ for an appropriate mapping $\phi^{(h)} : \mathcal{B} \rightarrow \mathcal{B}$. We recall that

$$h^* = \frac{1}{4v + L_{r^*}} \quad \text{and} \quad L_{r^*} = \sup_{r \in [-2r^*, 2r^*]} |f'(r)|.$$

Next, we prove that the mapping $\phi^{(h)}$ is continuous w.r.t. initial data.

Lemma 3 *Let Assumption (A) hold and $h \in (0, h^*)$. Then, $\phi^{(h)} : \mathcal{B} \rightarrow \mathcal{B}$ is Lipschitz continuous.*

Proof For any $u_n^{(h)}, v_n^{(h)} \in \mathcal{B}$, the next iterates $u_{n+1}^{(h)}, v_{n+1}^{(h)} \in \mathcal{B}$ by Lemma 1. For their difference we have

$$\begin{aligned} \|u_{n+1}^{(h)} - v_{n+1}^{(h)}\| &\leq \|u_n^{(h)} - v_n^{(h)}\| + h\|A(u_{n+1}^{(h)} - v_{n+1}^{(h)})\| \\ &\quad + h\|F(u_{n+1}^{(h)}) - F(v_{n+1}^{(h)})\|. \end{aligned}$$

Moreover, the estimates in the proof of Lemma 2 imply that

$$\|u_{n+1}^{(h)} - v_{n+1}^{(h)}\| \leq \|u_n^{(h)} - v_n^{(h)}\| + h(4v + L_{r^*})\|u_{n+1}^{(h)} - v_{n+1}^{(h)}\|,$$

and consequently,

$$\|u_{n+1}^{(h)} - v_{n+1}^{(h)}\| \leq \frac{1}{1 - h(4v + L_{r^*})} \|u_n^{(h)} - v_n^{(h)}\|,$$

provided that $h < h^*$. $\square$

3.1 Existence of the Numerical Attractor

Due to Lemmata 1, 2 and 3, for every $h \in (0, h^*)$ the IES (6), restricted to $\mathcal{B}$, generates a discrete semi dynamical system $\phi^{(h)}(n) : \mathcal{B} \rightarrow \mathcal{B}$, $n \in \mathbb{N}_0$, defined by

$$\phi^{(h)}(n)u_o = u_n^{(h)}(u_o), \quad u_o \in \mathcal{B},$$

where $u_n^{(h)}(u_o) = u_n^{(h)}$ denotes the unique solution of (6) with initial data $u_o \in \mathcal{B}$.

Remark 3 Let $r \geq r^*$, i.e., $\mathcal{B} \subset \mathbb{B}_r(0) \subset \ell^2$. Then, one can prove as in the previous subsection that there exists $h_r \in (0, h^*]$ such that the IES (6) restricted to $\mathbb{B}_r(0)$ generates a discrete semi dynamical system $\Phi^{(h)}(n) : \mathbb{B}_r(0) \rightarrow \mathbb{B}_r(0)$, $n \in \mathbb{N}$, for all $h \in (0, h_r)$. Moreover, the estimate in Lemma 1 implies that $\mathcal{B}$ is an absorbing set for the semi dynamical system $\{\phi^{(h)}(n)\}_{n \in \mathbb{N}}$.

For the following analysis it is important that $\mathcal{B}$ is a common absorbing set for all step sizes $h \in (0, h^*)$ under consideration.

In this subsection we show that $\{\phi^{(h)}(n)\}_{n \in \mathbb{N}_0}$ possesses an attractor $\mathcal{A}^{(h)}$ in $\mathcal{B} \subset \ell^2$, namely, the *numerical attractor* corresponding to the ODE (3). Since the restricted phase space $\mathcal{B}$ is bounded, it suffices to prove that the semi dynamical system $\{\phi^{(h)}(n)\}_{n \in \mathbb{N}_0}$ is asymptotically compact. To this end, in the following lemma we derive uniform bounds for the tail ends of solutions. Such estimates were obtained in the continuous time setting in [2].

Lemma 4 Assume that (A) holds, let $h \in (0, h^*)$ and $u_o \in \mathcal{B}$. Then, for every $\varepsilon > 0$ there exist $N^{(h)}(\varepsilon)$ and $I(\varepsilon)$ in $\mathbb{N}$ such that the solution of (6) satisfies

$$\sum_{|i| > I(\varepsilon)} |u_{n,i}^{(h)}|^2 \leq \varepsilon \quad \forall n \geq N^{(h)}(\varepsilon).$$

Proof Define a smooth function $\eta : \mathbb{R}_+ \rightarrow [0, 1]$ satisfying

$$\eta(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2. \end{cases}$$

Then, there exists a constant C such that $|\eta'(s)| \leq C$ for all $s \geq 0$. Let κ be a fixed large integer (to be specified later) and for any $u_n^{(h)} = (u_{n,i}^{(h)})_{i \in \mathbb{Z}}$ set $v_n^{(h)} = (v_{n,i}^{(h)})_{i \in \mathbb{Z}}$ with $v_{n,i}^{(h)} = \eta\left(\frac{|i|}{\kappa}\right)u_{n,i}^{(h)}$. Taking the inner product of (6) with $v_{n+1}^{(h)}$ in ℓ^2 results in

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{\kappa}\right) |u_{n+1,i}^{(h)}|^2 &= \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{\kappa}\right) u_{n,i}^{(h)} u_{n+1,i}^{(h)} + h \langle Au_{n+1}^{(h)}, v_{n+1}^{(h)} \rangle \\ &\quad + h \langle F(u_{n+1}^{(h)}) + g, v_{n+1}^{(h)} \rangle. \end{aligned} \quad (8)$$

The terms on the right hand side of the equation are estimated as follows. First, by the triangle inequality, it follows that

$$\sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{\kappa}\right) u_{n,i}^{(h)} u_{n+1,i}^{(h)} \leq \frac{1}{2} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{\kappa}\right) |u_{n+1,i}^{(h)}|^2 + \frac{1}{2} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{\kappa}\right) |u_{n,i}^{(h)}|^2.$$

