Journal of Dynamical and Control Systems (2023) 29:521–568
https://doi.org/10.1007/s10883-022-09604-2

A Gradient Flow Equation for Optimal Control Problems With End-point Cost

A. Scagliotti[1]

Received: 25 July 2021 / Revised: 29 May 2022 / Accepted: 4 June 2022 / Published online: 7 July 2022

Abstract
In this paper, we consider a control system of the form $\dot{x} = F(x)u$, linear in the control variable u. Given a fixed starting point, we study a finite-horizon optimal control problem, where we want to minimize a weighted sum of an end-point cost and the squared 2-norm of the control. This functional induces a gradient flow on the Hilbert space of admissible controls, and we prove a convergence result by means of the Lojasiewicz-Simon inequality. Finally, we show that, if we let the weight of the end-point cost tend to infinity, the resulting family of functionals is Γ-convergent, and it turns out that the limiting problem consists in joining the starting point and a minimizer of the end-point cost with a horizontal length-minimizer path.

Keywords Gradient flow · Optimal control · End-point cost · Lojasiewicz-Simon inequality · Γ-convergence.

Mathematics Subject Classification (2010) Primary 34H05 · 35K55 · 49J45 · Secondary 53C17

1 Introduction

In this paper, we consider a control system of the form

$$\dot{x} = F(x)u, \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ is a Lipschitz-continuous function, and $u \in \mathbb{R}^k$ is the control variable. If $k \le n$, for every $x \in \mathbb{R}^n$, we may think of the columns $\{F^i(x)\}_{i=1,\dots,k}$ of the matrix $F(x)$ as an ortho-normal frame of vectors, defining a sub-Riemannian structure on $\mathbb{R}^n$. For a thorough introduction to the topic, we refer the reader to the monograph [4]. In our framework, $\mathcal{U} := L^2([0, 1], \mathbb{R}^k)$ will be the space of admissible controls, equipped with

✉ A. Scagliotti
ascaglio@sissa.it

[1] Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy

the usual Hilbert space structure. Given a base-point $x_0 \in \mathbb{R}^n$, for every $u \in \mathcal{U}$, we consider the absolutely continuous trajectory $x_u : [0, 1] \to \mathbb{R}^n$ that solves

$$\begin{cases} \dot{x}_u(s) = F(x_u(s))u(s) \ \text{ for a.e. } s \in [0, 1], \\ x_u(0) = x_0. \end{cases} \tag{1.2}$$

For every $\beta > 0$ and $x_0 \in \mathbb{R}^n$, we define the the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ as follows:

$$\mathcal{F}^\beta(u) := \frac{1}{2}||u||^2_{\mathcal{U}} + \beta a(x_u(1)), \tag{1.3}$$

where $a : \mathbb{R}^n \to \mathbb{R}_+$ is a non-negative C^1-regular function, and $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of Eq. 1.2 corresponding to the control $u \in \mathcal{U}$. In this paper we want to investigate the gradient flow induced by the functional $\mathcal{F}^\beta$ on the Hilbert space $\mathcal{U}$, i.e., the evolution equation

$$\partial_t U_t = -\mathcal{G}^\beta[U_t], \tag{1.4}$$

where $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ is the vector field on the Hilbert space $\mathcal{U}$ that represents the differential $d\mathcal{F}^\beta : \mathcal{U} \to \mathcal{U}^*$ through the Riesz's isometry. In other words, for every $u \in \mathcal{U}$, we denote by $d_u\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}$ the differential of $\mathcal{F}^\beta$ at u, and $\mathcal{G}^\beta[u]$ is defined as the only element of $\mathcal{U}$ such that the identity

$$\langle \mathcal{G}^\beta[u], v\rangle_{L^2} = d_u\mathcal{F}^\beta(v) \tag{1.5}$$

holds for every $v \in \mathcal{U}$. In order to avoid confusion, we use different letters to denote the time variable in the control system Eq. 1.2 and in the evolution equation Eq. 1.4. Namely, the variable $s \in [0, 1]$ will be exclusively used for the control system Eq. 1.2, while $t \in [0, +\infty)$ will be employed only for the gradient flow Eq. 1.4 and the corresponding trajectories. Moreover, when dealing with operators taking values in a space of functions, we express the argument using the square brackets.

The first part of the paper is devoted to the formulation of the gradient flow equation Eq. 1.4. In particular, we first study the differentiability of the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$, then we introduce the vector field $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ as the representation of its differential, and finally we show that, under suitable assumptions, $\mathcal{G}^\beta$ is locally Lipschitz-continuous. As a matter of fact, it turns out that Eq. 1.4 can be treated as an infinite-dimensional ODE, and we prove that, for every initial datum $U_0 = u_0$, the gradient flow equation Eq. 1.4 admits a unique continuously differentiable solution $U : [0, +\infty) \to \mathcal{U}$. In the central part of this contribution, we focus on the asymptotic behavior of the curves that solve Eq. 1.4. The main result states that, if the application $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ that defines the linear-control system Eq. 1.1 is real-analytic as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ that provides the end-point term in Eq. 1.3, then, for every $u_0 \in H^1([0, 1], \mathbb{R}^k) \subset \mathcal{U}$, the curve $t \mapsto U_t$ that solves the gradient flow equation Eq. 1.4 with initial datum $U_0 = u_0$ satisfies

$$\lim_{t\to+\infty} ||U_t - u_\infty||_{L^2} = 0, \tag{1.6}$$

where $u_\infty \in \mathcal{U}$ is a critical point for $\mathcal{F}^\beta$. To establish this fact we first show that the functional $\mathcal{F}^\beta$ satisfies the Lojasiewicz-Simon inequality. Finally, in the last part of this work, we prove a Γ-convergence result concerning the family of functionals $(\mathcal{F}^\beta)_{\beta\in\mathbb{R}_+}$. In particular, we show that, when $\beta \to +\infty$, the limiting problem consists in minimizing the L^2-norm of the controls that steer the initial point x_0 to the set $\{x \in \mathbb{R}^n : a(x) = 0\}$. This fact can be applied, for example, to approximate the problem of finding a sub-Riemannian length-minimizer curve that joins two assigned points.

We report below in detail the organization of the sections.

In Section 2, we introduce the linear-control system Eq. 1.1 and we establish some preliminary results that will be used throughout the paper. In particular, in Subsection 2.2, we focus on the first variation of a trajectory when a perturbation of the corresponding control occurs. In Subsection 2.3, we study the second variation of the trajectories at the final evolution instant.

In Section 3, we prove that, for every intial datum $u_0 \in \mathcal{U}$, the evolution equation Eq. 1.4 gives a well-defined Cauchy problem whose solutions exist for every $t \geq 0$. To see that, we use the results obtained in Subsection 2.2 to introduce the vector field $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ satisfying Eq. 1.5 and to prove that it is Lipschitz-continuous when restricted to the bounded subsets of $\mathcal{U}$. Combining this fact with the theory of ODEs in Banach spaces (see, e.g., [10]), it descends that, for every choice of the initial datum $U_0 = u_0$, the evolution equation Eq. 1.4 admits a unique and locally defined solution $U : [0, \alpha) \to \mathcal{U}$, with $\alpha > 0$. Using the particular structure of the gradient flow Eq. 1.4, we finally manage to extend these solutions for every positive time.

In Section 4, we show that, if the Cauchy datum u_0 has Sobolev regularity (i.e., $u_0 \in H^m([0, 1], \mathbb{R}^k) \subset \mathcal{U}$ for some positive integer m), then the curve $t \mapsto U_t$ that solves Eq. 1.4 and satisfies $U_0 = u_0$ is pre-compact in $\mathcal{U}$. The key-observation lies in the fact that, under suitable regularity assumptions on $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ and $a : \mathbb{R}^n \to \mathbb{R}_+$, the Sobolev space $H^m([0, 1], \mathbb{R}^k)$ is invariant for the gradient flow Eq. 1.4. Moreover, we obtain that, when the Cauchy datum belongs to $H^m([0, 1], \mathbb{R}^k)$, the curve $t \mapsto U_t$ that solves Eq. 1.4 is bounded in the H^m-norm.

In Section 5, we establish the Lojasiewicz-Simon inequality for the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$, under the assumption that $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ and $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic. We recall that the first result on the Lojasiewicz inequality dates back to 1963, when in [11] Lojasiewicz proved that, if $f : \mathbb{R}^d \to \mathbb{R}$ is a real-analytic function, then for every $x \in \mathbb{R}^d$ there exist $\gamma \in (1, 2]$, $C > 0$ and $r > 0$ such that

$$|f(y) - f(x)| \leq C|\nabla f(y)|_2^\gamma \tag{1.7}$$

for every $y \in \mathbb{R}^d$ satisfying $|y - r|_2 < r$. This kind of inequalities are ubiquitous in several branches of Mathematics. For example, as suggested by Lojasiewicz in [11], Eq. 1.7 can be employed to study the convergence of the solutions of

$$\dot{x} = -\nabla f(x).$$

Another important application can be found in [12], where Polyak studied the convergence of the gradient descent algorithm for strongly convex functions using a particular instance of Eq. 1.7, which is sometimes called Polyak-Lojasiewicz inequality. In [13], Simon extended Eq. 1.7 to real-analytic functionals defined on Hilbert spaces, and he employed it to establish convergence results for evolution equations. For further details, see also the lecture notes [14]. The infinite-dimensional version of Eq. 1.7 is often called Lojasiewicz-Simon inequality. For a complete survey on the topic, we refer the reader to the paper [7]. Following this approach, the Lojasiewicz-Simon inequality for the functional $\mathcal{F}^\beta$ is the cornerstone for the convergence result of the subsequent section.

In Section 6, we prove that, if the Cauchy datum belongs to $H^m([0, 1], \mathbb{R}^k)$ for an integer $m \geq 1$, the corresponding gradient flow trajectory converges to a critical point of $\mathcal{F}^\beta$. This result requires that both $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ and $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic. Indeed, we use the Lojasiewicz-Simon inequality for $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ to show that the solutions of Eq. 1.4 with Sobolev-regular initial datum have finite length. This fact immediately yields Eq. 1.6.

In Section 7, we study the behavior of the minimization problem Eq. 1.3 when the positive parameter β tends to infinity. We address this problem using the tools of the Γ-convergence (see [8] for a complete introduction to the subject). In particular, we consider

$\mathcal{U}_\rho := \{u \in \mathcal{U} : ||u||_{L^2} \leq \rho\}$ and we equip it with the topology of the weak convergence of $\mathcal{U}$. For every $\beta > 0$, we introduce the restrictions $\mathcal{F}^\beta_\rho := \mathcal{F}^\beta|_{\mathcal{U}_\rho}$, and we show that there exists a functional $\mathcal{F}_\rho : \mathcal{U}_\rho \to \mathbb{R}_+ \cup \{+\infty\}$ such that the family $\left(\mathcal{F}^\beta_\rho\right)_{\beta \in \mathbb{R}_+}$ Γ-converges to $\mathcal{F}_\rho$ as $\beta \to +\infty$. In the case $a : \mathbb{R}^n \to \mathbb{R}_+$ admits a unique point $x_1 \in \mathbb{R}^n$ such that $a(x_1) = 0$, then the limiting problem of minimizing the functional $\mathcal{F}_\rho$ consists in finding (if it exists) a control $u \in \mathcal{U}_\rho$ with minimal L^2-norm such that the corresponding curve $x_u : [0, 1] \to \mathbb{R}^n$ defined by Eq. 1.2 satisfies $x_u(1) = x_1$. The final result of Section 7 guarantees that the minimizers of $\mathcal{F}^\beta_\rho$ provide L^2-strong approximations of the minimizers of $\mathcal{F}_\rho$.

2 Framework and Preliminary Results

In this paper, we consider control systems on $\mathbb{R}^n$ with linear dependence in the control variable $u \in \mathbb{R}^k$, i.e., of the form

$$\dot{x} = F(x)u, \tag{2.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ is a Lipschitz-continuous function. We use the notation F^i for $i = 1, \dots, k$ to indicate the vector fields on $\mathbb{R}^n$ obtained by taking the columns of F, and we denote by $L > 0$ the Lipschitz constant of these vector fields, i.e., we set

$$L := \sup_{i=1,\dots,k} \sup_{x,y\in\mathbb{R}^n} \frac{|F^i(x) - F^i(y)|_2}{|x - y|_2}. \tag{2.2}$$

We immediately observe that Eq. 2.2 implies that the vector fields $F^1, \dots, F^k$ have sub-linear growth, i.e., there exists $C > 0$ such that

$$\sup_{i=1,\dots,k} |F^i(x)| \leq C(|x|_2 + 1) \tag{2.3}$$

for every $x \in \mathbb{R}^n$. Moreover, for every $i = 1, \dots, k$, if F^i is differentiable at $y \in \mathbb{R}^n$, then from Eq. 2.2 we deduce that

$$\left| \frac{\partial F^i(y)}{\partial x} \right|_2 \leq L. \tag{2.4}$$

We define $\mathcal{U} := L^2([0, 1], \mathbb{R}^k)$ as the space of admissible controls, and we endow $\mathcal{U}$ with the usual Hilbert space structure, induced by the scalar product

$$\langle u, v\rangle_{L^2} = \int_0^1 \langle u(s), v(s)\rangle_{\mathbb{R}^k}\, ds. \tag{2.5}$$

Given $x_0 \in \mathbb{R}^n$, for every $u \in \mathcal{U}$, let $x_u : [0, 1] \to \mathbb{R}^n$ be the absolutely continuous curve that solves the following Cauchy problem:

$$\begin{cases} \dot{x}_u(s) = F(x_u(s))u(s) \text{ for a.e. } s \in [0, 1], \\ x_u(0) = x_0. \end{cases} \tag{2.6}$$

We recall that, under the condition Eq. 2.2, the existence and uniqueness of the solution of Eq. 2.6 is guaranteed by Carathéodory Theorem (see, e.g., [9, Theorem 5.3]). We insist on the fact that in this paper the Cauchy datum $x_0 \in \mathbb{R}^n$ is assumed to be assigned.

In the remainder of this section, we introduce auxiliary results that will be useful in the other sections. In Subsection 2.1, we recall some results concerning Sobolev spaces in one-dimensional domains. In Sections 2.2 and 2.3, we investigate the properties of the solutions of Eq. 2.6.

2.1 Sobolev Spaces in One Dimension

In this subsection, we recall some results for one-dimensional Sobolev spaces. Since in this paper we work only in Hilbert spaces, we shall restrict our attention to the Sobolev exponent $p = 2$, i.e., we shall state the results for the Sobolev spaces $H^m := W^{m,2}$ with $m \geq 1$. For a complete discussion on the topic, the reader is referred to [6, Chapter 8]. Throughout the paper we use the convention $H^0 := L^2$. For every $m \geq 1$, the function $u \in L^2([a,b],\mathbb{R}^d)$ belongs to the Sobolev space $H^m([a,b],\mathbb{R}^d)$ if and only if, for every integer $1 \leq \ell \leq m$ there exists $u^{(\ell)} \in L^2([a,b],\mathbb{R}^d)$, the ℓ-th Sobolev derivative of u. We recall that, for every $m \geq 1$, $H^m([a,b],\mathbb{R}^d)$ is a Hilbert space (see, e.g., [6, Proposition 8.1]) when it is equipped with the norm $||\cdot||_{H^m}$ induced by the scalar product $\langle u, v\rangle_{H^m} := \langle u, v\rangle_{L^2} + \sum_{\ell=1}^m \int_a^b \langle u^{(\ell)}(s), v^{(\ell)}(s)\rangle_{\mathbb{R}^d}\, ds$. We recall that a linear and continuous application $T : E_1 \to E_2$ between two Banach spaces E_1, E_2 is *compact* if, for every bounded set $B \subset E_1$, the image $T(B)$ is pre-compact with respect to the strong topology of E_2. In the following result, we list three classical compact inclusions.

Theorem 2.1 *For every $m \geq 1$, the following inclusions are compact:*

$$H^m([a,b],\mathbb{R}^d) \hookrightarrow L^2([a,b],\mathbb{R}^d), \tag{2.7}$$

$$H^m([a,b],\mathbb{R}^d) \hookrightarrow C^0([a,b],\mathbb{R}^d), \tag{2.8}$$

$$H^m([a,b],\mathbb{R}^d) \hookrightarrow H^{m-1}([a,b],\mathbb{R}^d), \tag{2.9}$$

Finally, we recall the notion of *weak convergence*. For every $m \geq 0$ (we set $H^0 := L^2$), if $(u_n)_{n\geq 1}$ is a sequence in $H^m([0,1],\mathbb{R}^d)$ and $u \in H^m([0,1],\mathbb{R}^d)$, then the sequence $(u_n)_{n\geq 1}$ weakly converges to u if and only if

$$\lim_{n\to\infty} \langle v, u_n\rangle_{H^m} = \langle v, u\rangle_{H^m}$$

for every $v \in H^m([0,1],\mathbb{R}^d)$, and we write $u_n \rightharpoonup_{H^m} u$ as $n \to \infty$. Finally, in view of the compact inclusion Eq. 2.9 and of [6, Remark 6.2], for every $m \geq 1$, if a sequence $(u_n)_{n\geq 1}$ in $H^m([0,1],\mathbb{R}^d)$ satisfies $u_n \rightharpoonup_{H^m} u$ as $n \to \infty$, then

$$\lim_{n\to\infty} ||u_n - u||_{H^{m-1}} = 0.$$

2.2 General Properties of the Linear-Control System Eq. 2.1

In this subsection, we investigate basic properties of the solutions of Eq. 2.6, with a particular focus on the relation between the admissible control $u \in \mathcal{U}$ and the corresponding trajectory x_u. We postpone the most technical proofs of this subsection to Appendix A. We recall that, for every $u \in \mathcal{U} := L^2([0,1],\mathbb{R}^k)$, the following inequality holds:

$$||u||_{L^1} = \int_0^1 \sum_{i=1}^k |u^i(s)|\, ds \leq \sqrt{k} \sqrt{\int_0^1 \sum_{i=1}^k |u^i(s)|^2\, ds} = \sqrt{k}||u||_{L^2}. \tag{2.10}$$

We first show that, for every admissible control $u \in \mathcal{U}$, the corresponding solution of Eq. 2.6 is bounded in the C^0-norm. In our framework, given a continuous function $f : [0,1] \to \mathbb{R}^n$, we set

$$||f||_{C^0} := \sup_{s\in[0,1]} |f(s)|_2.$$

Lemma 2.2 *Let $u \in \mathcal{U}$ be an admissible control, and let $x_u : [0,1] \to \mathbb{R}^n$ be the solution of the Cauchy problem* Eq. 2.6 *corresponding to the control u. Then, the following inequality holds:*

$$||x_u||_{C^0} \leq \left(|x_0|_2 + \sqrt{k} C ||u||_{L^2} \right) e^{\sqrt{k} C ||u||_{L^2}}, \tag{2.11}$$

where $C > 0$ is the constant of sub-linear growth prescribed by Eq. 2.3.

