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Summability of Transseries Solution of Non-integrable Hamiltonian System

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Abstract
This paper studies the summability of the transseries solution of a nonintegrable Hamiltonian system. Since our system has a resonance and is not integrable a general transseries theory does not work well as far as the author knows. In order to construct a formal transseries solution and prove its summability our main idea is to use the superintegrability of a Hamiltonian system in a class of transseries. More precisely we first show the superintegrability of a Hamiltonian system in the category of transseries via the key Lemmas 1 and 4 which follow. By virtue of the superintegrability we show the existence of a formal transseries solution. Then its summability is proved via the superintegrability. We note that the argument based on the superintegrability is elementary.

Keywords Nonintegrability · Hamiltonian system · Transseries · Superintegrability

Mathematics Subject Classification (2010) 35C10 · 45E10

1 Introduction

Let $(q_1, q_2, q_3, \cdots, q_n)$, $(p_1, p_2, p_3, \cdots, p_n)$ be the variables in some neighborhood of the origin of $\mathbb{R}^{2n}$ or $\mathbb{C}^{2n}$ $(n \geq 1)$ with respect to a standard symplectic form. Let $\mathbb{Z}_+$ be the set of nonnegative integers. In this paper we study the summability of the transseries solution of the C^∞-integrable and C^ω-nonintegrable Hamiltonian system studied by Bolsinov, Taimanov, Gorni and Zampieri. (cf. [3] and [6]). Consider the Hamitonian system with n degrees of freedom

$$\dot{q}_j = \nabla_{p_j} H, \quad \dot{p}_j = -\nabla_{q_j} H, \quad j = 1, 2, \ldots, n, \tag{1}$$

for the Hamiltonian H whose precise definition is given in the next section.

Starting from the pioneering work of Ecalle, [5] a transseries attracts more attention in mathematical physics and dynamical systems. (See also [7, 8] and the references therein). It

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is also noted in [4] that a transseries solution gives a general solution. In order to construct a transseries solution we first construct a formal transseries solution. Then the summability theory gives the solution of the equation via the sum of a formal transseries.

In this paper we give the rigorous proof of the above assertion by using a superintegrability. More precisely we first show a superintegrability in the category of transseries. By virtue of the superintegrability one can construct a formal transseries solution and show its summability. We consider a transserial first integral since the nonintegrability in the analytic category implies that Eq. 1 does not have n C^{ω}-first integrals. On the other hand, a superintegrability holds in the category of transseries. Namely, we can construct $2n-1$ functionally independent first integrals of Eq. 1 containing transseries. The idea of a transserial superintegrability for a nonintegrable Hamiltonian system seems natural although no rigorous proof seems to be known.

This paper is organized as follows. In Section 2 we formulate the existence of a formal transseries solution as Theorem 1. In Section 3 we prove a superintegrability in the category of formal transseries. In Section 4 we prove Theorem 1 by virtue of the superintegrability. In Section 5 we state the summability of the formal transseries solution constructed in Theorem 1 as Theorem 5. In Section 6 we prove the summability of formal first integrals and show the superintegrability. We prove Theorem 5 in Section 7.

2 Formal Transseries Solution

We set $q = (q_2, \dots, q_n)$ and write $(q_1, q_2, \dots, q_n) = (q_1, q)$. Similarly, we write $(p_1, p_2, \dots, p_n) = (p_1, p)$, where $p = (p_2, \dots, p_n)$. Let $H = H(q_1, p_1, q, p)$ be a smooth function. Define the Hamiltonian vector field χ_H by

$$\chi_H := \{H, \cdot\} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right), \tag{2}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. We say that ϕ is the first integral of χ_H if $\chi_H \phi = 0$. Let $\Omega \subset \mathbb{R}^{2n}$ or $\Omega \subset \mathbb{C}^{2n}$ be an open set. Let $C^{\omega} \equiv C^{\omega}(\Omega)$ be the set of functions analytic in Ω. We say that χ_H is C^{ω}-Liouville integrable in Ω if there exist first integrals $\phi_j \in C^{\omega}(\Omega)$ $(j = 1, 2, \dots, n)$ which are Poisson commuting, i.e., $\{\phi_j, \phi_k\} = 0$, $\{H, \phi_k\} = 0$, and that are functionally independent on an open set $\Omega' \subset \Omega$ with Ω' being dense in Ω. If $\phi_j \in C^{\infty}$ $(j = 1, 2, \dots, n)$, then we say C^{∞}-Liouville integrable. We say that χ_H is superintegrable if χ_H has $2n-1$ functionally independent first integrals on an open set. We say that v is the formal first integral of χ_H if $\chi_H v = 0$ as a formal power series.

Let H_0 and H_1 be given, respectively, by

$$H_0 = q_1^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j, \tag{3}$$

$$H_1 = \sum_{j=2}^{n} q_j^2 B_j(q_1, q_1^{2\sigma} p_1, q), \tag{4}$$

where $B_j(q_1, s, q)$'s are holomorphic in some neighborhood of the origin and $\sigma \geq 1$ is an integer. Define $H := H_0 + H_1$. Assume

$$B_{\nu} \equiv B_{\nu}(q_1, q_1^{2\sigma} p_1, q) = B_{\nu,0}(q_1, q) + q_1^{2\sigma} p_1 B_{\nu,1}(q_1, q), \quad \nu = 2, \dots, n, \tag{5}$$

where $B_{\nu,0}$ and $B_{\nu,1}$ are analytic at $(q_1, q) = (0, 0)$. Suppose that the Poincaré condition holds

$$Re\,\lambda_j > 0, \quad j = 2, 3, \ldots, n. \tag{6}$$

Assume the nonresonance condition

$$\sum_{\nu=2}^{n} \lambda_\nu k_\nu - \lambda_j \neq 0, \quad \forall\, k_\nu \in \mathbb{Z}_+,\ \nu = 2, \ldots, n,\ j = 2, \ldots, n. \tag{7}$$

Set $\lambda = (\lambda_2, \ldots, \lambda_n)$. Consider the formal power series solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ whose component has the following form

$$\sum_{k \geq k_0,\ \ell \geq \ell_0} c_{k,\ell} t^{-\frac{\ell}{2\sigma-1}} e^{\lambda k t}, \tag{8}$$

where $k = (k_2, \ldots, k_n)$, $\lambda k = \lambda_2 k_2 + \cdots + \lambda_n k_n$, and where $c_{k,\ell}$'s are complex constants and k_0 is a multiinteger and $\ell_0 \geq 0$ is an integer. The series Eq. 8 is called a transseries. As for the general definition of a transseries and its property we refer to Ecalle [5], Costin, [4] and Kuik, [7] and the references therein.

Our first aim is to construct the formal transseries solution of Eq. 1. In [1] a formal transseries solution similar to Eq. 8 is constructed for the first order system of ordinary differential equations under the nonresonance condition. See also the related works [4] and [7]. Now, we have

Theorem 1 *Suppose that Eqs. 5, 6 and 7 are satisfied. Then there exists a neighborhood of* $t = \infty$*,* Ω_1 *such that there exists a formal transseries solution* $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ *of Eq. 1 at every point of* $\{t | Re(\lambda_j t) < 0, j = 2, \ldots, n\} \cap \Omega_1$.

Example 1 The following Hamiltonian is the local counterpart of the Hamiltonian studied by Taimanov related with the nonintegrability of a geodesic flow (cf. [3, 6])

$$H_1 := c q_1^{4\sigma} p_1^2 + \sum_{j=2}^{n} B_j(q_1) p_j^2, \tag{9}$$

where c is a constant and $B_j(q_1)$ is an analytic function in some neighborhood of $q_1 = 0$. For H_0 in Eq. 3, we define $H := H_0 + H_1$. χ_H is C^∞-Liouville integrable at the origin. (cf. [10]). We know that it is not C^ω-Liouville integrable under a certain condition. (cf. [9]).

In this paper we study the superintegrability and the solvability of a Hamiltonian system including H in the category of transseries. In the next section we show that the system is superintegrable in the class of transseries. By virtue of the superintegrability we prove the solvability of a Hamiltonian system in the class of transseries. As for the property of the solution we see from Theorem 1 that the solution has a formal transseries expansion. Moreover, the formal transseries is a true solution in a certain sector since it is summable by Theorem 5 which follows. The study of the global behavior of the summed transseries solution is left as a future problem.

3 Superintegrability in the Category of Transseries

3.1 Formal First Integral and Superintegrability

In order to prove Theorem 1 we first show the superintegrability of Eq. 1 in the category of formal transseries. For this purpose, we first construct $2n - 1$ functionally independent first integrals. We recall that the formal power series v is said to be the formal first integral of the Hamiltonian vector field χ_H if $\chi_H v = 0$ as a formal power series. As for the precise definition of functionally independent first integrals in a formal series we need to introduce some notation. For $c \in \mathbb{C}$ and $\alpha = (\alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^{n-1}$, define

$$E_c \equiv E_c(q_1) = \exp\left(\frac{cq_1^{-2\sigma+1}}{2\sigma - 1}\right), \quad E^\alpha = E_{\lambda_2}^{\alpha_2} \cdots E_{\lambda_n}^{\alpha_n}. \tag{10}$$

We denote by e_j the j-th unit vector, $e_j = (0, \cdots, 1, \cdots 0)$, $j = 2, 3, \ldots, n$.

We construct the first integral v of χ_H given by

$$v = \phi^{(\alpha)}(q_1, p_1, q, p)E^\alpha, \tag{11}$$

where $\phi^{(\alpha)}(q_1, p_1, q, p)$ is a formal power series in q_1, q, p_1 and p of the following form.

i) Case $\alpha = 0$. We have

$$\phi^{(0)} \equiv \phi_j^{(0)} = p_j q_j + U_{0,j} + q_1^{2\sigma} p_1 U_{1,j}, \quad j = 2, \ldots, n, \tag{12}$$

where

$$U_{0,j} = U_{0,j}(q_1, q) = \sum_{\nu=0}^{\infty} U_{0,j,\nu}(q) q_1^\nu, \tag{13}$$

$$U_{1,j} = U_{1,j}(q_1, q) = \sum_{\nu=0}^{\infty} U_{1,j,\nu}(q) q_1^\nu, \tag{14}$$

are formal power series in q_1 with coefficients analytic in q.

ii) Case $\alpha = e_j$, $(2 \le j \le n)$. We have

$$\phi^{(e_j)} = p_j q_j^2 (1 + U_{2,j}) + U_{0,j} + q_1^{2\sigma} p_1 U_{1,j}, \quad j = 2, \ldots, n, \tag{15}$$

where $U_{0,j}$, $U_{1,j}$ and $U_{2,j}$ are formal power series in q_1 with coefficients analytic in q. Here the formal power series $U_{0,j}$ and $U_{1,j}$ may not coincide with the power series in i), respectively.

iii) Case $\alpha = -e_j$, $(2 \le j \le n)$. We have

$$\phi^{(-e_j)} = p_j (1 + U_{2,j}) + U_{0,j} + q_1^{2\sigma} p_1 U_{1,j}, \quad j = 2, \ldots, n, \tag{16}$$

where $U_{0,j}$, $U_{1,j}$ and $U_{2,j}$ are formal power series in q_1 with coefficients analytic in q. Here $U_{0,j}$, $U_{1,j}$ and $U_{2,j}$ may not coincide with the power series in ii), respectively. And $U_{0,j}$ and $U_{1,j}$ may not coincide with the ones in i), respectively.

Definition 1 We say that v in Eq. 11 is the formal first integral of χ_H if the following conditions are satisfied.

(i) $\chi_H v = 0$ as a formal power series.

(ii) If $\alpha = 0$, then $\phi^{(0)} \equiv \phi_j^{(0)}$, $(j = 2, \ldots, n)$ satisfies Eqs. 12, 13 and 14 with $U_{0,j,\nu}(q)$'s and $U_{1,j,\nu}(q)$'s analytic in some neighborhood of the origin $q = 0$ independent of ν and j. If $\alpha = e_j$ (resp. $\alpha = -e_j$), $(j = 2, \ldots, n)$, then $\phi^{(\alpha)}$ has the form Eq. 15 (resp. Eq. 16), with $U_{0,j}$'s, $U_{1,j}$'s and $U_{2,j}$'s satisfying the same conditions as the case $\alpha = 0$.

We say that the formal series $U_{0,j}$ in Eq. 13 is Gevrey of order s (in short, s- Gevrey), for some $s \geq 0$, if there exist a neighborhood of the origin $q = 0$, Ω_0 and constants $C > 0$, $K > 0$ for which

$$\sup_{q \in \Omega_0} |U_{0,j,\nu}(q)| \leq C K^{\nu} \Gamma(1 + s\nu),$$

hold for all $\nu \geq 0$, where Γ denotes the Gamma function. If both $U_{0,j}$ and $U_{1,j}$ are s-Gevrey, then we say that $\phi_j^{(0)}$ is s-Gevrey. We say that $\phi^{(e_j)}$ (resp. $\phi^{(-e_j)}$) is s-Gevrey if $U_{0,j}$, $U_{1,j}$ and $U_{2,j}$ are s-Gevrey.