Second, since $\langle Au, v \rangle = -\langle Bu, Bv \rangle$ for all $u, v \in \ell^2$, we have

$$\begin{aligned} \langle Au_{n+1}^{(h)}, v_{n+1}^{(h)} \rangle &= -\langle Bu_{n+1}^{(h)}, Bv_{n+1}^{(h)} \rangle \\ &= -v \sum_{i \in \mathbb{Z}} \left(u_{n+1,i+1}^{(h)} - u_{n+1,i}^{(h)} \right) \left[\eta \left(\frac{|i|}{\kappa} \right) \left(u_{n+1,i+1}^{(h)} - u_{n+1,i}^{(h)} \right) \right. \\ &\quad \left. + \left(\eta \left(\frac{|i+1|}{\kappa} \right) - \eta \left(\frac{|i|}{\kappa} \right) \right) u_{n+1,i+1}^{(h)} \right] \\ &\leq -v \sum_{i \in \mathbb{Z}} \left(\eta \left(\frac{|i+1|}{\kappa} \right) - \eta \left(\frac{|i|}{\kappa} \right) \right) \left(u_{n+1,i+1}^{(h)} - u_{n+1,i}^{(h)} \right) u_{n+1,i+1}^{(h)}. \end{aligned}$$

Then, by the smoothness of η , there exists $\zeta_i \in \left(\frac{|i|}{\kappa}, \frac{|i+1|}{\kappa} \right)$ such that

$$\begin{aligned} &\left| v \sum_{i \in \mathbb{Z}} \left(\eta \left(\frac{|i+1|}{\kappa} \right) - \eta \left(\frac{|i|}{\kappa} \right) \right) \left(u_{n+1,i+1}^{(h)} - u_{n+1,i}^{(h)} \right) u_{n+1,i+1}^{(h)} \right| \\ &\leq v \sum_{i \in \mathbb{Z}} \frac{|\eta'(\zeta_i)|}{\kappa} \left| u_{n+1,i+1}^{(h)} - u_{n+1,i}^{(h)} \right| \left| u_{n+1,i+1}^{(h)} \right| \\ &\leq \frac{vC}{\kappa} \sum_{i \in \mathbb{Z}} \left(\left| u_{n+1,i+1}^{(h)} \right|^2 + \left| u_{n+1,i}^{(h)} \right| \left| u_{n+1,i+1}^{(h)} \right| \right) \leq \frac{2vCr^{*2}}{\kappa}, \end{aligned}$$

where we used the positive invariance of $\mathcal{B} = \mathbb{B}_{r^*}(0)$.

Moreover, to estimate the last term in (8) we use assumption (2) and Young's inequality to obtain

$$\begin{aligned} \langle F(u_{n+1}^{(h)}) + g, v_{n+1}^{(h)} \rangle &= \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) \left(f \left(u_{n+1,i}^{(h)} \right) u_{n+1,i}^{(h)} + g u_{n+1,i}^{(h)} \right) \\ &\leq -\frac{\alpha}{2} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) |u_{n+1,i}^{(h)}|^2 + \frac{1}{2\alpha} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) g_i^2. \end{aligned}$$

Inserting these inequalities into (8) yields

$$\begin{aligned} &(1 + \alpha h) \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) |u_{n+1,i}^{(h)}|^2 \\ &\leq \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) |u_{n+1,i}^{(h)}|^2 + h \left(\frac{4vC(r^*)^2}{\kappa} + \frac{1}{\alpha} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) g_i^2 \right). \end{aligned} \quad (9)$$

Since $g = (g_i)_{i \in \mathbb{Z}} \in \ell^2$, for every $\varepsilon > 0$ there exists $I(\varepsilon) > 0$ such that

$$\frac{4vC(r^*)^2}{\kappa} + \frac{1}{\alpha} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) g_i^2 < \frac{\varepsilon \alpha}{2} \quad \forall \kappa \geq I(\varepsilon).$$

Hence, by iterating estimate (9) it follows that

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) |u_{n,i}^{(h)}|^2 &\leq \frac{1}{1 + \alpha h} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) |u_{n-1,i}^{(h)}|^2 + \frac{\varepsilon}{2} \frac{\alpha h}{1 + \alpha h} \\ &\leq \frac{1}{(1 + \alpha h)^n} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) |u_{0,i}^{(h)}|^2 + \frac{\varepsilon}{2} \frac{h\alpha}{1 + h\alpha} \sum_{j=0}^{n-1} \frac{1}{(1 + h\alpha)^j} \\ &\leq \frac{\|u_0\|^2}{(1 + \alpha h)^n} + \frac{\varepsilon}{2} \leq \frac{(r^*)^2}{(1 + \alpha h)^n} + \frac{\varepsilon}{2}, \end{aligned}$$

since $u_o \in \mathcal{B}$. Finally, let

$$N^{(h)}(\varepsilon) := \log_{1+h\alpha} \left(\frac{2(r^*)^2}{\varepsilon} \right).$$

Then, we have for all $n \geq N^{(h)}(\varepsilon)$

$$\sum_{|i| \geq 2\kappa} |u_{n,i}|^2 \leq \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{\kappa} \right) |u_{n,i}^{(h)}|^2 < \varepsilon \quad \forall \kappa \geq I(\varepsilon),$$

which implies the desired assertion. $\square$

Theorem 2 For all step sizes $h \in (0, h^*)$ the discrete semi dynamical system $\{\phi^{(h)}(n)\}_{n \in \mathbb{N}_0}$ generated by the implicit Euler scheme (6) possesses an attractor $\mathcal{A}^{(h)}$ in $\mathcal{B} \subset \ell^2$.