Proof This estimate follows from Eq. 2.3 as a direct application of Grönwall inequality. □

In the following proposition, we prove that the solution of the Cauchy problem Eq. 2.6 has a continuous dependence on the admissible control.

Proposition 2.3 *Let us consider $u, v \in \mathcal{U}$ and let $x_u, x_{u+v} : [0,1] \to \mathbb{R}^n$ be the solutions of the Cauchy problem* Eq. 2.6 *corresponding, respectively, to the controls u and $u + v$. Then, for every $R > 0$ there exists $L_R > 0$ such that the inequality*

$$||x_{u+v} - x_u||_{C^0} \leq L_R ||v||_{L^2} \tag{2.12}$$

holds for every $u, v \in \mathcal{U}$ such that $||u||_{L^2}, ||v||_{L^2} \leq R$.

Proof See Appendix A □

The previous result shows that the map $u \mapsto x_u$ is Lipschitz-continuous when restricted to any bounded set of the space of admissible controls $\mathcal{U}$. We remark that Proposition 2.3 holds under the sole assumption that the controlled vector fields $F^1, \ldots, F^k : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz-continuous. In the next result, by requiring that the controlled vector fields are C^1-regular, we compute the first order variation of the solution of Eq. 2.6 resulting from a perturbation in the control.

Proposition 2.4 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are C^1-regular. For every $u, v \in \mathcal{U}$, for every $\varepsilon \in (0, 1]$, let $x_u, x_{u+\varepsilon v} : [0,1] \to \mathbb{R}^n$ be the solutions of* Eq. 2.6 *corresponding, respectively, to the admissible controls u and $u + \varepsilon v$. Then, we have that*

$$||x_{u+\varepsilon v} - x_u - \varepsilon y_u^v||_{C^0} = o(\varepsilon) \text{ as } \varepsilon \to 0, \tag{2.13}$$

where $y_u^v : [0,1] \to \mathbb{R}^n$ is the solution of the following affine system:

$$\dot{y}_u^v(s) = F(x_u(s))v(s) + \left(\sum_{i=1}^{k} u^i(s) \frac{\partial F^i(x_u(s))}{\partial x} \right) y_u^v(s) \tag{2.14}$$

for a.e. $s \in [0,1]$, and with $y_u^v(0) = 0$.

Proof See Appendix A. □

Let us assume that $F^1, \ldots, F^k$ are C^1-regular. For every admissible control $u \in \mathcal{U}$, let us define $A_u \in L^2([0,1], \mathbb{R}^{n \times n})$ as

$$A_u(s) := \sum_{i=1}^{k} \left(u^i(s) \frac{\partial F^i(x_u(s))}{\partial x} \right) \tag{2.15}$$

for a.e. $s \in [0,1]$. For every $u \in \mathcal{U}$, let us introduce the absolutely continuous curve $M_u : [0,1] \to \mathbb{R}^{n\times n}$, defined as the solution of the following linear Cauchy problem:

$$\begin{cases} \dot{M}_u(s) = A_u(s)M_u(s) \text{ for a.e. } s \in [0,1], \\ M_u(0) = \mathrm{Id}. \end{cases} \tag{2.16}$$

The existence and uniqueness of the solution of Eq. 2.16 descends once again from the Carathéodory Theorem. We can prove the following result.

Lemma 2.5 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^1-regular. For every admissible control $u \in \mathcal{U}$, let $M_u : [0,1] \to \mathbb{R}^{n\times n}$ be the solution of the Cauchy problem* Eq. 2.16. *Then, for every $s \in [0,1]$, $M_u(s)$ is invertible, and the following estimates hold:*

$$|M_u(s)|_2 \le C_u, \quad |M_u^{-1}(s)|_2 \le C_u, \tag{2.17}$$

where

$$C_u = e^{\sqrt{k}L||u||_{L^2}}.$$

Proof See Appendix B. □

Using the curve $M_u : [0,1] \to \mathbb{R}^{n\times n}$ defined by Eq. 2.16, we can rewrite the solution of the affine system Eq. 2.14 for the first-order variation of the trajectory. Indeed, for every $u, v \in \mathcal{U}$, a direct computation shows that the function $y_u^v : [0,1] \to \mathbb{R}^n$ that solves Eq. 2.14 can be expressed as

$$y_u^v(s) = \int_0^s M_u(s)M_u^{-1}(\tau)F(x_u(\tau))v(\tau)\,d\tau \tag{2.18}$$

for every $s \in [0,1]$. Using Eq. 2.18 we can prove an estimate of the norm of y_u^v.

Lemma 2.6 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^1-regular. Let us consider $u, v \in \mathcal{U}$, and let $y_u^v : [0,1] \to \mathbb{R}^n$ be the solution of the affine system* Eq. 2.14 *with $y_u^v(0) = 0$. Then, for every $R > 0$ there exists $C_R > 0$ such that the following inequality holds*

$$|y_u^v(s)|_2 \le C_R||v||_{L^2} \tag{2.19}$$

for every $s \in [0,1]$ and for every $u \in \mathcal{U}$ satisfying $||u||_{L^2} \le R$.

Proof Using the expression Eq. 2.18, from Eqs. 2.17, 2.11, and 2.3, we directly deduce the thesis. □

Let us introduce the end-point map associated to the control system Eq. 2.6. For every $s \in [0,1]$, let us consider the map $P_s : \mathcal{U} \to \mathbb{R}^n$ defined as

$$P_s : u \mapsto P_s(u) := x_u(s), \tag{2.20}$$

where $x_u : [0,1] \to \mathbb{R}^n$ is the solution of Eq. 2.6 corresponding to the admissible control $u \in \mathcal{U}$. Using the results obtained before, it follows that the end-point map is differentiable.

Proposition 2.7 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^1-regular. For every $s \in [0,1]$, let $P_s : \mathcal{U} \to \mathbb{R}^n$ be the end-point map defined by* Eq. 2.20. *Then, for every $u \in \mathcal{U}$, P_s is Gateaux differentiable at u, and the differential*

$D_u P_s = (D_u P_s^1, \dots, D_u P_s^n) : \mathcal{U} \to \mathbb{R}^n$ is a linear and continuous operator. Moreover, using the Riesz's isometry, for every $u \in \mathcal{U}$ and for every $s \in [0,1]$, every component of the differential $D_u P_s$ can be represented as follows:

$$D_u P_s^j(v) = \int_0^1 \left\langle g_{s,u}^j(\tau), v(\tau) \right\rangle_{\mathbb{R}^k} d\tau, \tag{2.21}$$

where, for every $j = 1, \dots, n$, the function $g_{s,u}^j : [0,1] \to \mathbb{R}^k$ is defined as

$$g_{s,u}^j(\tau) = \begin{cases} \left((\mathbf{e}^j)^T M_u(s) M_u^{-1}(\tau) F(x_u(\tau)) \right)^T & \tau \in [0,s], \\ 0 & \tau \in (s,1], \end{cases} \tag{2.22}$$

where the column vector $\mathbf{e}^j$ is the j-th element of the standard basis $\{\mathbf{e}^1, \dots, \mathbf{e}^n\}$ of $\mathbb{R}^n$.

Proof For every $s \in [0,1]$, Proposition 2.4 guarantees that the end-point map $P_s : \mathcal{U} \to \mathbb{R}^n$ is Gateaux differentiable at every point $u \in \mathcal{U}$. In particular, for every $u, v \in \mathcal{U}$ and for every $s \in [0,1]$ the following identity holds:

$$D_u P_s(v) = y_u^v(s). \tag{2.23}$$

Moreover, Eq. 2.18 shows that the differential $D_u P_s : \mathcal{U} \to \mathbb{R}^n$ is linear, and Lemma 2.6 implies that it is continuous. The representation follows as well from Eq. 2.18. □

Remark 2.8 In the previous proof we used Lemma 2.6 to deduce for every $u \in \mathcal{U}$ the continuity of the linear operator $D_u P_s : \mathcal{U} \to \mathbb{R}^n$. Actually, Lemma 2.6 is slightly more informative, since it implies that for every $R > 0$ there exists $C_R > 0$ such that

$$|D_u P_s(v)|_2 \leq C_R ||v||_{L^2} \tag{2.24}$$

for every $v \in \mathcal{U}$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. As a matter of fact, we deduce that

$$||g_{s,u}^j||_{L^2} \leq C_R \tag{2.25}$$

for every $j = 1, \dots, n$, for every $s \in [0,1]$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$.

Remark 2.9 It is interesting to observe that, for every $s \in (0,1]$ and for every $u \in \mathcal{U}$, the function $g_{s,u}^j : [0,1] \to \mathbb{R}^k$ that provides the representation the jth component of $D_u P_s$ is absolutely continuous on the interval $[0,s]$, being the product of absolutely continuous matrix-valued curves. Indeed, on one hand, $\tau \mapsto F(x_u(\tau))$ is absolutely continuous, being the composition of a C^1-regular function with the absolutely continuous curve $\tau \mapsto x_u(\tau)$ (see, e.g., [6, Corollary 8.11]). On the other hand, $\tau \mapsto M_u^{-1}(\tau)$ is absolutely continuous as well, since it can be expressed as the solution of a linear system (see Eq. A.8).

We now prove that for every $s \in [0,1]$ the differential of the end-point map $u \mapsto D_u P_s$ is Lipschitz-continuous on the bounded subsets of $\mathcal{U}$. This result requires further regularity assumptions on the controlled vector fields. We first establish an auxiliary result concerning the matrix-valued curve that solves Eq. 2.16.

Lemma 2.10 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular. For every $u, w \in \mathcal{U}$, let $M_u, M_{u+w} : [0,1] \to \mathbb{R}^{n \times n}$ be the solutions of* Eq. 2.16 *corresponding to the admissible controls u and $u + w$, respectively. Then, for* *every $R > 0$, there exists $L_R > 0$ such that, for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$, we have*

$$|M_{u+w}(s) - M_u(s)|_2 \leq L_R ||w||_{L^2}, \tag{2.26}$$

and

$$\left| M_{u+w}^{-1}(s) - M_u^{-1}(s) \right|_2 \leq L_R ||w||_{L^2} \tag{2.27}$$

for every $s \in [0, 1]$.

Proof See Appendix A. □

We are now in position to prove the regularity result on the differential of the end-point map.

Proposition 2.11 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular. Then, for every $R > 0$, there exists $L_R > 0$ such that, for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$, the following inequality holds*

$$|D_{u+w} P_s(v) - D_u P_s(v)|_2 \leq L_R ||w||_{L^2} ||v||_{L^2} \tag{2.28}$$

for every $s \in [0, 1]$ and for every $v \in \mathcal{U}$.

Proof See Appendix A. □

2.3 Second Differential of the End-point Map

In this subsection, we study the second-order variation of the end-point map $P_s : \mathcal{U} \to \mathbb{R}^n$ defined in Eq. 2.20. The main results reported here will be stated in the case $s = 1$, which corresponds to the final evolution instant of the control system Eq. 2.6. However, they can be extended (with minor adjustments) also in the case $s \in (0, 1)$. Similarly as done in Subsection 2.2, we show that, under proper regularity assumptions on the controlled vector fields $F^1, \ldots, F^k$, the end-point map $P_1 : \mathcal{U} \to \mathbb{R}^n$ is C^2-regular. Therefore, for every $u \in \mathcal{U}$, we can consider the second differential $D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$, which turns out to be a bilinear and symmetric operator. For every $\nu \in \mathbb{R}^n$, we provide a representation of the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$, and we prove that it is a compact self-adjoint operator.

Before proceeding, we introduce some notations. We set $\mathcal{V} := L^2([0, 1], \mathbb{R}^n)$, and we equip it with the usual Hilbert space structure. In order to avoid confusion, in the present subsection, we denote by $|| \cdot ||_{\mathcal{U}}$ and $|| \cdot ||_{\mathcal{V}}$ the norms of the Hilbert spaces $\mathcal{U}$ and $\mathcal{V}$, respectively. We use a similar convention for the respective scalar products, too. Moreover, given an application $\mathcal{R} : \mathcal{U} \to \mathcal{V}$, for every $u \in \mathcal{U}$, we use the notation $\mathcal{R}[u] \in \mathcal{V}$ to denote the image of u through $\mathcal{R}$. Then, for $s \in [0, 1]$, we write $\mathcal{R}[u](s) \in \mathbb{R}^n$ to refer to the value of (a representative of) the function $\mathcal{R}[u]$ at the point s. More generally, we adopt this convention for every function-valued operator.

It is convenient to introduce a linear operator that will be useful to derive the expression of the second differential of the end-point map. Assuming that the controlled fields $F^1, \ldots, F^k$ are C^1-regular, for every $u \in \mathcal{U}$ we define $\mathcal{L}_u : \mathcal{U} \to \mathcal{V}$ as follows:

$$\mathcal{L}_u[v](s) := y_u^v(s) \tag{2.29}$$

for every $s \in [0, 1]$, where $y_u^v : [0, 1] \to \mathbb{R}^n$ is the curve introduced in Proposition 2.4 that solves the affine system Eq. 2.14. Recalling Eq. 2.18, we have that the identity

$$\mathcal{L}_u[v](s) = \int_0^s M_u(s) M_u^{-1}(\tau) F(x_u(\tau)) v(\tau)\, d\tau \tag{2.30}$$

holds for every $s \in [0, 1]$ and for every $v \in \mathcal{U}$, and this shows that $\mathcal{L}_u$ is a linear operator. Moreover, using the standard Hilbert space structure of $\mathcal{U}$ and of $\mathcal{V}$, for every $u \in \mathcal{U}$ we can introduce the adjoint of $\mathcal{L}_u$, namely the linear operator $\mathcal{L}_u^* : \mathcal{V} \to \mathcal{U}$ that satisfies

$$\left\langle \mathcal{L}_u^*[w], v \right\rangle_{\mathcal{U}} = \left\langle \mathcal{L}_u[v], w \right\rangle_{\mathcal{V}} \tag{2.31}$$

for every $v \in \mathcal{U}$ and $w \in \mathcal{V}$.

Remark 2.12 We recall a result in functional analysis concerning the norm of the adjoint of a bounded linear operator. For further details, see [6, Remark 2.16]. Given two Banach spaces E_1, E_2, let $\mathscr{L}(E_1, E_2)$ be the Banach space of the bounded linear operators from E_1 to E_2, equipped with the norm induced by E_1 and E_2. Let E_1^*, E_2^* be the dual spaces of E_1, E_2, respectively, and let $\mathscr{L}(E_2^*, E_1^*)$ be defined as above. Therefore, if $A \in \mathscr{L}(E_1, E_2)$, then the adjoint operator satisfies $A^* \in \mathscr{L}(E_2^*, E_1^*)$, and the following identity holds:

$$||A^*||_{\mathscr{L}(E_2^*, E_1^*)} = ||A||_{\mathscr{L}(E_1, E_2)}.$$

If E_1, E_2 are Hilbert spaces, using the Riesz's isometry it is possible to write A^* as an element of $\mathscr{L}(E_2, E_1)$, and the identity of the norms is still satisfied.

We now show that $\mathcal{L}_u$ and $\mathcal{L}_u^*$ are bounded and compact operators.

Lemma 2.13 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are C^1-regular. Then, for every $u \in \mathcal{U}$, the linear operators $\mathcal{L}_u : \mathcal{U} \to \mathcal{V}$ and $\mathcal{L}_u^* : \mathcal{V} \to \mathcal{U}$ defined, respectively, by* Eqs. 2.29 *and* 2.31 *are bounded and compact.*

Proof See Appendix B. □

In the next result, we study the local Lipschitz-continuity of the correspondence $u \mapsto \mathcal{L}_u$.

Lemma 2.14 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular. Then, for every $R > 0$, there exists $L_R > 0$ such that*

$$||\mathcal{L}_{u+w}[v] - \mathcal{L}_u[v]||_{\mathcal{V}} \le L_R ||w||_{\mathcal{U}} ||v||_{\mathcal{U}} \tag{2.32}$$

for every $v \in \mathcal{U}$ and for every $u, w \in \mathcal{U}$ such that $||u||_{\mathcal{U}}, ||w||_{\mathcal{U}} \le R$.

Proof See Appendix B □

Remark 2.15 From Lemma 2.14 and Remark 2.12, it follows that the correspondence $u \mapsto \mathcal{L}_u^*$ is as well Lipschitz-continuous on the bounded sets of $\mathcal{U}$.

If the vector fields $F^1, \ldots, F^k$ are C^2-regular, we write $\frac{\partial^2 F^1}{\partial x^2}, \ldots, \frac{\partial^2 F^k}{\partial x^2}$ to denote their second differential. In the next result, we investigate the second-order variation of the solutions produced by the control system Eq. 2.6.

