

# Stability Analysis of a Reaction–Diffusion Equation with Spatiotemporal Delay and Dirichlet Boundary Condition

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**Abstract** In this paper, we concentrate on the study of a reaction–diffusion equation with spatiotemporal delay and homogeneous Dirichlet boundary condition. It is shown that a positive spatially nonhomogeneous equilibrium can bifurcate from the trivial equilibrium. Moreover, the stability of the bifurcated positive equilibrium is investigated. And we prove that, for the given spatiotemporal delay, the bifurcated equilibrium is stable under some conditions, and Hopf bifurcation cannot occur.

**Keywords** Reaction–diffusion · Spatiotemporal delay · Stability analysis

## 1 Introduction

For single population models, a prototypical delayed reaction–diffusion equation is the following diffusive Hutchinson equation:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = d \frac{\partial^2 u(x, t)}{\partial x^2} + \lambda u(x, t) (1 - u(x, t - \tau)), & x \in (0, \pi), t > 0, \\ u(x, t) = 0, & x = 0, \pi, t > 0 \end{cases} \quad (1.1)$$

Busenberg and Huang [4] showed that for  $\lambda > d$  but close to  $d$ , a large delay  $\tau$  can make the unique spatially nonhomogeneous positive equilibrium of Eq. (1.1) unstable through a Hopf bifurcation. Related work can also be found in References [11, 12, 18, 19, 22].

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Dedicated to Professor John Mallet-Paret on the occasion of his 60th birthday.

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It has been pointed out that the diffusion and time delay are always not independent of each other for a delayed reaction–diffusion model, (see References [2, 3, 8–10, 23]). For example, Britton [3] introduced the following model:

$$\frac{\partial u(x, t)}{\partial t} = d\Delta u(x, t) + \lambda u(x, t) (1 + \alpha u - (1 + \alpha)g * *u), \tag{1.2}$$

where

$$g * *u = \int_{-\infty}^t \int_{\Omega} g(x, y, t - s)u(y, s)dyds, \tag{1.3}$$

and analyzed the traveling waves on unbounded domain. Then Gourley and Britton [8] proposed a predator–prey system with spatiotemporal delay. In [9], Gourley and So introduced a food-limited population model as follows:

$$\frac{\partial u(x, t)}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + \lambda u(x, t) \left( \frac{1 - au(x, t) - b(f * u)(x, t)}{1 + acu(x, t) + bc(f * u)(x, t)} \right), \quad x \in (0, \pi), \tag{1.4}$$

where

$$(f * u)(x, t) = \int_0^\infty \int_0^\pi G(x, y, s) f(s)u(y, t - s)dyds. \tag{1.5}$$

Here for Neumann boundary condition,  $G(x, y, t)$  is the solution of the following equation:

$$\begin{cases} \frac{\partial G}{\partial t} = d \frac{\partial^2 G}{\partial y^2}, & y \in (0, \pi), \\ \frac{\partial G}{\partial y} = 0, & y = 0, \pi, \\ G(x, y, 0) = \delta(x - y), \end{cases} \tag{1.6}$$

and hence

$$G(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^\infty e^{-dk^2t} \cos kx \cos ky.$$

Similarly, for Dirichlet boundary condition,

$$G(x, y, t) = \frac{2}{\pi} \sum_{k=1}^\infty e^{-dk^2t} \sin kx \sin ky. \tag{1.7}$$

For Neumann boundary condition, Gourley and So [9] analyzed the stability and the Hopf bifurcation of the spatially homogeneous positive equilibrium, and for Dirichlet boundary condition, they proved the existence of the positive spatially nonhomogeneous equilibrium, which bifurcates from the trivial equilibrium. However the stability of the bifurcated nonhomogeneous equilibrium has not been investigated. And it remains open that whether Hopf bifurcation could occur for Dirichlet boundary condition. We also remark that there are many results on the traveling wave solutions of reaction–diffusion models with nonlocal delay, (see References [1, 7, 14–17, 20] and the references therein).