Proof The phase space $\mathcal{B}$ is bounded and consequently, it suffices to verify that the discrete time semi dynamical system $\phi^{(h)}$ is asymptotically compact. To this end let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{B}$ and $(n_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{N}$ converging to ∞ . Since $\mathcal{B}$ is bounded and positively invariant, there exists $\hat{u} \in \ell^2$ and a subsequence of $\{\phi^{(h)}(n_k)u_k\}_{k \in \mathbb{N}}$, again denoted by $\{\phi^{(h)}(n_k)u_k\}_{k \in \mathbb{N}}$, such that

$$\phi^{(h)}(n_k)u_k \rightharpoonup \hat{u} \quad \text{weakly in } \ell^2 \text{ as } k \rightarrow \infty. \quad (10)$$

We will show that the convergence above is, in fact, strong in ℓ^2 .

Let $\varepsilon > 0$. By Lemma 4 there exist $I_1(\varepsilon)$ and $N_1^{(h)}(\varepsilon)$ in $\mathbb{N}$ such that

$$\sum_{|i| > I_1(\varepsilon)} (\phi^{(h)}(n_k)u_k)_i^2 < \frac{\varepsilon^2}{8} \quad \forall k \geq N_1^{(h)}(\varepsilon).$$

Moreover, since $\hat{u} \in \ell^2$, there exists $I_2(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{|i| > I_2(\varepsilon)} \hat{u}_i^2 < \frac{\varepsilon^2}{8}.$$

On the other hand, due to the weak convergence (10), it follows that

$$((\phi^{(h)}(n_k)u_k)_i)_{|i| \leq I(\varepsilon)} \rightarrow (\hat{u}_i)_{|i| \leq I(\varepsilon)} \quad \text{in } \mathbb{R}^{2I(\varepsilon)+1}, \text{ as } k \rightarrow \infty,$$

where $I(\varepsilon) = \max\{I_1(\varepsilon), I_2(\varepsilon)\}$. Hence, there exists $N_2^{(h)}(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{|i| \leq I(\varepsilon)} ((\phi^{(h)}(n_k)u_k)_i - u_i)^2 < \frac{\varepsilon^2}{2} \quad \forall k \geq N_2^{(h)}(\varepsilon).$$

Finally, for $k \geq \max \{N_1^{(h)}(\varepsilon), N_2^{(h)}(\varepsilon)\}$ we obtain

$$\begin{aligned} & \|\phi^{(h)}(\mathbf{n}_k)u_k - u\|^2 \\ &= \sum_{|i| \leq I(\varepsilon)} ((\phi^{(h)}(\mathbf{n}_k)u_k)_i - u_i)^2 + \sum_{|i| > I(\varepsilon)} ((\phi^{(h)}(\mathbf{n}_k)u_k)_i - u_i)^2 \\ &< \frac{\varepsilon^2}{2} + 2 \sum_{|i| > I(\varepsilon)} \left((\phi^{(h)}(\mathbf{n}_k)u_k)_i^2 + u_i^2 \right) < \varepsilon^2, \end{aligned}$$

which implies that $\{\phi^{(h)}(\mathbf{n}_k)u_k\}_{k \in \mathbb{N}}$ converges strongly to $\hat{u}$ in ℓ^2 as $k \rightarrow \infty$. $\square$

3.2 Discretisation Error Estimate

In this subsection we derive estimates for the discretisation error which are needed to prove the upper semicontinuity of the numerical attractors. To this end, we first consider the local discretisation error, i.e., the error for one time step of the IES, assuming that the numerical scheme and the lattice system start at the same point.

Let $\mathcal{F}(u) := Au + F(u) + g$ be the vector field of the lattice ODE (3) and introduce the total derivative of $\mathcal{F}$ along the solution,

$$D\mathcal{F}(u) := ((D\mathcal{F}(u))_i)_{i \in \mathbb{Z}}.$$

In this case, the Fréchet derivative of the Nemytskii operator F can be expressed in terms of the derivatives of the original function f (see Jentzen and Kloeden [8]), and we obtain

$$\begin{aligned} D\mathcal{F}(u) &= (A + \text{diag}(f'(u_i)))\mathcal{F}(u) \\ &= (A + \text{diag}(f'(u_i)))(Au + F(u) + g). \end{aligned} \quad (11)$$

Lemma 5 For all $h \in (0, h^*)$ the local discretisation error for the IES (6), restricted to $\mathcal{B}$, is of order 2, i.e.,

$$\|u(h; u_n^{(h)}) - u_{n+1}^{(h)}\| \leq C_{\mathcal{B}} h^2,$$

for some constant $C_{\mathcal{B}} > 0$.

Proof Note that the Taylor expansion of the ODE (3) in ℓ^2 “starting” from $u(t_{n+1}) = u(t_{n+1}; u_o)$ at t_{n+1} and going back $-h$ in time to $u(t_n) = u(t_n; u_o)$ reads

$$u(t_n) = u(t_{n+1}) + (-h)\mathcal{F}(u(t_{n+1})) + \frac{1}{2}(-h)^2 D\mathcal{F}(u(\theta_h)), \quad (12)$$

for some $\theta_h \in (t_n, t_{n+1})$. For each $n \in \mathbb{N}_0$ write

$$\Delta_n(h) := u(t_n; u_o) - u_n^{(h)}.$$

Then, subtracting Eq. (6) from Eq. (12) gives

$$\Delta_{n+1}(h) = \Delta_n(h) + h \left(A\Delta_{n+1}(h) + (F(u(t_{n+1})) - F(u_n^{(h)})) \right) - \frac{1}{2}h^2 D\mathcal{F}(u(\theta_h)).$$