Then we have

Theorem 2 *Assume Eqs. 5, 6 and 7. Then χ_H has the formal first integrals, $\phi_j^{(0)}$, $\phi^{(e_j)} E^{e_j}$ and $\phi^{(-e_j)} E^{-e_j}$, $(j = 2, \ldots, n)$, which are $(2\sigma - 1)$- Gevrey. If $\alpha = 0$, then we have $U_{0,j} = O(|q|^2)$ and $U_{1,j} = O(|q|^2)$ as $q \to 0$. If $\alpha = e_j$, then we have $U_{0,j} = O(|q|^3)$, $U_{1,j} = O(|q|^3)$ and $U_{2,j} = O(|q|^2)$ as $q \to 0$. If $\alpha = -e_j$, then we have $U_{0,j} = O(|q|^2)$, $U_{1,j} = O(|q|^2)$ and $U_{2,j} = O(|q|)$ as $q \to 0$.*

Consider the system of formal first integrals $v_j^{(\alpha)} = v_{j,0}^{(\alpha)} E^{\alpha} + \tilde{v_j^{(\alpha)}} E^{\alpha}$, H $(\alpha = 0, -e_j; j = 2, \ldots, n-1)$ with $v_{j,0}^{(\alpha)}$ being analytic and independent of q_1 and $\tilde{v_j^{(\alpha)}} = O(q_1)$. We say that the system of functions H, $v_j^{(\alpha)}$ $(\alpha = 0, -e_j; j = 2, \ldots, n-1)$ is functionally independent if the Jacobian of the functions $v_{j,0}^{(\alpha)} E^{\alpha}$ $(\alpha = 0, -e_j; j = 2, \ldots, n-1)$, $H|_{q_1=0}$ does not vanish.

In order to prove Theorem 2 we introduce some notation. By definition we have, for $\mathcal{L} := \{H_0, \cdot\}$ and $R := \{H_1, \cdot\}$,

$$\mathcal{L} = q_1^{2\sigma} \frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1} p_1 \frac{\partial}{\partial p_1} + \sum_{j=2}^{n} \lambda_j \left(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right), \tag{17}$$

$$R = \sum_{j=2}^{n} \left(-2q_j B_j \frac{\partial}{\partial p_j} + q_j^2 (\partial_{p_1} B_j) \frac{\partial}{\partial q_1} - q_j^2 (\partial_{q_1} B_j) \frac{\partial}{\partial p_1} - q_j^2 \nabla_q B_j \cdot \frac{\partial}{\partial p} \right). \tag{18}$$

By using the formula

$$\partial_{p_1} B_j = B_{j,1} q_1^{2\sigma}, \quad q_1^{2\sigma} (\partial/\partial q_1) E^{\alpha} = -(\sum_{j=2}^{n} \lambda_j \alpha_j) E^{\alpha} = -\langle \lambda, \alpha \rangle E^{\alpha},$$

we have

$$\mathcal{L}(\phi^{(\alpha)} E^{\alpha}) = E^{\alpha} \left(q_1^{2\sigma} \frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1} p_1 \frac{\partial}{\partial p_1} + \sum_{j=2}^{n} \lambda_j \left(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} - \alpha_j \right) \right) \phi^{(\alpha)}, \tag{19}$$

and

$$R(\phi^{(\alpha)} E^{\alpha}) = E^{\alpha} \left(-\langle \lambda, \alpha \rangle \sum_{j=2}^{n} q_j^2 B_{j,1} + R \right) \phi^{(\alpha)}. \tag{20}$$

It follows that if $v = E^{\alpha}\phi^{(\alpha)}$ is the first integral of χ_H, then $\phi^{(\alpha)}$ satisfies

$$\left(q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}p_1\frac{\partial}{\partial p_1} + \sum_{j=2}^{n}\lambda_j\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j} - \alpha_j\right)\right)\phi^{(\alpha)} + \left(-\sum_{j=2}^{n}\langle\lambda,\alpha\rangle q_j^2 B_{j,1} + R\right)\phi^{(\alpha)} = 0. \tag{21}$$

Let $\alpha = 0$. We look for the equations which $U_{0,j}$ and $U_{1,j}$ in Eq. 12 satisfy. Substitute Eqs. 12 and 5 into Eq. 21. For the sake of simplicity, we set $U_0 := U_{0,j}$ and $U_1 := U_{1,j}$. We consider the terms which appear from $p_j q_j$ in $\phi^{(\alpha)}$. The nonvanishing terms appearing from $R(p_j q_j)$ are given by

$$-2q_j^2 B_j - \sum_{\nu=2}^{n} q_\nu^2(\partial_{q_j}B_\nu)q_j = -2q_j^2(B_{j,0}+q_1^{2\sigma}p_1 B_{j,1}) - \sum_{\nu=2}^{n} q_\nu^2 q_j\partial_{q_j}(B_{\nu,0}+q_1^{2\sigma}p_1 B_{\nu,1}). \tag{22}$$

Consider the terms which contain p_1. The terms containing p_1 which appear from the differentiation $\mathcal{L}$ in Eq. 21 are given by

$$q_1^{2\sigma}p_1\left(q_1^{2\sigma}\frac{\partial U_1}{\partial q_1} + 2\sigma q_1^{2\sigma-1}U_1 - 2\sigma q_1^{2\sigma-1}U_1 + \sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_1}{\partial q_\nu}\right). \tag{23}$$

Next, the terms containing p_1 that appear from R are given by the second and the third terms of R:

$$\begin{aligned} &-q_1^{2\sigma}p_1\left(\sum_{\nu=2}^{n} q_\nu^2 q_1^{2\sigma}(\partial_{q_1}B_{\nu,1})U_1 + \sum_{\nu=2}^{n} q_\nu^2 2\sigma q_1^{2\sigma-1}B_{\nu,1}U_1\right) \\ &+q_1^{2\sigma}p_1\left(\sum_{\nu=2}^{n} q_\nu^2 2\sigma q_1^{2\sigma-1}B_{\nu,1}U_1 + \sum_{\nu} q_\nu^2 B_{\nu,1}q_1^{2\sigma}\frac{\partial U_1}{\partial q_1}\right) \\ &= q_1^{2\sigma}p_1\left(-\sum_{\nu=2}^{n} q_\nu^2 q_1^{2\sigma}(\partial_{q_1}B_{\nu,1})U_1 + \sum_{\nu} q_\nu^2 B_{\nu,1}q_1^{2\sigma}\frac{\partial U_1}{\partial q_1}\right). \end{aligned} \tag{24}$$

These expressions give the terms which contain p_1.

Next, we calculate the terms which do not contain p_1 in Eq. 21 in a similar way. By substituting Eq. 12 and 5 into Eq. 21 we have

$$\begin{aligned} &q_1^{2\sigma}p_1\left(q_1^{2\sigma}\frac{\partial U_1}{\partial q_1} + \sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_1}{\partial q_\nu} - \sum_{\nu=2}^{n} q_\nu^2 q_1^{2\sigma}(\partial_{q_1}B_{\nu,1})U_1\right. \\ &\left.+\sum_{\nu} q_\nu^2 B_{\nu,1}q_1^{2\sigma}\frac{\partial U_1}{\partial q_1} - 2q_j^2 B_{j,1} - \sum_{\nu} q_\nu^2 q_j\partial_{q_j}B_{\nu,1}\right) \\ &+q_1^{2\sigma}\frac{\partial U_0}{\partial q_1} + \sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_0}{\partial q_\nu} - \sum_{\nu} q_\nu^2(\partial_{q_1}B_{\nu,0})q_1^{2\sigma}U_1 \\ &+\sum_{\nu} q_\nu^2 q_1^{2\sigma}B_{\nu,1}\frac{\partial U_0}{\partial q_1} - 2q_j^2 B_{j,0} - \sum_{\nu} q_\nu^2(q_j\partial_{q_j}B_{\nu,0}) = 0. \end{aligned} \tag{25}$$

Hence we have

$$q_1^{2\sigma}\frac{\partial U_1}{\partial q_1}+\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_1}{\partial q_\nu}-\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}(\partial_{q_1}B_{\nu,1})U_1 + \sum_{\nu}q_\nu^2 B_{\nu,1}q_1^{2\sigma}\frac{\partial U_1}{\partial q_1}-2q_j^2 B_{j,1}-\sum_{\nu}q_\nu^2 q_j\partial_{q_j}B_{\nu,1}=0. \tag{26}$$

$$q_1^{2\sigma}\frac{\partial U_0}{\partial q_1}+\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_0}{\partial q_\nu}+\sum_{\nu}q_\nu^2 q_1^{2\sigma}B_{\nu,1}\frac{\partial U_0}{\partial q_1} -2q_j^2 B_{j,0}-\sum_{\nu}q_\nu^2(q_j\partial_{q_j}B_{\nu,0})-\sum_{\nu}q_\nu^2(\partial_{q_1}B_{\nu,0})q_1^{2\sigma}U_1=0. \tag{27}$$

We next consider the case $\alpha=-e_j$. The calculation is the same as for the case $\alpha=0$. Set $U_0:=U_{0,j}$, $U_1:=U_{1,j}$ and $U_2:=U_{2,j}$. Then we have

$$q_1^{2\sigma}\frac{\partial U_2}{\partial q_1}+\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_2}{\partial q_\nu}+\lambda_j q_j^2 B_{j,1}U_2 + \sum_{\nu}q_\nu^2 B_{\nu,1}q_1^{2\sigma}\frac{\partial U_2}{\partial q_1}+\lambda_j q_j^2 B_{j,1}=0. \tag{28}$$

$$q_1^{2\sigma}\frac{\partial U_1}{\partial q_1}+\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_1}{\partial q_\nu}+\lambda_j U_1-\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}(\partial_{q_1}B_{\nu,1})U_1+\lambda_j q_j^2 B_{j,1}U_1 -2q_jB_{j,1}U_2-q_j^2(\partial_{q_j}B_{j,1})U_2-\sum_{\nu}q_\nu^2 q_j^2(\partial_{q_j}B_{\nu,1})-2q_j^2 B_{j,1}=0. \tag{29}$$

$$q_1^{2\sigma}\frac{\partial U_0}{\partial q_1}+\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_0}{\partial q_\nu}+\lambda_j U_0+\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}B_{\nu,1}\frac{\partial U_0}{\partial q_1}+\lambda_j q_j^2 B_{j,1}U_0 -\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}(\partial_{q_1}B_{\nu,0})U_1-2q_jB_{j,0}U_2 -\sum_{\nu=2}^{n}q_\nu^2(\partial_{q_j}B_{\nu,0})U_2-\sum_{\nu=2}^{n}q_\nu^2 q_j(\partial_{q_j}B_{\nu,0})-2q_j^2 B_{j,0}=0. \tag{30}$$

Next we consider the case $\alpha=e_j$. Set $U_0:=U_{0,j}$, $U_1:=U_{1,j}$ and $U_2:=U_{2,j}$. Then we have

$$q_1^{2\sigma}\frac{\partial U_2}{\partial q_1}+\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_2}{\partial q_\nu}-\lambda_j q_j^2 B_{j,1}U_2-\lambda_j q_j^2 B_{j,1}+\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}B_{\nu,1}\frac{\partial U_2}{\partial q_1}=0. \tag{31}$$

$$q_1^{2\sigma}\frac{\partial U_1}{\partial q_1}+\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_1}{\partial q_\nu}-\lambda_j U_1+\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}B_{\nu,1}\frac{\partial U_1}{\partial q_1}-\lambda_j q_j^2 B_{j,1}U_1 -\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}(\partial_{q_1}B_{\nu,1})U_1-2q_j^3 B_{j,1}U_2 -\sum_{\nu=2}^{n}q_\nu^2 q_j^2(\partial_{q_j}B_{\nu,1})U_2-\sum_{\nu=2}^{n}q_\nu^2 q_j^2(\partial_{q_j}B_{\nu,1})-2q_j^3 B_{j,1}=0. \tag{32}$$

$$
\begin{aligned}
&q_1^{2\sigma}\frac{\partial U_0}{\partial q_1}+\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial U_0}{\partial q_\nu}-\lambda_j U_0+\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}B_{\nu,1}\frac{\partial U_0}{\partial q_1}-\lambda_j q_j^2 B_{j,1}U_0\\
&-2q_j^3 B_{j,0}U_2-\sum_{\nu=2}^{n}q_\nu^2 q_1^{2\sigma}(\partial_{q_1}B_{\nu,0})U_1\\
&-\sum_{\nu=2}^{n}q_\nu^2 q_j^2(\partial_{q_j}B_{\nu,0})U_2-\sum_{\nu=2}^{n}q_\nu^2 q_j^2(\partial_{q_j}B_{\nu,0})-2q_j^3 B_{j,0}=0. \qquad (33)
\end{aligned}
$$

In order to prove Theorem 2 we prepare a lemma. Let $R_j > 0$ $(j = 2, \dots, n)$ be given. Define $V_0 := \prod_{j=2}^{n}\{z_j \mid |z_j| < R_j\}$. Let $\mathcal{O}(V_0)$ be the set of holomorphic functions in V_0 continuous up to the boundary. Define $M_0(q) := \prod_{j=2}^{n}(R_j - |q_j|)$. For $f \in \mathcal{O}(V_0)$ we define the norm $\|f\|$ and the weighted norm $|||f|||$ by $\|f\| := \sup_{q\in V_0}|f(q)|$ and $|||f||| := \sup_{q\in V_0}|f(q)M_0(q)|$, respectively. Equipped with the norm $|||\cdot|||$, the space $\mathcal{O}(V_0)$ is a Banach space.