Motivated by the above work of [9], we analyze the following reaction–diffusion equation with spatiotemporal delay and homogeneous Dirichlet boundary condition:

$$\begin{cases} \frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} \\ \quad + \lambda u F(u, \int_0^\infty \int_0^\pi G(x, y, s) f(s) u(y, t - s) dy ds), & x \in (0, \pi), t > 0, \\ u(x, t) = 0, & x = 0, \pi, t > 0, \end{cases} \tag{1.8}$$

where  $G(x, y, t)$  is defined as in Eq. (1.7), and  $f(t)$  is the delay kernel, satisfying  $f(t) \geq 0$  for  $t \geq 0$ , and  $\int_0^\infty f(t) dt = 1$ . Here we choose

$$f(t) = \frac{1}{\Gamma(m)\beta^m} t^{m-1} e^{-t/\beta}, \tag{1.9}$$

or

$$f(t) = \delta(t - \tau), \tag{1.10}$$

for simplicity, where  $\beta, \tau > 0$ , and  $m$  is a positive integer. The first one is the Gamma distribution delay kernel, and when  $m = 1, 2$ , it is referred to as the “weak” and “strong” generic delay kernels respectively. The second one is the Dirac distribution delay kernel. We recall that the Laplace transform of  $f(t)$  is:

$$L(\lambda) = \int_0^\infty e^{-\lambda s} f(s) ds,$$

and denote

$$M_k = L(dk^2), \quad \text{for } k = 1, 2, \dots \tag{1.11}$$

In this paper, we prove that a positive spatially nonhomogeneous equilibrium can bifurcate from the trivial equilibrium and then focus on the stability of this bifurcated positive equilibrium. Actually, in Sect. 2, we prove that, for the given spatiotemporal delay, the bifurcated “small” positive spatially nonhomogeneous equilibrium is stable, and the Hopf bifurcation cannot occur. Due to the difficulties for Dirichlet boundary problem, we prove the above result when  $\lambda$  is near the bifurcation point  $d$ . The existence and stability of the positive equilibrium remain open when  $\lambda$  is large.

Throughout the paper, we suppose that the function  $F(x, y)$  is smooth,  $F(0, 0) = 1$ , denote

$$r_1 = \frac{\partial F}{\partial x}(0, 0), \quad r_2 = \frac{\partial F}{\partial y}(0, 0). \tag{1.12}$$

Moreover, we denote the spaces  $X = H^2(0, \pi) \cap H_0^1(0, \pi), Y = L^2(0, \pi)$ , and define the complexification of a space  $Z$  to be  $Z_{\mathbb{C}} := Z \oplus iZ = \{x_1 + ix_2 \mid x_1, x_2 \in Z\}$ , the domain of a linear operator  $L$  by  $\mathcal{D}(L)$  and the range of  $L$  by  $\mathcal{R}(L)$ . For the complex-valued Hilbert space  $Y_{\mathbb{C}}$ , we use the standard inner product  $\langle u, v \rangle = \int_{\Omega} \bar{u}(x)v(x) dx$ .

### 2 Main Results and Proofs

We first study the existence of the positive steady state solutions of Eq. (1.8), which is a solution of the following elliptic equation:

$$\begin{cases} d \frac{\partial^2 u(x)}{\partial x^2} + \lambda u(x) F \left( u(x), \int_0^\infty \int_0^\pi G(x, y, s) f(s) u(y) dy ds \right) = 0, & x \in (0, \pi), \\ u(x) = 0, & x = 0, \pi. \end{cases} \tag{2.1}$$

It is well-known that the following decompositions are satisfied:

$$X = \mathcal{N} \left( d \frac{\partial^2}{\partial x^2} + d \right) \oplus X_1, \quad Y = \mathcal{N} \left( d \frac{\partial^2}{\partial x^2} + d \right) \oplus Y_1,$$

where

$$\mathcal{N} \left( d \frac{\partial^2}{\partial x^2} + d \right) = \text{span}\{\sin x\}, \quad X_1 = \left\{ y \in X : \int_0^\pi y(x) \sin x dx = 0 \right\},$$

and

$$Y_1 = \mathcal{R} \left( d \frac{\partial^2}{\partial x^2} + d \right) = \left\{ y \in Y : \int_0^\pi y(x) \sin x dx = 0 \right\}.$$

Similar to [4, 18], we also use the implicit function theorem to prove the existence of the positive equilibrium near  $\lambda = d$ .