Taking the inner product of the above equation with $\Delta_{n+1}(h)$ in ℓ^2 and using the fact that $\langle A\Delta_{n+1}(h), \Delta_{n+1}(h) \rangle \leq 0$, we obtain

$$\begin{aligned} \|\Delta_{n+1}(h)\|^2 &= \langle \Delta_n(h), \Delta_{n+1}(h) \rangle + h \langle A\Delta_{n+1}(h), \Delta_{n+1}(h) \rangle \\ &\quad + h \langle F(u(t_{n+1})) - F(u_{n+1}^{(h)}), \Delta_{n+1}(h) \rangle + \frac{1}{2}h^2 \langle D\mathcal{F}(u(\theta_h)), \Delta_{n+1}(h) \rangle \\ &\leq \|\Delta_{n+1}(h)\| \|\Delta_n(h)\| + h \|\Delta_{n+1}(h)\| \|F(u(t_{n+1})) - F(u_{n+1}^{(h)})\| \\ &\quad + \frac{1}{2}h^2 \|\Delta_{n+1}(h)\| \|D\mathcal{F}(u(\theta_h))\|. \end{aligned}$$

Dividing the above inequality by $\|\Delta_{n+1}(h)\|$ gives

$$\|\Delta_{n+1}(h)\| \leq \|\Delta_n(h)\| + h \|F(u(t_{n+1})) - F(u_{n+1}^{(h)})\| + \frac{1}{2}h^2 \|D\mathcal{F}(u(\theta_h))\|. \quad (13)$$

Since the set $\mathcal{B}$ is positive invariant for both, the time continuous dynamical system φ and the IES, we have $u(t_{n+1}), u_{n+1}^{(h)} \in \mathcal{B}$, and hence, it follows as in the proof of Lemma 2 that

$$\|F(u(t_{n+1})) - F(u_{n+1}^{(h)})\| \leq L_{r^*} \|u(t_{n+1}) - u_{n+1}^{(h)}\| = L_{r^*} \|\Delta_{n+1}(h)\|. \quad (14)$$

To estimate the last term in (13) we use the expression (11) for the total derivative. By assumption (2), the positive invariance of $\mathcal{B}$ and the boundedness of the operator A it follows that

$$\begin{aligned} \|D\mathcal{F}(u(\theta_h))\| &\leq \|A + \text{diag}(f'(u_i(\theta_h)))\| (\|Au(\theta_h)\| + \|F(u(\theta_h))\| + \|g\|) \\ &\leq C_{\mathcal{B}}, \end{aligned} \quad (15)$$

for some constant $C_{\mathcal{B}} > 0$ depending on the radius of $\mathcal{B}$. Inserting the inequalities (14) and (15) into (13) yields

$$\|\Delta_{n+1}(h)\| \leq \|\Delta_n(h)\| + hL_{r^*} \|\Delta_{n+1}(h)\| + h^2C_{\mathcal{B}}.$$

Thus, for all $h < h^* < \frac{1}{L_{r^*}}$ we obtain the estimate

$$\|\Delta_{n+1}(h)\| \leq \frac{1}{1 - hL_{r^*}} \|\Delta_n(h)\| + C_{\mathcal{B}}h^2. \quad (16)$$

To estimate the local discretisation error in $\mathcal{B}$, set $u(t_n; u_o) = u_n^{(h)}$. Thus $\Delta_n(h) = 0$ and $\Delta u_{n+1}(h) = u(h; u_n^{(h)}) - u_{n+1}^{(h)}$ and therefore,

$$\|u(h; u_n^{(h)}) - u_{n+1}^{(h)}\| \leq C_{\mathcal{B}}h^2.$$

□

To derive an estimate for the global discretisation error

$$\Delta_n(h) := u(t_n; u_o) - u_n^{(h)}, \quad n \in \mathbb{N}_0,$$

we iterate the inequality (16) and obtain

$$\begin{aligned} \|\Delta_n(h)\| &\leq \frac{1}{(1 - hL_{r^*})^n} \|\Delta_0(h)\| + C_{\mathcal{B}}h^2 \sum_{j=0}^{n-1} \frac{1}{(1 - hL_{r^*})^j} \\ &\leq \frac{1}{(1 - hL_{r^*})^n} \|\Delta_0(h)\| + C_{T, \mathcal{B}}h. \end{aligned}$$

Then, using that $\Delta_0(h) = 0$ the global discretisation error estimate on $\mathcal{B}$ immediately follows,

$$\|\Delta_n(h)\| = \|u(t_n; u_0) - u_n^{(h)}\| \leq C_{T,\mathcal{B}}h \quad \forall 0 \leq t_n \leq T, \quad (17)$$

for some constant $C_{T,\mathcal{B}} > 0$.

4 Upper Semi Continuous Convergence of the Numerical Attractors

In this section we prove the upper semicontinuity of the numerical attractors $\mathcal{A}^{(h)}$ of the IES. In particular, we first show that the attractors $\mathcal{A}^{(h)}$ converge to the attractor $\mathcal{A}$ of the lattice system when $h \rightarrow 0^+$ (see Theorem 3). Subsequently, we consider finite dimensional approximations in $\mathbb{R}^{2m+1}$ for the IES (6). We prove the convergence of the attractors of the finite dimensional approximated IES, denoted by $\mathcal{A}_m^{(h)}$, to the numerical attractor $\mathcal{A}^{(h)}$ as $m \rightarrow \infty$ (see Theorem 4). Combining both results it follows that the finite dimensional numerical attractors $\mathcal{A}_m^{(h)}$ converge to the global attractor $\mathcal{A}$ of the lattice system as $h \rightarrow 0^+$ and $m \rightarrow \infty$.

4.1 Upper Semicontinuous Convergence of $\mathcal{A}^{(h)}$ to $\mathcal{A}$

First, we show that the numerical attractors $\mathcal{A}^{(h)}$ converge to the global attractor $\mathcal{A}$ of the ODE system (3) as $h \rightarrow 0^+$.

Theorem 3 *Let Assumption (A) hold. Then,*

$$\text{dist}_{\ell^2}(\mathcal{A}^{(h)}, \mathcal{A}) \rightarrow 0 \quad \text{as } h \rightarrow 0^+,$$

where $\text{dist}_{\ell^2}(\cdot, \cdot)$ denotes the Hausdorff semidistance in ℓ^2 .