Proposition 2.16 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular. For every $u, v, w \in \mathcal{U}$, for every $\varepsilon \in (0, 1]$, let $y_u^v, y_{u+\varepsilon w}^v : [0, 1] \to \mathbb{R}^n$ be the solutions of* Eq. 2.14 *corresponding to the first-order variation v and to the admissible controls u and $u + \varepsilon w$, respectively. Therefore, we have that*

$$\sup_{||v||_{L^2}\leq 1} ||y_{u+\varepsilon w}^v - y_u^v - \varepsilon z_u^{v,w}||_{C^0} = o(\varepsilon) \text{ as } \varepsilon \to 0, \tag{2.33}$$

where $z_u^{v,w} : [0, 1] \to \mathbb{R}^n$ is the solution of the following affine system:

$$\dot{z}_u^{v,w}(s) = \sum_{i=1}^k \left[v^i(s) \frac{\partial F^i(x_u(s))}{\partial x} y_u^w(s) + w^i(s) \frac{\partial F^i(x_u(s))}{\partial x} y_u^v(s) \right] \tag{2.34}$$

$$+ \sum_{i=1}^k u^i(s) \frac{\partial^2 F^i(x_u(s))}{\partial x^2} (y_u^v(s), y_u^w(s)) \tag{2.35}$$

$$+ \sum_{i=1}^k u^i(s) \frac{\partial F^i(x_u(s))}{\partial x} z_u^{v,w}(s) \tag{2.36}$$

with $z_u^{v,w}(0) = 0$, and where $y_u^v, y_u^w : [0, 1] \to \mathbb{R}^n$ are the solutions of Eq. 2.14 *corresponding to the admissible control u and to the first-order variations v and w, respectively.*

Proof The proof of this result follows using the same kind of techniques and computations as in the proof of Proposition 2.4. □

Remark 2.17 Similarly as done in Eq. 2.18 for the first-order variation, we can express the solution of the affine system Eqs. 2.34–2.36 through an integral formula. Indeed, for every $u, v, w \in \mathcal{U}$, for every $s \in [0, 1]$ we have that

$$z_u^{v,w}(s) = \int_0^s M_u(s) \quad M_u^{-1}(\tau) \left(\sum_{i=1}^k v^i(\tau) \frac{\partial F^i(x_u(\tau))}{\partial x} \mathcal{L}_u[w](\tau) \right. \tag{2.37}$$

$$+ \sum_{i=1}^k w^i(\tau) \frac{\partial F^i(x_u(\tau))}{\partial x} \mathcal{L}_u[v](\tau) \tag{2.38}$$

$$\left. + \sum_{i=1}^k u^i(\tau) \frac{\partial^2 F^i(x_u(\tau))}{\partial x^2} (\mathcal{L}_u[v](\tau), \mathcal{L}_u[w](\tau)) \right) d\tau, \tag{2.39}$$

where we used the linear operator $\mathcal{L}_u : \mathcal{U} \to \mathcal{V}$ defined in Eq. 2.29. From the previous expression it follows that, for every $u, v, w \in \mathcal{U}$, the roles of v and w are interchangeable, i.e., for every $s \in [0, 1]$ we have that $z_u^{v,w}(s) = z_u^{w,v}(s)$. Moreover, we observe that, for every $s \in [0, 1]$ and for every $u \in \mathcal{U}$, $z_u^{v,w}(s)$ is bilinear with respect to v and w.

We are now in position to introduce the second differential of the end-point map $P_s : \mathcal{U} \to \mathbb{R}^n$ defined in Eq. 2.20. In view of the applications in the forthcoming sections, we shall focus on the case $s = 1$, i.e., we consider the map $P_1 : \mathcal{U} \to \mathbb{R}^n$. Before proceeding, for every $u \in \mathcal{U}$ we define the symmetric and bilinear map $\mathcal{B}_u : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ as follows

$$\mathcal{B}_u(v, w) := z_u^{v,w}(1). \tag{2.40}$$

Proposition 2.18 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular. Let $P_1 : \mathcal{U} \to \mathbb{R}^n$ be the end-point map defined by* Eq. 2.20, *and, for every $u \in \mathcal{U}$, let $D_u P_1 : \mathcal{U} \to \mathbb{R}^n$ be its differential. Then, the correspondence $u \mapsto D_u P_1$ is Gateaux differentiable at every $u \in \mathcal{U}$, namely*

$$\lim_{\varepsilon \to 0} \sup_{||v||_{L^2} \le 1} \left| \frac{D_{u+\varepsilon w} P_1(v) - D_u P_1(v)}{\varepsilon} - \mathcal{B}_u(v, w) \right|_2 = 0, \tag{2.41}$$

where $B_u : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ is the bilinear, symmetric and bounded operator defined in Eq. 2.40.

Proof In view of Eq. 2.23, for every $u, v, w \in \mathcal{U}$ and for every $\varepsilon \in (0, 1]$, we have that $D_u P_1(v) = y_u^v(1)$ and $D_{u+\varepsilon w} P_1(v) = y_{u+\varepsilon w}^v(1)$. Therefore, Eq. 2.41 follows directly from Eq. 2.33 and from Eq. 2.40. The symmetry and the bilinearity of $\mathcal{B}_u : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ descend from the observations in Remark 2.17. Finally, we have to show that, for every $u \in \mathcal{U}$, there exists $C > 0$ such that

$$|\mathcal{B}_u(v, w)|_2 \le C ||v||_{L^2} ||w||_{L^2}$$

for every $v, w \in \mathcal{U}$. Recalling Eq. 2.40 and the integral expression Eqs. 2.37–2.39, the last inequality follows from the estimate Eq. B.1, from Lemma 2.5, from Proposition 2.2 and from the C^2-regularity of $F^1, \dots, F^k$. □

In view of the previous result, for every $u \in \mathcal{U}$, we use $D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ to denote the second differential of the end-point map $P_1 : \mathcal{U} \to \mathbb{R}^n$. Moreover, for every $u, v, w \in \mathcal{U}$ we have the following identities:

$$D_u^2 P_1(v, w) = \mathcal{B}_u(v, w) = z_u^{v,w}(1). \tag{2.42}$$

Remark 2.19 It is possible to prove that the correspondence $u \mapsto D_u^2 P_1$ is continuous. In particular, under the further assumption that the controlled vector fields $F^1, \dots, F^k$ are C^3-regular, the application $u \mapsto D_u^2 P_1$ is Lipschitz-continuous on the bounded subsets of $\mathcal{U}$. Indeed, taking into account Eq. 2.42 and Eqs. 2.37–2.39, this fact follows from Lemma 2.10, from Lemma 2.14 and from the regularity of $F^1, \dots, F^k$.

For every $\nu \in \mathbb{R}^n$ and for every $u \in \mathcal{U}$, we can consider the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$, which is defined as

$$\nu \cdot D_u^2 P_1(v, w) := \langle \nu, D_u^2 P_1(v, w) \rangle_{\mathbb{R}^n}. \tag{2.43}$$

We conclude this section by showing that, using the scalar product of $\mathcal{U}$, the bilinear form defined in Eq. 2.43 can be represented as a self-adjoint compact operator. Before proceeding, it is convenient to introduce two auxiliary linear operators. In this part we assume that the vector fields $F^1, \dots, F^k$ are C^2-regular. For every $u \in \mathcal{U}$ let us consider the application $\mathcal{M}_u^\nu : \mathcal{U} \to \mathcal{V}$ defined as follows:

$$\mathcal{M}_u^\nu[v](\tau) := \left(M_u(1) M_u^{-1}(\tau) \sum_{i=1}^k v^i(\tau) \frac{\partial F^i(x_u(\tau))}{\partial x} \right)^T \nu \tag{2.44}$$

for a.e. $\tau \in [0, 1]$, where $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of Eq. 2.6 and $M_u : [0, 1] \to \mathbb{R}^{n \times n}$ is defined in Eq. 2.16. We recall that, for every $i = 1, \dots, k$ and for every $y \in \mathbb{R}^n$,

$\frac{\partial^2 F^i(y)}{\partial x^2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric and bilinear function. Hence, for every $i = 1, \dots, k$, for every $u \in \mathcal{U}$, and for every $\tau \in [0, 1]$, we have that the application

$$(\eta_1, \eta_2) \mapsto \nu^T M_u(1) M_u^{-1}(\tau) \frac{\partial^2 F^i(x_u(\tau))}{\partial x^2}(\eta_1, \eta_2)$$

is a symmetric and bilinear form on $\mathbb{R}^n$. Therefore, for every $i = 1, \dots, k$, for every $u \in \mathcal{U}$, and for every $\tau \in [0, 1]$, we introduce the symmetric matrix $S_u^{\nu,i}(\tau) \in \mathbb{R}^{n \times n}$ that satisfies the identity

$$\left\langle S_u^{\nu,i}(\tau)\eta_1, \eta_2 \right\rangle_{\mathbb{R}^n} = \nu^T M_u(1) M_u^{-1}(\tau) \frac{\partial^2 F^i(x_u(\tau))}{\partial x^2}(\eta_1, \eta_2)$$

for every $\eta_1, \eta_2 \in \mathbb{R}^n$. We define the linear operator $\mathcal{S}_u^\nu : C^0([0, 1], \mathbb{R}^n) \to \mathcal{V}$ as follows:

$$\mathcal{S}_u^\nu[v](\tau) := \sum_{i=1}^k u^i(\tau) S_u^{\nu,i}(\tau) v(\tau) \tag{2.45}$$

for every $v \in C^0([0, 1], \mathbb{R}^n)$ and for a.e. $\tau \in [0, 1]$.

In the next result, we prove that the linear operators introduced above are both continuous.

Lemma 2.20 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular. Therefore, for every $u \in \mathcal{U}$ and for every $\nu \in \mathbb{R}^n$, the linear operators $\mathcal{M}_u^\nu : \mathcal{U} \to \mathcal{V}$ and $\mathcal{S}_u^\nu : C^0([0, 1], \mathbb{R}^n) \to \mathcal{V}$ defined, respectively, in* Eqs. 2.44 *and* 2.45 *are continuous.*

Proof See Appendix B. □

We are now in position to represent the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ through the scalar product of $\mathcal{U}$. Indeed, recalling Eqs. 2.43 and 2.42, from Eqs. 2.37–2.39 for every $u \in \mathcal{U}$, we obtain that

$$\begin{aligned}\nu \cdot D_u^2 P_1(v, w) &= \left\langle M_u^\nu[v], \mathcal{L}_u[w] \right\rangle_{\mathcal{V}} + \left\langle M_u^\nu[w], \mathcal{L}_u[v] \right\rangle_{\mathcal{V}} + \left\langle \mathcal{S}_u^\nu \mathcal{L}_u[v], \mathcal{L}_u[w] \right\rangle_{\mathcal{V}} \\ &= \left\langle \mathcal{L}_u^* M_u^\nu[v], w \right\rangle_{\mathcal{U}} + \left\langle (\mathcal{M}_u^\nu)^* \mathcal{L}_u[v], w \right\rangle_{\mathcal{U}} + \left\langle \mathcal{L}_u^* \mathcal{S}_u^\nu \mathcal{L}_u[v], w \right\rangle_{\mathcal{U}}\end{aligned}$$

for every $v, w \in \mathcal{U}$, where $(\mathcal{M}_u^\nu)^* : \mathcal{V} \to \mathcal{U}$ is the adjoint of the linear operator $\mathcal{M}_u^\nu : \mathcal{U} \to \mathcal{V}$. Recalling Remark 2.12, we have that $(\mathcal{M}_u^\nu)^*$ is a bounded linear operator. This shows that the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ can be represented by the linear operator $\mathcal{N}_u^\nu : \mathcal{U} \to \mathcal{U}$, i.e.,

$$\nu \cdot D_u^2 P_1(v, w) = \left\langle \mathcal{N}_u^\nu[v], w \right\rangle_{\mathcal{U}} \tag{2.46}$$

for every $v, w \in \mathcal{U}$, where

$$\mathcal{N}_u^\nu := \mathcal{L}_u^* M_u^\nu + (\mathcal{M}_u^\nu)^* \mathcal{L}_u + \mathcal{L}_u^* \mathcal{S}_u^\nu \mathcal{L}_u. \tag{2.47}$$

We conclude this section by proving that $\mathcal{N}_u^\nu : \mathcal{U} \to \mathcal{U}$ is a bounded, compact, and self-adjoint operator.

Proposition 2.21 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular. For every $u \in \mathcal{U}$ and for every $\nu \in \mathbb{R}^n$, let $\mathcal{N}_u^\nu : \mathcal{U} \to \mathcal{U}$ be the linear operator that represents the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ through the identity* Eq. 2.46. *Then, $\mathcal{N}_u^\nu$ is continuous, compact, and self-adjoint.*

Proof We observe that the term $\mathcal{L}_u^* M_u^\nu + (\mathcal{M}_u^\nu)^* \mathcal{L}_u$ at the right-hand side of Eq. 2.47 is continuous, since it is obtained as the sum and the composition of continuous linear operators, as shown in Lemma 2.13 and Lemma 2.20. Moreover, it is also compact, since both $\mathcal{L}_u$ and $\mathcal{L}_u^*$ are, in virtue of Lemma 2.13. Finally, the fact that $\mathcal{L}_u^* M_u^\nu + (\mathcal{M}_u^\nu)^* \mathcal{L}_u$ is self-adjoint is immediate. Let us consider the last term at the right-hand side of Eq. 2.47, i.e., $\mathcal{L}_u^* \mathcal{S}_u^\nu \mathcal{L}_u$. We first observe that $\mathcal{S}_u^\nu \mathcal{L}_u : \mathcal{U} \to \mathcal{V}$ is continuous, owing to Lemma 2.20 and the inequality Eq. B.1. Recalling that $\mathcal{L}_u^* : \mathcal{V} \to \mathcal{U}$ is compact, the composition $\mathcal{L}_u^* \mathcal{S}_u^\nu \mathcal{L}_u : \mathcal{U} \to \mathcal{U}$ is compact as well. Once again, the operator is clearly self-adjoint. □

3 Gradient Flow: Well-posedness and Global Definition

For every $\beta > 0$, we consider the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ defined as follows:

$$\mathcal{F}^\beta(u) := \frac{1}{2}||u||_{L^2}^2 + \beta a(x_u(1)), \tag{3.1}$$

where $a : \mathbb{R}^n \to \mathbb{R}_+$ is a non-negative C^1-regular function, and, for every $u \in \mathcal{U}$, $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of the Cauchy problem Eq. 2.6 corresponding to the admissible control $u \in \mathcal{U}$. In this section, we want to study the gradient flow induced by the functional $\mathcal{F}^\beta$ on the Hilbert space $\mathcal{U}$. In particular, we establish a result that guarantees existence, uniqueness and global definition of the solutions of the gradient flow equation associated to $\mathcal{F}^\beta$. In this section, we adopt the approach of the monograph [10], where the theory of ODEs in Banach spaces is developed.

We start from the notion of differentiable curve with values in $\mathcal{U}$. We stress that in the present paper the time variable t is exclusively employed for curves taking values in $\mathcal{U}$. In particular, we recall that we use $s \in [0, 1]$ to denote the time variable only in the control system Eq. 2.6 and in the related objects (e.g., admissible controls, controlled trajectories, etc.). Given a curve $U : (a, b) \to \mathcal{U}$, we say that it is (strongly) differentiable at $t_0 \in (a, b)$ if there exists $u \in \mathcal{U}$ such that

$$\lim_{t \to t_0} \left|\left| \frac{U_t - U_{t_0}}{t - t_0} - u \right|\right|_{L^2} = 0. \tag{3.2}$$

In this case, we use the notation $\partial_t U_{t_0} := u$. In the present section, we study the well-posedness in $\mathcal{U}$ of the evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}^\beta[U_t], \\ U_0 = u_0, \end{cases} \tag{3.3}$$

where $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ is the representation of the differential $d\mathcal{F}^\beta : \mathcal{U} \to \mathcal{U}^*$ through the Riesz isomorphism, i.e.,

$$\langle \mathcal{G}^\beta[u], v \rangle_{L^2} = d_u \mathcal{F}^\beta(v) \tag{3.4}$$

for every $u, v \in \mathcal{U}$. More precisely, for every initial datum $u_0 \in \mathcal{U}$ we prove that there exists a curve $t \mapsto U_t$ that solves Eq. 3.3, that it is unique and that it is defined for every $t \geq 0$.

We first show that $d_u \mathcal{F}^\beta$ can be actually represented as an element of $\mathcal{U}$, for every $u \in \mathcal{U}$. We immediately observe that this problem reduces to study the differential of the end-point cost, i.e., the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$, defined as

$$\mathcal{E}(u) := a(x_u(1)), \tag{3.5}$$

where $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of Eq. 2.6 corresponding to the admissible control $u \in \mathcal{U}$.

Lemma 3.1 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^1-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. Then, the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$ is Gateaux differentiable at every $u \in \mathcal{U}$. Moreover, using the Riesz's isomorphism, for every $u \in \mathcal{U}$, the differential $d_u\mathcal{E} : \mathcal{U} \to \mathbb{R}$ can be represented as follows:*

$$d_u\mathcal{E}(v) = \int_0^1 \sum_{j=1}^n \left(\frac{\partial a(x_u(1))}{\partial x^j} \left\langle g^j_{1,u}(\tau), v(\tau) \right\rangle_{\mathbb{R}^k} \right) d\tau \tag{3.6}$$

for every $v \in \mathcal{U}$, where, for every $j = 1, \dots, n$, the function $g^j_{1,u} \in \mathcal{U}$ is defined as in Eq. 2.22.

Proof See Appendix C. □

Remark 3.2 Similarly as done in Remark 2.8, we can provide a uniform estimate of the norm of $d_u\mathcal{E}$ when u varies on a bounded set. Indeed, recalling Lemma 2.2 and the fact that $a : \mathbb{R}^n \to \mathbb{R}_+$ is C^1-regular, for every $R > 0$ there exists $C'_R > 0$ such that

$$\left| \frac{\partial a(x_u(1))}{\partial x^j} \right| \leq C'_R$$

for every $j = 1, \dots, n$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. Combining the last inequality with Eqs. C.1 and 2.24, we deduce that there exists $C_R > 0$ such that for every $||u||_{L^2} \leq R$ the estimate

$$|d_u\mathcal{E}(v)|_2 \leq C_R ||v||_{L^2} \tag{3.7}$$

holds for every $v \in \mathcal{U}$.

Remark 3.3 We observe that, for every $u, v \in \mathcal{U}$, we can rewrite Eq. 3.6 as follows

$$d_u\mathcal{E}(v) = \int_0^1 \left\langle F^T(x_u(\tau))\lambda_u^T(\tau), v(\tau) \right\rangle_{\mathbb{R}^k} d\tau, \tag{3.8}$$

where $\lambda_u : [0, 1] \to (\mathbb{R}^n)^*$ is an absolutely continuous curve defined for every $s \in [0, 1]$ by the relation

$$\lambda_u(s) := \nabla a(x_u(1)) \cdot M_u(1) M_u^{-1}(s), \tag{3.9}$$

where $M_u : [0, 1] \to \mathbb{R}^{n\times n}$ is defined as in Eq. 2.16, and $\nabla a(x_u(1))$ is understood as a row vector. Recalling that $s \mapsto M_u^{-1}(s)$ solves Eq. A.8, it turns out that $s \mapsto \lambda_u(s)$ is the solution of the following linear Cauchy problem:

$$\begin{cases} \dot\lambda_u(s) = -\lambda_u(s) \sum_{i=1}^k \left(u^i(s) \frac{\partial F^i(x_u(s))}{\partial x} \right) & \text{for a.e. } s \in [0, 1], \\ \lambda_u(1) = \nabla a(x_u(1)). \end{cases} \tag{3.10}$$

Finally, Eq. 3.8 implies that, for every $u \in \mathcal{U}$, we can represent $d_u\mathcal{E}$ with the function $h_u : [0, 1] \to \mathbb{R}^k$ defined as

$$h_u(s) := F^T(x_u(s))\lambda_u^T(s) \tag{3.11}$$

for a.e. $s \in [0, 1]$. We observe that Eq. 3.7 and the Riesz's isometry imply that for every $R > 0$ there exists $C_R > 0$ such that

$$||h_u||_{L^2} \leq C_R \tag{3.12}$$

for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. We further underline that the representation $h_u : [0, 1] \to \mathbb{R}^k$ of the differential $d_u\mathcal{E}$ is actually absolutely continuous, similarly as observed in Remark 2.9 for the representations of the components of the differential of the end-point map.