**Theorem 2.1** *Suppose that  $r_1, r_2$ , and  $M_1$  satisfy*

$$r_1 + r_2 M_1 < 0, \tag{2.2}$$

where  $r_1, r_2$  and  $M_1$  are defined as in Eqs. (1.11), (1.12). Then there exists  $\lambda^* > d$ , and a continuously differentiable mapping  $\lambda \mapsto (\xi_\lambda, \alpha_\lambda)$  from  $[d, \lambda^*]$  to  $X_1 \times \mathbb{R}^+$  such that Eq. (1.8) has a positive equilibrium solution

$$u_\lambda = \alpha_\lambda (\lambda - d) [\sin x + (\lambda - d) \xi_\lambda] \tag{2.3}$$

for  $\lambda \in (d, \lambda^*]$ . Moreover

$$\alpha_d = - \frac{\int_0^\pi \sin^2 x dx}{d (r_1 + r_2 M_1) \int_0^\pi \sin^3 x dx}, \tag{2.4}$$

and  $\xi_d \in X_1$  is the unique solution of the equation

$$\left( d \frac{\partial^2}{\partial x^2} + d \right) \xi + \sin x [1 + \alpha_d d (r_1 + r_2 M_1) \sin x] = 0. \tag{2.5}$$

*Proof* Since  $r_1, r_2$  and  $M_1$  satisfy Eq. (2.2), we obtain that  $\alpha_d$  is well defined and positive. Because  $d \frac{\partial^2}{\partial x^2} + d$  is bijective from  $X_1$  to  $Y_1$ ,  $\xi_d$  is uniquely defined. Since  $X_1$  is compactly imbedded into  $C^\alpha([0, \pi])$  for  $0 < \alpha < 1$ , we can define a mapping  $m : X_1 \times \mathbb{R}^3 \rightarrow Y$  by

$$m(\xi, \alpha, \lambda) = \left( d \frac{\partial^2}{\partial x^2} + d \right) \xi + \sin x + (\lambda - d) \xi + \lambda [\sin x + (\lambda - d) \xi] m_1(\xi, \alpha, \lambda),$$

where

$$m_1(\xi, \alpha, \lambda) = \begin{cases} \frac{F(u(x), \int_0^\infty \int_0^\pi G(x, y, s) f(s) u(y) dy ds) - 1}{\lambda - d}, & \lambda \neq d, \\ \alpha r_1 \sin x + \alpha r_2 \int_0^\infty \int_0^\pi G(x, y, s) f(s) \sin y dy ds, & \lambda = d, \end{cases} \tag{2.6}$$

for  $u(x) = \alpha(\lambda - d) [\sin x + (\lambda - d)\xi]$ .

Noticing that  $G(x, y, t)$  and  $f(t)$  are defined as in Eq. (1.7) and Eqs. (1.9), (1.10) respectively, we have

$$m(\xi, \alpha, d) = \left( d \frac{\partial^2}{\partial x^2} + d \right) \xi + \sin x + \alpha d (r_1 + r_2 M_1) \sin^2 x.$$

From Eqs. (2.4) and (2.5), we have that  $m(\xi_d, \alpha_d, d) = 0$ . And the Fréchet derivative of  $m$  with respect to  $(\xi, \alpha)$  at  $(\xi_d, \alpha_d, d)$  is:

$$D_{(\xi, \alpha)} m(\xi_d, \alpha_d, d)[\eta, \epsilon] = \left( d \frac{\partial^2}{\partial x^2} + d \right) \eta + \epsilon d (r_1 + r_2 M_1) \sin^2 x.$$

Since  $r_1, r_2$  and  $M_1$  satisfy Eq. (2.2), we see that  $D_{(\xi, \alpha)} m(\xi_d, \alpha_d, d)$  is bijective from  $X_1 \times \mathbb{R}$  to  $Y$ . And the implicit function theorem implies that there exists a  $\lambda^* > d$ , and a continuously differentiable mapping  $\lambda \mapsto (\xi_\lambda, \alpha_\lambda) \in X_1 \times \mathbb{R}^+$  such that

$$m(\xi_\lambda, \alpha_\lambda, \lambda) = 0, \quad \text{for } \lambda \in [d, \lambda^*].$$

Substituting  $u_\lambda(x) = \alpha_\lambda(\lambda - d)[\sin x + (\lambda - d)\xi_\lambda] \in X$  into Eq. (2.1), we find that it satisfies Eq. (2.1). And so the proof is complete. □

It is shown above that the positive spatially nonhomogeneous equilibrium  $u_\lambda$  can bifurcate from the trivial equilibrium, and the equilibrium  $u_\lambda$  depends on the delay kernel  $f(t)$ , which was firstly found in [9] for the food-limited population model. This is different from the case without spatiotemporal kernel  $G(x, y, t)$ , where the steady state solution is independent of the delay kernel  $f(t)$ .