Proof Assume that the assertion is false. Then, there exist $\varepsilon_0 > 0$ and a sequence $h_j \rightarrow 0^+$ as $j \rightarrow \infty$ such that

$$\text{dist}_{\ell^2}(\mathcal{A}^{(h_j)}, \mathcal{A}) \geq \varepsilon_0 \quad \forall j \in \mathbb{N}.$$

On the one hand, since $\mathcal{A}^{(h_j)}$ and $\mathcal{A}$ are compact, for every $j \in \mathbb{N}$ there exists $a_j \in \mathcal{A}^{(h_j)}$ such that

$$\text{dist}_{\ell^2}(a_j, \mathcal{A}) = \text{dist}_{\ell^2}(\mathcal{A}^{(h_j)}, \mathcal{A}) \geq \varepsilon_0.$$

On the other hand, due to the invariance of $\mathcal{A}^{(h_j)}$, there exists $b_j \in \mathcal{A}^{(h_j)}$ and $N_j \in \mathbb{N}$ such that the N_j -th iterate of the IES (6) starting at b_j with step size h_j equals a_j , i.e., $u_{N_j}^{(h_j)}(b_j) = a_j$. Denote by $u(N_j h_j; b_j)$ the solution of the original ODE (3) starting at b_j , evaluated at $T_j := N_j h_j$. Then,

$$\begin{aligned} \text{dist}_{\ell^2}(\mathcal{A}^{(h_j)}, \mathcal{A}) &= \text{dist}_{\ell^2}(a_j, \mathcal{A}) = \text{dist}_{\ell^2}(u_{N_j}^{(h_j)}(b_j), \mathcal{A}) \\ &\leq \|u_{N_j}^{(h_j)}(b_j) - u(N_j h_j; b_j)\| + \text{dist}_{\ell^2}(u(N_j h_j; b_j), \mathcal{A}). \end{aligned} \quad (18)$$

Since $\mathcal{A}$ is the global attractor for the original lattice ODE system (3), there exists $T(\varepsilon_0) > 0$ and for each $j \in \mathbb{N}$ an $M_j \in \mathbb{N}$ such that $T(\varepsilon_0) \leq M_j h_j < T(\varepsilon_0) + 1$ and

$$\text{dist}_{\ell^2}(u(M_j h_j; b_j), \mathcal{A}) \leq \frac{\varepsilon_0}{4}. \quad (19)$$

Due to the global discretisation error estimate (17) and the fact that $b_j \in \mathcal{A}^{(h_j)} \subset \mathcal{B}$, we have

$$\left\| u_{M_j}^{(h_j)}(b_j) - u(M_j h_j; b_j) \right\| \leq C_{T(\varepsilon_0+1), \mathcal{B}} h_j \leq \frac{\varepsilon_0}{4}, \quad (20)$$

if j is sufficiently large. Finally, inserting (19) and (20) into (18) immediately implies that there exists $j \in \mathbb{N}$ such that

$$\text{dist}_{\ell^2}(\mathcal{A}^{(h_j)}, \mathcal{A}) \leq \frac{\varepsilon_0}{2},$$

which contradicts the hypothesis and completes the proof. $\square$

4.2 Finite Dimensional Approximations for the IES

We now consider finite dimensional approximations for the IES (6). Namely, for m in $\mathbb{N}$ we analyse the $(2m+1)$ -dimensional implicit system of difference equations

$$\begin{aligned} u_{n,-m}^{(h)} &= u_{n-1,-m}^{(h)} + hv \left(u_{n,m}^{(h)} - 2u_{n,-m}^{(h)} + u_{n,-m+1}^{(h)} \right) + hf(u_{n,-m}^{(h)}) + hg_{-m} \\ u_{n,-m+1}^{(h)} &= u_{n-1,-m+1}^{(h)} + hAu_{n,-m+1}^{(h)} + hf(u_{n,-m+1}^{(h)}) + hg_{-m+1} \\ &\vdots \\ u_{n,m-1}^{(h)} &= u_{n-1,m-1}^{(h)} + hAu_{n,m-1}^{(h)} + hf(u_{n,m-1}^{(h)}) + hg_{m-1} \\ u_{n,m}^{(h)} &= u_{n-1,m}^{(h)} + hv \left(u_{n,m}^{(h)} - 2u_{n,m}^{(h)} + u_{n,-m}^{(h)} \right) + hf(u_{n,m}^{(h)}) + hg_m \end{aligned}$$

with initial data $(u_{-m}^{(h)}, \dots, u_m^{(h)})(0) = (u_{o,-m}, \dots, u_{o,m}) \in \mathbb{R}^{2m+1}$.

Remark 4 The approximating system is obtained by truncating the infinite dimensional system (6) and assuming the periodic boundary conditions $u_{n,m}^{(h)} = u_{n,-m-1}^{(h)}$ and $u_{n,-m}^{(h)} = u_{n,m+1}^{(h)}$ for every $n \in \mathbb{N}$. The use of periodic boundary conditions was shown by Bates et al. [2] to give the appropriate truncated system of ordinary differential equations corresponding to the lattice system (1).

In order to simplify the notation, from now on we use $\mathbf{x}_n^{(h)} \in \mathbb{R}^{2m+1}$ to denote the $(2m+1)$ -dimensional truncation of $u_n^{(h)} \in \ell^2$. Notice that $\mathbf{x}_n^{(h)}$ actually depends on m , so when the dependence on m is explicitly needed we write $\mathbf{x}_n^{(h)} = \mathbf{x}_n^{(h,m)}$.

Remark 5 Every $\mathbf{x} = (u_i)_{|i| \leq m} \in \mathbb{R}^{2m+1}$ can be naturally extended to an element $u = (u_i)_{i \in \mathbb{Z}}$ in ℓ^2 , by setting $u_i = 0$ for all $|i| > m$.