Under the assumption that the controlled vector fields $F^1, \dots, F^k$ and the function $a : \mathbb{R}^n \to \mathbb{R}_+$ are C^2-regular, we can show that the differential $u \mapsto d_u\mathcal{E}$ is Lipschitz-continuous on bounded sets.

Lemma 3.4 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. Then, for every $R > 0$ there exists $L_R > 0$ such that*

$$||h_{u+w} - h_u||_{L^2} \leq L_R ||w||_{L^2} \tag{3.13}$$

for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$, where h_{u+w}, h_u are the representations, respectively, of $d_{u+w}\mathcal{E}$ and $d_u\mathcal{E}$ provided by Eq. 3.11.

Proof See Appendix C. □

Remark 3.5 In Lemma 3.1 we have computed the Gateaux differential $d_u\mathcal{E}$ of the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}$. The continuity of the map $u \mapsto d_u\mathcal{E}$ implies that the Gateaux differential coincides with the Fréchet differential (see, e.g., [5, Theorem 1.9]).

Using Lemma 3.1 and Remark 3.3, we can provide an expression for the representation map $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ defined in Eq. 3.4. Indeed, for every $\beta > 0$ we have that

$$\mathcal{G}^\beta[u] = u + \beta h_u, \tag{3.14}$$

where $h_u : [0, 1] \to \mathbb{R}^k$ is defined in Eq. 3.11. Before proving that the solution of the gradient flow Eq. 3.3 exists and is globally defined, we report the statement of a local existence and uniqueness result for the solution of ODEs in infinite-dimensional spaces.

Theorem 3.6 *Let $(E, ||\cdot||_E)$ be a Banach space, and, for every $u_0 \in E$ and $R > 0$, let $B_R(u_0)$ be the set*

$$B_R(u_0) := \{u \in E : ||u - u_0||_E \leq R\}.$$

Let $\mathcal{K} : E \to E$ be a continuous map such that

(i) $||\mathcal{K}[u]||_E \leq M$ *for every* $u \in B_R(u_0)$;
(ii) $||\mathcal{K}[u_1] - \mathcal{K}[u_2]||_E \leq L||u_1 - u_2||_E$ *for every* $u_1, u_2 \in B_R(u_0)$.

For every $t_0 \in \mathbb{R}$, let us consider the following Cauchy problem:

$$\begin{cases} \partial_t U_t = \mathcal{K}[U_t], \\ U_{t_0} = u_0. \end{cases} \tag{3.15}$$

Then, setting $\alpha := \frac{R}{M}$, the equation Eq. 3.15 *admits a unique and continuously differentiable solution $t \mapsto U_t$, which is defined for every $t \in \mathcal{I} := [t_0 - \alpha, t_0 + \alpha]$ and satisfies $U_t \in B_R(u_0)$ for every $t \in \mathcal{I}$.*

Proof This result descends directly from [10, Theorem 5.1.1]. □

In the following result, we show that, whenever it exists, any solution of Eq. 3.3 is bounded with respect to the L^2-norm.

Lemma 3.7 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. For every initial datum $u_0 \in \mathcal{U}$, let $U : [0, \alpha) \to \mathcal{U}$ be a continuously differentiable solution of the Cauchy problem* Eq. 3.3. *Therefore, for every $R > 0$, there exists $C_R > 0$ such that, if $||u_0||_{L^2} \le R$, then*

$$||U_t||_{L^2} \le C_R$$

for every $t \in [0, \alpha)$.

Proof Recalling Eq. 3.3 and using the fact that both $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ and $t \mapsto U_t$ are differentiable, we observe that

$$\frac{d}{dt}\mathcal{F}^\beta(U_t) = d_{U_t}\mathcal{F}^\beta(\partial_t U_t) = \langle \mathcal{G}^\beta[U_t], \partial_t U_t \rangle_{L^2} = -||\partial_t U_t||^2_{L^2} \le 0 \tag{3.16}$$

for every $t \in [0, \alpha)$, and this immediately implies that

$$\mathcal{F}^\beta(U_t) \le \mathcal{F}^\beta(U_0)$$

for every $t \in [0, \alpha)$. Moreover, from the definition of the functional $\mathcal{F}^\beta$ given in Eq. 3.1 and recalling that the end-point term is non-negative, it follows that $\frac{1}{2}||u||^2_{L^2} \le \mathcal{F}^\beta(u)$ for every $u \in \mathcal{U}$. Therefore, combining these facts, if $||u_0||_{L^2} \le R$, we deduce that

$$\frac{1}{2}||U_t||^2_{L^2} \le \sup_{||u_0||_{L^2} \le R} \mathcal{F}^\beta(u_0) \le \frac{1}{2}R^2 + \sup_{||u_0||_{L^2} \le R} a(x_{u_0}(1))$$

for every $t \in [0, \alpha)$. Finally, using Lemma 2.2 and the continuity of the terminal cost $a : \mathbb{R}^n \to \mathbb{R}_+$, we deduce the thesis. □

We are now in position to prove that the gradient flow equation Eq. 3.3 admits a unique and globally defined solution.

Theorem 3.8 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. For every $u_0 \in \mathcal{U}$, let us consider the Cauchy problem* Eq. 3.3 *with initial datum $U_0 = u_0$. Then,* Eq. 3.3 *admits a unique, globally defined and continuously differentiable solution $U : [0, +\infty) \to \mathcal{U}$.*

Proof Let us fix the initial datum $u_0 \in \mathcal{U}$, and let us set $R := ||u_0||_{L^2}$. Let $C_R > 0$ be the constant provided by Lemma 3.7. Let us introduce $R' := C_R + 1$ and let us consider

$$B_{R'}(0) := \{u \in \mathcal{U} : ||u||_{L^2} \le R'\}.$$

We observe that, for every $\bar{u} \in \mathcal{U}$ such that $||\bar{u}||_{L^2} \le C_R$, we have that

$$B_1(\bar{u}) \subset B_{R'}(0), \tag{3.17}$$

where $B_1(\bar{u}) := \{u \in \mathcal{U} : ||u - \bar{u}||_{L^2} \le 1\}$. Recalling that the vector field that generates the gradient flow Eq. 3.3 has the form $\mathcal{G}^\beta[u] = u + \beta h_u$ for every $u \in \mathcal{U}$, from Eq. 3.12, we deduce that there exists $M_{R'} > 0$ such that

$$||\mathcal{G}^\beta[u]||_{L^2} \le M_{R'} \tag{3.18}$$

for every $u \in B_{R'}(0)$. On the other hand, Lemma 3.4 implies that there exists $L_{R'} > 0$ such that

$$||\mathcal{G}^\beta[u_1] - \mathcal{G}^\beta[u_2]||_{L^2} \le L_{R'}||u_1 - u_2||_{L^2} \tag{3.19}$$

for every $u_1, u_2 \in B_{R'}(0)$. Recalling the inclusion Eqs. 3.17, 3.18, and 3.19 guarantee that the hypotheses of Theorem 3.6 are satisfied in the ball $B_1(\bar{u})$, for every $\bar{u}$ satisfying $||\bar{u}||_{L^2} \leq C_R$. This implies that, for every $t_0 \in \mathbb{R}$, the evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}^\beta[U_t], \\ U_{t_0} = \bar{u}, \end{cases} \tag{3.20}$$

admits a unique and continuously differentiable solution defined in the interval $[t_0 - \alpha, t_0 + \alpha]$, where we set $\alpha := \frac{1}{M_{R'}}$. In particular, if we choose $t_0 = 0$ and $\bar{u} = u_0$ in Eq. 3.20, we deduce that the gradient flow equation Eq. 3.3 with initial datum $U_0 = u_0$ admits a unique and continuously differentiable solution $t \mapsto U_t$ defined in the interval $[0, \alpha]$. We shall now prove that we can extend this local solution to every positive time. In virtue of Lemma 3.7, we obtain that the local solution $t \mapsto U_t$ satisfies

$$||U_t||_{L^2} \leq C_R \tag{3.21}$$

for every $t \in [0, \alpha]$. Therefore, if we set $t_0 = \frac{\alpha}{2}$ and $\bar{u} = U_{\frac{\alpha}{2}}$ in Eq. 3.20, recalling that, if $||\bar{u}||_{L^2} \leq C_R$, then Eq. 3.20 admits a unique solution defined in $[t_0 - \alpha, t_0 + \alpha]$, it turns out that the curve $t \mapsto U_t$ that solves Eq. 3.3 with Cauchy datum $U_0 = u_0$ can be uniquely defined for every $t \in [0, \frac{3}{2}\alpha]$. Since Lemma 3.7 guarantees that Eq. 3.21 holds whenever the solution $t \mapsto U_t$ exists, we can repeat recursively the argument and we can extend the domain of the solution to the whole half-line $[0, +\infty)$. □

We observe that Theorem 3.6 suggests that the solution of the gradient flow equation Eq. 3.3 could be defined also for negative times. In the following result we investigate this fact.

Corollary 3.9 *Under the same assumptions of Theorem 3.8, for every $R_2 > R_1 > 0$, there exists $\alpha > 0$ such that, if $||u_0||_{L^2} \leq R_1$, then the solution $t \mapsto U_t$ of the Cauchy problem* Eq. 3.3 *with initial datum $U_0 = u_0$ is defined for every $t \in [-\alpha, +\infty)$. Moreover, $||U_t||_{L^2} \leq R_2$ for every $t \in [-\alpha, 0]$.*

Proof The fact that the solutions are defined for every positive time descends from Theorem 3.8. Recalling the expression of $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ provided by Eq. 3.14, from Eq. 3.12 it follows that, for every $R_2 > 0$, there exists M_{R_2} such that

$$||\mathcal{G}^\beta[u]||_{L^2} \leq M_{R_2}$$

for every $u \in B_{R_2}(0) := \{u \in \mathcal{U} : ||u||_{L^2} \leq R_2\}$. On the other hand, in virtue of Lemma 3.4, we deduce that there exists L_{R_2} such that

$$||\mathcal{G}^\beta[u_1] - \mathcal{G}^\beta[u_2]||_{L^2} \leq L_{R_2}||u_1 - u_2||_{L^2}$$

for every $u_1, u_2 \in B_{R_2}(0)$. We further observe that, for every $u_0 \in \mathcal{U}$ such that $||u_0||_{L^2} \leq R_1$, we have the inclusion $B_R(u_0) := \{u \in \mathcal{U} : ||u - u_0|| \leq R\} \subset B_{R_2}(0)$, where we set $R := R_2 - R_1$. Therefore, the previous inequalities guarantee that the hypotheses of Theorem 3.6 are satisfied in $B_R(u_0)$, whenever $||u_0||_{L^2} \leq R_1$. Finally, in virtue of Theorem 3.6 and the inclusion $B_R(u_0) \subset B_{R_2}(0)$, we obtain the thesis with

$$\alpha = \frac{R_2 - R_1}{M_{R_2}}.$$

□

4 Pre-compactness of Gradient Flow Trajectories

In Section 3, we considered the $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ defined in Eq. 3.1 and we proved that the gradient flow equation Eq. 3.3 induced on $\mathcal{U}$ by $\mathcal{F}^\beta$ admits a unique solution $U : [0, +\infty) \to \mathcal{U}$, for every Cauchy datum $U_0 = u_0 \in \mathcal{U}$. The aim of the present section is to investigate the pre-compactness in $\mathcal{U}$ of the gradient flow trajectories $t \mapsto U_t$. In order to do that, we first show that, under suitable regularity assumptions on the vector fields $F^1, \ldots, F^k$ and on the function $a : \mathbb{R}^n \to \mathbb{R}_+$, for every $t \geq 0$, the value of the solution $U_t \in \mathcal{U}$ has the same Sobolev regularity as the initial datum u_0. The key-fact is that, when $F^1, \ldots, F^k$ are C^r-regular with $r \geq 2$ and $a : \mathbb{R}^n \to \mathbb{R}_+$ is of class C^2, the map $\mathcal{G}^\beta : H^m([0,1], \mathbb{R}^k) \to H^m([0,1], \mathbb{R}^k)$ is locally Lipschitz continuous, for every non-negative integer $m \leq r - 1$. This implies that the gradient flow equation Eq. 3.3 can be studied as an evolution equation in the Hilbert space $H^m([0,1], \mathbb{R}^k)$.

The following result concerns the curve $\lambda_u : [0,1] \to (\mathbb{R}^n)^*$ defined in Eq. 3.9.

Lemma 4.1 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. For every $R > 0$, there exists $C_R > 0$ such that, for every $u \in \mathcal{U}$ satisfying $||u||_{L^2} \leq R$, the following inequality holds*

$$||\lambda_u||_{C^0} \leq C_R, \tag{4.1}$$

where the curve $\lambda_u : [0,1] \to (\mathbb{R}^n)^*$ is defined as in Eq. 3.9. *Moreover, for every $R > 0$, there exists $L_R > 0$ such that, for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$, for the corresponding curves $\lambda_u, \lambda_{u+w} : [0,1] \to (\mathbb{R}^n)^*$ the following inequality holds:*

$$||\lambda_{u+w} - \lambda_u||_{C^0} \leq L_R ||w||_{L^2}. \tag{4.2}$$

Proof Recalling the definition of λ_u given in Eq. 3.9, we have that

$$|\lambda_u(s)|_2 \leq |\nabla a(x_u(1))|_2 |M_u(1)|_2 |M_u^{-1}(s)|_2$$

for every $s \in [0,1]$, where $x_u : [0,1] \to \mathbb{R}^n$ is solution of Eq. 2.6 corresponding to the control $u \in \mathcal{U}$. Lemma 2.2 implies that there exists $C'_R > 0$ such that $|\nabla a(x_u(1))|_2 \leq C'_R$ for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. Combining this with Eq. 2.17, we deduce Eq. 4.1.

To prove Eq. 4.2, we first observe that the C^2-regularity of $a : \mathbb{R}^n \to \mathbb{R}_+$ and Proposition 2.3 imply that, for every $R > 0$, there exists $L'_R > 0$ such that

$$|\nabla_{x_{u+w}(1)} a - \nabla_{x_u(1)} a|_2 \leq L'_R ||w||_{L^2}$$

for every $u, w \in \mathcal{U}$ such that $||u||_{L^2}, ||w||_{L^2} \leq R$. Therefore, recalling Eq. 2.17 and Eqs. 2.26–2.27, we deduce Eq. 4.2 by applying the triangular inequality to the identity

$$|\lambda_{u+w}(s) - \lambda_u(s)|_2 = |\nabla_{x_{u+w}(1)} a \cdot M_{u+w}(1) M_{u+w}^{-1}(s) - \nabla_{x_u(1)} a \cdot M_u(1) M_u^{-1}(s)|_2$$

for every $s \in [0,1]$. □

We recall the notion of *Lie bracket* of vector fields. Let $G^1, G^2 : \mathbb{R}^n \to \mathbb{R}^n$ be two vector fields such that $G^1 \in C^{r_1}(\mathbb{R}^n, \mathbb{R}^n)$ and $G^2 \in C^{r_2}(\mathbb{R}^n, \mathbb{R}^n)$, with $r_1, r_2 \geq 1$, and let us set $r := \min(r_1, r_2)$. Then, the *Lie bracket of G^1 and G^2* is the vector field $[G^1, G^2] : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:

$$[G^1, G^2](y) = \frac{\partial G^2(y)}{\partial x} G^1(y) - \frac{\partial G^1(y)}{\partial x} G^2(y).$$

We observe that $[G^1, G^2] \in C^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$. In the following result, we establish some estimates for vector fields obtained via iterated Lie brackets.

Lemma 4.2 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^m-regular, with $m \geq 2$. For every compact $K \subset \mathbb{R}^n$, there exist $C > 0$ and $L > 0$ such that, for every $j_1, \dots, j_m = 1, \dots, k$, the vector field*

$$G := [F^{j_m}, [\dots, [F^{j_3}, [F^{j_2}, F^{j_1}]] \dots] : \mathbb{R}^n \to \mathbb{R}^n$$

satisfies the following inequalities:

$$|G(x)|_2 \leq C \tag{4.3}$$

for every $x \in K$, and

$$|G(x) - G(y)|_2 \leq L|x - y|_2 \tag{4.4}$$

for every $x, y \in K$.

Proof The thesis follows immediately from the fact that the vector field G is C^1-regular. □

The next result is the cornerstone this section. It concerns the regularity of the function $h_u : [0, 1] \to \mathbb{R}^k$ introduced in Eq. 3.11. We recall that, for every $u \in \mathcal{U}$, h_u is the representation of the differential $d_u\mathcal{E}$ through the scalar product of $\mathcal{U}$, where the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$ is defined as in Eq. 3.5. We recall the convention $H^0([0, 1], \mathbb{R}^k) = L^2([0, 1], \mathbb{R}^k) = \mathcal{U}$.

Lemma 4.3 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^r-regular with $r \geq 2$, and that the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost is C^2-regular. For every $u \in \mathcal{U}$, let $h_u : [0, 1] \to \mathbb{R}^k$ be the representation of the differential $d_u\mathcal{E} : \mathcal{U} \to \mathbb{R}$ provided by* Eq. 3.11. *For every integer $1 \leq m \leq r - 1$, if $u \in H^{m-1}([0, 1], \mathbb{R}^k) \subset \mathcal{U}$, then $h_u \in H^m([0, 1], \mathbb{R}^k)$. Moreover, for every integer $1 \leq m \leq r - 1$, for every $R > 0$ there exist $C^m_R > 0$ and $L^m_R > 0$ such that*

$$||h_u||_{H^m} \leq C^m_R \tag{4.5}$$

for every $u \in H^{m-1}([0, 1], \mathbb{R}^k)$ such that $||u||_{H^{m-1}} \leq R$, and

$$||h_{u+w} - h_u||_{H^m} \leq L^m_R ||w||_{H^{m-1}} \tag{4.6}$$

for every $u, w \in H^{m-1}([0, 1], \mathbb{R}^k)$ such that $||u||_{H^{m-1}}, ||w||_{H^{m-1}} \leq R$.