In the following, we will analyze the linear stability of  $u_\lambda$  and always assume that  $r_1, r_2$  and  $M_1$  satisfy Eq. (2.2). Moreover, we assume that  $\lambda \in (d, \lambda^*]$  unless otherwise specified. Firstly we linearize Eq. (1.8) at  $u_\lambda(x)$ :

$$\begin{cases} \frac{\partial v(x, t)}{\partial t} = A(\lambda)v + \lambda u_\lambda p_1^\lambda(x)v(x, t) \\ + \lambda u_\lambda(x) p_2^\lambda(x) \int_0^\infty \int_0^\pi G(x, y, s) f(s) v(y, t - s) dy ds, & x \in (0, \pi), t > 0, \\ v(x, t) = 0, & x = 0, \pi, t > 0. \end{cases} \tag{2.7}$$

Here  $A(\lambda) : \mathcal{D}(A(\lambda)) \rightarrow Y$  is a linear operator defined by

$$A(\lambda) = d\Delta + \lambda F \left( u_\lambda, \int_0^\infty \int_0^\pi G(x, y, s) f(s) u_\lambda(y) dy ds \right),$$

with domain  $\mathcal{D}(A(\lambda)) = X$ , and

$$\begin{aligned} p_1^\lambda(x) &= \frac{\partial F}{\partial x} \left( u_\lambda, \int_0^\infty \int_0^\pi G(x, y, s) f(s) u_\lambda(y) dy ds \right), \\ p_2^\lambda(x) &= \frac{\partial F}{\partial y} \left( u_\lambda, \int_0^\infty \int_0^\pi G(x, y, s) f(s) u_\lambda(y) dy ds \right). \end{aligned} \tag{2.8}$$

From [9, 21], we see that  $\mu$  is an eigenvalue of system (2.7) if and only if  $\mu \in S(\lambda)$ . Here the set  $S(\lambda)$  is defined by

$$S(\lambda) = \{\mu \in \mathbb{C} : \Delta(\lambda, \mu)\psi = 0, \text{ for some } \psi \in X_{\mathbb{C}} \setminus \{0\}\},$$

where

$$\begin{aligned} \Delta(\lambda, \mu)\psi := & A(\lambda)\psi + \lambda u_{\lambda} p_1^{\lambda}(x)\psi \\ & + \lambda u_{\lambda} p_2^{\lambda}(x) \int_0^{\infty} \int_0^{\pi} G(x, y, s) f(s)\psi(y)e^{-\mu s} dy ds - \mu\psi. \end{aligned}$$

In the following we will give a profile of the set  $S(\lambda)$  to determine the stability of the steady state solution  $u_{\lambda}$ . In Proposition 2.9 of [6], the authors dealt with the stability of a positive equilibrium in a reaction–diffusion equation with a nonlocal reaction term. And we find that the method can also be used here to deal with the stability of a positive equilibrium in a reaction–diffusion equation with spatiotemporal delay.

**Theorem 2.2** *Suppose that  $r_1, r_2$  and  $M_1$  satisfy Eq. (2.2). Then there exists  $\bar{\lambda} > d$ , where  $\bar{\lambda} \leq \lambda^*$ , such that for any  $\lambda \in (d, \bar{\lambda}]$ ,*

$$S(\lambda) \subset \{x + iy : x, y \in \mathbb{R}, x < 0\}.$$

*Proof* To the contrary, there exists a sequence  $\{\lambda^n\}_{n=1}^{\infty}$ , such that  $\lambda^n > d$  for  $n \geq 1$ ,  $\lim_{n \rightarrow \infty} \lambda^n = d$ , and for any  $n \geq 1$ , the corresponding eigenvalue problem

$$\begin{cases} A(\lambda^n)\psi + \lambda^n u_{\lambda^n} p_1^{\lambda^n}(x)\psi \\ + \lambda^n u_{\lambda^n} p_2^{\lambda^n}(x) \int_0^{\infty} \int_0^{\pi} G(x, y, s) f(s)\psi(y)e^{-\mu s} dy ds = \mu\psi, & x \in (0, \pi), \\ \psi(x) = 0, & x = 0, \pi, \end{cases} \quad (2.9)$$

has an eigenvalue  $\mu_{\lambda^n}$  with nonnegative real part. And without loss of generality, we assume the associated eigenfunction  $\psi_{\lambda^n}$  with respect to  $\mu_{\lambda^n}$  satisfies  $\|\psi_{\lambda^n}\|_{Y_{\mathbb{C}}} = 1$ . For each  $n \geq 1$ ,  $\psi_{\lambda^n}$  can be decomposed as  $\psi_{\lambda^n} = c_{\lambda^n} u_{\lambda^n} + \phi_{\lambda^n}$ , where  $c_{\lambda^n} \in \mathbb{C}$ ,  $u_{\lambda^n}$  is the positive solution of Eq. (1.8) for  $\lambda = \lambda^n$  and satisfies Eq. (2.3), and  $\phi_{\lambda^n} \in X_{\mathbb{C}}$  satisfies  $\langle \phi_{\lambda^n}, u_{\lambda^n} \rangle = 0$ . Since