Let $\lambda := (\lambda_2, \dots, \lambda_n)$ and $\alpha = (\alpha_2, \dots, \alpha_n)$. We consider the equation

$$
Lu \equiv \left(\sum_{\nu=2}^{n}\lambda_\nu q_\nu\frac{\partial}{\partial q_\nu}-\lambda\cdot\alpha\right)u = f \in \mathcal{O}(V_0), \quad f = O(|q|). \qquad (34)
$$

Then we have

Lemma 1 *Let $\alpha = 0, \pm e_j$, $j = 2, \dots, n$. Assume Eqs. 6 and 7. Then there exists a constant $K > 0$ such that, for every $f \in \mathcal{O}(V_0)$ with $f = O(|q|)$ there exist a unique holomorphic solution u of Eq. 34 in $\mathcal{O}(V_0)$ such that $|||u||| \le K\|f\|$.*

Proof Expand $f(q) = \sum_{\eta\neq 0} f_\eta q^\eta$. Then the unique analytic solution of Eq. 34 is given by $u(q) = \sum_{\eta\neq 0} f_\eta q^\eta(\lambda\cdot(\eta-\alpha))^{-1}$. Take $0 < R_j' < R_j$ for $j = 2, \dots, n$. Let $|q_j| < R_j' < R_j$. By Cauchy's theorem we have

$$
u(q) = \sum_{\eta\neq 0} q^\eta(\lambda\cdot(\eta-\alpha))^{-1}(2\pi i)^{-n+1}\int_{|z_j|=R_j'}\frac{f(z)}{z^{\eta+e}}dz, \qquad (35)
$$

where $e = (1, \dots, 1)$. By Eqs. 6 and 7 there exists $K_1 > 0$ independent of η such that $|\lambda\cdot(\eta-\alpha)|^{-1} \le K_1$ for all η. Hence $|u(q)|$ is estimated by

$$
\begin{aligned}
&K_1(2\pi)^{-n+1}\|f\|\sum_{\eta\neq 0}\prod_j\left((|q_j|/R_j')^{\eta_j}2\pi\right)\\
&\le K_1\|f\|\prod_j\frac{1}{1-|q_j|/R_j'}\\
&\le K_1\|f\|\prod_j\frac{R_j'}{R_j'-|q_j|}\le K_1\|f\|(\prod_j R_j)\prod_j\frac{1}{R_j'-|q_j|}. \qquad (36)
\end{aligned}
$$

We let R_j' with $R_j' < R_j$ tend to R_j. Then $|u(q)|M_0(q)$ is estimated by $K_1\|f\|(\prod_j R_j)$.

3.2 Proof of Theorem 2

Proof *(Proof of Theorem 2)* Let $\kappa = 2\sigma - 1$. We show that $\phi^{(\alpha)}$ with $\alpha = 0$ is a formal first integral having the κ^{-1}-Gevrey estimate. Consider $\sum_\nu q_\nu \partial_{q_\nu} u = g$. By Lemma 1 we have

$$\|u\| \leq \|M_0^{-1} M_0 u\| \leq C_2 \||u|\| \leq C_2 K \|g\|,$$

where $C_2 = \sup_q |M_0(q)^{-1}|$.

Consider Eq. 12. For simplicity we denote $U_{0,j}$ and $U_{1,j}$ by U_0 and U_1, respectively. Determine U_1 by Eq. 26 and define

$$f(q_1, q) := -2q_j^2 B_{j,1} - \sum_\nu q_\nu^2 q_j \partial_{q_j} B_{\nu,1}.$$

Expand the functions $f(q_1, q)$, U_1, $q_1^{2\sigma} \sum q_\nu^2 (\partial_{q_1} B_{\nu,1})$ and $q_1^{2\sigma-1} \sum q_\nu^2 B_{\nu,1}$ in the power series of q_1

$$f(q_1, q) = \sum_{\ell=0}^{\infty} f_\ell(q) q_1^\ell, \; U_1 = \sum_\ell u_\ell(q) q_1^\ell, \tag{37}$$

$$q_1^{2\sigma} \sum_{\nu=2}^{n} q_\nu^2 (\partial_{q_1} B_{\nu,1}) = \sum_{\ell \geq 2\sigma} a_\ell(q) q_1^\ell, \tag{38}$$

$$q_1^{2\sigma-1} \sum_{\nu=2}^{n} q_\nu^2 B_{\nu,1} = \sum_{\ell \geq 2\sigma-1} b_\ell(q) q_1^\ell. \tag{39}$$

By substituting the expansions Eqs. 37, 38, 39 into 26 and by comparing the power of q_1^ℓ we have

$$(\ell - \kappa) u_{\ell-\kappa} + \sum_{\nu=2}^{n} \lambda_\nu q_\nu \frac{\partial u_\ell}{\partial q_\nu} + \sum_{m+k=\ell, m \geq 2\sigma} a_m u_k + \sum_{m+k=\ell, m \geq \kappa} k b_m u_k = f_\ell. \tag{40}$$

Hence, we can easily show that U_1 has the κ^{-1}-Gevrey estimate. We can similarly show that U_0 has the κ^{-1}-Gevrey estimate. Hence $\phi^{(\alpha)}$ with $\alpha = 0$ is a formal first integral with the κ^{-1}-Gevrey estimate. We can similarly show that $\phi^{(\alpha)}$'s with $\alpha = \pm e_j$ $(j = 2, \ldots, n)$ have the same property.

4 Proof of Theorem 1

4.1 Preparatory Lemma

Define

$$\mathcal{C} := \left\{ z \in \mathbb{C} | \, Re\,(\overline{\lambda_j} z^{2\sigma-1}) > 0 \quad j = 2, \ldots, n \right\}. \tag{41}$$

Let $\phi_j^{(0)}$ and $\phi^{(-e_j)} E^{-e_j}$ $(j = 2, \ldots, n)$ be the formal first integrals given by Theorem 2 with $q_1 = z$.

Let C_j, $\tilde{C}_j$ and C_0 be constants. For $z \in \mathcal{C}$, we solve the system of equations for q, p, p_1

$$\phi_j^{(0)} = C_j, \; \phi^{(-e_j)} E^{-e_j} = \tilde{C}_j, \; H = C_0, \quad j = 2, \ldots, n, \tag{42}$$

where $H = H_0 + H_1$ is given by

$$H = z^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j + \sum_{j=2}^{n} q_j^2 B_j(z, z^{2\sigma} p_1, q). \tag{43}$$

Here the unknown quantities are

$$q = q(z, T),\ \ p = p(z, T),\ \ p_1 = p_1(z, T), \tag{44}$$

where

$$q = \sum_{n=0}^{\infty} c_n z^n,\, c_n = c_n(T^{-1}), \quad T = (T_j)_j, \quad T_j = \tilde{C}_j E^{e_j}, \tag{45}$$

is a formal series in z with $c_n(T^{-1})$ convergent in T in some neighborhood of $T = \infty$. The Taylor series of p has the same form as q. (cf. Eq. 54 which follows). Concerning p_1 we have

$$p_1 z^{2\sigma} = \sum_{n=0}^{\infty} \rho_n z^n,\, \rho_n = \rho_n(T^{-1}), \tag{46}$$

with $\rho_n(T^{-1})$ convergent in T in some neighborhood of $T = \infty$. By Eqs. 12 and 16 we have

$$p_j q_j + \tilde{A}_j(z, z^{2\sigma} p_1, q) = C_j, \quad j = 2, \ldots, n, \tag{47}$$

$$p_j(1 + D_j(z, q)) + \tilde{D}_j(z, z^{2\sigma} p_1, q) = T_j, \quad j = 2, \ldots, n, \tag{48}$$

$$H = C_0. \tag{49}$$

Then we have

Lemma 2 *Assume Eq. 6. Then Eqs. 47, 48 and 49 has the formal solution (q, p, p_1) for $z \in \mathcal{C}$ given by Eqs. 44, 45 and 46.*

Proof We prove the proposition in two steps. By Theorem 2 we have $\tilde{A}_j = \tilde{A}_j(z, z^{2\sigma} p_1, q) = O(|q|^2)$, $\tilde{D}_j = \tilde{D}_j(z, z^{2\sigma} p_1, q) = O(|q|)$ and $D_j(z, q) = O(|q|^2)$. Moreover, $\tilde{A}_j$ and $\tilde{D}_j$ are the polynomials of $p_1 z^{2\sigma}$ of degree 1 whose coefficients are independent of p.

Step 1. Since $E_\lambda(z) = \exp(\frac{\lambda z^{-2\sigma+1}}{2\sigma-1})$ and

$$Re\,(\lambda z^{-2\sigma+1}) = Re\,(\overline{\lambda}\overline{z}^{-2\sigma+1}) = Re\,(\overline{\lambda} z^{2\sigma-1}/|z|^{2(2\sigma-1)}),$$

we see that $z \in \mathcal{C}$ if and only if $Re\,(\lambda_j z^{-2\sigma+1}) > 0$ for $j = 2, \ldots, n$. Hence, we have $T \to \infty$ when $z \to 0$ in $z \in \mathcal{C}$. Set

$$D_j = \sum_k q_k D_{j,0,k}(q) + z\tilde{D}_{j,1}(z, q), \tag{50}$$

where $D_{j,0,k}(q)$ is analytic at $q = 0$ and where $\tilde{D}_{j,1}(z, q)$ is a formal power series in z with coefficients analytic in q. Then $(1 + z\tilde{D}_{j,1})^{-1}$ exists as a formal power series.

By Eq. 48 we have

$$p_j \sum_k q_k D_{j,0,k}(q) + p_j(1 + z\tilde{D}_{j,1}) + \tilde{D}_j = T_j. \tag{51}$$

Since $(1 + z\tilde{D}_{j,1})^{-1}$ exists as a formal power series, we have

$$p_j \sum_k q_k D_{j,0,k}(q)(1 + z\tilde{D}_{j,1})^{-1} + p_j = T_j(1 + z\tilde{D}_{j,1})^{-1} - \tilde{D}_j(1 + z\tilde{D}_{j,1})^{-1}. \tag{52}$$

We decompose

$$p_j \sum_k q_k D_{j,0,k}(q)(1 + z\tilde{D}_{j,1})^{-1} = E_{j,0} + E_{j,1}, \tag{53}$$

where $E_{j,0}$ either vanishes identically or is unbounded when $T_j \to \infty$ and $E_{j,1} = O(1)$ when $T_j \to \infty$. Indeed, by Eqs. 45 and 46 we have $q_j = O(1)$ and $z^{2\sigma} p_1 = O(1)$. Hence $p_j q_j = O(1)$ by Eq. 47. By Eq. 48 we have $p_j \to \infty$. It follows that $q_j \to 0$. Hence we have Eq. 53. Define

$$p_j = \tilde{p_j} + \tilde{T}_j, \quad \tilde{T}_j := T_j(1 + z\tilde{D}_{j,1})^{-1} - E_{j,0}. \tag{54}$$

Then Eq. 52 is written in

$$\tilde{p_j} = -E_{j,1} - \tilde{D}_j(1 + z\tilde{D}_{j,1})^{-1} =: F_{j,0}. \tag{55}$$

Next, let $\tilde{A}_j(z, z^{2\sigma} p_1, q) = \tilde{A}_{j,0}(z, q) + z^{2\sigma} p_1 \tilde{A}_{j,1}(z, q)$. Substitute Eqs. 47 into 49. Then we have

$$\begin{aligned} &C_0 - \sum \lambda_\nu C_\nu + \sum \lambda_\nu \tilde{A}_{\nu,0}(z, q) + z^{2\sigma} p_1 \sum \lambda_\nu \tilde{A}_{\nu,1}(z, q) \\ &= z^{2\sigma} p_1 + \sum_\nu q_\nu^2 (B_{\nu,0} + z^{2\sigma} p_1 B_{\nu,1}). \end{aligned} \tag{56}$$

Hence we have

$$z^{2\sigma} p_1 (1 + \sum q_\nu^2 B_{\nu,1} - \sum \lambda_\nu \tilde{A}_{\nu,1}) = \sum \lambda_\nu \tilde{A}_{\nu,0} - \sum q_\nu^2 B_{\nu,0} + C_0 - \sum \lambda_\nu C_\nu. \tag{57}$$

If $c_0 = q(0)$ is sufficiently small, then $(1 + \sum q_\nu^2 B_1 - \sum \lambda_\nu \tilde{A}_{\nu,1})^{-1}$ exists as a formal series. Hence we have

$$\begin{aligned} &z^{2\sigma} p_1 = \\ &\left(\sum \lambda_\nu \tilde{A}_{\nu,0} - \sum q_\nu^2 B_{\nu,0} + C_0 - \sum \lambda_\nu C_\nu\right) \left(1 + \sum q_\nu^2 B_1 - \sum \lambda_\nu \tilde{A}_{\nu,1}\right)^{-1}. \end{aligned} \tag{58}$$

The right-hand side of Eq. 58 is a formal power series in z whose coefficients are analytic in q at $q = 0$. Note that, if q is a formal power series in z with a sufficiently small constant term c_0, then the right-hand side of Eq. 58 is a formal power series in z as well. We also note that, by substituting $z^{2\sigma} p_1$ in $F_{j,0}$, Eq. 55, $F_{j,0}$ is a formal power series in z with coefficients analytic in q. If q is a formal power series in z whose constant term c_0 is sufficiently small, then $F_{j,0}$ is also a formal power series in z.