Proof It is sufficient to prove the thesis in the case $m = r - 1$, for every integer $r \geq 2$. When $r = 2$, $m = 1$, we have to prove that, for every $u \in \mathcal{U}$, the function $h_u : [0, 1] \to \mathbb{R}^k$ is in H^1. Recalling Eq. 3.11, we have that, for every $j = 1, \dots, k$, the jth component of h_u is given by the product

$$h^j_u(s) = \lambda_u(s) \cdot F^j(x_u(s))$$

for every $s \in [0, 1]$, where $\lambda_u : [0, 1] \to (\mathbb{R}^n)^*$ was defined in Eq. 3.9. Since both $s \mapsto \lambda_u(s)$ and $s \mapsto F^j(x_u(s))$ are in H^1, then their product is in H^1 as well (see, e.g., [6, Corollary 8.10]). Therefore, since $\lambda_u : [0, 1] \to (\mathbb{R}^n)^*$ solves Eq. 3.10, we can compute

$$\dot{h}^j_u(s) = \lambda_u(s) \cdot \sum_{i=1}^{k} [F^i, F^j]_{x_u(s)} u^i(s) \tag{4.7}$$

for every $j = 1, \dots, k$ and for a.e. $s \in [0, 1]$. In virtue of Eqs. 4.1, 2.11 and 4.3, for every $R > 0$, there exists $C'_R > 0$ such that

$$|\dot{h}_u^j(s)| \le C'_R |u(s)|_1$$

for a.e. $s \in [0, 1]$, for every $j = 1, \dots, k$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \le R$. Recalling Eq. 2.10, we deduce that

$$||\dot{h}_u^j||_{L^2} \le \sqrt{k} C'_R ||u||_{L^2} \tag{4.8}$$

for every $j = 1, \dots, k$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \le R$. Finally, using Eq. 3.12, we obtain that Eq. 4.5 holds for $r = 2, m = 1$. To prove Eq. 4.6, we observe that, for every $j = 1, \dots, k$ and for every $u, w \in \mathcal{U}$ we have

$$\begin{aligned}
&\left|\dot{h}_{u+w}^j(s) - \dot{h}_u^j(s)\right| \\
&\le |\lambda_{u+w}(s) - \lambda_u(s)|_2 \sum_{i=1}^k \left|[F^i, F^j]_{x_{u+w}(s)}\right|_2 |u^i(s) + w^i(s)| \\
&\quad + |\lambda_u(s)|_2 \sum_{i=1}^k \left|[F^i, F^j]_{x_{u+w}(s)} - [F^i, F^j]_{x_u(s)}\right|_2 |u^i(s) + w^i(s)| \\
&\quad + |\lambda_u(s)|_2 \sum_{i=1}^k \left|[F^i, F^j]_{x_u(s)}\right|_2 |w^i(s)|
\end{aligned}$$

for a.e. $s \in [0, 1]$. In virtue of Lemma 4.1, Lemma 2.2, Proposition 2.3 and Lemma 4.2, for every $R > 0$ there exist $L'_R > 0$ and $C''_R > 0$ such that for every $j = 1, \dots, k$ the inequality

$$\left|\dot{h}_{u+w}^j(s) - \dot{h}_u^j(s)\right| \le L'_R ||w||_{L^2} |u(s) + w(s)|_1 + C''_R |w(s)|_1$$

holds for a.e. $s \in [0, 1]$ and for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \le R$. Using Eq. 2.10, the previous inequality implies that there exists $L''_R > 0$ such that

$$||\dot{h}_{u+w}^j - \dot{h}_u^j||_{L^2} \le L''_R ||w||_{L^2} \tag{4.9}$$

for every $u, w \in \mathcal{U}$ such that $||u||_{L^2}, ||w||_{L^2} \le R$. Recalling Eq. 3.13, we conclude that Eq. 4.6 holds for $r = 2, m = 1$.

For $r = 3, m = 2$, we have to prove that, for every $u \in H^1([0, 1], \mathbb{R}^k)$, the function h_u belongs to $H^2([0, 1], \mathbb{R}^k)$. This follows if we show that $\dot{h}_u \in H^1([0, 1], \mathbb{R}^k)$ for for every $u \in H^1([0, 1], \mathbb{R}^k)$. Using the identity Eq. 4.7, we deduce that, whenever $u \in H^1([0, 1], \mathbb{R}^k)$, $\dot{h}_u^j$ is the product of three H^1-regular functions, for every $j = 1, \dots, k$. Therefore, using again [6, Corollary 8.10], we deduce that $\dot{h}_u^j$ is H^1-regular as well. From Eq. 4.7, for every $j = 1, \dots, k$, we have that

$$\begin{aligned}
\ddot{h}_u^j(s) &= \lambda_u(s) \cdot \sum_{i_1, i_2 = 1}^k [F^{i_2}, [F^{i_1}, F^j]]_{x_u(s)} u^{i_1}(s) u^{i_2}(s) \\
&\quad + \lambda_u(s) \cdot \sum_{i_1 = 1}^k [F^{i_1}, F^j]_{x_u(s)} \dot{u}^{i_1}(s)
\end{aligned}$$

for a.e. $s \in [0,1]$. Using Lemma 4.1, Lemma 2.2, Lemma 4.2, and recalling Theorem 2.1, we obtain that, for every $R > 0$ there exist $C'_R, C''_R > 0$ such that

$$||\ddot{h}^j_u(s)||_{L^2} \leq C'_R + C''_R||\dot{u}(s)||_{L^2} \tag{4.10}$$

for a.e. $s \in [0,1]$, for every $j = 1,\dots,k$ and for every $u \in H^1([0,1],\mathbb{R}^k)$ such that $||u||_{H^1} \leq R$. Therefore, combining Eqs. 3.12, 4.8 and 4.10, the inequality Eq. 4.5 follows for the case $r = 3, m = 2$. In view of Eqs. 3.13 and 4.9, in order to prove Eq. 4.6 for $r = 3, m = 2$ it is sufficient to show that, for every $R > 0$ there exists $L'_R > 0$ such that

$$||\ddot{h}^j_{u+w} - \ddot{h}^j_u||_{L^2} \leq L'_R||w||_{H^1} \tag{4.11}$$

for every $u, w \in H^1([0,1],\mathbb{R}^k)$ such that $||u||_{H^1}, ||w||_{H^1} \leq R$. The inequality Eq. 4.11 can be deduced with an argument based on the triangular inequality, similarly as done in the case $r = 2, m = 1$.

The same strategy works for every $r \geq 4$. □

The main consequence of Lemma 4.3 is that, when the map $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ defined in Eq. 3.14 is restricted to $H^m([0,1],\mathbb{R}^k)$, the restriction $\mathcal{G}^\beta : H^m \to H^m$ is bounded and Lipschitz continuous on bounded sets.

Proposition 4.4 *Let us assume that the vector fields $F^1,\dots,F^k$ defining the control system* Eq. 2.6 *are C^r-regular with $r \geq 2$, and that the function $a : \mathbb{R}^n \to \mathbb{R}$ designing the end-point cost is C^2-regular. For every $\beta > 0$, let $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ be the representation map defined in* Eq. 3.4. *Then, for every integer $1 \leq m \leq r-1$, we have that*

$$\mathcal{G}^\beta(H^m([0,1],\mathbb{R}^k)) \subset H^m([0,1],\mathbb{R}^k).$$

Moreover, for every integer $1 \leq m \leq r-1$ and for every $R > 0$ there exists $C^m_R > 0$ such that

$$||\mathcal{G}^\beta[u]||_{H^m} \leq C^m_R \tag{4.12}$$

for every $u \in H^m([0,1],\mathbb{R}^k)$ such that $||u||_{H^m} \leq R$, and there exists $L^m_R > 0$ such that

$$||\mathcal{G}^\beta[u+w] - \mathcal{G}^\beta[u]||_{H^m} \leq L^m_R||w||_{H^m} \tag{4.13}$$

for every $u, w \in H^m([0,1],\mathbb{R}^k)$ such that $||u||_{H^m}, ||w||_{H^m} \leq R$.

Proof Recalling that for every $u \in \mathcal{U}$ we have

$$\mathcal{G}^\beta[u] = u + \beta h_u,$$

the thesis follows directly from Lemma 4.3. □

Proposition 4.4 suggests that, when the vector fields $F^1,\dots,F^k$ are C^r-regular with $r \geq 2$, we can restrict the gradient flow equation Eq. 3.3 to the Hilbert spaces $H^m([0,1],\mathbb{R}^k)$, for every integer $1 \leq m \leq r-1$. Namely, for every integer $1 \leq m \leq r-1$, we shall introduce the application $\mathcal{G}^\beta_m : H^m([0,1],\mathbb{R}^k) \to H^m([0,1],\mathbb{R}^k)$ defined as the restriction of $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ to H^m, i.e.,

$$\mathcal{G}^\beta_m := \mathcal{G}^\beta|_{H^m}. \tag{4.14}$$

For every integer $m \geq 1$, given a curve $U : (a,b) \to H^m([0,1],\mathbb{R}^k)$, we say that it is (strongly) differentiable at $t_0 \in (a,b)$ if there exists $u \in H^m([0,1],\mathbb{R}^k)$ such that

$$\lim_{t\to t_0} \left|\left| \frac{U_t - U_{t_0}}{t - t_0} - u \right|\right|_{H^m} = 0. \tag{4.15}$$

In this case, we use the notation $\partial_t U_{t_0} := u$. For every $\ell = 1, \ldots, m$ and for every $t \in (a, b)$, we shall write $U_t^{(\ell)} \in H^{m-\ell}([0, 1], \mathbb{R}^k)$ to denote the ℓ-th Sobolev derivative of the function $U_t : s \mapsto U_t(s)$, i.e.,

$$\int_0^1 \left\langle U_t(s), \phi^{(\ell)}(s) \right\rangle_{\mathbb{R}^k} ds = (-1)^\ell \int_0^1 \left\langle U_t^{(\ell)}(s), \phi(s) \right\rangle_{\mathbb{R}^k} ds$$

for every $\phi \in C_c^\infty([0, 1], \mathbb{R}^k)$. It is important to observe that, for every order of derivation $\ell = 1, \ldots, m$, Eq. 4.15 implies that

$$\lim_{t \to t_0} \left|\left| \frac{U_t^{(\ell)} - U_{t_0}^{(\ell)}}{t - t_0} - u^{(\ell)} \right|\right|_{L^2} = 0,$$

and we use the notation $\partial_t U_{t_0}^{(\ell)} := u^{(\ell)}$. In particular, for every $\ell = 1, \ldots, m$, it follows that

$$\frac{d}{dt} ||U_t^{(\ell)}||_{L^2}^2 = 2 \int_0^1 \langle \partial_t U_t^{(\ell)}(s), U_t^{(\ell)}(s) \rangle_{\mathbb{R}^k} ds = 2 \langle \partial_t U_t^{(\ell)}, U_t^{(\ell)} \rangle_{L^2}. \tag{4.16}$$

In the next result, we study the following evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}_m^\beta[U_t], \\ U_0 = u_0, \end{cases} \tag{4.17}$$

with $u_0 \in H^m([0, 1], \mathbb{R}^k)$, and where $\mathcal{G}_m^\beta : H^m([0, 1], \mathbb{R}^k) \to H^m([0, 1], \mathbb{R}^k)$ is defined as in Eq. 4.14. Before establishing the existence, uniqueness and global definition result for the Cauchy problem Eq. 4.17, we study the evolution of the semi-norms $||U_t^{(\ell)}||_{L^2}$ for $\ell = 1, \ldots, m$ along its solutions.

Lemma 4.5 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are C^r-regular with $r \geq 2$, and that the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost is C^2-regular. For every integer $1 \leq m \leq r - 1$ and for every inital datum $u_0 \in H^m([0, 1], \mathbb{R}^k)$, let $U : [0, \alpha) \to H^m([0, 1], \mathbb{R}^k)$ be a continuously differentiable solution of the Cauchy problem* Eq. 4.17. *Therefore, for every $R > 0$, there exists $C_R > 0$ such that, if $||u_0||_{H^m} \leq R$, then*

$$||U_t||_{H^m} \leq C_R \tag{4.18}$$

for every $t \in [0, \alpha)$.

Proof It is sufficient to prove the statement in the case $r \geq 2, m = r - 1$. We shall use an induction argument on r.

Let us consider the case $r = 2, m = 1$. We observe that if $U : [0, \alpha) \to H^1([0, 1], \mathbb{R}^k)$ is a solution of Eq. 4.17 with $m = 1$, then it solves as well the Cauchy problem Eq. 3.3 in $\mathcal{U}$. Therefore, recalling that $||u_0||_{L^2} \leq ||u_0||_{H^1}$, in virtue of Lemma 3.7, for every $R > 0$, there exists $C_R' > 0$ such that, if $||u_0||_{H^1} \leq R$, we have that

$$||U_t||_{L^2} \leq C_R' \tag{4.19}$$

for every $t \in [0, \alpha)$. Hence, it is sufficient to provide an upper bound to the semi-norm $||U_t^{(1)}||_{L^2}$. From Eq. 4.16 and from the fact that $t \mapsto U_t$ solves Eq. 4.17 for $m = 1$, it follows that

$$\begin{aligned} \frac{d}{dt}||U_t^{(1)}||_{L^2}^2 &= 2\langle \partial_t U_t^{(1)}, U_t^{(1)} \rangle_{L^2} = -2 \int_0^1 \left\langle U_t^{(1)}(s) + \beta h_{U_t}^{(1)}(s), U_t^{(1)}(s) \right\rangle_{\mathbb{R}^k} ds \\ &\leq -2||U_t^{(1)}||_{L^2}^2 + 2\beta ||h_{U_t}^{(1)}||_{L^2} ||U_t^{(1)}||_{L^2} \\ &\leq -||U_t^{(1)}||_{L^2}^2 + \beta^2 ||h_{U_t}^{(1)}||_{L^2}^2 \end{aligned}$$

for every $t \in [0, \alpha)$, where $h_{U_t} : [0, 1] \to \mathbb{R}^k$ is the absolutely continuous curve defined in Eq. 3.11, and $h_{U_t}^{(1)}$ is its Sobolev derivative. Combining Eq. 4.19 with Eq. 4.5, we obtain that there exists $C_R^1 > 0$ such that

$$\frac{d}{dt} \left\| U_t^{(1)} \right\|_{L^2}^2 \leq - \left\| U_t^{(1)} \right\|_{L^2}^2 + \beta^2 C_R^1$$

for every $t \in [0, \alpha)$. This implies that

$$\left\| U_t^{(1)} \right\|_{L^2} \leq \max \left\{ \left\| U_0^{(1)} \right\|_{L^2}, \beta \sqrt{C_R^1} \right\}$$

for every $t \in [0, \alpha)$. This proves the thesis in the case $r = 2, m = 1$.

Let us prove the induction step. We shall prove the thesis in the case $r, m = r - 1$. Let $U : [0, \alpha) \to H^m([0, 1], \mathbb{R}^k)$ be a solution of Eq. 4.17 with $m = r - 1$. We observe that $t \mapsto U_t$ solves as well

$$\begin{cases} \partial_t U_t = -\mathcal{G}_{m-1}^\beta [U_t], \\ U_0 = u_0. \end{cases}$$

Using the inductive hypothesis and that $||u_0||_{H^{m-1}} \leq ||u_0||_{H^m}$, for every $R > 0$ there exists $C_R' > 0$ such that, if $||u_0||_{H^m} \leq R$, we have that

$$||U_t||_{H^{m-1}} \leq C_R' \tag{4.20}$$

for every $t \in [0, \alpha)$. Hence, it is sufficient to provide an upper bound to the semi-norm $||U_t^{(m)}||_{L^2}$. Recalling Eq. 4.16, the same computation as before yields

$$\frac{d}{dt} \left\| U_t^{(m)} \right\|_{L^2}^2 \leq - \left| U_t^{(m)} \right|_{L^2}^2 + \beta^2 \left\| h_{U_t}^{(m)} \right|_{L^2}^2$$

for every $t \in [0, \alpha)$. Combining Eq. 4.20 with Eq. 4.5, we obtain that there exists $C_R^1 > 0$ such that

$$\frac{d}{dt} \left\| U_t^{(m)} \right|_{L^2}^2 \leq - \left\| U_t^{(m)} \right\|_{L^2}^2 + \beta^2 C_R^1$$

for every $t \in [0, \alpha)$. This yields Eq. 4.18 for the inductive case $r, m = r - 1$. □

We are now in position to prove that the Cauchy problem Eq. 4.17 admits a unique and globally defined solution. The proof of the following result follows the lines of the proof of Theorem 3.8.

Theorem 4.6 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are C^r-regular with $r \geq 2$, and that the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost is C^2-regular. Then, for every integer $1 \leq m \leq r - 1$ and for every inital datum $u_0 \in H^m([0, 1], \mathbb{R}^k)$, the evolution equation* Eq. 4.17 *admits a unique, globally defined and* *continuously differentiable solution* $U : [0, +\infty) \to H^m([0, 1], \mathbb{R}^k)$*. Moreover, there exists* $C_{u_0} > 0$ *such that*

$$||U_t||_{H^m} \leq C_{u_0} \tag{4.21}$$

for every $t \in [0, +\infty)$.

Proof It is sufficient to prove the statement in the case $r \geq 2, m = r - 1$. In virtue of Lemma 4.5 and Proposition 4.4, the global existence of the solution of Eq. 4.17 follows from a *verbatim* repetition of the argument of the proof of Theorem 3.8. Finally, Eq. 4.21 descends directly from Lemma 4.5. □

Remark 4.7 We insist on the fact that, under the regularity assumptions of Theorem 4.6, if the initial datum u_0 is H^m-Sobolev regular with $m \leq r - 1$, then the solution $U : [0, +\infty) \to \mathcal{U}$ of Eq. 3.3 does coincide with the solution of Eq. 4.17. In other words, let us assume that the hypotheses of Theorem 4.6 are met, and let us consider the evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}^\beta[U_t], \\ U_0 = u_0, \end{cases} \tag{4.22}$$

where $u_0 \in H^m([0, 1], \mathbb{R}^k)$, with $m \leq r - 1$. Owing to Theorem 3.8, it follows that Eq. 4.22 admits a unique solution $U : [0, +\infty) \to \mathcal{U}$. We claim that $t \mapsto U_t$ solves as well the evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}_m^\beta[U_t], \\ U_0 = u_0. \end{cases} \tag{4.23}$$

Indeed, Theorem 4.6 implies that Eq. 4.23 admits a unique solution $\tilde{U} : [0, +\infty) \to H^m([0, 1], \mathbb{R}^k)$. Moreover, any solution of Eq. 4.23 is also a solution of Eq. 4.22; therefore, we must have $U_t = \tilde{U}_t$ for every $t \geq 0$ by the uniqueness of the solution of Eq. 4.22. Hence, it follows that, if the controlled vector fields $F^1, \dots, F^k$ and the function $a : \mathbb{R}^n \to \mathbb{R}_+$ are regular enough, then for every $t \in [0, +\infty)$, each point of the gradient flow trajectory U_t solving Eq. 4.22 has the same Sobolev regularity as the initial datum.