$$A(\lambda^n)u_{\lambda^n} = 0, \text{ and } \langle A(\lambda^n)\phi_{\lambda^n}, u_{\lambda^n} \rangle = \langle \phi_{\lambda^n}, A(\lambda^n)u_{\lambda^n} \rangle,$$

substituting  $\mu = \mu_{\lambda^n}$  and  $\psi = \psi_{\lambda^n} = c_{\lambda^n} u_{\lambda^n} + \phi_{\lambda^n}$  into the first equation of Eq. (2.9), multiplying  $\psi_{\lambda^n} = c_{\lambda^n} u_{\lambda^n} + \phi_{\lambda^n}$  and integrating, we obtain that

$$\langle A(\lambda^n)\phi_{\lambda^n}, \phi_{\lambda^n} \rangle = -T_{\lambda^n} + \mu_{\lambda^n}, \quad (2.10)$$

where

$$\begin{aligned} T_{\lambda^n} = & \lambda^n \left\langle \psi_{\lambda^n}, u_{\lambda^n} p_1^{\lambda^n}(x)\psi_{\lambda^n} \right\rangle \\ & + \lambda^n \left\langle \psi_{\lambda^n}, u_{\lambda^n} p_2^{\lambda^n}(x) \int_0^{\infty} \int_0^{\pi} G(x, y, s) f(s)\psi_{\lambda^n}(y)e^{-\mu_{\lambda^n} s} dy ds \right\rangle. \end{aligned}$$

For  $\lambda \in (d, \lambda^*]$ ,  $u_{\lambda}(x)$  is bounded which implies that  $\|p_i^{\lambda}(x)\|_{\infty} < E$  ( $i = 1, 2$ ) for some  $E > 0$ . Noticing that  $\mu_{\lambda^n}$  has nonnegative real part, we have that

$$\begin{aligned} |T_{\lambda^n}| \leq & E \|u_{\lambda^n}\|_{\infty} \max_n \lambda^n + \frac{2E \max_n \lambda^n}{\pi} \|u_{\lambda^n}\|_{\infty} \|\psi\|_{L^1}^2 \sum_{k=1}^{\infty} \int_0^{\infty} e^{-dk^2 s} f(s) ds \\ \leq & E \|u_{\lambda^n}\|_{\infty} \max_n \lambda^n + 2E \max_n \lambda^n \|u_{\lambda^n}\|_{\infty} \sum_{k=1}^{\infty} M_k. \end{aligned} \quad (2.11)$$

Since  $f(t)$  is defined as in Eqs. (1.9), (1.10), we have  $\sum_{k=1}^{\infty} M_k < \infty$ , which implies that  $\lim_{n \rightarrow \infty} |T_{\lambda^n}| = 0$ . Since  $u_{\lambda^n}$  is the principal eigenfunction of  $A(\lambda^n)$  with principal eigenvalue 0, we see that  $\langle A(\lambda^n)\phi_{\lambda^n}, \phi_{\lambda^n} \rangle \leq 0$ , which implies

$$0 \leq \operatorname{Re}(\mu_{\lambda^n}) \leq \operatorname{Re}T_{\lambda^n} \leq |T_{\lambda^n}|, \tag{2.12}$$

and hence  $\lim_{n \rightarrow \infty} \operatorname{Re}(\mu_{\lambda^n}) = 0$ . Similarly, we have

$$|\operatorname{Im}(\mu_{\lambda^n})| = |\operatorname{Im}T_{\lambda^n}| \leq |T_{\lambda^n}|, \tag{2.13}$$

which implies  $\lim_{n \rightarrow \infty} \operatorname{Im}(\mu_{\lambda^n}) = 0$ . Since

$$|\langle A(\lambda^n)\phi_{\lambda^n}, \phi_{\lambda^n} \rangle| \geq |\lambda_2(\lambda^n)| \cdot \|\phi_{\lambda^n}\|_{Y_C}^2,$$

where  $\lambda_2(\lambda^n)$  is the second eigenvalue of  $A(\lambda^n)$ , we have

$$|\lambda_2(\lambda^n)| \cdot \|\phi_{\lambda^n}\|_{Y_C}^2 \leq |T_{\lambda^n}| + |\mu_{\lambda^n}|. \tag{2.14}$$