We look for the equation of q. By Eq. 55 we have $p_j = \tilde{T}_j + \tilde{p_j} = \tilde{T}_j + F_{j,0}(z, q)$. $(\tilde{T}_j + F_{j,0})^{-1}$ exists as a formal series in z with coefficients analytic in q at $q = 0$. By Eq. 47 we have

$$q_j = (\tilde{T}_j + F_{j,0})^{-1} C_j - (\tilde{T}_j + F_{j,0})^{-1} \tilde{A}_j(z, z^2 p_1, q) =: G_j. \tag{59}$$

Substitute $z^2 p_1$ in Eq. 58 into G_j. Then G_j is a formal power series in z whose coefficients are analytic in q and $\xi = (\xi_j)_j$, $\xi_j := T_j^{-1}$ at $q = 0$ and $\xi = 0$, respectively. Indeed, by the form of G_j in Eq. 59 we consider $(\tilde{T}_j + F_{j,0})^{-1}$. Since $z\tilde{D}_{j,1} = zO(|q|^2)$ is small, we have $1 + z\tilde{D}_{j,1} \neq 0$. Hence we have

$$\begin{aligned} (\tilde{T}_j + F_{j,0})^{-1} &= T_j^{-1} \left((1 + z\tilde{D}_{j,1})^{-1} - T_j^{-1} E_{j,0} + T_j^{-1} F_{j,0}\right)^{-1} \\ &= \xi_j \left((1 + z\tilde{D}_{j,1})^{-1} - \xi_j E_{j,0} + \xi_j F_{j,0}\right)^{-1}. \end{aligned} \tag{60}$$

The right-hand side of Eq. 60 is holomorphic in ξ_j in some neighborhood of $\xi_j = 0$. On the other hand, the second term of G_j, $-(\tilde{T}_j + F_{j,0})^{-1}\tilde{A}_j(z, z^{2\sigma}p_1, q)$ is holomorphic in ξ_j. Hence G_j is holomorphic in ξ_j at $\xi_j = 0$. Similarly, we see that G_j is holomorphic in q at $q = 0$.

Note that, if we substitute q with a formal power series in z whose constant term c_0 is sufficiently small, then G_j is a formal power series in z. Therefore we consider

$$q_j = G_j(z, q, \xi), \quad j = 2, \dots, n. \tag{61}$$

Step 2. Let $q_j = \sum_{n=0}^{\infty} c_{j,n}z^n,\ c_{j,n} = c_{j,n}(\xi)$ be the Taylor series of q_j. Compare the constant term of both sides of Eq. 61. We have

$$c_{j,0} = G_j(0, c_0, \xi). \tag{62}$$

In order to estimate $G_j(0, c_0, \xi)$ by Eq. 59 we consider $F_{j,0}$. By Eqs. 55, 43 and 58 and $D_{j,0}(q_2) = O(1)$ we see that $F_{j,0} = O(1)$. On the other hand, the constant part of $\tilde{A}_j$ is $O(c_0^2)$. The second term of $G_j(0, c_0, \xi)$ in the expression Eq. 59 is $O(c_0^2\xi)$. Hence, by Eq. 59 we have $G_j(0, c_0, \xi) = O(\xi) + O(c_0^2\xi)$. By the implicit function theorem there exists a solution c_0 being arbitrarily small if T_j is sufficiently large.

Suppose that the coefficient $c_\nu(\xi)$'s are determined for $\nu = 0, \dots, n-1$. Then the coefficient of $c_n(\xi)$ of the Taylor series of q is determined by the relation

$$c_n = (G_j)_{q_2}(0, c_0, \xi)c_n + \tilde{R}_j, \tag{63}$$

where $\tilde{R}_j$ is the function of c_ν with $\nu \le n-1$. Since $(G_j)_{q_2}(0, c_0, \xi)$ is arbitrarily small if c_0 and ξ are sufficiently small, we determine c_n if c_0 is sufficiently small. The κ- Gevrey estimate of q is shown by differentiating Eq. 61 and by recalling that G_j is κ- Gevrey series with respect to z. This ends the proof.

Remark 1 In Eq. 62 the value $c_{0,j}$ depends on T. We can verify that $c_{0,j}T_j$ does not vanish for some j as T_j tends to infinity.

4.2 Construction of Formal Transseries Solution

Let z satisfy $\dot{z} = z^{2\sigma}$. Namely

$$t = -\frac{z^{1-2\sigma}}{2\sigma - 1}. \tag{64}$$

Let $q \equiv q(z)$, $p_1 \equiv p_1(z)$ and $p \equiv p(z)$ be the formal series given by Lemma 2. From Eq. 64 one can deduce the transseries representations of $q = q(z)$, $p_1 = p_1(z)$ and $p = p(z)$ in the t-variable, by changing the variable in a transseries given by Lemma 2. The exponential part is given by $e^{\lambda k t}$ for $k \ge -1$. For the sake of simplicity, we write the transseries with the same letter $q \equiv q(t)$, $p_1 \equiv p_1(t)$ and $p \equiv p(t)$. Then we have

Theorem 3 *Suppose that Eqs. 5, 6 and 7 are satisfied. Then there exists a neighborhood of $t = \infty$, Ω_1 such that there exists a formal transseries solution $q_1(t)$ of $\dot{q}_1 = H_{p_1}$ at every point of $\{t|Re(\lambda_j t) < 0,\ j = 2, \dots, n\} \cap \Omega_1$ for which $(q_1(t), q(t), p_1(t), p(t))$ is the formal transseries solution of Eq. 1.*

Theorem 1 follows from Theorem 3. Set $\lambda = (\lambda_2, \dots, \lambda_n)$ and
$Re\,\lambda = (Re\,\lambda_2, \dots, Re\,\lambda_n)$. Then we have

Lemma 3 *Assume Eq. 6. Then there exists a real number θ such that*

$$Re\,(e^{i\theta}\lambda_j) > 0, \quad j = 2, \dots, n, \tag{65}$$

and, for every pair $\alpha, \beta \in \mathbb{Z}_+^{n-1} \setminus 0$ we have $\alpha \neq \beta$ if and only if

$$\alpha \cdot Re\,(e^{i\theta}\lambda) \neq \beta \cdot Re\,(e^{i\theta}\lambda). \tag{66}$$

Moreover, $\alpha \cdot Re\,(e^{i\theta}\lambda)$ does not accumulate on a finite value when $|\alpha| \to \infty$, $\alpha \in \mathbb{Z}_+^{n-1} \setminus 0$.

Proof The last statement follows from Eq. 65 easily. We prove Eq. 66. Set $m = \alpha - \beta$ and consider $m \cdot Re\,(e^{i\theta}\lambda) \neq 0$, namely $Re\,(e^{i\theta}m \cdot \lambda) \neq 0$. The last condition is equivalent to $\theta \neq -\arg(m \cdot \lambda) \pm \pi/2$ for $m \in \mathbb{Z}^{n-1} \setminus 0$ and $m \neq 0$. The condition holds for some sufficiently small real number θ.

Multiplying the equation with $e^{i\theta}$, if necessary, we assume that Eq. 66 with $\theta = 0$ holds in the following.

Proof *(Proof of Theorem 3)* The proof is done in four steps.

Step 1. By definition we have

$$H_{p_1} = q_1^{2\sigma} + q_1^{2\sigma} \sum_{\nu=2}^{n} q_\nu^2 B_{\nu,1}(q_1, q). \tag{67}$$

We construct the solution q_1 of $\dot{q}_1 = H_{p_1}$ in the form

$$q_1 = \sum_{m \geq 0, n \geq 0} a_{m,n} t^{-n/(2\sigma-1)} e^{\lambda m t} = \sum_{m \geq 0} A_m(t) e^{\lambda m t}. \tag{68}$$

We note

$$\frac{d}{dt}\left(A_m(t)e^{\lambda m t}\right) = e^{\lambda m t}\left(\frac{d}{dt}A_m + \lambda m A_m\right). \tag{69}$$

Insert Eq. 68 into $\dot{q}_1 = H_{p_1}$ and compare the terms with $m = 0$. Note also that the expansion of q does not contain the terms with $m = 0$. Hence we have

$$\frac{d}{dt}A_0 = A_0^{2\sigma}. \tag{70}$$

By Eq. 70 we have

$$A_0 = (1 - 2\sigma)^{-1/(2\sigma-1)} t^{-1/(2\sigma-1)}. \tag{71}$$

By Lemma 3 we line up all multiindices $m \in \mathbb{Z}_+^{n-1} \setminus 0$ in the ascending series $m_1, m_2, \dots$ such that $0 < m_j \cdot Re\lambda < m_{j+1} \cdot Re\lambda$ $(j = 1, 2, \dots)$. Next we determine A_1 for $m = m_1$. By considering the terms of the order $O(e^{\lambda m_1 t})$ in $\dot{q}_1 = H_{p_1}$ we obtain

$$\frac{d}{dt}A_1 + \lambda \cdot m_1 A_1 = 2\sigma A_0^{2\sigma-1} A_1 = \frac{2\sigma}{(1 - 2\sigma)t} A_1. \tag{72}$$

First we see that the constant term of A_1 vanishes. Then, by the inductive arguments we see that A_1 vanishes.

Next we consider A_2 for $m = m_2$. We compare the terms of order $e^{\lambda m_2 t}$ of both sides of $\dot{q}_1 = H_{p_1}$. The term of order $e^{\lambda m_2 t}$ in $\dot{q}_1 - q_1^{2\sigma}$ is given by

$$\frac{d}{dt}A_2 + \lambda \cdot m_2 A_2 - 2\sigma A_0^{2\sigma-1} A_2 = \frac{d}{dt}A_2 + \lambda \cdot m_2 A_2 - \frac{2\sigma}{(1 - 2\sigma)t} A_2. \tag{73}$$

We write $q_j = \sum_{m=1}^{\infty} \alpha_{j,m}(t)e^{\lambda mt}$. Since $q_j = O(e^{\lambda_j t})$ and

$$A_0^{2\sigma} = (1-2\sigma)^{-1-\frac{1}{2\sigma-1}} t^{-1-\frac{1}{2\sigma-1}},$$

the terms of order $e^{\lambda m_2 t}$ appearing from $q_1^{2\sigma} \sum_\nu q_\nu^2 B_{\nu,1}(q_1, q)$ are written as

$$A_0^{2\sigma} F_2(t) = (1-2\sigma)^{-1-1/(2\sigma-1)} t^{-1-1/(2\sigma-1)} F_2(t), \tag{74}$$

with $F_2(t)$ being determined by $A_0(t)$ and has the nonpositive power of t. Note that A_2 does not appear in Eq. 74. By Eqs. 73 and 74 A_2 satisfies

$$\frac{d}{dt}A_2 + \lambda \cdot m_2 A_2 - \frac{2\sigma}{(1-2\sigma)t} A_2 = (1-2\sigma)^{-1-1/(2\sigma-1)} t^{-1-1/(2\sigma-1)} F_2(t). \tag{75}$$

One can determine A_2 from Eq. 75 by the same argument as the case of A_1. One also sees that $A_2 = O(t^{-1-1/(2\sigma-1)})$.

The general case m_ν can be shown by the same argument. The term containing A_ν is given by

$$\frac{dA_\nu}{dt} + m_\nu \cdot \lambda A_\nu + \frac{2\sigma}{2\sigma-1}\frac{A_\nu}{t}. \tag{76}$$

On the other hand, by substituting the expansion of q_1 into $q_1^{2\sigma}$ there appears the polynomial of $A_0, \ldots, A_{\nu-1}$ in the inhomogeneous term. These terms are $O(t^{-1-1/(2\sigma-1)})$. Similarly, the polynomial of $A_0, \ldots, A_{\nu-1}$ appears from $H_{p_1} - q_1^{2\sigma}$ since the term q_ν^2 exists. By the definition of H these terms are $O(t^{-1-1/(2\sigma-1)})$. By constructing the formal series of the negative power of t inductively like in the case A_2, we can determine the formal expansion of A_ν inductively. We easily see that $A_\nu = O(t^{-1-1/(2\sigma-1)})$ for $\nu \geq 0$.