We now prove a pre-compactness result for the gradient flow trajectories. We recall that we use the convention $H^0 = L^2$.

Corollary 4.8 *Under the same assumptions of Theorem 4.6, let us consider* $u_0 \in H^m([0, 1], \mathbb{R}^k)$ *with the integer m satisfying* $1 \leq m \leq r - 1$*. Let* $U : [0, +\infty) \to \mathcal{U}$ *be the solution of the Cauchy problem* Eq. 3.3 *with initial condition* $U_0 = u_0$*. Then, the trajectory* $\{U_t : t \geq 0\}$ *is pre-compact in* $H^{m-1}([0, 1], \mathbb{R}^k)$.

Proof As observed in Remark 4.7, we have that the solution $U : [0, +\infty) \to \mathcal{U}$ of Eq. 3.3 satisfies $U_t \in H^m([0, 1], \mathbb{R}^k)$ for every $t \geq 0$, and that it solves Eq. 4.17 as well. In virtue of Theorem 2.1, the inclusion $H^m([0, 1], \mathbb{R}^k) \hookrightarrow H^{m-1}([0, 1], \mathbb{R}^k)$ is compact for every integer $m \geq 1$; therefore, from Eq. 4.21, we deduce the thesis. □

5 Lojasiewicz-Simon Inequality

In this section, we show that when the controlled vector fields $F^1, \dots, F^k$ and the function $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic, then the cost functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ satisfies the

Lojasiewicz-Simon inequality. This fact will be of crucial importance for the convergence proof of the next section. For a complete survey on the Lojasiewicz-Simon inequality, we refer the reader to the paper [7].

In this section, we prove that the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ defined in Eq. 3.1 satisfies the Lojasiewicz-Simon inequality for every $\beta > 0$. We first show that, when the function $a : \mathbb{R}^n \to \mathbb{R}_+$ involved in the definition of the end-point cost Eq. 3.5 and the controlled vector fields $F^1, \dots, F^k$ are real-analytic, $\mathcal{F}^\beta$ is real-analytic as well, for every $\beta > 0$. We recall the notion of real-analytic application defined on a Banach space. For an introduction to the subject, see, for example, [15].

Definition 5.1 Let E_1, E_2 be Banach spaces, and let us consider an application $\mathcal{T} : E_1 \to E_2$. The function $\mathcal{T}$ is said to be *real-analytic at* $e_0 \in E_1$ if for every $N \geq 1$ there exists a continuous and symmetric multi-linear application $l_N \in \mathscr{L}((E_1)^N, E_2)$ and if there exists $r > 0$ such that, for every $e \in E_1$ satisfying $||e - e_0||_{E_1} < r$, we have

$$\sum_{N=1}^{\infty} ||l_N||_{\mathscr{L}((E_1)^N, E_2)} \, ||e - e_0||_{E_1}^N < +\infty$$

and

$$\mathcal{T}(e) - \mathcal{T}(e_0) = \sum_{N=1}^{\infty} l_N (e - e_0)^N,$$

where, for every $N \geq 1$, we set $l_N(e-e_0)^N := l_N(e-e_0, \dots, e-e_0)$. Finally, $\mathcal{T} : E_1 \to E_2$ is *real-analytic on* E_1 if it is real-analytic at every $e_0 \in E_1$.

In the next result, we provide the conditions that guarantee that $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}$ is real-analytic.

Proposition 5.2 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are real-analytic, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost* Eq. 3.5. *Therefore, for every $\beta > 0$, the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ defined in* Eq. 3.1 *is real-analytic.*

Proof Since $\mathcal{F}^\beta(u) = \frac{1}{2}||u||_{L^2} + \beta\mathcal{E}(u)$ for every $u \in \mathcal{U}$, the proof reduces to show that the end-point cost $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$ is real-analytic. Recalling the definition of $\mathcal{E}$ given in Eq. 3.5 and the end-point map $P_1 : \mathcal{U} \to \mathbb{R}^n$ introduced in Eq. 2.20, we have that the former can be expressed as the composition

$$\mathcal{E} = a \circ P_1.$$

In the proof of [4, Proposition 8.5] it is shown that P_1 is smooth as soon as $F^1, \dots, F^k$ are C^∞-regular, and the expression of the Taylor expansion of P_1 at every $u \in \mathcal{U}$ is provided. In [2, Proposition 2.1], it is proved that, when $a : \mathbb{R}^n \to \mathbb{R}_+$ and the controlled vector fields are real-analytic, the Taylor series of $a \circ P_1$ is actually convergent. □

The previous result implies that the differential $d\mathcal{F}^\beta : \mathcal{U} \to \mathcal{U}^*$ is real-analytic.

Corollary 5.3 *Under the same assumptions as in Proposition 5.2, for every $\beta > 0$, the differential $d\mathcal{F}^\beta : \mathcal{U} \to \mathcal{U}^*$ is real-analytic.*

Proof Owing to Proposition 5.2, the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ is real-analytic. Using this fact, the thesis follows from [15, Theorem 2, p.1078]. □

Another key-step in view of the Lojasiewicz-Simon inequality is the study of the Hessian of the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$. In our framework, the Hessian of $\mathcal{F}^\beta$ at a point $u \in \mathcal{U}$ is the bounded linear operator $\mathrm{Hess}_u \mathcal{F}^\beta : \mathcal{U} \to \mathcal{U}$ that satisfies the identity:

$$\langle \mathrm{Hess}_u \mathcal{F}^\beta[v], w\rangle_{L^2} = d_u^2 \mathcal{F}^\beta(v, w) \tag{5.1}$$

for every $v, w \in \mathcal{U}$, where $d_u^2\mathcal{F}^\beta : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is the second differential of $\mathcal{F}^\beta$ at the point u. In the next proposition we prove that, for every $u \in \mathcal{U}$, $\mathrm{Hess}_u \mathcal{F}^\beta$ has finite-dimensional kernel. We stress on the fact that, unlike the other results of the present section, we do not have to assume that $F^1, \dots, F^k$ and $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic to study the kernel of $\mathrm{Hess}_u \mathcal{F}^\beta$.

Proposition 5.4 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ defining the end-point cost* Eq. 3.5. *For every $u \in \mathcal{U}$, let* $\mathrm{Hess}_u \mathcal{F}^\beta : \mathcal{U} \to \mathcal{U}$ *be the linear operator that represents the second differential $d_u^2\mathcal{F}^\beta : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ through the identity* Eq. 5.1. *Then, the the kernel of* $\mathrm{Hess}_u \mathcal{F}^\beta$ *is finite-dimensional.*

Proof For every $u \in \mathcal{U}$, we have that

$$d_u^2 \mathcal{F}^\beta(v, w) = \langle v, w\rangle_{L^2} + \beta d_u^2 \mathcal{E}(v, w)$$

for every $v, w \in \mathcal{U}$. Therefore, we are reduced to study the second differential of the end-point cost $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$. Recalling its definition in Eq. 3.5 and applying the chain-rule, we obtain that

$$d_u^2 \mathcal{E}(v, w) = [D_u P_1(v)]^T\, \nabla^2 a(x_u(1))\, [D_u P_1(w)] + (\nabla a(x_u(1)))^T \cdot D_u^2 P_1(v, w), \tag{5.2}$$

where $P_1 : \mathcal{U} \to \mathbb{R}^n$ is the end-point map defined in Eq. 2.20, and where the curve $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of Eq. 2.6 corresponding to the control $u \in \mathcal{U}$. We recall that, for every $y \in \mathbb{R}^n$, we understand $\nabla a(y)$ as a row vector. Let us set $\nu_u := (\nabla a(x_u(1)))^T$ and $H_u := \nabla^2 a(x_u(1))$, where $H_u : \mathbb{R}^n \to \mathbb{R}^n$ is the self-adjoint linear operator associated to the Hessian of $a : \mathbb{R}^n \to \mathbb{R}_+$ at the point $x_u(1)$. Therefore, we can write

$$d_u^2 \mathcal{E}(v, w) = \langle \big(D_u P_1^* \circ H_u \circ D_u P_1\big)[v], w\rangle_{L^2} + \nu_u \cdot D_u^2 P_1(v, w) \tag{5.3}$$

for every $v, w \in \mathcal{U}$, where $D_u P_1^* : \mathbb{R}^n \to \mathcal{U}$ is the adjoint of the differential $D_u P_1 : \mathcal{U} \to \mathbb{R}^n$. Moreover, recalling the definition of the linear operator $\mathcal{N}_u^\nu : \mathcal{U} \to \mathcal{U}$ given in Eq. 2.46, we have that

$$\nu_u \cdot D_u^2 P_1(v, w) = \langle \mathcal{N}_u^{\nu_u}[v], w\rangle_{L^2}$$

for every $v, w \in \mathcal{U}$. Therefore, we obtain

$$d_u^2 \mathcal{E}(v, w) = \langle \mathrm{Hess}_u \mathcal{E}[v], w\rangle_{L^2} \tag{5.4}$$

for every $v, w \in \mathcal{U}$, where $\mathrm{Hess}_u \mathcal{E} : \mathcal{U} \to \mathcal{U}$ is the linear operator that satisfies the identity:

$$\mathrm{Hess}_u \mathcal{E} = D_u P_1^* \circ H_u \circ D_u P_1 + \mathcal{N}_u^{\nu_u}.$$

We observe that $\mathrm{Hess}_u \mathcal{E}$ is a self-adjoint compact operator. Indeed, $N_u^{\nu_u}$ is self-adjoint and compact in virtue of Proposition 2.21, while $D_u P_1^* \circ H_u \circ D_u P_1$ has finite-rank and it self-adjoint as well. Combining Eqs. 5.2 and 5.4, we deduce that

$$\mathrm{Hess}_u \mathcal{F}^\beta = \mathrm{Id} + \beta \mathrm{Hess}_u \mathcal{E}, \tag{5.5}$$

where $\mathrm{Id} : \mathcal{U} \to \mathcal{U}$ is the identity. Finally, using the Fredholm alternative (see, e.g., [6, Theorem 6.6]), we deduce that the kernel of $\mathrm{Hess}_u \mathcal{F}^\beta$ is finite-dimensional. □

We are now in position to prove that the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ satisfies the Lojasiewicz-Simon inequality.

Theorem 5.5 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are real-analytic, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ defining end-point cost* Eq. 3.5. *For every $\beta > 0$ and for every $u \in \mathcal{U}$, there exists $r > 0$, $C > 0$ and $\gamma \in (1, 2]$ such that*

$$\left|\mathcal{F}^\beta(v) - \mathcal{F}^\beta(u)\right| \leq C \left\|d_v \mathcal{F}^\beta\right\|_{\mathcal{U}^*}^\gamma \tag{5.6}$$

for every $v \in \mathcal{U}$ such that $||v - u||_{L^2} < r$.

Proof If $u \in \mathcal{U}$ is not a critical point for $\mathcal{F}^\beta$, i.e., $d_u \mathcal{F}^\beta \neq 0$, then there exists $r_1 > 0$ and $\kappa > 0$ such that

$$||d_v \mathcal{F}^\beta||_{\mathcal{U}^*}^2 \geq \kappa$$

for every $v \in \mathcal{U}$ satisfying $||v - u||_{L^2} < r_1$. On the other hand, by the continuity of $\mathcal{F}^\beta$, we deduce that there exists $r_2 > 0$ such that

$$\left|\mathcal{F}^\beta(v) - \mathcal{F}^\beta(u)\right| \leq \kappa$$

for every $v \in \mathcal{U}$ satisfying $\|v - u\|_{L^2} < r_2$. Combining the previous inequalities and taking $r := \min\{r_1, r_2\}$, we deduce that, when $d_u \mathcal{F}^\beta \neq 0$, Eq. 5.6 holds with $\gamma = 2$.

The inequality Eq. 5.6 in the case $d_u \mathcal{F}^\beta = 0$ follows from [7, Corollary 3.11]. We shall now verify the assumptions of this result. First of all, [7, Hypothesis 3.2] is satisfied, being $\mathcal{U}$ an Hilbert space. Moreover, [7, Hypothesis 3.4] follows by choosing $W = \mathcal{U}^*$. In addition, we recall that $d\mathcal{F}^\beta : \mathcal{U} \to \mathcal{U}^*$ is real-analytic in virtue of Corollary 5.3, and that $\mathrm{Hess}_u \mathcal{F}^\beta$ has finite-dimensional kernel owing to Proposition 5.4. These facts imply that the conditions (1)–(4) of [7, Corollary 3.11] are verified if we set $X = \mathcal{U}$ and $Y = \mathcal{U}^*$. □

6 Convergence of the Gradient Flow

In this section, we show that the gradient flow trajectory $U : [0 + \infty) \to \mathcal{U}$ that solves Eq. 3.3 is convergent to a critical point of the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}$, provided that the Cauchy datum $U_0 = u_0$ satisfies $u_0 \in H^1([0, 1], \mathbb{R}^k) \subset \mathcal{U}$. The Lojasiewicz-Simon inequality established in Theorem 5.5 will play a crucial role in the proof of the convergence result. Indeed, we use this inequality to show that the trajectories with Sobolev-regular initial datum have finite length. In order to satisfy the assumptions of Theorem 5.5, we need to assume throughout the section that the controlled vector fields $F^1, \dots, F^k$ and the function $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic.

We first recall the notion of the Riemann integral of a curve that takes values in $\mathcal{U}$. For general statements and further details, we refer the reader to [10, Section 1.3]. Let us consider a continuous curve $V : [a, b] \to \mathcal{U}$. Therefore, using [10, Theorem 1.3.1], we can define

$$\int_a^b V_t \, dt := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} V_{\frac{b-a}{n} k}.$$

We immediately observe that the following inequality holds:

$$\left|\left|\int_a^b V_t \, dt\right|\right|_{L^2} \leq \int_a^b ||V_t||_{L^2} \, dt. \tag{6.1}$$

Moreover, [10, Theorem 1.3.4] guarantees that, if the curve $V : [a, b] \to \mathcal{U}$ is continuously differentiable, then we have:

$$V_b - V_a = \int_a^b \partial_t V_\theta \, d\theta, \tag{6.2}$$

where $\partial_t V_\theta$ is the derivative of the curve $t \mapsto V_t$ defined as in Eq. 3.2 and computed at the instant $\theta \in [a, b]$. Finally, combining Eqs. 6.2 and 6.1, we deduce that

$$||V_b - V_a||_{L^2} \leq \int_a^b ||\partial_t V_\theta||_{L^2} \, d\theta. \tag{6.3}$$

We refer to the quantity at the right-hand side of Eq. 6.3 as *the length of the continuously differentiable curve* $V : [a, b] \to \mathcal{U}$.

Let $U : [0, +\infty) \to \mathcal{U}$ be the solution of the gradient flow equation Eq. 3.3 with initial datum $u_0 \in \mathcal{U}$. We say that $u_\infty \in \mathcal{U}$ is a *limiting point* for the curve $t \mapsto U_t$ if there exists a sequence $(t_j)_{j\geq 1}$ such that $t_j \to +\infty$ and $||U_{t_j} - u_\infty||_{L^2} \to 0$ as $j \to \infty$. In the next result, we study the length of $t \mapsto U_t$ in a neighborhood of a limiting point.

Proposition 6.1 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *are real-analytic, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. Let $U : [0, +\infty) \to \mathcal{U}$ be the solution of the Cauchy problem* Eq. 3.3 *with initial datum $U_0 = u_0$, and let $u_\infty \in \mathcal{U}$ be any of its limiting points. Then, there exists $r > 0$ such that the portion of the curve that lies in $B_r(u_\infty)$ has finite length, i.e.,*

$$\int_{\mathcal{I}} ||\partial_t U_\theta||_{L^2} \, d\theta < \infty, \tag{6.4}$$

where $\mathcal{I} := \{t \geq 0 : U_t \in B_r(u_\infty)\}$, and $B_r(u_\infty) := \{u \in \mathcal{U} : ||u - u_\infty||_{L^2} < r\}$.

Proof Let $u_\infty \in \mathcal{U}$ be a limiting point of $t \mapsto U_t$, and let $(\bar{t}_j)_{j\geq 1}$ be a sequence such that $\bar{t}_j \to +\infty$ and $||U_{\bar{t}_j} - u_\infty||_{L^2} \to 0$ as $j \to \infty$. The same computation as in Eq. 3.16 implies that the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ is decreasing along the trajectory $t \mapsto U_t$, i.e.,

$$\mathcal{F}^\beta(U_{t'}) \leq \mathcal{F}^\beta(U_t) \tag{6.5}$$

for every $t' \geq t \geq 0$. In addition, using the continuity of $\mathcal{F}^\beta$, it follows that $\mathcal{F}^\beta(U_{\bar{t}_j}) \to \mathcal{F}^\beta(u_\infty)$ as $j \to \infty$. Combining these facts, we have that

$$\mathcal{F}^\beta(U_t) - \mathcal{F}^\beta(u_\infty) \geq 0 \tag{6.6}$$

for every $t \geq 0$. Moreover, owing to Theorem 5.5, we deduce that there exist $C > 0$, $\gamma \in (1, 2]$ and $r > 0$ such that

$$|\mathcal{F}^\beta(v) - \mathcal{F}^\beta(u_\infty)| \leq \frac{1}{C} ||d_v \mathcal{F}^\beta||_{\mathcal{U}^*}^\gamma \tag{6.7}$$

for every $v \in B_r(u_\infty)$. Let $t_1 \geq 0$ be the infimum of the instants such that $U_t \in B_r(u_\infty)$, i.e.,

$$t_1 := \inf_{t\geq 0}\{U_t \in B_r(u_\infty)\}.$$

We observe that the set where we take the infimum is nonempty, in virtue of the convergence $||U_{\bar{t}_j} - u_\infty||_{L^2} \to 0$ as $j \to \infty$. Then, there exists $t_1' \in (t_1, +\infty]$ such that $U_t \in B_r(u_\infty)$

for every $t \in (t_1, t_1')$, and we take the supremum $t_1' > t_1$ such that the previous condition is satisfied, i.e.,

$$t_1' := \sup_{t'>t_1} \{U_t \in B_r(u_\infty), \forall t \in (t_1, t')\}.$$

If $t_1' < \infty$, we set

$$t_2 := \inf_{t \geq t_1'} \{U_t \in B_r(u_\infty)\},$$

and

$$t_2' := \sup_{t'>t_2} \{U_t \in B_r(u_\infty), \forall t \in (t_2, t')\}.$$

We repeat this procedure (which terminates in a finite number of steps if and only if there exits $\bar{t} > 0$ such that $U_t \in B_r(u_\infty)$ for every $t \geq \bar{t}$), and we obtain a family of intervals $\{(t_j, t_j')\}_{j=1,\dots,N}$, where $N \in \mathbb{N} \cup \{\infty\}$. We observe that $\bigcup_{j=1}^N (t_j, t_j') = \mathcal{I}$, where we set $\mathcal{I} := \{t \geq 0 : U_t \in B_r(u_\infty)\}$.