And the continuity of the eigenvalues of  $A(\lambda)$  with respect to  $\lambda$  implies that

$$\lim_{n \rightarrow \infty} \lambda_2(\lambda^n) = -\lambda_2 + d < 0,$$

where  $\lambda_2 = -4d$  is the second eigenvalue of  $-d \frac{\partial^2}{\partial x^2}$ . Noticing that  $\lim_{n \rightarrow \infty} |T_{\lambda^n}| = \lim_{n \rightarrow \infty} |\mu_{\lambda^n}| = 0$ , from Eq. (2.14) we have that  $\lim_{n \rightarrow \infty} \|\phi_{\lambda^n}\|_{Y_C} = 0$ . Because of  $\psi_{\lambda^n} = c_{\lambda^n}u_{\lambda^n} + \phi_{\lambda^n}$  and  $\|\psi_{\lambda^n}\|_{L^2} = 1$ , we see that

$$\lim_{n \rightarrow \infty} |c_{\lambda^n}|(\lambda^n - d) \lim_{n \rightarrow \infty} \left\| \frac{u_{\lambda^n}}{\lambda^n - d} \right\|_{Y_C} = 1,$$

which implies  $\lim_{n \rightarrow \infty} |c_{\lambda^n}|(\lambda^n - d) > 0$ . We denote

$$\begin{aligned} T_{\lambda^n}^1 &= \lambda^n \left\langle \psi_{\lambda^n}, u_{\lambda^n} p_1^{\lambda^n}(x) \psi_{\lambda^n} \right\rangle, \quad \text{and} \\ T_{\lambda^n}^2 &= \lambda^n \left\langle \psi_{\lambda^n}, u_{\lambda^n} p_2^{\lambda^n}(x) \int_0^\infty \int_0^\pi G(x, y, s) f(s) \psi_{\lambda^n}(y) e^{-\mu_{\lambda^n} s} dy ds \right\rangle, \end{aligned}$$

and then  $T_{\lambda^n} = T_{\lambda^n}^1 + T_{\lambda^n}^2$ . We first calculate that:

$$\begin{aligned} \frac{T_{\lambda^n}^2}{\lambda^n - d} &= |c_{\lambda^n}|^2 (\lambda^n - d)^2 \lambda^n \gamma_1^n + c_{\lambda^n} (\lambda^n - d) \lambda^n \gamma_2^n \\ &\quad + \overline{c_{\lambda^n}} (\lambda^n - d) \lambda^n \gamma_3^n + \lambda^n \gamma_4^n, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} \gamma_1^n &= \int_0^\pi \int_0^\infty \int_0^\pi G(x, y, s) f(s) e^{-\mu_{\lambda^n} s} p_2^{\lambda^n}(x) \frac{u_{\lambda^n}^2(x) u_{\lambda^n}(y)}{(\lambda^n - d)^3} dy ds dx, \\ \gamma_2^n &= \int_0^\pi \int_0^\infty \int_0^\pi G(x, y, s) f(s) e^{-\mu_{\lambda^n} s} p_2^{\lambda^n}(x) \frac{\overline{\phi_{\lambda^n}}(x) u_{\lambda^n}(x) u_{\lambda^n}(y)}{(\lambda^n - d)^2} dy ds dx, \\ \gamma_3^n &= \int_0^\pi \int_0^\infty \int_0^\pi G(x, y, s) f(s) e^{-\mu_{\lambda^n} s} p_2^{\lambda^n}(x) \frac{\phi_{\lambda^n}(y) u_{\lambda^n}^2(x)}{(\lambda^n - d)^2} dy ds dx, \\ \gamma_4^n &= \int_0^\pi \int_0^\infty \int_0^\pi G(x, y, s) f(s) e^{-\mu_{\lambda^n} s} p_2^{\lambda^n}(x) \frac{\overline{\phi_{\lambda^n}}(x) \phi_{\lambda^n}(y) u_{\lambda^n}(x)}{\lambda^n - d} dy ds dx. \end{aligned} \tag{2.16}$$