Step 2. Let $q = q(t)$, $p_1 = p_1(t)$ and $p = p(t)$ be the formal solution of Eqs. 47, 48 and 49, where we use Eq. 64. Let $q_1 = q_1(t)$ be the solution of $\dot{q}_1 = H_{p_1}$ given in Step 1. Let $\phi_j^{(0)}$ and $\phi^{(-e_j)}E^{(-e_j)}$ $(j = 2, \ldots, n)$ be the formal first integral given by Theorem 2.

We shall verify that the substitution of the transseries $q_1(t)$, $q(t)$, $p_1(t)$ and $p(t)$ into the formal first integral is well defined. Since the first integral is the linear function of p_1 and p, the assertion is trivial. Next, consider the formal expansions of q_1 and $q = \sum_{n=0}^{\infty} q_1^n a_n(q)$. We recall that $a_n(q)$'s are analytic in some neighborhood of the origin $q = 0$ independent of n. Let

$$q = q^{(0)} + \sum_{\nu=0}^{\infty} b_{m_\nu}(t) E_\lambda^{-m_\nu} \equiv q^{(0)} + \tilde{q}, \tag{77}$$

where $m_0 = 0$ and $b_0(t)$ does not contain the constant term and consists of negative powers of t. On the other hand, $b_{m_\nu}(t)$ consists of nonnegative powers of t. Note also that the constant term $q^{(0)}$ can be taken sufficiently small. By Step 1, $q_1(t)$ also has an expansion like Eq. 77 without a constant term.

Since $a_n(q)$ is analytic in some neighborhood of the origin independent of n and $\tilde{q}(t)$ does not have a constant term, $a_n(q^{(0)} + \tilde{q})$ is well defined and is analytic at $\tilde{q} = 0$. Similarly, by noting that $q_1(t)$ does not have a constant term, the substitution $q_1 = q_1(t)$, $q = q(t)$ in $\sum_{n=0}^{\infty} q_1^n a_n(q)$ is also well defined.

Step 3. Substitute $z = q_1(t)$ in Eq. 49 and differentiate it with respect to t. Then we have

$$\dot{q}_1 H_{q_1} + \dot{q} H_q + \dot{p}_1 H_{p_1} + \dot{p} H_p = 0. \tag{78}$$

Substitute the formal series $q_1 = q_1(t)$, $q = q(t)$, $p_1 = p_1(t)$ and $p = p(t)$ into Eq. 78. Since H is the first integral of the Hamiltonian vector field we get the trivial relation

$$H_{p_1} H_{q_1} + H_p H_q - H_{q_1} H_{p_1} - H_q H_p = 0. \tag{79}$$

By subtracting Eqs. 79 from 78 and by using $\dot{q_1} = H_{p_1}$ we obtain

$$(\dot{q} - H_p)H_q + (\dot{p}_1 + H_{q_1})H_{p_1} + (\dot{p} + H_q)H_p = 0. \tag{80}$$

Let $G^{(j)}$ and $\tilde{G}^{(j)}E^{-e_j}$ $(j = 2, \dots, n)$ be the formal first integrals given by Theorem 2; $G^{(j)} = \phi_j^{(0)}$, $\tilde{G}^{(j)} = \phi^{(-e_j)}$ $(j = 2, \dots, n)$, respectively. By applying the same argument to $G^{(j)}$ we have

$$\dot{q_1}(G^{(j)})_{q_1} + \dot{q}(G^{(j)})_q + \dot{p}_1(G^{(j)})_{p_1} + \dot{p}(G^{(j)})_p = 0, \;\; j = 2, \dots, n. \tag{81}$$

On the other hand, since $G^{(j)}$ is the first integral of the Hamiltonian vector field of H we have

$$H_{p_1}(G^{(j)})_{q_1} + H_p(G^{(j)})_q - H_{q_1}(G^{(j)})_{p_1} - H_q(G^{(j)})_p = 0, \;\; j = 2, \dots, n. \tag{82}$$

By subtracting Eqs. 82 from 54 and by using $\dot{q_1} = H_{p_1}$ we have

$$(\dot{q} - H_p)(G^{(j)})_q + (\dot{p}_1 + H_{q_1})(G^{(j)})_{p_1} + (\dot{p} + H_q)(G^{(j)})_p = 0, \tag{83}$$

where $j = 2, \dots, n$. Similarly, we have

$$(\dot{q} - H_p)(\tilde{G}^{(j)})_q + (\dot{p}_1 + H_{q_1})(\tilde{G}^{(j)})_{p_1} + (\dot{p} + H_q)(\tilde{G}^{(j)})_p = 0, \tag{84}$$

where $j = 2, \dots, n$.

Step 4. We prove

$$\dot{p}_1 + H_{q_1} = 0, \quad \dot{p} + H_q = 0, \quad \dot{q} - H_p = 0, \tag{85}$$

where the equalities are understood in the sense of transseries. By definition we expand

$$\dot{p}_1 + H_{q_1} = \sum_{k=0}^{\infty} (\dot{p}_1 + H_{q_1})_k(t) e^{\lambda m_k t}, \tag{86}$$

$$\dot{q} - H_p = \sum_{k=1}^{\infty} (\dot{q} - H_p)_k(t) e^{\lambda m_k t}, \tag{87}$$

$$\dot{p}_j + H_{q_j} = \sum_{k=0}^{\infty} (\dot{p}_j + H_{q_j})_k(t) e^{(-\lambda_j + \lambda m_k)t}, \quad j = 2, \dots, n, \tag{88}$$

where $m_0 = 0$ and $(\dot{p}_1 + H_{q_1})_k(t)$ denotes the coefficients of $e^{\lambda m_k t}$ in the expansion of $\dot{p}_1 + H_{q_1}$. The expansion Eqs. 87 follows from 59. The relation Eqs. 88 follows from 53 and 54.

We show

$$(\dot{p}_1 + H_{q_1})_k(t) = 0, \quad k = 0, 1, 2, \dots, \tag{89}$$

$$(\dot{p}_j + H_{q_j})_k(t) = 0, \quad k = 0, 1, 2, \dots, \;\; j = 2, \dots, n, \tag{90}$$

$$(\dot{q} - H_p)_k(t) = 0, \quad k = 1, 2, \dots \tag{91}$$

Clearly, Eqs. 89, 90 and 91 imply 85.

Let $G^{(j)}$ and $\tilde{G}^{(j)}E^{-e_j}$ be the formal first integrals given in Step 3. Then, by the result of Step 3 we have Eqs. 80, 83 and 84. On the other hand, we have, by definition,

$$G^{(j)} = q_j p_j + U_{0,j} + q_1^{2\sigma} p_1 U_{1,j}, \quad j = 2, \dots, n, \tag{92}$$

$$\tilde{G}^{(j)} = p_j(1 + \tilde{U}_{2,j}) + \tilde{U}_{0,j} + q_1^{2\sigma} p_1 \tilde{U}_{1,j}, \quad j = 2, \dots, n. \tag{93}$$

By Eqs. 3, 92 and 93 we have

$$(H_q, H_{p_1}, H_p) = \Big(\lambda_2 p_2 + \partial_{q_2}(\sum_\nu q_\nu^2 B\nu), \ldots, \lambda_n p_n + \partial_{q_n}(\sum_\nu q_\nu^2 B\nu), \tag{94}$$
$$q_1^{2\sigma} + q_1^{2\sigma}\sum_\nu B_{\nu,1}q_\nu^2, \lambda_2 q_2, \ldots, \lambda_n q_n\Big),$$

$$((G^{(j)})_q, (G^{(j)})_{p_1}, (G^{(j)})_p) = \Big(\partial_q(q_j p_j + \tilde{U}_{0,j}) + q_1^{2\sigma} p_1 \partial_q \tilde{U}_{1,j}, q_1^{2\sigma}\tilde{U}_{1,j}, \partial_p(q_j p_j)\Big), \tag{95}$$

$$((\tilde{G}^{(j)})_q, (\tilde{G}^{(j)})_{p_1}, (\tilde{G}^{(j)})_p) \tag{96}$$
$$= \Big(p_j\partial_q\tilde{U}_{2,j} + \partial_q\tilde{U}_{0,j} + q_1^{2\sigma} p_1\partial_q\tilde{U}_{1,j}, q_1^{2\sigma}\tilde{U}_{1,j}, \partial_p(p_j(1+\tilde{U}_{2,j}))\Big),$$

where $j = 2, \ldots, n$. By Theorem 2 we have

$$U_{0,j} = O(|q|^2),\ U_{1,j} = O(|q|^2),\ \tilde{U}_{0,j} = O(|q|^2),$$
$$\tilde{U}_{1,j} = O(|q|^2),\ \tilde{U}_{2,j} = O(|q|). \tag{97}$$

We compare the terms with the order $O(e^{-\lambda_j t})$ of both sides of Eq. 84. By the proof of Theorem 2 we have

$$q_j = O(e^{\lambda_j t}),\ p_j = O(e^{-\lambda_j t}),\ p_1 = O(1), \quad Re\,(\lambda_j t) \to -\infty. \tag{98}$$

We have

$$(\tilde{G}^{(j)})_q(\dot{q} - H_p) = O(1),\ (\tilde{G}^{(j)})_{p_1}(\dot{p}_1 + H_{q_1}) = O(1),$$
$$(\tilde{G}^{(j)})_{p_j}(\dot{p}_j + H_{q_j}) = O(e^{-\lambda_j t}).$$

Thus the term with the order $e^{-\lambda_j t}$ of both sides of Eq. 84 is given by $(\dot{p}_j + H_{q_j})_0$. Therefore we have $(\dot{p}_j + H_{q_j})_0 = 0$ for $j = 2, \ldots, n$.

Next, compare terms with the order $O(1)$ of both sides of Eq. 83. We have $(G^{(j)})_q(\dot{q} - H_p) = O(1)$ and the term $(\dot{q}_j - H_{p_j})_\ell (p_j)_0$ appears for some $\ell \geq 1$ with $\lambda m_\ell = \lambda_j$. The second term has the order $(G^{(j)})_{p_1}(\dot{p}_1 + H_{q_1}) = o(1)$. The third term has the order $(G^{(j)})_p(\dot{p} + H_q) = O(1)$ and the term $(\dot{p}_j + H_{q_j})_0(q_j)_\nu$ appears for some ν, which vanishes by induction. Hence, we obtain $(\dot{q}_j - H_{p_j})_\ell(p_j)_0 = 0$. Since $(p_j)_0 \neq 0$ by Theorem 2 we have $(\dot{q}_j - H_{p_j})_\ell = 0$ for $j = 2, \ldots, n$. We note that ℓ satisfies $\lambda m_\ell = \lambda_j$ and there appears no lower order term than $e^{\lambda_j t}$ in $\dot{q}_j - H_{p_j}$ by definition.

Next we compare terms with the order $O(1)$ of both sides of Eq. 80. The term $(\dot{q}_j - H_{p_j})_\ell(p_j)_0$ appears from $(\dot{q} - H_p)H_q$. It vanishes by induction. Next the term $(\dot{p}_1 + H_{q_1})_0(q_1^{2\sigma})_0$ appears from $(\dot{p}_1 + H_{q_1})H_{p_1}$.On the other hand, the term $(\dot{p}_j + H_{q_j})_0(q_j)_\nu$ appears from $(\dot{p} + H_q)H_p$, which vanishes by induction. Hence we obtain $(\dot{p}_1 + H_{q_1})_0(q_1^{2\sigma})_0 = 0$. Since $(q_1^{2\sigma})_0 \neq 0$ by Step 1 we have $(\dot{p}_1 + H_{q_1})_0 = 0$.

We proceed in the same way. Namely, we compare the terms with the order $O(1)$ of both sides of Eq. 84. Then we obtain $(\dot{p}_j + H_{q_j})_1 = 0$ for $j = 2, \ldots, n$. Next we compare terms with the order $O(e^{\lambda m_1 t})$ of both sides of Eq. 83. Then we obtain $(\dot{q} - H_p)_2 = 0$ and so on. Next we consider terms with the order $O(e^{\lambda m_1 t})$ of both sides of Eq. 80. We have $(\dot{p}_1 + H_{q_1})_1 = 0$. By induction we have Eqs. 89, 90 and 91.