Without loss of generality, we may assume that $\mathcal{I}$ is a set of infinite Lebesgue measure. Indeed, if this is not the case, we would have the thesis:

$$\int_{\mathcal{I}} ||\partial_t U_\theta||_{L^2}\, d\theta = \int_{\mathcal{I}} ||\mathcal{G}^\beta[U_\theta]||_{L^2}\, d\theta < \infty,$$

since $||\mathcal{G}^\beta[u]||_{L^2}$ is bounded on the bounded subsets of $\mathcal{U}$, as shown in Eq. 3.18. Therefore, we focus on the case when the Lebesgue measure of $\mathcal{I}$ is infinite. Let us introduce the following sequence:

$$\tau_0 = t_1, \quad \tau_1 = t_1', \quad \tau_2 = \tau_1 + (t_2' - t_2), \quad \dots, \quad \tau_j = \tau_{j-1} + (t_j' - t_j), \quad \dots, \tag{6.8}$$

where $t_1, t_1', \dots$ are the extremes of the intervals $\{(t_j, t_j')\}_{j=1,\dots,N}$ constructed above. Finally, we define the function $\sigma : [\tau_0, +\infty) \to [\tau_0, +\infty)$ as follows:

$$\sigma(t) := \begin{cases} t & \text{if } \tau_0 \leq t < \tau_1, \\ t - \tau_1 + t_2 & \text{if } \tau_1 \leq t < \tau_2, \\ t - \tau_2 + t_3 & \text{if } \tau_2 \leq t < \tau_3, \\ \dots & \dots \end{cases} \tag{6.9}$$

We observe that $\sigma : [\tau_0, +\infty) \to [\tau_0, +\infty)$ is piecewise affine and it is monotone increasing. In particular, we have that

$$\sigma(\tau_j) = t_{j+1} \geq t_j' = \lim_{t \to \tau_j^-} \sigma(t). \tag{6.10}$$

Moreover, from Eq. 6.8 and from the definition of the intervals $\left\{\left(t_j, t_j'\right)\right\}_{j\geq 1}$, it follows that

$$U_{\sigma(t)} \in B_r(u_\infty) \tag{6.11}$$

for every $t \in [\tau_0, +\infty)$. Let us define the function $g : [\tau_0, +\infty) \to \mathbb{R}_+$ as follows:

$$g(t) := \mathcal{F}^\beta(U_{\sigma(t)}) - \mathcal{F}^\beta(u_\infty), \tag{6.12}$$

where we used Eq. 6.6 to deduce that g is always non-negative. From Eq. 6.9, we obtain that the restriction $g|_{(\tau_j, \tau_{j+1})}$ is C^1-regular, for every $j \geq 0$. Therefore, using the fact that $\dot{\sigma}|_{(\tau_j, \tau_{j+1})} \equiv 1$, we compute

$$\dot{g}(t) = \frac{d}{dt}\left(\mathcal{F}^\beta(U_{\sigma(t)}) - \mathcal{F}^\beta(u_\infty)\right) = -d_{U_{\sigma(t)}}\mathcal{F}^\beta\left(\mathcal{G}^\beta[U_{\sigma(t)}]\right)$$

for every $t \in (\tau_j, \tau_{j+1})$ and for every $j \geq 0$. Recalling that $\mathcal{G}^\beta : \mathcal{U} \to \mathcal{U}$ is the Riesz's representation of the differential $d\mathcal{F}^\beta : \mathcal{U} \to \mathcal{U}^*$, it follows that

$$\dot{g}(t) = -\left\| d_{U_{\sigma(t)}} \mathcal{F}^\beta \right\|^2_{\mathcal{U}^*} \tag{6.13}$$

for every $t \in (\tau_j, \tau_{j+1})$ and for every $j \geq 0$. Moreover, owing to the Lojasiewicz-Simon inequality Eq. 6.7, from Eq. 6.11 we deduce that

$$\dot{g}(t) \leq -C g^{\frac{2}{\gamma}}(t) \tag{6.14}$$

for every $t \in (\tau_j, \tau_{j+1})$ and for every $j \geq 0$. Let $h : [\tau_0, \infty) \to [0, +\infty)$ be the solution of the Cauchy problem

$$\dot{h} = -C h^{\frac{2}{\gamma}}, \quad h(\tau_0) = g(\tau_0), \tag{6.15}$$

whose expression is

$$h(t) = \begin{cases} \left(h(\tau_0)^{1-\frac{2}{\gamma}} + \frac{(2-\gamma)C}{\gamma}(t - \tau_0) \right)^{-1-\frac{2\gamma-2}{2-\gamma}} & \text{if } \gamma \in (1,2), \\ h(\tau_0) e^{-Ct} & \text{if } \gamma = 2, \end{cases}$$

for every $t \in [\tau_0, \infty)$. Using the fact that $g|_{(\tau_0,\tau_1)}$ is C^1-regular, in view of Eq. 6.14, we deduce that

$$g(t) \leq h(t), \tag{6.16}$$

for every $t \in [\tau_0, \tau_1)$. We shall now prove that the previous inequality holds for every $t \in [\tau_0, +\infty)$ using an inductive argument. Let us assume that Eq. 6.16 holds in the interval $[\tau_0, \tau_j)$, with $j \geq 1$. From the definition of g, combining Eqs. 6.5 and 6.10, we obtain that

$$g(\tau_j) \leq \lim_{t \to \tau_j^-} g(t) \leq \lim_{t \to \tau_j^-} h(t) = h(\tau_j). \tag{6.17}$$

Using that the restriction $g|_{(\tau_j,\tau_{j+1})}$ is C^1-regular, in virtue of Eqs. 6.14, 6.15, and 6.17, we extend the the inequality Eq. 6.16 to the interval $[\tau_0, \tau_{j+1})$. This shows that Eq. 6.16 is satisfied for every $t \in [\tau_0, +\infty)$.

We now prove that the portion of the trajectory that lies in $B_r(u_\infty)$ is finite. We observe that

$$\int_{\mathcal{I}} ||\partial_t U_\theta||_{L^2} \, d\theta = \int_{\mathcal{I}} ||\mathcal{G}^\beta(U_\theta)||_{L^2} \, d\theta = \int_{\mathcal{I}} ||d_{U_\theta} \mathcal{F}^\beta||_{\mathcal{U}^*} \, d\theta, \tag{6.18}$$

where we recall that $\mathcal{I} = \bigcup_{j=1}^N \left(t_j, t'_j\right)$. For every $j \geq 1$, in the interval (t_j, t'_j) we use the change of variable $\theta = \sigma(\vartheta)$, where σ is defined in Eq. 6.9. Using Eqs. 6.8 and 6.9, we observe that $\sigma^{-1}\left\{\left(t_j, t'_j\right)\right\} = (\tau_{j-1}, \tau_j)$ and that $\dot{\sigma}|_{(\tau_{j-1},\tau_j)} \equiv 1$. These facts yield

$$\int_{t_j}^{t'_j} ||d_{U_\theta} \mathcal{F}^\beta||_{\mathcal{U}^*} \, d\theta = \int_{\tau_{j-1}}^{\tau_j} ||d_{U_{\sigma(\vartheta)}} \mathcal{F}^\beta||_{\mathcal{U}^*} \, d\vartheta = \int_{\tau_{j-1}}^{\tau_j} \sqrt{-\dot{g}(\vartheta)} \, d\vartheta \tag{6.19}$$

for every $j \geq 1$, where we used Eq. 6.13 in the last identity. Therefore, combining Eqs. 6.18 and 6.19, we deduce that

$$\int_{\mathcal{I}} ||\partial_t U_\theta||_{L^2} \, d\theta = \int_{\tau_0}^{+\infty} \sqrt{-\dot{g}(\vartheta)} \, d\vartheta. \tag{6.20}$$

Then, the thesis reduces to prove that the quantity at the right-hand side of Eq. 6.20 is finite. Let $\delta > 0$ be a positive quantity whose value will be specified later. From the Cauchy-Schwarz inequality, it follows that

$$\int_{\tau_0}^{+\infty} \sqrt{-\dot{g}(\vartheta)}\, d\vartheta \leq \left(\int_{\tau_0}^{\infty} -\dot{g}(\vartheta)\vartheta^{1+\delta}\, d\vartheta \right)^{\frac{1}{2}} \left(\int_{\tau_0}^{\infty} \vartheta^{-1-\delta}\, d\vartheta \right)^{\frac{1}{2}}. \tag{6.21}$$

On the other hand, for every $j \geq 1$, using the integration by parts on each interval $(\tau_0, \tau_1), \ldots, (\tau_{j-1}, \tau_j)$, we have that

$$\begin{aligned} \int_{\tau_0}^{\tau_j} -\dot{g}(\vartheta)\vartheta^{1+\delta}\, d\vartheta &= \sum_{i=1}^{j} \left(\tau_{i-1}^{1+\delta} g(\tau_{i-1}) - \tau_i^{1+\delta} g(\tau_i^-) + (1+\delta) \int_{\tau_{i-1}}^{\tau_i} g(\vartheta)\vartheta^{\delta}\, d\vartheta \right) \\ &\leq \tau_0^{1+\delta} g(\tau_0) - \tau_j^{1+\delta} g(\tau_j^-) + (1+\delta) \int_{\tau_0}^{\tau_j} h(\vartheta)\vartheta^{\delta}\, d\vartheta \\ &\leq \tau_0^{1+\delta} g(\tau_0) + (1+\delta) \int_{\tau_0}^{\tau_j} h(\vartheta)\vartheta^{\delta}\, d\vartheta, \end{aligned}$$

where we introduced the notation $g(\tau_i^-) := \lim_{\vartheta \to \tau_i^-} g(\vartheta)$, and we used the first inequality of Eq. 6.17 and the fact that g is always non-negative. Finally, if the exponent γ in Eq. 6.7 satisfies $\gamma = 2$, we can choose any positive $\delta > 0$. On the other hand, if $\gamma \in (1, 2)$, we choose δ such that $0 < \delta < \frac{2\gamma-2}{2-\gamma}$. This choice guarantees that that

$$\lim_{j \to \infty} \int_{\tau_0}^{\tau_j} -\dot{g}(\vartheta)\vartheta^{1+\delta}\, d\vartheta = \int_{\tau_0}^{\infty} -\dot{g}(\vartheta)\vartheta^{1+\delta}\, d\vartheta < \infty,$$

and therefore, in virtue of Eqs. 6.21 and 6.20, we deduce the thesis. □

In the following corollary, we state an immediate (but important) consequence of Proposition 6.1.

Corollary 6.2 *Under the same assumptions as in Proposition 6.1, let the curve $U : [0, +\infty) \to \mathcal{U}$ be the solution of the Cauchy problem* Eq. 3.3 *with initial datum $U_0 = u_0$. If $u_\infty \in \mathcal{U}$ is a limiting point for the curve $t \mapsto U_t$, then the whole solution converges to u_∞ as $t \to \infty$, i.e.,*

$$\lim_{t \to \infty} ||U_t - u_\infty||_{L^2} = 0.$$

Moreover, the length of the whole solution is finite.

Proof We prove the statement by contradiction. Let us assume that $t \mapsto U_t$ is not converging to u_∞ as $t \to \infty$. Let $B_r(u_\infty)$ be the neighborhood of u_∞ given by Proposition 6.1. Diminishing $r > 0$ if necessary, we can find two sequences $\{t_j\}_{j\geq 0}$ and $\{t'_j\}_{j\geq 0}$ such that for every $j \geq 0$ the following conditions hold:

- $t_j < t'_j < t_{j+1}$;
- $||U_{t_j} - u_\infty||_{L^2} \leq \frac{r}{4}$;
- $\frac{r}{2} \leq ||U_{t'_j} - u_\infty||_{L^2} \leq r$;
- $U_t \in B_r(u_\infty)$ for every $t \in (t_j, t'_j)$.

We observe that $\bigcup_{j=1}^{\infty}(t_j, t'_j) \subset \mathcal{I}$, where $\mathcal{I} := \{t \geq 0 : U_t \in B_r(u_\infty)\}$. Moreover, the inequality Eq. 6.3 and the previous conditions imply that

$$\int_{t_j}^{t'_j} ||\partial_t U_\theta||_{\mathcal{U}}\, d\theta \geq ||U_{t'_k} - U_{t_k}||_{\mathcal{U}} \geq \frac{r}{4}$$

for every $j \geq 0$. However, this contradicts Eq. 6.4. Therefore, we deduce that $||U_t - u_\infty||_{\mathcal{U}} \to 0$ as $t \to \infty$. In particular, this means that there exists $\bar{t} \geq 0$ such that $U_t \in B_r(u_\infty)$ for every $t \geq \bar{t}$. This in turn implies that the whole trajectory has finite length, since

$$\int_0^{\bar{t}} ||\partial_t U_\theta||_{L^2}\, d\theta < +\infty.$$

□

We observe that in Corollary 6.2 we need to assume a priori that the solution of the Cauchy problem Eq. 3.3 admits a limiting point. However, for a general initial datum $u_0 \in \mathcal{U}$ we cannot prove that this is actually the case. On the other hand, if we assume more regularity on the Cauchy datum u_0, we can use the compactness results proved in Section 4. We recall the notation $H^0([0, 1], \mathbb{R}^k) =: \mathcal{U}$.

Theorem 6.3 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are real-analytic, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. Let $U : [0, +\infty) \to \mathcal{U}$ be the solution of the Cauchy problem* Eq. 3.3 *with initial datum $U_0 = u_0$, and let $m \geq 1$ be an integer such that u_0 belongs to $H^m([0, 1], \mathbb{R}^k)$. Then, there exists $u_\infty \in H^m([0, 1], \mathbb{R}^k)$ such that*

$$\lim_{t\to\infty} ||U_t - u_\infty||_{H^{m-1}} = 0. \tag{6.22}$$

Proof Let us consider $u_0 \in H^m([0, 1], \mathbb{R}^k)$ and let $U : [0, +\infty) \to \mathcal{U}$ be the solution of Eq. 3.3 satisfying $U_0 = u_0$. Owing to Theorem 4.6, we have that $U_t \in H^m([0, 1], \mathbb{R}^k)$ for every $t \geq 0$, and that the trajectory $\{U_t : t \geq 0\}$ $H^m([0, 1], \mathbb{R}^k)$. In addition, from Corollary 4.8, we deduce that $\{U_t : t \geq 0\}$ is pre-compact with respect to the strong topology of $H^{m-1}([0, 1], \mathbb{R}^k)$. Therefore, there exist $u_\infty \in H^{m-1}([0, 1], \mathbb{R}^k)$ and a sequence $(t_j)_{j\geq 1}$ such that we have $t_j \to +\infty$ and $||U_{t_j} - u_\infty||_{H^{m-1}} \to 0$ as $j \to \infty$. In particular, this implies that $||U_{t_j} - u_\infty||_{L^2} \to 0$ as $j \to \infty$. In virtue of Corollary 6.2, we deduce that $||U_t - u_\infty||_{L^2} \to 0$ as $t \to +\infty$. Using again the pre-compactness of the trajectory $\{U_t : t \geq 0\}$ with respect to the strong topology of $H^{m-1}([0, 1], \mathbb{R}^k)$, the previous convergence implies that $||U_t - u_\infty||_{H^{m-1}} \to 0$ as $t \to +\infty$.

To conclude, we have to show that $u_\infty \in H^m([0, 1], \mathbb{R}^k)$. Owing to the compact inclusion Eq. 2.9 in Theorem 2.1, and recalling that the trajectory $\{U_t : t \geq 0\}$ is pre-compact with respect to the weak topology of $H^m([0, 1], \mathbb{R}^k)$, the convergence Eq. 6.22 guarantees that $u_\infty \in H^m([0, 1], \mathbb{R}^k)$ and that $U_t \rightharpoonup_{H^m} u_\infty$ as $t \to +\infty$. □

In the next result, we study the regularity of the limiting points of the gradient flow trajectories.