To formulate the finite dimensional IES compactly in $\mathbb{R}^{2m+1}$, let $\Pi^{(m)}$ be the matrix

$$\Pi^{(m)} = v \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix} \in \mathbb{R}^{(2m+1) \times (2m+1)},$$

and set $\mathbf{x}_n^{(h)} = (u_{n,i}^{(h)})_{|i| \leq m}$, $\mathbf{f}^{(m)}(\mathbf{x}_n^{(h)}) = (f(u_{n,i}^{(h)}))_{|i| \leq m}$ and $\mathbf{g}^{(m)} = (g_i)_{|i| \leq m}$. Then, the truncated IES can be written as

$$\begin{aligned}\mathbf{x}_{n+1}^{(h)} &= \mathbf{x}_n^{(h)} + h\Pi^{(m)}\mathbf{x}_{n+1}^{(h)} + h\mathbf{f}^{(m)}(\mathbf{x}_{n+1}^{(h)}) + h\mathbf{g}^{(m)}, \quad n \in \mathbb{N}, \\ \mathbf{x}_0^{(h)} &= (u_{0,i})_{|i| \leq m} \in \mathbb{R}^{2m+1}.\end{aligned}\quad (21)$$

4.3 Finite Dimensional Numerical Attractors $\mathcal{A}_m^{(h)}$

As in the continuous time case, we restrict the truncated system (21) to a suitable bounded, positive invariant ball $\mathcal{B}_m \subset \mathbb{R}^{2m+1}$, namely,

$$\mathcal{B}_m = \left\{ \mathbf{x} \in \mathbb{R}^{2m+1} : |\mathbf{x}|^2 \leq (r^*)^2 = \frac{2\|g\|^2}{\alpha^2} + 1 \right\}.$$

Here and in the sequel, we denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^{2m+1}$, while we continue to use $\|\cdot\|$ for the ℓ^2 -norm.

One can show that for every $h \in (0, h^*)$ and $\mathbf{x}_0^{(h)} \in \mathcal{B}_m$ system (21) is globally well-posed, the solution takes values in $\mathcal{B}_m$ and hence, it generates a discrete semi dynamical system $\phi_m^{(h)}(n) : \mathcal{B}_m \rightarrow \mathcal{B}_m$, $n \in \mathbb{N}_0$, defined by

$$\phi_m^{(h)}(n)\mathbf{x}_0^{(h)} = \mathbf{x}_n^{(h)}(\mathbf{x}_0^{(h)}), \quad \mathbf{x}_0^{(h)} \in \mathcal{B}_m.$$

The details are similar to those for the infinite dimensional IES in Sect. 3.

Remark 6 The set $\mathcal{B}_m$ is, in fact, an absorbing set for the truncated IES and is contained in the corresponding basin of attraction that depends on both, m and the step size h . We recall that the numerical scheme need not to possess a globally attracting set even if the global attractor of the original system exists [7, Example 2.13].

First, we derive an estimate for the solutions, similarly to Lemma 1.

Lemma 6 Assume that Assumption (A) holds. Then, for every $h \in (0, h^*)$ and $\mathbf{x}_0^{(h)} \in \mathcal{B}_m$ the solution of (21) satisfies

$$\left| \mathbf{x}_{n+1}^{(h)} \right|^2 \leq \frac{1}{1+h\alpha} |\mathbf{x}_n^{(h)}|^2 + \frac{h}{1+h\alpha} \frac{\|g\|^2}{\alpha} \quad \forall n \in \mathbb{N}_0.$$

In particular, the set $\mathcal{B}_m$ is positive invariant for $\phi_m^{(h)}$.

Proof Notice that $-\Pi^{(m)} = P P^\top = P^\top P$, where P is the matrix

$$P = v^{1/2} \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & -1 & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} \in \mathbb{R}^{(2m+1) \times (2m+1)}.$$

Therefore, it follows that

$$\left(\Pi^{(m)} \mathbf{x}_{n+1}^{(h)} \right) \cdot \mathbf{x}_{n+1}^{(h)} = - \left| P \mathbf{x}_{n+1}^{(h)} \right|^2 \leq 0.$$

As in the proof of Lemma 1 we take the inner product $\cdot$ in $\mathbb{R}^{2m+1}$ of (21) with $\mathbf{x}_{n+1}^{(h)}$ and use (2) to obtain

$$\begin{aligned} |\mathbf{x}_{n+1}^{(h)}|^2 &= \mathbf{x}_n^{(h)} \cdot \mathbf{x}_{n+1}^{(h)} + h \left(\Pi^{(m)} \mathbf{x}_{n+1}^{(h)} \right) \cdot \mathbf{x}_{n+1}^{(h)} + h \left(\mathbf{f}^{(m)}(\mathbf{x}_{n+1}^{(h)}) + \mathbf{g}^{(m)} \right) \cdot \mathbf{x}_{n+1}^{(h)} \\ &\leq \frac{1}{2} \left(|\mathbf{x}_n^{(h)}|^2 + |\mathbf{x}_{n+1}^{(h)}|^2 \right) + h \sum_{|i| \leq m} f(\mathbf{x}_{n+1,i}^{(h)}) \mathbf{x}_{n+1,i}^{(h)} + \frac{h}{2} \left(\alpha |\mathbf{x}_{n+1}^{(h)}|^2 + \frac{1}{\alpha} |\mathbf{g}^{(m)}|^2 \right) \\ &\leq \frac{1-h\alpha}{2} |\mathbf{x}_{n+1}^{(h)}|^2 + \frac{1}{2} |\mathbf{x}_n^{(h)}|^2 + \frac{h}{2\alpha} \|\mathbf{g}\|^2, \end{aligned}$$

which implies the desired estimate.