4.3 Analytic Solution of Eq. 1

By the method of the proof of Theorem 3 we have the analogue of Theorem 3 in the analytic or in the class of formal series. Given analytic first integrals of Eq. 1, $G^{(j)}$, $\tilde{G}^{(j)}$ $(j = 2, \ldots, n)$. We say that H, $G^{(j)}$'s and $\tilde{G}^{(j)}$'s are functionally independent if the vectors $(H_{q_1}, H_q, H_{p_1}, H_p)$, $(G^{(j)}_{q_1}, G^{(j)}_q, G^{(j)}_{p_1}, G^{(j)}_p)$, $(\tilde{G}^{(j)}_{q_1}, \tilde{G}^{(j)}_q, \tilde{G}^{(j)}_{p_1}, \tilde{G}^{(j)}_p)$ $(j = 2, \ldots, n)$ are linearly independent in some domain, where $H_{q_1} = \partial H/\partial q_1$, $H_q = \nabla_q H$ and so on. Here ∇_q denotes the nabla with respect to q. Then we have

Theorem 4 *Suppose that H, $G^{(j)}$ and $\tilde{G}^{(j)}$ $(j = 2, \ldots, n)$ are functionally independent in some neighborhood of $(q_1^{(0)}, p_1^{(0)}, q^{(0)}, p^{(0)})$ with $q_1^{(0)} \neq 0$. Let $q(z)$, $p(z)$ and $p_1(z)$ be the formal transseries solution given by Lemma 2. Suppose that $q(z)$, $p(z)$ and $p_1(z)$ are analytic at $z = q_1^{(0)} \in \mathcal{C}$ such that $q(q_1^{(0)}) = q^{(0)}$, $p(q_1^{(0)}) = p^{(0)}$, $p_1(q_1^{(0)}) = p_1^{(0)}$. Assume that there exists an analytic solution $q_1 = q_1(t)$ of $\dot{q}_1 = H_{p_1}$ for $q = q(q_1)$, $p = p(q_1)$ and $p_1 = p_1(q_1)$. Then (q_1, q, p, p_1) ($q = q(q_1(t))$, $p_1 = p_1(q_1(t))$, $p = p(q_1(t))$) is an analytic solution of Eq. 1 in some neighborhood of t_0 with $q_1^{(0)} = q_1(t_0)$.*

Proof For simplicity we assume that $q_1^{(0)} = 0$. First we show that

$$\dot{q}(0) = H_p(0), \quad \dot{p}(0) = -H_q(0), \quad \dot{p}_1(0) = -H_{q_1}(0), \tag{99}$$

where the dot denotes the derivative with respect to t and $H_{p_2}(0) = H_{p_2}|_{t=0}$.

Substitute $z = q_1$ in Eq. 49 and differentiate the equation with respect to t. We obtain Eq. 78. Since H is the first integral of the Hamiltonian vector field we have the trivial relation Eq. 79. By subtracting Eqs. 79 from 78 and by using $\dot{q_1} = H_{p_1}$ we obtain Eq. 80.

Next, by applying the same argument to $G^{(j)}$ we have Eq. 81. On the other hand, since $G^{(j)}$ is the first integral of the Hamiltonian vector field of H we have Eq. 82. By subtracting Eqs. 82 from 81 and by using $\dot{q_1} = H_{p_1}$ we have Eq. 83. Similarly, we have Eq. 84. By Eqs. 80, 83 and 84 and the assumption on the functional independency of H, $G^{(j)}$'s and $\tilde{G}^{(j)}$'s we obtain Eq. 99.

Next, we differentiate Eqs. 80, 83 and 84 with respect to t. Then the functional independency of H, $G^{(j)}$'s and $\tilde{G}^{(j)}$'s and the inductive argument we see that $\dot{q}_2 - H_{p_2}$ vanishes up to first derivative. The same assertion holds for $\dot{p}_2 + H_{q_2}$ and $\dot{p}_1 + H_{q_1}$. By induction we see that all derivatives of $\dot{q}_2 - H_{p_2}$, $\dot{p}_2 + H_{q_2}$ and $\dot{p}_1 + H_{q_1}$ vanish.

5 Summability of the Formal Transseries Solution

5.1 Borel Summability

In order to state the result we begin with the definitions of the Borel summability of a formal first integral and a transseries.

Summability of Formal First Integral Consider the first integrals constructed in Theorem 2. We consider the formal series

$$\psi(q_1, q) = q_1^{\kappa} \sum_{n=0}^{\infty} v_n(q) q_1^n, \tag{100}$$

where $\kappa \geq 1$ is an integer and $v_n(q)$'s are holomorphic in $q \in V_0$ for some open sets V_0 independent of n. The formal κ- Borel transform $\hat{\mathcal{B}}_\kappa$ is defined by

$$\hat{\mathcal{B}}_\kappa(\psi)(\zeta, q) := \sum_{n=0}^{\infty} v_n(q) \frac{\zeta^n}{\Gamma(\frac{n+\kappa}{\kappa})}, \tag{101}$$

where ζ is the dual variable of q_1 and $\Gamma(z)$ is the Gamma function. For ϕ in Eq. 100 we have

$$\hat{\mathcal{B}}_\kappa(q_1^{\kappa+1} \frac{d}{dq_1} \psi)(\zeta, q) = \kappa \zeta^\kappa \hat{\mathcal{B}}_\kappa(\psi)(\zeta, q). \tag{102}$$

For the bisecting direction $d \in \mathbb{R}$ and the opening $\eta > 0$, define $S(d, \eta) := \{z \in \mathbb{C}; |\arg z - d| < \eta/2\}$. For the neighborhood $\Omega_0 \subset \mathbb{C}$ of the origin, define

$$\Sigma_0 := \Omega_0 \cup S(d, \eta). \tag{103}$$

We say that the formal power series $\psi(q_1, q)$ is κ-summable with respect to q_1 in the direction d if there exist $\theta > 0$ and a neighborhood Ω_1 of $\zeta = 0$ such that $\hat{\mathcal{B}}_\kappa(\psi)(\zeta, q)$ converges when $(\zeta, q) \in \Omega_1 \times V_0$ and $\hat{\mathcal{B}}_\kappa(\psi)(\zeta, q)$ can be analytically continued to $(\zeta, q) \in S(d, \eta) \times V_0$ and is of exponential type of order κ in $\zeta \in S(d, \eta)$. Namely, there exist $K_0 > 0$ and $K_2 > 0$ such that

$$|\hat{\mathcal{B}}_\kappa(\psi)(\zeta, q)| \leq K_0 e^{K_2 |\zeta|^\kappa}, \quad \zeta \in S(d, \eta), \; q \in V_0.$$

For simplicity, we denote the analytic continuation with the same notation. Then the κ- sum of the formal series $\psi(q_1, q)$, $\Psi(q_1, q)$ is defined by the Laplace transform

$$\Psi(q_1, q) := \int_0^{\infty e^{id}} e^{-(\zeta/q_1)^\kappa} \hat{\mathcal{B}}_\kappa(\psi)(\zeta, q) d\zeta^\kappa. \tag{104}$$

Summability of Transseries We give the definition of the Borel summability of the transseries Eq. 8. As for a general definition and properties we refer to [4] and the references therein.

Consider the transseries u given by Eq. 8. We write

$$u = \sum_{k \geq k_0, \ell \geq 0} c_{k,\ell} t^{-\ell/(2\sigma-1)} e^{\lambda k t} = \sum_{k \geq k_0} e^{\lambda k t} u_k(t), \tag{105}$$

where

$$u_k(t) = \sum_{j=0}^{2\sigma-2} t^{-j/(2\sigma-1)} u_{k,j}(t), \quad u_{k,j}(t) = \sum_{m=0}^{\infty} c_{k,m(2\sigma-1)+j} t^{-m}. \tag{106}$$

We say that u is κ- Borel summable in the direction d if there exist Σ_0 in Eq. 103 and the constant K_0 such that, for every j, $j = 0, \ldots, 2\sigma - 1$ and every integer $k \geq 0$ the formal κ-Borel transform of $f_{k,j}(t) := e^{\lambda k t} u_{k,j}(t)$, $\mathcal{B}_\kappa(f_{k,j})(\tau)$ is extended to a holomorphic function on Σ_0 of order κ uniformly in k, namely there exist $\exists C_k > 0$ satisfying $\sum_k C_k < \infty$ such that

$$|\mathcal{B}(f_{k,j})(\tau)| \leq C_k e^{K_0 |\tau|^\kappa}, \quad \forall \tau \in \Sigma_0, \tag{107}$$

where τ is the dual variable of t.

5.2 Statement of the Result

Let $(q_1(t), p_1(t), q(t), p(t))$ be the formal transseries solution given by Theorem 2. Then we have

Theorem 5 *Suppose that Eqs. 5, 6 and 7 are satisfied. Then the formal transseries solution* $(q_1(t), p_1(t), q(t), p(t))$ *is* $(2\sigma - 1)$*-Borel summable in every direction in* $\{t|Re(\lambda_j t) < 0, j = 2, \ldots, n\}$. *There exists a neighborhood of* $t = \infty$, Ω_1 *such that the Borel sum is the analytic transseries solution of Eq. 1 in the set* $\{t|Re(\lambda_j t) < 0, j = 2, \ldots, n\} \cap \Omega_1$.

In order to prove Theorem 5 we show a superintegrability by virtue of the Borel sum of formal first integrals.

5.3 Function Space

Let $c > 0$. Let $A_c^{(\kappa)}(\Sigma_0 \times V_0)$ be the set of all f holomorphic and of exponential growth of order κ in Σ_0 such that

$$\|f\|_c := \sup_{z\in\Sigma_0, q\in V_0} |f(z,q)e^{-cRe\,(z/d)^\kappa}(1+|z^\kappa|)^2| < \infty. \tag{108}$$

Equipped with the norm Eq. 108 the space $A_c^{(\kappa)}(\Sigma_0 \times V_0)$ is a Banach space.

For $f, g \in A_c^{(\kappa)}(\Sigma_0 \times V_0)$ we define the κ-convolution by

$$(f *_\kappa g)(\zeta, q) := \int_0^{\zeta^\kappa} f(\tau, q)g((\zeta^\kappa - \tau^\kappa)^{1/\kappa}, q)d\tau^\kappa. \tag{109}$$

We can easily verify that there exists a constant $K_0 > 0$ independent of f and g such that $\|f *_\kappa g\|_c \le K_0\|f\|_c\|g\|_c$ for every $f, g \in A_c^{(\kappa)}(\Sigma_0 \times V_0)$.

6 Summability of First Integral

6.1 Statement of Result

Set $\kappa = 2\sigma - 1$ and $\lambda := (\lambda_2, \ldots, \lambda_n)$. Let $\alpha = (\alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^{n-1}$ and $k = (k_2, \ldots, k_n) \in \mathbb{Z}_+^{n-1}$. Define

$$S_0(\alpha) := \left\{z \in \mathbb{C} \mid \kappa z^\kappa + \lambda \cdot (k - \alpha) = 0, \forall k \in \mathbb{Z}_+^{n-1} \setminus \{0\}\right\}. \tag{110}$$

Let $B_{\nu,0}$ and $B_{\nu,1}$ be given by Eq. 5. Assume

$$B_{\nu,0}(q_1, q) = O(q_1^\kappa), \quad B_{\nu,1}(q_1, q) = O(q_1^\kappa), \quad \nu = 2, \ldots, n. \tag{111}$$

Then we have

Theorem 6 *Assume Eqs. 5, 6, 7 and 111. Let* $v = E^\alpha\phi^{(\alpha)}$ $(\alpha = 0, \pm e_j, j = 2, \ldots, n)$ *be the formal first integrals given by Theorem 2. Then* $\phi^{(\alpha)}$ *is* κ*-summable with respect to* q_1 *in every direction* d *such that* $d \notin S_0(\alpha)$.

Remark We can drop Eq. 111 if we use Lemma 2 of [2].

We give the corollary of Theorem 6. Assume Eq. 6. Let $0 \le \theta_0 \le \theta_1 < 2\pi$ be such that the closure of $\{-\lambda \cdot k \mid k \in \mathbb{Z}_+^{n-1} \setminus \{0\}\}$ is contained in the sector $\{z \mid \theta_0 \le \arg z \le \theta_1\}$. Then we have

Corollary 1 *Assume Eqs. 5, 6, 7 and 111. Suppose* $\sigma = 1$. *Then there exist neighborhoods* $\Omega_1, \tilde{\Omega}_1, \Omega_0, \tilde{\Omega}_0$ *of* $q_1 = 0$, $p_1 = 0$, $q = 0$ *and* $p = 0$, *respectively such that the Hamiltonian system for* H *is Liouville-integrable in the domain* $(q_1, p_1, q, p) \in \Omega_1 \times \tilde{\Omega}_1 \times \Omega_0 \times \tilde{\Omega}_0$ *with* $q_1 \in \{z \mid \theta_1 - \pi/2 < \arg z < \theta_0 + 5\pi/2\}$.

Proof By Eq. 6 we have $\theta_1 - \theta_0 < \pi$. We show that the first integrals $\phi^{(-e_j)}E^{-e_j}$ $(j = 2, \ldots, n)$ and H are functionally independent first integrals. By the definition of $S_0(-e_j)$ and by Theorem 6 together with $\kappa = 2\sigma - 1 = 1$, the summability of the formal first integrals holds for the directions $\theta_1 \le \arg z \le \theta_0 + 2\pi$. Hence the Borel sums of formal first integrals exist in the sector given in the theorem. Elementary calculations show that $\nabla\phi^{(-e_j)}$ $(j = 2, \ldots, n)$ and ∇H are linearly independent when $|q|$ is sufficiently small.