Since  $\lim_{n \rightarrow \infty} \|\phi_{\lambda^n}\|_{Y_C} = 0$ , we have  $\lim_{n \rightarrow \infty} \|\phi_{\lambda^n}\|_{L^1} = 0$ , which implies that each of  $\gamma_i^n (i = 2, 3, 4)$  goes to zero as  $n \rightarrow \infty$ . Since  $G(x, y, t)$  and  $f(t)$  are defined as in Eq. (1.7) and Eqs. (1.9), (1.10) respectively,  $\mu_{\lambda^n}$  has nonnegative real parts with  $\lim_{n \rightarrow \infty} |\mu_{\lambda^n}| = 0$ , and  $\lim_{n \rightarrow \infty} p_2^{\lambda^n}(x) = r_1$  uniformly, we have

$$\lim_{n \rightarrow \infty} \gamma_1^n = r_2 \alpha_d^3 M_1 \int_0^\pi \sin^3 x dx.$$

Then we have

$$\begin{aligned} \frac{T_{\lambda^n}^1}{\lambda^n - d} &= |c_{\lambda^n}|^2 (\lambda^n - d)^2 \lambda^n \int_0^\pi \frac{p_1^{\lambda^n}(x) u_{\lambda^n}^3(x)}{(\lambda^n - d)^3} dx \\ &+ c_{\lambda^n} (\lambda^n - d) \lambda^n \int_0^\pi \frac{\overline{\phi_{\lambda^n}}(x) u_{\lambda^n}^2(x) p_1^{\lambda^n}(x)}{(\lambda^n - d)^2} dx \\ &+ \overline{c_{\lambda^n}} (\lambda^n - d) \lambda^n \int_0^\pi \frac{p_1^{\lambda^n}(x) \phi_{\lambda^n}(x) u_{\lambda^n}^2(x)}{(\lambda^n - d)^2} dx \\ &+ \lambda^n \int_0^\pi \frac{p_1^{\lambda^n}(x) |\phi_{\lambda^n}(x)|^2 u_{\lambda^n}(x)}{\lambda^n - d} dx. \end{aligned} \tag{2.17}$$

Similarly, since  $\lim_{n \rightarrow \infty} \|\phi_{\lambda^n}\|_{Y_C} = 0$ , we have that each of the last three terms of Eq. (2.17) goes to zero, and

$$\lim_{n \rightarrow \infty} \int_0^\pi \frac{p_1^{\lambda^n}(x) u_{\lambda^n}^3(x)}{(\lambda^n - d)^3} dx = r_1 \alpha_d^3 \int_0^\pi \sin^3 x dx.$$

So

$$\lim_{n \rightarrow \infty} \frac{T_{\lambda^n}}{\lambda^n - d} = \alpha_d^3 (dr_1 + dr_2 M_1) \int_0^\pi \sin^3 x dx \lim_{n \rightarrow \infty} |c_{\lambda^n}|^2 (\lambda^n - d)^2 < 0.$$

Therefore, for sufficiently large  $n$ ,  $\text{Re}(T_{\lambda^n}) < 0$ , and consequently,

$$\text{Re}(\mu_{\lambda^n}) = \langle A(\lambda^n) \phi_{\lambda^n}, \phi_{\lambda^n} \rangle + \text{Re}(T_{\lambda^n}) < 0, \tag{2.18}$$

which is a contradiction with  $\text{Re}(\mu_{\lambda^n}) \geq 0$  for  $n \geq 1$ . Hence we have

$$S(\lambda) \subset \{x + iy : x, y \in \mathbb{R}, x < 0\}.$$

And the proof is complete. □

From Theorem 2.2, we can easily have the local stability of the positive steady state  $u_\lambda$ :

**Theorem 2.3** *Suppose that  $r_1, r_2$  and  $M_1$  satisfy Eq. (2.2). Then there exists  $\bar{\lambda} > d$ , where  $\bar{\lambda} \leq \lambda^*$ , such that for any  $\lambda \in (d, \bar{\lambda}]$ , the positive steady state  $u_\lambda$  of Eq. (1.8) is locally asymptotically stable.*

We remark that for the Gamma distribution delay kernel, from Theorem 2.3, we have that:

**Proposition 2.4** *Assume that  $f(t)$  is given by Eq. (1.9). Then for any fixed  $\beta$  and  $m$  satisfying*

$$r_1 + \frac{r_2}{(d\beta + 1)^m} < 0,$$

*there exists  $\bar{\lambda} > d$  such that for any  $\lambda \in (d, \bar{\lambda}]$ , Eq. (3.1) has a positive steady state  $u_\lambda$ . Moreover,  $u_\lambda$  is locally asymptotically stable for  $\lambda \in (d, \bar{\lambda}]$ .*



And for the Dirac distribution delay kernel, we have that:

**Proposition 2.5** *Assume that  $f(t)$  is given by Eq. (1.10). Then for any fixed  $\tau > 0$  satisfying*

$$r_1 + e^{-d\tau}r_2 < 0,$$

*there exists  $\bar{\lambda} > d$  such that for any  $\lambda \in (d, \bar{\lambda}]$ , Eq. (3.1) has a positive steady state  $u_\lambda$ . Moreover,  $u_\lambda$  is locally asymptotically stable for  $\lambda \in (d, \bar{\lambda}]$ .*

### 3 Applications and Discussions

In this section, we apply Theorem 2.3 to model (1.2) and model (1.4). We first consider model (1.4). In this case we see that  $r_1 = -a - ac < 0$ ,  $r_2 = -b - bc < 0$ , and Eq. (2.2) is satisfied. From the simulation in [9, see Figs. 4, 5], we see that the bifurcated positive steady state may be stable in some conditions. Actually, from Theorem 2.3, we have that the bifurcated steady state solution is stable near  $\lambda = d$ , and this result supplements the work of Gourley and So [9] for Dirichlet boundary problem.

**Theorem 3.1** *Assume that  $f(t)$  is given by Eq. (1.9) or Eq. (1.10). Then there exists  $\bar{\lambda} > d$  such that for any  $\lambda \in (d, \bar{\lambda}]$ , Eq. (1.4) has a positive steady state  $u_\lambda$ . Moreover,  $u_\lambda$  is locally asymptotically stable for  $\lambda \in (d, \bar{\lambda}]$ .*

Then we consider model (1.2) for one dimensional domain  $\Omega = (0, \pi)$  and  $\alpha = 0$ . Following [9], we choose  $g(x, y, s) = G(x, y, s)f(s)$ , where  $G$  is defined as in Eq. (1.7). Then for the homogeneous Dirichlet boundary condition, Eq. (1.2) has the following form:

$$\begin{cases} \frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} \\ \quad + \lambda u \left( 1 - \int_0^\infty \int_0^\pi G(x, y, s) f(s) u(y, t - s) dy ds \right), & x \in (0, \pi), t > 0, \\ u(x, t) = 0, & x = 0, \pi, t > 0. \end{cases} \quad (3.1)$$

In this case  $r_1 = 0$ ,  $r_2 = -1$  and Eq. (2.2) is also satisfied. Then we have:

**Theorem 3.2** *Assume that  $f(t)$  is given by Eq. (1.9) or Eq. (1.10). Then there exists  $\bar{\lambda} > d$  such that for any  $\lambda \in (d, \bar{\lambda}]$ , Eq. (3.1) has a positive steady state  $u_\lambda$ . Moreover,  $u_\lambda$  is locally asymptotically stable for  $\lambda \in (d, \bar{\lambda}]$ .*

We remark that if the delay kernel  $f(t)$  is given by Eq. (1.9), then for any fixed  $\beta$  and  $m$ , the bifurcated positive equilibrium of Eq. (1.4) is stable near  $\lambda = d$  and Hopf bifurcation cannot occur. For a delayed differential equations with a Gamma distribution delay kernel, the effect of  $\beta$  and  $m$  on the stability and bifurcations of the equilibrium has been investigated [5, 13]. However, when  $f(s)$  is given by Eq. (1.9), we see that from Theorem 2.1, the bifurcated equilibrium of Eq. (1.4) depends on  $\beta$  and  $m$ . Hence it is difficult to study the effect of  $\beta$  and  $m$  on the stability of the bifurcated equilibrium. Moreover, if the delay kernel  $f(t)$  is given by Eq. (1.10), then for any fixed  $\tau > 0$ , the bifurcated positive equilibrium of Eq. (1.1) is stable near  $\lambda = d$  and Hopf bifurcation cannot occur. We remark that for local delay effect, where  $G(x, y, s) = \delta(x - y)$  and  $f(s) = \delta(s - \tau)$ , Busenberg and Huang [4] have showed that, when  $\lambda > d$  but close to  $d$ , a large delay  $\tau$  can make the unique spatially nonhomogeneous positive equilibrium of Eq. (1.1) unstable through a Hopf bifurcation. However, if  $G(x, y, t)$  is defined as in Eq. (1.7) and  $f(t) = \delta(t - \tau)$ , the bifurcated steady state solution of Eq. (1.1) depends on  $\tau$ , and hence it is also difficult to study the effect of  $\tau$  on the stability of the bifurcated equilibrium. These all await future investigation.

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