Moreover, if $\mathbf{x}_n^{(h)} \in \mathcal{B}_m$, i.e., $|\mathbf{x}_n^{(h)}|^2 \leq (r^*)^2 = \frac{2\|\mathbf{g}\|^2}{\alpha^2} + 1$, then

$$|\mathbf{x}_{n+1}^{(h)}|^2 \leq \frac{1}{1+h\alpha} \left(\frac{2\|\mathbf{g}\|^2}{\alpha^2} + 1 \right) + \frac{h}{1+h\alpha} \frac{\|\mathbf{g}\|^2}{\alpha} \leq (r^*)^2,$$

which proves the positive invariance of $\mathcal{B}_m$. $\square$

Since the phase space $\mathcal{B}_m \subset \mathbb{R}^{2m+1}$ is compact, it immediately follows that the truncated IES possesses an attractor $\mathcal{A}_m^{(h)} \subset \mathcal{B}_m$.

Lemma 7 *Let Assumption (A) hold and $h \in (0, h^*)$. Then, the discrete semi dynamical system $\{\phi_m^{(h)}(\mathbf{n})\}_{\mathbf{n} \in \mathbb{N}}$ generated by the finite dimensional IES (21) possesses an attractor $\mathcal{A}_m^{(h)} \subset \mathcal{B}_m$.*

4.4 Main Convergence Results

In this subsection we show that for any fixed $h \in (0, h^*)$, the finite dimensional attractors $\mathcal{A}_m^{(h)}$, naturally embedded into ℓ^2 , tend to the numerical attractor $\mathcal{A}^{(h)}$ as $m \rightarrow \infty$.

Lemma 8 *Let Assumption (A) hold and $h \in (0, h^*)$. Then, for every $\varepsilon > 0$ there exists $I(\varepsilon) \in \mathbb{N}$ such that*

$$\sum_{I(\varepsilon) \leq |i| \leq m} (a_i)^2 \leq \varepsilon \quad \forall a = (a_i)_{|i| \leq m} \in \mathcal{A}_m^{(h)}.$$

Proof Let η be the smooth function defined in the proof of Lemma 4 and $\kappa \leq m$ be a fixed integer to be determined. For $\mathbf{x}_n^{(h)} = (u_{n,i}^{(h)})_{|i| \leq m}$ we set $\mathbf{y}_n^{(h)} = \left(\eta\left(\frac{|i|}{\kappa}\right) u_{n,i}^{(h)} \right)_{|i| \leq m}$, for each $\mathbf{n} \in \mathbb{N}_0$.

Taking the inner product of Eq.(21) with $\mathbf{y}_n^{(h)}$ in $\mathbb{R}^{2m+1}$, using the fact that $\Pi^{(m)} \mathbf{x}_n^{(h)} \cdot \mathbf{y}_n^{(h)} = P \mathbf{x}_n^{(h)} \cdot P \mathbf{y}_n^{(h)}$, and then following exactly the same arguments as in the proof of Lemma 4, it follows that for every $\varepsilon > 0$ there exist $I(\varepsilon)$ and $N^{(h)}(\varepsilon)$ in $\mathbb{N}$ such that for $\kappa \geq I(\varepsilon)$ we have

$$\sum_{2\kappa \leq |i| \leq m} \left(u_{n,i}^{(h)} \right)^2 \leq \sum_{|i| \leq m} \eta\left(\frac{|i|}{\kappa}\right) \left(u_{n,i}^{(h)} \right)^2 \leq \varepsilon \quad \forall \mathbf{n} \geq N^{(h)}(\varepsilon).$$

Since the global attractor $\mathcal{A}_m^{(h)}$ is invariant under $\phi_m^{(h)}$, for any $a \in \mathcal{A}_m^{(h)}$ and $N \in \mathbb{N}$ there exists $\mathbf{x}_0 \in \mathcal{A}_m^{(h)}$ such that $a = \mathbf{x}_N^{(h)}(\mathbf{x}_0) = \left(u_{N,i}^{(h)}\right)_{|i| \leq m}$. This implies that

$$\sum_{I(\varepsilon) \leq |i| \leq m} a_i^2 = \sum_{I(\varepsilon) \leq |i| \leq m} \left(u_{N,i}^{(h)}\right)^2 \leq \varepsilon \quad \forall N \geq N^{(h)}(\varepsilon),$$

which completes the proof. $\square$

Lemma 9 *Let Assumption (A) hold, $h \in (0, h^*)$ be fixed and $a_m \in \mathcal{A}_m^{(h)}$, $m \in \mathbb{N}$. Then, there exists a subsequence $(a_{m_k})_{k \in \mathbb{N}}$ and an element $a_* \in \mathcal{A}^{(h)}$ such that a_{m_k} converges to a_* in ℓ^2 as $k \rightarrow \infty$.*

Proof Since h is fixed and the explicit dependence on m is important in this lemma, we write $\mathbf{x}_n^{(h,m)}$ as $\mathbf{x}_n^{(m)}$ for every $\mathbf{x}_n^{(h,m)} = (u_{n,i}^{(h)})_{|i| \leq m}$ and all $n \in \mathbb{N}_0$. Given $a_m \in \mathcal{A}_m^{(h)}$, let

$$\mathbf{x}_n^{(m)} = \phi_m^{(h)}(n)a_m$$

be the solution of the ODE (21) with initial data $\mathbf{x}_0^{(h)} = (u_{0,i})_{|i| \leq m} = a_m$. Since $a_m \in \mathcal{A}_m^{(h)}$, it follows that $\mathbf{x}_n^{(m)} \in \mathcal{A}_m^{(h)}$, and it is an entire solution, i.e., defined for all $n \in \mathbb{Z}$. For each $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ let $u_n^{(m)}$ be the natural extension of $\mathbf{x}_n^{(m)}$ in ℓ^2 , i.e.,

$$u_n^{(m)} = \left(\dots, 0, \dots, 0, u_{n,-m}^{(m)}, \dots, u_{n,0}^{(m)}, \dots, u_{n,m}^{(m)}, 0, \dots, 0, \dots\right).$$