6.2 Estimate of a Linearized Operator

In order to prove Theorem 6 we prepare a lemma. Let $R_j > 0$ $(j = 2, \ldots, n)$ be given. Define $V_0 := \prod_{j=2}^{n}\{z_j | |z_j| < R_j\}$. Let $\hat{f} \in A_c^{(\kappa)}(\Sigma_0 \times V_0)$ be given. Consider the equation of $\hat{u} \in A_c^{(\kappa)}(\Sigma_0 \times V_0)$

$$L\hat{u} := \left(\kappa\zeta^{\kappa} + \sum_{\nu=2}^{n} \lambda_\nu q_\nu \frac{\partial}{\partial q_\nu}\right)\hat{u} = \hat{f}, \quad \zeta \in \Sigma_0. \tag{112}$$

For $\hat{f} \in A_c^{(\kappa)}(\Sigma_0 \times V_0)$ we define the weighted norm $|||\hat{f}|||_c$ by $|||\hat{f}|||_c := \|M_0(q)\hat{f}\|_c$, where $M_0(q)$ is given in Lemma 1. Then we have

Lemma 4 *Let $\alpha = 0, \pm e_j$, $j = 2, \ldots, n$. Suppose that Eqs. 6 and 7 and $\Sigma_0 \cap S_0(\alpha) = \emptyset$ are satisfied. Assume that $\hat{f} = O(|q|)$ when $q \to 0$. Then Eq. 112 has a unique solution $\hat{u}$ in $A_c^{(\kappa)}(\Sigma_0 \times V_0)$. Moreover, there exists $K > 0$ independent of $\zeta \in \Sigma_0$ such that $|||\hat{u}|||_c \le K\|\hat{f}\|_c$ and $|||\zeta^\kappa \hat{u}|||_c \le K\|\hat{f}\|_c$ hold for every $\hat{f} \in A_c^{(\kappa)}(\Sigma_0 \times V_0)$.*

The proof of Lemma 4 is almost identical with that of Lemma 1.

6.3 Proof of Theorem 6

Proof The proof is done in three steps.

Step 1. Let d be the direction not contained in $S_0(\alpha)$. Define Σ_0 by Eq. 103. By Eqs. 6 and 7 we have $\Sigma_0 \cap S_0(\alpha) = \emptyset$ if we take $\eta > 0$ and $\Omega \equiv \Omega_0$ in Eq. 103 sufficiently small.

We prove the case $\alpha = 0$. The proof of other cases $\alpha = \pm e_j$ $(j = 2, \ldots, n)$ is similar to the case $\alpha = 0$. By Eq. 12 it is sufficient to prove the κ-summability of U_0 and U_1. Since the proof is similar we consider U_1. Let $\sum_{\nu=0}^{\infty} U_{1,j,\nu} q_1^{\nu}$ be the formal series solution of Eq. 26. Define

$$U_1 = V + u_0, \quad u_0 := \sum_{\nu=0}^{\kappa-1} U_{1,j,\nu} q_1^{\nu}. \tag{113}$$

Then we have $V = O(q_1^{\kappa})$. Substitute Eqs. 113 into 26. Then, by denoting the inhomogeneous term by $f = O(q_1^{\kappa})$ we have

$$q_1^{2\sigma}\frac{\partial V}{\partial q_1} + \sum_{\nu=2}^{n} \lambda_\nu q_\nu \frac{\partial V}{\partial q_\nu} - \sum_{\nu=2}^{n} q_\nu^2 q_1^{2\sigma}(\partial_{q_1} B_{\nu,1})V \tag{114}$$
$$+ \sum_{\nu} q_\nu^2 B_{\nu,1} q_1^{2\sigma} \frac{\partial V}{\partial q_1} + f(q_1, q) = 0.$$

Step 2. Recalling Eq. 111 we apply the κ-Borel transform to Eq. 114. Let ζ be the dual variable of q_1. Let $\hat{V}$ and $\hat{f}$ be the κ-Borel transform of V and f, respectively. Then we have

$$L\hat{V} - \sum_{\nu=2}^{n} q_\nu^2 \hat{B}_2 * \hat{V} + \sum_\nu q_\nu^2 \kappa \hat{B}_1 * \zeta^\kappa \hat{V} + \hat{f}(\zeta, q) = 0, \tag{115}$$

where

$$L\hat{V} := \left(\kappa \zeta^\kappa + \sum_{\nu=2}^{n} \lambda_\nu q_\nu \frac{\partial}{\partial q_\nu} \right) \hat{V}, \tag{116}$$

and $\hat{B}_1 := \widehat{B_{\nu,1}}$ and $\hat{B}_2 := \widehat{q_1^{2\sigma} \partial_{q_1} B_{\nu,1}}$ are the κ-Borel transform of $B_{\nu,1}$ and $q_1^{2\sigma} \partial_{q_1} B_{\nu,1}$, respectively.

Define $L\hat{V} =: \hat{W}$. Since L^{-1} exists, by Lemma 4 we have

$$\hat{W} - \sum_\nu q_\nu^2 \hat{B}_2 * L^{-1}\hat{W} + \sum_\nu q_\nu^2 \kappa \hat{B}_1 * \zeta^\kappa L^{-1}\hat{W} + \hat{f}(\zeta, q) = 0. \tag{117}$$

Define

$$L_1\hat{W} := \kappa \hat{B}_1 * \zeta^\kappa L^{-1}\hat{W}, \quad L_2\hat{W} := \hat{B}_2 * L^{-1}\hat{W}. \tag{118}$$

Then, by Eq. 117 we obtain

$$\hat{W} + \sum_\nu q_\nu^2 (L_1 - L_2)\hat{W} + \hat{f}(\zeta, q) = 0. \tag{119}$$

Step 3. We construct the approximate sequence

$$\hat{W}_0 := -\hat{f}(\zeta, q), \quad \hat{W}_j := -\sum_\nu q_\nu^2 (L_1 - L_2)\hat{W}_{j-1}, \ j = 1, 2, \ldots \tag{120}$$

If the limit $\hat{W} := \sum_{j=0}^\infty \hat{W}_j$ exists in $\mathcal{A}_c^{(\kappa)}$ with R_j replaced by $R'_j < R_j$, $r_j - R'_j$ small, then $\hat{W}$ is the desired solution.

We show the convergence. By the estimate of a convolution and Eq. 118 we have

$$\begin{aligned} \|L_1\hat{W}\|_c &= \|M_0^{-1} M_0 L_1 \hat{W}\|_c \le K_1 \|M_0 L_1 \hat{W}\|_c = K_1 \||L_1\hat{W}|\|_c \\ &\le K_1 K_2 \||\zeta^\kappa L^{-1}\hat{W}|\|_c \le K_1 K_2 K \|\hat{W}\|_c, \end{aligned} \tag{121}$$

where $K_1 = \sup_q |M_0(q)^{-1}|$, $K_2 = \|\kappa \hat{B}_1\|_c$. Similarly, we have $\|L_2\hat{W}\|_c \le K_1 K_2 K \|\hat{W}\|_c$. Let $\epsilon > 0$ be given. By taking $R_j > 0$ sufficiently small we have

$$\|\hat{W}_j\|_c \le K_1 K_2 K \epsilon \|\hat{W}_{j-1}\|_c, \quad j = 1, 2, \ldots \tag{122}$$

The estimate Eq. 122 implies the desired convergence of the sequence.

7 Proof of Theorem 5

Proof *(Proof of Theorem 5)* We prove the theorem by five steps.

Step 1. Consider Eqs. 47, 48 and 49. These equations are equivalent to Eqs. 55, 58 and 61. Hence, if we show the summability of q we have the summability of p and p_1 as well.

Set $\kappa = 2\sigma - 1$. Let t and z satisfy $t = -\kappa^{-1} z^{-\kappa}$. Let z_0 be such that $Re(\lambda_j z_0^{-\kappa}) > 0$. Define Σ_0 by Eq. 103 with $d = \arg z_0$. We show that there exist constants $C_0 > 0$, $C_1 > 0$

and $\eta > 0$ such that, for Σ_0 given by Eq. 103 with $d = \arg z_0$ we have

$$\left|\kappa z^{\kappa} + \lambda_j k\right| \geq C_0 |z|^{\kappa}, \quad \forall z, \ |z| > C_1, z \in \Sigma_0, \tag{123}$$

$$\left|\kappa z^{\kappa} + \lambda_j k\right| > C_0, \quad \forall z, \ |z| \leq C_1, \ z \in \Sigma_0, \tag{124}$$

for $j = 2, \ldots, n$ and $k = 1, 2, \ldots$. Indeed, we have

$$\left|\kappa z^{\kappa} \lambda_j^{-1} + k\right| = \left|\frac{\kappa}{\lambda_j z^{-\kappa}} + k\right| = \left|-\frac{1}{\lambda_j t} + k\right| > C_0/|\lambda_j|, \tag{125}$$

for some $C_0 > 0$ and all t, $Re(\lambda_j t) < 0$, $j = 2, \ldots, n$, $k = 1, 2, \ldots$ Eqs. 124 follows from 125.

We show Eq. 123. We have

$$\left|\kappa z^{\kappa} \lambda_j^{-1} + k\right| \geq \left|Re\left(\kappa z^{\kappa} \lambda_j^{-1}\right) + k\right| \geq Re\left(\kappa z^{\kappa} \lambda_j^{-1}\right) \geq \epsilon_1 |z|^{\kappa}, \tag{126}$$

for some $\epsilon_1 > 0$ and all $z \in \Sigma_0$ and all $k = 1, 2, \ldots$, $j = 2, \ldots, n$.

Step 2. We solve Eq. 61. Let ζ be the dual variable of z. Let

$$q_0 = \sum_{n=0}^{\infty} c_n(\xi) z^n \tag{127}$$

be the formal series solution of Eq. 61. Define $\tilde{q}$ by

$$q = \tilde{q} + \rho, \quad \rho = \sum_{n=0}^{\kappa-1} c_n(\xi) z^n. \tag{128}$$

Clearly we have $\tilde{q} = O(z^{\kappa})$. Let $G = (G_j)$ be given by Eq. 59. Next, consider

$$G(z, q, \xi) = \sum_{n=0}^{\infty} z^n s_n(q, \xi) = \sum_{n,\ell} s_{n,\ell}(\xi) z^n q^{\ell},$$

where the Taylor expansions of $s_n(q, \xi)$'s with respect to q converge in some neighborhood of the origin $q = 0$ and $\xi = 0$ independent of n. Then we consider

$$\sum_n z^n s_n(\tilde{q} + \rho, \xi) - \sum_{n=0}^{\kappa-1} c_n(\xi) z^n = \sum_{n,\ell} s_{n,\ell}(\xi) z^n (\tilde{q} + \rho)^{\ell} - \sum_{n=0}^{\kappa-1} c_n(\xi) z^n. \tag{129}$$

We note that Eq. 129 is $O(z^{\kappa})$ by the definition of formal series. By the formula $(\tilde{q} + \rho)^{\ell} = \sum_{\beta} \tilde{q}^{\beta} \rho^{\ell-\beta} \ell!/(\beta!(\ell-\beta)!)$ the formal κ-Borel transform of the left-hand side of Eq. 129 converges. Denoting the right-hand side of Eq. 129 by $G(z, \tilde{q}, \xi)$ for the sake of simplicity and rewriting $\tilde{q}$ as q we consider

$$q = G(z, q, \xi). \tag{130}$$

We note that the κ- Borel transform of $\frac{\partial G}{\partial q}(z, 0, \xi)$ in $A_c^{(\kappa)}$ is sufficiently small for sufficiently small ξ by the definition of G.

Step 3. $G(z, q, \xi)$ is a formal power series in z with coefficients being holomorphic in ξ and q in some neighborhood of the origin $\xi = 0$, $q = 0$ which is uniform among the coefficients. By expanding the coefficients in the power series of ξ and q and rearranging them we obtain the series of ξ and q whose coefficients are formal series in z. We show that the coefficients of the series of G with respect to ξ and q are summable in z which are uniform among the coefficients. We denote the uniform summability property by (P).

By the definition of $G = (G_j)$ it is sufficient to show that (P) holds for the first integrals constructed in Theorem 2. Indeed, since G_j is given by Eq. 59, it is sufficient to show that $(\tilde{T}_j + F_{j,0})^{-1}$ and $\tilde{A}_j(z, z^2 p_1, q)$ satisfy (P). The latter term satisfies (P) if the first integral satisfies (P). As for the former one, consider $F_{j,0}$ in Eq. 55. The terms $\tilde{D}_j$ and $\tilde{D}_{j,1}$ satisfy (P) by the assumption on the first integral. By Eq. 53 we see that $E_{j,1}$ satisfies (P). Next, by Eq. 54 $\tilde{T}_j$ satisfies (P). It follows that $(\tilde{T}_j + F_{j,0})^{-1}$ satisfies (P). Hence G_j satisfies (P).

Let $C(z, q, \xi)$ be any formal first integral constructed in Theorem 2. For every pair of multiintegers $m \geq 0, n \geq 0$ we consider the coefficient of $q^m \xi^n$ of the Taylor series of $C(z, q, \xi)$

$$C_{m,n}(z) = \frac{1}{(2\pi i)^2} \int\int_{|w_j|=\epsilon_1, |s_\nu|=\epsilon_2} \frac{C(z, w, s)}{w^{m+1} s^{n+1}} dw ds, \tag{131}$$

where $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are small constants. Let $\hat{C}(\zeta, w, \xi)$ be the formal Borel transform of $C(z, w, \xi)$ with respect to z, where ζ is the dual variable of z. By the formal Borel transform of Eq. 131 we have

$$\hat{C}_{m,n}(\zeta) = \frac{1}{(2\pi i)^2} \int\int_{|w_j|=\epsilon_1, |s_\nu|=\epsilon_2} \frac{\hat{C}(\zeta, w, s)}{w^{m+1} s^{n+1}} dw ds. \tag{132}$$

Since $C(z, w, s)$ is Borel summable, there exist Σ_0 in Eq. 103 and the neighborhoods V_0 and V_1 of $q = 0$ and $\xi = 0$, respectively, such that $\hat{C}(\zeta, w, s)$ is holomorphic in $(\zeta, w, s) \in \Sigma_0 \times V_0 \times V_1$. Moreover, $\hat{C}(\zeta, w, s)$ is of exponential order of one in $\zeta \in \Sigma_0$ for every $(w, s) \in V_0 \times V_1$. By the scale change of variables $q \mapsto \epsilon q$ and $\xi \mapsto \epsilon \xi$ we may assume that V_0 and V_1 contain a disk with sufficiently large radius. Therefore, by Eq. 132 we have the summability of $C_{m,n}(z)$ uniformly in m and n. In the following we assume the condition.

Step 4. We prove the summability of q as a transseries. It is sufficient to show the summability with respect to the variable z instead of t. Expand $c_n(\xi)$ in Eq. 127 in the power series of ξ and consider

$$q(z) = \sum_{j \geq 0} \xi^j q_j(z). \tag{133}$$

By Eq. 128 it is sufficient to show the summability of q in Eq. 130. Note that, by the definition of the summability of a transseries it is sufficient to show the uniform summability of q_j's and the convergence of the sum Eq. 133 with q_j replaced by its Borel sum.

If $j = 0$, then the summability of $q_0 \equiv 0$ is trivial. Suppose that the uniform summability of q_j for $j = 0, \ldots, k-1$ holds. Namely, the formal Borel transform of q_j, $\hat{q}_j$ is holomorphic in Σ_0 and has the same exponential order for $j = 0, \ldots, k-1$. Consider q_k. Substitute Eqs. 133 into 130.

Since G is analytic at $q = 0$ we consider the term

$$C_\ell(\xi, z)(\sum_{j, |j|>0} q_j \xi^j)^\ell, \tag{134}$$

where $\ell \geq 0$ is a multiinteger and $C_\ell(\xi, z)$ is analytic in ξ and is a formal power series in z. Expand $C_\ell(\xi, z)$ in a power series in ξ, $C_\ell(\xi, z) = \sum_{|\nu| \geq 1} K_{\ell,\nu}(z) \xi^\nu$. We introduce the weight ϵ_0^j in front of q_j by the scale change $\xi \mapsto \epsilon_0 \xi$ (cf. step 4), where $\epsilon_0 > 0$ is a sufficiently small number. Then the coefficient of ξ^k appearing from $G(z, q, \xi)$ is given by

$$\sum \frac{K_{\ell,\nu}(z)\ell!\epsilon_0^{|k|}}{m_1!\cdots m_\mu!} q_{j_1}^{m_1} q_{j_2}^{m_2} \cdots q_{j_\mu}^{m_\mu}, \tag{135}$$

where the summation is taken over the pair of multiintegers, $m_1, \ldots, m_\mu$ satisfying

$$m_1 + \cdots + m_\mu = \ell, \quad j_1|m_1| + j_2|m_2| + \cdots + j_\mu|m_\mu| = k - \nu, \tag{136}$$

where μ is an integer and $j_1, \ldots, j_\mu \geq 0$ are multiintegers. By the result of Step 3 $K_{\ell,\nu}(z)$ is uniformly summable in ℓ and ν and $\sum_{\ell,\nu} \|K_{\ell,\nu}\| < \infty$. By Eqs. 135 and 130 we see that the formal Borel transform of $q_k(z)$, $\hat{q}_k(\zeta)$ is holomorphic in Σ_0 and has the same exponential order as $\hat{q}_j$'s.

It remains to estimate $\|q_k\|$, where $\|q_k\|$ is a certain maximal norm. Suppose that

$$\|q_j\| \leq K_1\epsilon_2^{|j|}, \quad |j| < |k|, \tag{137}$$

for some positive constants K_1 and ϵ_2, where ϵ_2 is chosen sufficiently small. Take $\epsilon_0 \leq 1$ and $2\epsilon_0 < \epsilon_2$. We have

$$\begin{aligned} &\sum \frac{\ell!}{m_1!\cdots m_\mu!}(\|q_{j_1}\|)^{m_1}(\|q_{j_2}\|)^{m_2}\cdots(\|q_{j_\mu}\|)^{m_\mu} \\ &\leq \sum \frac{\ell!}{m_1!\cdots m_\mu!}(K_1\epsilon_2^{|j_1|})^{|m_1|}(K_1\epsilon_2^{|j_2|})^{|m_2|}\cdots(K_1\epsilon_2^{|j_\mu|})^{|m_\mu|} \\ &\leq (K_1 \sum_{j,|j|>0} \epsilon_2^{|j|})^{|\ell|} \leq (CK_1\epsilon_2)^{|\ell|}, \end{aligned} \tag{138}$$

where the summation is taken over all combinations satisfying Eq. 136 and where C satisfies $\sum_{j,|j|>0} \epsilon_2^{|j|} \leq C\epsilon_2$.

Then the term Eq. 135 is estimated by

$$\epsilon_2^k \sum_{\ell,\nu}(CK_1\epsilon_2)^{|\ell|}\|K_{\ell,\nu}\|. \tag{139}$$

By taking ϵ_2 sufficiently small we have

$$\sum_{\ell\neq 0,\nu}(CK_1\epsilon_2)^{|\ell|}\|K_{\ell,\nu}\| \leq \frac{K_1}{2}. \tag{140}$$

On the other hand, we may assume $\|K_{0,\nu}\| \leq K_1/2$ since $\nu \geq 1$. Hence Eq. 139 is estimated by $K_1\epsilon_2^n$, which proves the convergence of the sum.

Step 5. We prove the summability of q_1. If we prove the summability of q_1 and q, then we have the summability of p_1 and p as well by Eqs. 58 and 55, respectively. Let $q(z)$ be given by Eq. 133. Consider

$$\frac{dq_1}{dt} = H_{p_1} = q_1^{2\sigma}\left(1 + \sum q_\nu^2 B_{1,\nu}(q_1, q)\right). \tag{141}$$

Introduce z by Eq. 64.

Let q be given by Eq. 133. Set

$$q = q^{(0)} + \tilde{q}, \quad q^{(0)} = \sum_{|k|>0} q_k^{(0)}(z)\xi^k, \tag{142}$$

where $q^{(0)}$ is the polynomial of z with degree $\kappa - 1$ and analytic in ξ at $\xi = 0$. Expand

$$\tilde{q} = \sum_{|k|>0} \tilde{q}_k(z)\xi^k. \tag{143}$$

By what we have proved in the above, the κ- Borel transform of $\tilde{q}_k(z)$ exists for every k. Substitute Eq. 142 into the right-hand side of Eq. 141. Since B_1 is the polynomial in q by assumption we see that the right-hand side of Eq. 141 is the polynomial of $\tilde{q}$ whose coefficients are the polynomials of z and analytic in q_1 and ξ.

Substitute the Taylor series of q_1

$$q_1(z) = \sum_{|j|\ge 0} \xi^j q_{1,j}(z), \tag{144}$$

into Eq. 141 and compare the coefficients of the power ξ^k with $k = 0$ of both sides of Eq. 141. We have $q_{1,0} = A_0$, where A_0 is given by Eq. 71. One can easily verify that $A_0 = z$ by the definition of z. By Eq. 72 we have $q_{1,1} = A_1 = 0$. Let $k \ge 2$. By Eq. 76 and $A_k = q_{1,k}$ we see that the left-hand side of Eq. 141 yields $\frac{q_{1,k}}{dt} + k\lambda q_{1,k}$. It is equal to

$$z^{\kappa+1}\frac{q_{1,k}}{dz} + k\lambda q_{1,k}.$$

Therefore $q_{1,k}$'s satisfy the recurrence relation

$$z^{\kappa+1}\frac{d}{dz}q_{1,k} + k\lambda q_{1,k} = c_0 z^{\kappa} q_{1,k} + f\left(z, \xi, q_{1,j}, \tilde{q}_\ell;\ j < k, \ell < k\right), \tag{145}$$

where f is the polynomial of $q_{1,j}$'s, $\tilde{q}_\ell$'s and z and analytic in ξ at $\xi = 0$, and where c_0 is a constant. By the scale change of z we may assume that c_0 is sufficiently small.

In order to see the form of f, note that the partial Taylor expansion of the right-hand side of Eq. 141 has the form

$$C_{\ell,\nu}(z, \xi)(\sum_{|j|>0} q_{1,j}\xi^j)^{2\sigma+\ell}(\sum_{|k|>0} \tilde{q}_k\xi^k)^{\nu}, \tag{146}$$

for some non-negative integer ℓ and a multiinteger ν with $|\nu| \ge 2$. Expand

$$C_{\ell,\nu}(z, \xi) = \sum_{s\ge 0} C_{\ell,\nu,s}(z)\xi^s. \tag{147}$$

Then, by comparing the coefficients of the power ξ^k in Eq. 146 we have

$$\sum C_{\ell,\nu,s}(z)\frac{\nu!}{\alpha_1!\cdots\alpha_i!}\frac{(2\sigma+\ell)!}{\mu_1!\cdots\mu_m!}(q_{1,j_1})^{\mu_1}\cdots(q_{1,j_m})^{\mu_m}(\tilde{q}_{k_1})^{\alpha_1}\cdots(\tilde{q}_{k_i})^{\alpha_i}, \tag{148}$$

where the summation is taken over the combinations

$$\begin{aligned}&\mu_1 + \mu_2 + \cdots + \mu_m = 2\sigma + \ell, \quad \alpha_1 + \alpha_2 + \cdots + \alpha_i = \nu \\ &j_1\mu_1 + j_2\mu_2 + \cdots + j_m\mu_m + k_1|\alpha_1| + k_2|\alpha_2| + \cdots + k_i|\alpha_i| = k - s,\ 0 \le s \le k.\end{aligned} \tag{149}$$

By Step 1 of the proof of Theorem 3 we have $q_{1,j} = O(t^{-1-1/\kappa}) = O(z^{\kappa+1})$. Hence the formal κ-Borel transform of $q_{1,j}$ is well defined. By the formal κ-Borel transform of Eq. 145 we have

$$(\kappa\zeta^{\kappa} + k\lambda)\widehat{q_{1,k}} = c_0\left(1 * \widehat{q_{1,k}}\right) + f^*\left(\zeta, \xi, \widehat{q_{1,j}}, \widehat{\tilde{q}_\ell};\ j < k, \ell < k\right), \tag{150}$$

where $\widehat{q_{1,k}}$ is the formal κ- Borel transform of $q_{1,k}$ and $*$ denotes the convolution. f^* denotes the formal κ- Borel transform of f, where the product is understood as the convolution product. Define $v := \widehat{q_{1,k}}$ and denote the second term of the right-hand side of Eq. 150 by g_k, for simplicity. Then Eq. 150 is written as

$$(\kappa\zeta^{\kappa} + k\lambda)v = c_0\,(1 * v) + g_k. \tag{151}$$

We define the sequence $v_0, v_1, \cdots$ by

$$(\kappa\zeta^{\kappa} + k\lambda)v_0 = g_k \tag{152}$$

$$(\kappa\zeta^{\kappa} + k\lambda)v_j = c_0(1 * v_{j-1}), \quad j = 1, 2, \ldots \tag{153}$$

If the limit $v := v_0 + v_1 + \cdots$ exists, then v gives the solution of Eq. 150. By definition we have the summability of $q_{1,k}$. It is easy to see the convergence of v.

In order to see the summability of q_1 as a transseries we need to show the uniform summmability of $q_{1,k}$'s in k. Since we have the uniform summability of q_k's, we can inductively show that g_k is holomorphic in the domain uniform in k with exponential growth order being uniform in k. In view of the definition of the approximate sequence of v in Eqs. 152 and 153 the same property holds for $v = \widehat{q_{1,k}}$.

Next we show the convergence of $q_1 = \sum_k q_{1,k}\xi^k$. In order to have the estimate of $\|q_{1,k}\|$ it is sufficient to estimate $\|g_k\|$ in view of the definition of the approximate sequence of $q_{1,k}$. The estimate of g_k follows from the expression Eq. 148. Since the expression Eq. 148 has the same form as Eq. 135, we can estimate Eq. 148 by the same argument as the estimate of Eq. 135 in the step 4. Hence we can show the estimate of $q_{1,k}$ by the same calculation in proving Eq. 137. Therefore we have the summability of q_1 as a transseries. This ends the proof.

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