Denoting by $\tilde{\mathcal{B}}_m$ the natural embedding of $\mathcal{B}_m \subset \mathbb{R}^{2m+1}$ in ℓ^2 we observe that $\tilde{\mathcal{B}}_m \subset \mathcal{B}$ for all $m \in \mathbb{N}$, and hence, $(u_n^{(m)})_{m \in \mathbb{N}, n \in \mathbb{Z}}$ is uniformly bounded in ℓ^2 . Consequently, by a diagonal argument there exists $u_n^* = (u_{n,i}^*)_{i \in \mathbb{Z}} \in \ell^2$ and a subsequence $(u_n^{(m_j)})_{j \in \mathbb{N}}$ converging weakly to u_n^* in ℓ^2 ,

$$u_n^{(m_j)} \rightharpoonup u_n^* \quad \text{as } j \rightarrow \infty \quad \forall n \in \mathbb{Z}.$$

Similarly, as in the proof of Theorem 2, we can show that the weak convergence is, in fact, strong, i.e., for all $n \in \mathbb{Z}$ we have

$$\|u_n^{(m_j)} - u_n^*\| \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad (22)$$

and consequently, $u_n^* \in \mathcal{B}$ for all $n \in \mathbb{Z}$.

Next, we prove that u_n^* is an entire solution of the infinite dimensional IES (6) for all $n \in \mathbb{Z}$. For fixed $k \in \mathbb{Z}$ let $m_j > |k|$. Since $u_n^{(m_j)}$ is a solution of the truncated IES (21), we have

$$u_{n,k}^{(m_j)} = u_{n-1,k}^{(m_j)} + h\nu \left(u_{n,k-1}^{(m_j)} - 2u_{n,k}^{(m_j)} + u_{n,k+1}^{(m_j)}\right) + hf(u_{n,k}^{(m_j)}) + hg_k, \quad (23)$$

for all $n \in \mathbb{Z}$. By estimate (4) it follows that

$$\|f(u_n^{(m_j)}) - f(u_n^*)\| \leq L_{r^*} \|u_n^{(m_j)} - u_n^*\|.$$

Hence, taking the limit $j \rightarrow \infty$ in Eq. (23) and using (22) we obtain

$$u_{n,k}^* = u_{n-1,k}^* + h\nu \left(u_{n,k-1}^* - 2u_{n,k}^* + u_{n,k+1}^*\right) + hf(u_{n,k}^*) + hg_k, \quad n \in \mathbb{Z}.$$

Consequently, u_n^* , $n \in \mathbb{Z}$, is a bounded, entire solution of (6), which implies that it is contained in the attractor $\mathcal{A}^{(h)}$. In particular, we have $u_0^* \in \mathcal{A}^{(h)}$, and by (22) it follows that

$$\|u_0^{(m_j)} - u_0^*\| = \|a_{(m_j)} - u_0^*\| \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

which concludes the proof of the lemma. $\square$

We are now ready to state our main result.

Theorem 4 *Let Assumption (A) hold and $h \in (0, h^*)$ be fixed. Then, the attractors $\mathcal{A}_m^{(h)}$ of the truncated IES (21) converge to the attractor of the IES (6),*

$$\lim_{m \rightarrow \infty} \text{dist}_{\ell^2}(\mathcal{A}_m^{(h)}, \mathcal{A}^{(h)}) = 0.$$

Proof Suppose that the assertion is false. Then, there exist $\varepsilon_0 > 0$ and sequences $a_{m_j} \in \mathcal{A}_{m_j}^{(h)}$ and $m_j \rightarrow \infty$ such that

$$\text{dist}_{\ell^2}(a_{m_j}, \mathcal{A}^{(h)}) \geq \varepsilon_0 \quad \forall j \in \mathbb{N}. \quad (24)$$

However, since $a_{m_j} \in \mathcal{A}_{m_j}^{(h)}$ and by Lemma 9, there exists a subsequence $(a_{m_{j_k}})_{k \in \mathbb{N}}$ of $(a_{m_j})_{j \in \mathbb{N}}$ such that

$$\text{dist}_{\ell^2}(a_{m_{j_k}}, \mathcal{A}^{(h)}) \rightarrow 0 \quad \text{as } m_{j_k} \rightarrow \infty.$$

This contradicts (24) and completes the proof. $\square$

Combining Theorem 3 and Theorem 4 the convergence of the approximated numerical attractor $\mathcal{A}_m^{(h)}$ to the attractor $\mathcal{A}$ of the lattice system follows immediately.

Corollary 1 *The finite dimensional numerical attractors $\mathcal{A}_m^{(h)}$ of (21) converge upper semi-continuously to the global attractor $\mathcal{A}$ of the system (3),*

$$\lim_{h \rightarrow 0^+} \lim_{m \rightarrow \infty} \text{dist}_{\ell^2}(\mathcal{A}_m^{(h)}, \mathcal{A}) = 0.$$

5 Closing Remarks

We constructed finite dimensional numerical approximations for the global attractor $\mathcal{A}$ and showed that the numerical attractors converge to the analytical attractor $\mathcal{A}$ as the time step tends to zero. This can be done either by applying a one step numerical scheme to a finite

Fig. 1 Convergence paths for the approximated numerical attractor to the analytical attractor

dimensional system of ODEs approximating the lattice system (1) or by applying a one step numerical scheme to (1) in ℓ_2 and then approximating it by a finite dimensional counterpart (see Figure 1).

Two different paths for the convergence of the finite dimensional numerical attractors $\mathcal{A}_m^{(h)}$ to the analytical attractor $\mathcal{A}$ are illustrated in Figure 1. Bates et al. proved the convergence IV in [2], while the convergence III follows by a general result of Kloeden and Lorenz in [11] on the discretisation of attractors of ordinary differential equations (see also [7], Chapter 9). In this work we proved the convergence via paths I and II. Compared to the alternative convergence proof via paths III and IV, our approach avoids the intermediate finite dimensional truncated system of ODEs corresponding to the lattice system (1) and leads to simpler proofs.

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