Theorem 6.4 *Let us assume that the vector fields $F^1, \ldots, F^k$ defining the control system* Eq. 2.6 *are real-analytic, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. Let $U : [0, +\infty) \to \mathcal{U}$ be the solution of the Cauchy problem* Eq. 3.3 *with initial datum $U_0 = u_0$, and let $u_\infty \in \mathcal{U}$ be any of its limiting points. Then, u_∞ is a critical point for the functional $\mathcal{F}^\beta$, i.e., $d_{u_\infty}\mathcal{F}^\beta = 0$. Moreover, $u_\infty \in H^m([0, 1], \mathbb{R}^k)$ for every integer $m \geq 1$.*

Proof By Corollary 6.2, we have that the solution $t \mapsto U_t$ converges to u_∞ as $t \to +\infty$ with respect to the strong topology of $\mathcal{U}$. Let us consider the radius $r > 0$ prescribed by Proposition 6.1. If $d_{u_\infty}\mathcal{F}^\beta \neq 0$, taking a smaller $r > 0$ if necessary, we have that there exists $\varepsilon > 0$ such that $||d_u\mathcal{F}^\beta||_{\mathcal{U}^*} \geq \varepsilon$ for every $u \in B_r(u_\infty)$. Recalling that $||U_t - u_\infty||_{\mathcal{U}} \to 0$ as $t \to +\infty$, then there exists $\bar{t} \geq 0$ such that $U_t \in B_r(u_\infty)$ and for every $t \geq \bar{t}$. On the other hand, this fact implies that $||\partial_t U_t||_{\mathcal{U}} = ||d_{U_t}\mathcal{F}^\beta||_{\mathcal{U}^*} \geq \varepsilon$ for every $t \geq \bar{t}$, but this contradicts Eq. 6.4, i.e., the fact that the length of the trajectory is finite. Therefore, we deduce that $d_{u_\infty}\mathcal{F}^\beta = 0$. As regards the regularity of u_∞, we observe that $d_{u_\infty}\mathcal{F}^\beta = 0$ implies that $\mathcal{G}^\beta[u_\infty] = 0$, which in turn gives

$$u_\infty = -\beta h_{u_\infty},$$

where the function $h_{u_\infty} : [0, 1] \to \mathbb{R}^k$ is defined as in Eq. 3.11. Owing to Lemma 4.3, we deduce that the right-hand side of the previous equality has regularity H^{m+1} whenever $u_\infty \in H^m$, for every integer $m \geq 0$. Using a bootstrapping argument, this implies that $u_\infty \in H^m([0, 1], \mathbb{R}^k)$, for every integer $m \geq 1$. □

Remark 6.5 We can give a further characterization of the critical points of the functional $\mathcal{F}^\beta$. Let $\hat{u}$ be such that $d_{\hat{u}}\mathcal{F}^\beta = 0$. Therefore, as seen in the proof of Theorem 6.4, we have that the identity

$$\hat{u}(s) = -\beta h_{\hat{u}}(s)$$

is satisfied for every $s \in [0, 1]$. Recalling the definition of $h_{\hat{u}} : [0, 1] \to \mathbb{R}^k$ given in Eq. 3.11, we observe that the previous relation yields

$$\hat{u}(s) = \arg\max_{u \in \mathbb{R}^k} \left\{ -\beta \lambda_{\hat{u}}(s) F(x_{\hat{u}}(s))u - \frac{1}{2}|u|_2^2 \right\}, \tag{6.23}$$

where $x_{\hat{u}} : [0, 1] \to \mathbb{R}^n$ solves

$$\begin{cases} \dot{x}_{\hat{u}}(s) = F(x_{\hat{u}}(s))\hat{u}(s) \text{ for a.e. } s \in [0, 1], \\ x_{\hat{u}}(0) = x_0, \end{cases} \tag{6.24}$$

and $\lambda_{\hat{u}} : [0, 1] \to (\mathbb{R}^n)^*$ satisfies

$$\begin{cases} \dot{\lambda}_{\hat{u}}(s) = -\lambda_{\hat{u}}(s) \sum_{i=1}^k \left(\hat{u}^i(s) \frac{\partial F^i(x_{\hat{u}}(s))}{\partial x} \right) \text{ for a.e. } s \in [0, 1], \\ \lambda_{\hat{u}}(1) = \nabla a(x_{\hat{u}}(1)). \end{cases} \tag{6.25}$$

Recalling the Pontryagin Maximum Principle (see, e.g., [3, Theorem 12.10]), from Eqs. 6.23–6.25 we deduce that the curve $x_{\hat{u}} : [0, 1] \to \mathbb{R}^n$ is a normal Pontryagin extremal for the following optimal control problem:

$$\begin{cases} \min_{u \in \mathcal{U}} \left\{ \frac{1}{2}||u||_{L^2}^2 + \beta a(x_u(1)) \right\}, \\ \text{subject to } \begin{cases} \dot{x}_u = F(x_u)u, \\ x_u(0) = x_0. \end{cases} \end{cases}$$

7 Γ-convergence

In this section, we study the behavior of the functionals $(\mathcal{F}^\beta)_{\beta \in \mathbb{R}_+}$ as $\beta \to +\infty$ using the tools of the Γ-convergence. More precisely, we show that the problem of minimizing the functional $\mathcal{F}^\beta : \mathcal{U} \to \mathbb{R}_+$ converges as $\beta \to +\infty$ (in the sense of Γ-convergence) to a limiting minimization problem. A classical consequence of this fact is that the minimizers of the functionals $(\mathcal{F}^\beta)_{\beta\in\mathbb{R}_+}$ can provide an approximation of the solutions of the limiting problem. Moreover, in the present case, the limiting functional has an important geometrical meaning, since it is related to the search of sub-Riemannian length-minimizing paths that connect an initial point to a target set. The results obtained in this section hold under mild regularity assumptions on the vector fields $F^1, \dots, F^k$ and on the end-point cost $a : \mathbb{R}^n \to \mathbb{R}_+$. Finally, for a complete introduction to the theory of Γ-convergence, we refer the reader to the monograph [8].

In this section, we shall work with the weak topology of the Hilbert space $\mathcal{U} := L^2([0, 1], \mathbb{R}^k)$. We first establish a preliminary result. We consider a L^2-weakly convergent sequence $(u_m)_{m\geq 1} \subset \mathcal{U}$, and we study the convergence of the sequence $(x_m)_{m\geq 1}$, where, for every $m \geq 1$, the curve $x_m : [0, 1] \to \mathbb{R}^n$ is the solution of the Cauchy problem Eq. 2.6 corresponding to the admissible control u_m.

Lemma 7.1 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *satisfy the Lipschitz-continuity condition* Eq. 2.2. *Let us consider a sequence $(u_m)_{m\geq 1} \subset \mathcal{U}$ such that $u_m \rightharpoonup_{L^2} u_\infty$ as $m \to \infty$. For every $m \in \mathbb{N} \cup \{\infty\}$, let $x_m : [0, 1] \to \mathbb{R}^n$ be the solution of* Eq. 2.6 *corresponding to the control u_m. Then, we have that*

$$\lim_{m\to\infty} ||x_m - x_\infty||_{C^0} = 0.$$

Proof Being the sequence $(u_m)_{m\geq 1}$ weakly convergent, we deduce that there exists $R > 0$ such that $||u_m||_{L^2} \leq R$ for every $m \geq 1$. The estimate established in Lemma 2.2 implies that there exists $C_R > 0$ such that

$$||x_m||_{C^0} \leq C_R, \tag{7.1}$$

for every $m \geq 1$. Moreover, using the sub-linear growth inequality Eq. 2.3, we have that there exists $C > 0$ such that

$$|\dot{x}_m(s)| \leq \sum_{j=1}^{k} |F^j(x_m(s)|_2 |u_m^j(s)| \leq C(1 + C_R) \sum_{j=1}^{k} |u_m^j(s)|,$$

for a.e. $s \in [0, 1]$. Then, recalling that $||u_m||_{L^2} \leq R$ for every $m \geq 1$, we deduce that

$$||\dot{x}_m||_{L^2} \leq C(1 + C_R)kR \tag{7.2}$$

for every $m \geq 1$. Combining Eqs. 7.1 and 7.2, we obtain that the sequence $(x_m)_{m\geq 1}$ is pre-compact with respect to the weak topology of $H^1([0, 1], \mathbb{R}^n)$. Our goal is to prove that the set of the H^1-weak limiting points of the sequence $(x_m)_{m\geq 1}$ coincides with $\{x_\infty\}$, i.e., that the whole sequence $x_m \rightharpoonup_{H^1} x_\infty$ as $m \to \infty$. Let $\hat{x} \in H^1([0, 1], \mathbb{R}^n)$ be any H^1-weak limiting point of the sequence $(x_m)_{m\geq 1}$, and let $(x_{m_\ell})_{\ell\geq 1}$ be a sub-sequence such that $x_{m_\ell} \rightharpoonup_{H^1} \hat{x}$ as $\ell \to \infty$. Recalling Eq. 2.8 in Theorem 2.1, we have that the inclusion $H^1([0, 1], \mathbb{R}^n) \hookrightarrow C^0([0, 1], \mathbb{R}^n)$ is compact, and this implies that

$$x_{m_\ell} \to_{C^0} \hat{x} \tag{7.3}$$

as $\ell \to \infty$. From Eq. 7.3 and the assumption Eq. 2.2, for every $j = 1, \dots, k$ it follows that

$$||F^j(x_{m_l}) - F^j(\hat{x})||_{C^0} \to 0 \tag{7.4}$$

as $\ell \to \infty$. Let us consider a smooth and compactly supported test function $\phi \in C_c^\infty([0,1],\mathbb{R}^n)$. Therefore, recalling that x_{m_ℓ} is the solution of the Cauchy problem Eq. 2.6 corresponding to the control $u_{m_\ell} \in \mathcal{U}$, we have that

$$\int_0^1 x_{m_\ell}(s)\cdot\dot{\phi}(s)\,ds = -\sum_{j=1}^k \int_0^1 \Big(F^j(x_{m_\ell}(s))\cdot\phi(s)\Big)\,u^j_{m_\ell}(s)\,ds$$

for every $\ell \geq 1$. Thus, passing to the limit as $\ell \to \infty$ in the previous identity, we obtain

$$\int_0^1 \hat{x}(s)\cdot\dot{\phi}(s)\,ds = -\sum_{j=1}^k \int_0^1 \Big(F^j(\hat{x}(s))\cdot\phi(s)\Big)\,u^j_{\infty}(s)\,ds. \tag{7.5}$$

Indeed, the convergence of the right-hand side is guaranteed by Eq. 7.3. On the other hand, for every $j=1,\dots,k$, from Eq. 7.4 we deduce the strong convergence $F^j(x_{m_\ell})\cdot\phi \to_{L^2} F^j(\hat{x})\cdot\phi$ as $\ell\to\infty$, while $u^j_{m_\ell} \rightharpoonup_{L^2} u^j_\infty$ as $\ell\to\infty$ by the hypothesis. Finally, observing that Eq. 7.3 gives $\hat{x}(0)=x_0$, we deduce that

$$\begin{cases} \dot{\hat{x}}(s) = F(\hat{x}(s))u_\infty(s), \text{ for a.e. } s\in[0,1],\\ \hat{x}(0) = x_0, \end{cases}$$

that implies $\hat{x}\equiv x_\infty$. This argument shows that $x_m \rightharpoonup_{H^1} x_\infty$ as $m\to\infty$. Finally, the thesis follows using again the compact inclusion Eq. 2.8. □

The standard theory of Γ-convergence requires the domain of the functionals to be a metric space, or, more generally, to be equipped with a first-countable topology (see [1, Chapter 12]). Since the weak topology of $\mathcal{U}$ is first-countable (and metrizable) only on the bounded subsets of $\mathcal{U}$, we shall restrict the functionals $(\mathcal{F}^\beta)_{\beta\in\mathbb{R}_+}$ to the set

$$U_\rho := \{u\in\mathcal{U} : ||u||_{L^2}\leq\rho\},$$

where $\rho>0$. We set

$$\mathcal{F}^\beta_\rho := \mathcal{F}^\beta|_{\mathcal{U}_\rho},$$

where $\mathcal{F}^\beta : \mathcal{U}\to\mathbb{R}_+$ is defined in Eq. 3.1. Using Lemma 7.1 we deduce that for every $\beta>0$ and $\rho>0$ the functional $\mathcal{F}^\beta_\rho : \mathcal{U}_\rho \to \mathbb{R}_+$ admits a minimizer.

Proposition 7.2 *Let us assume that the vector fields $F^1,\dots,F^k$ defining the control system* Eq. 2.6 *satisfy the Lipschitz-continuity condition* Eq. 2.2*, and that the function $a:\mathbb{R}^n\to\mathbb{R}_+$ designing the end-point cost is continuous. Then, for every $\beta>0$ and $\rho>0$ there exists $\hat{u}\in\mathcal{U}_\rho$ such that*

$$\mathcal{F}^\beta_\rho(\hat{u}) = \inf_{\mathcal{U}_\rho}\mathcal{F}^\beta_\rho.$$

Proof Let us set $\beta>0$ and $\rho>0$. If we show that $\mathcal{F}^\beta_\rho : \mathcal{U}_\rho\to\mathbb{R}_+$ is sequentially coercive and sequentially lower semi-continuous, then the thesis will follow from the Direct Method of calculus of variations (see, e.g., [8, Theorem 1.15]). The sequential coercivity is immediate, since the domain $\mathcal{U}_\rho$ is sequentially compact, for every $\rho>0$. Let $(u_m)_{m\geq1}\subset\mathcal{U}_\rho$ be a sequence such that $u_m \rightharpoonup_{L^2} u_\infty$ as $m\to\infty$. On one hand, in virtue of Lemma 7.1, we have that

$$\lim_{m\to\infty} a(x_m(1)) = a(x_\infty(1)), \tag{7.6}$$

where for every $m \in \mathbb{N} \cup \{\infty\}$ the curve $x_m : [0, 1] \to \mathbb{R}^n$ is the solution of the Cauchy problem Eq. 2.6 corresponding to the admissible control u_m. On the other hand, the L^2-weak convergence implies that

$$||u_\infty||_{L^2} \leq \liminf_{m\to\infty} ||u_m||_{L^2}. \tag{7.7}$$

Therefore, combining Eqs. 7.6 and 7.7, we deduce that the functional $\mathcal{F}_\rho^\beta$ is lower semi-continuous. □

Before proceeding to the main result of the section, we recall the definition of Γ-convergence.

Definition 7.3 The family of functionals $(\mathcal{F}_\rho^\beta)_{\beta\in\mathbb{R}_+}$ is said to Γ-converge to a functional $\mathcal{F}_\rho : \mathcal{U}_\rho \to \mathbb{R}_+ \cup \{+\infty\}$ with respect to the weak topology of $\mathcal{U}$ as $\beta \to +\infty$ if the following conditions hold:

– for every $(u_\beta)_{\beta\in\mathbb{R}_+} \subset \mathcal{U}_\rho$ such that $u_\beta \rightharpoonup_{L^2} u$ as $\beta \to +\infty$ we have

$$\liminf_{\beta\to+\infty} \mathcal{F}_\rho^\beta(u_\beta) \geq \mathcal{F}_\rho(u); \tag{7.8}$$

– for every $u \in \mathcal{U}$ there exists a sequence $(u_\beta)_{\beta\in\mathbb{R}_+} \subset \mathcal{U}_\rho$ called *recovery sequence* such that $u_\beta \rightharpoonup_{L^2} u$ as $\beta \to +\infty$ and such that

$$\limsup_{\beta\to+\infty} \mathcal{F}_\rho^\beta(u_\beta) \leq \mathcal{F}_\rho(u). \tag{7.9}$$

If Eqs. 7.8 and 7.9 are satisfied, then we write $\mathcal{F}_\rho^\beta \to_\Gamma \mathcal{F}_\rho$ as $\beta \to +\infty$.

Remark 7.4 Let us assume that $\mathcal{F}_\rho^\beta \to_\Gamma \mathcal{F}_\rho$ as $\beta \to \infty$, and let us consider a non-decreasing sequence $(\beta_m)_{m\geq 1}$ such that $\beta_m \to +\infty$ as $m \to \infty$. For every $u \in \mathcal{U}_\rho$ and for every sequence $(u_{\beta_m})_{m\geq 1} \subset \mathcal{U}_\rho$ such that $u_{\beta_m} \rightharpoonup_{L^2} u$ as $m \to \infty$, we have that

$$\mathcal{F}_\rho(u) \leq \liminf_{m\to\infty} \mathcal{F}_\rho^{\beta_m}(u_{\beta_m}). \tag{7.10}$$

Indeed, it is sufficient to "embed" the sequence $(u_{\beta_m})_{m\geq 1}$ into a sequence $(u_\beta)_{\beta\in\mathbb{R}_+}$ such that $u_\beta \rightharpoonup_{L^2} u$ as $\beta \to +\infty$, and to observe that

$$\liminf_{\beta\to+\infty} \mathcal{F}^\beta(u_\beta) \leq \liminf_{m\to\infty} \mathcal{F}_\rho^{\beta_m}(u_{\beta_m}).$$

Combining the last inequality with the lim inf condition Eq. 7.8, we obtain Eq. 7.10.

Let $a : \mathbb{R}^n \to \mathbb{R}_+$ be the non-negative function that defines the end-point cost, and let us assume that the set $D := \{x \in \mathbb{R}^n : a(x) = 0\}$ is non-empty. Let us define the functional $\mathcal{F}_\rho : \mathcal{U}_\rho \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$\mathcal{F}_\rho(u) := \begin{cases} \frac{1}{2}||u||_{L^2}^2 & \text{if } x_u(1) \in D, \\ +\infty & \text{otherwise,} \end{cases} \tag{7.11}$$

where $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of Eq. 2.6 corresponding to the control u.

Remark 7.5 A situation relevant for applications occurs when the set D is reduced to a single point, i.e., $D = \{x_1\}$ with $x_1 \in \mathbb{R}^n$. Indeed, in this case the minimization of the limiting functional $\mathcal{F}_\rho$ is equivalent to find a horizontal energy-minimizing path that connect x_0 (i.e., the Cauchy datum of the control system Eq. 2.6) to x_1. This in turn coincides with the problem of finding a sub-Riemannian length-minimizing curve that connect x_0 to x_1 (see [4, Lemma 3.64]).

We now prove the Γ-convergence result, i.e., we show that $\mathcal{F}_\rho^\beta \to_\Gamma \mathcal{F}_\rho$ as $\beta \to \infty$ with respect to the weak topology of $\mathcal{U}$.

Theorem 7.6 *Let us assume that the vector fields $F^1, \dots, F^k$ defining the control system* Eq. 2.6 *satisfy the Lipschitz-continuity condition* Eq. 2.2*, and that the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost is continuous. Given $\rho > 0$, let us consider $\mathcal{F}_\rho^\beta : \mathcal{U}_\rho \to \mathbb{R}_+$ with $\beta > 0$. Let $\mathcal{F}_\rho : \mathcal{U}_\rho \to \mathbb{R}_+ \cup \{+\infty\}$ be defined as in* Eq. 7.11*. Then, the functionals $(\mathcal{F}_\rho^\beta)_{\beta \in \mathbb{R}_+}$ Γ-converge to $\mathcal{F}_\rho$ as $\beta \to +\infty$ with respect to the weak topology of $\mathcal{U}$.*

Remark 7.7 If $\rho > 0$ is not large enough, it may happen that no control in $\mathcal{U}_\rho$ steers x_0 to D, i.e., $x_u(1) \notin D$ for every $u \in \mathcal{U}_\rho$. In this case, the Γ-convergence result is still valid, and the Γ-limit satisfies $\mathcal{F}_\rho \equiv +\infty$. We can easily avoid this uninteresting situation when system Eq. 2.1 is controllable. Indeed, using the controllability assumption, we deduce that there exists a control $\tilde{u} \in \mathcal{U}$ such that the corresponding trajectory $x_{\tilde{u}}$ satisfies $x_{\tilde{u}}(1) \in D$. On the other hand, we have that

$$\inf_{u \in \mathcal{U}} \mathcal{F}^\beta(u) \leq \mathcal{F}^\beta(\tilde{u})$$

for every $\beta > 0$. Moreover, using the fact that $x_{\tilde{u}}(1) \in D$ and recalling the definition of $\mathcal{F}^\beta$ in Eq. 3.1, we have that

$$\mathcal{F}^\beta(\tilde{u}) = \frac{1}{2}||\tilde{u}||_{L^2}^2$$

for every $\beta > 0$. The fact that the end-point cost $a : \mathbb{R}^n \to \mathbb{R}_+$ is non-negative implies that $\mathcal{F}^\beta(u) > \mathcal{F}^\beta(\tilde{u})$ whenever $||u||_{L^2} > ||\tilde{u}||_{L^2}$. Setting $\rho = ||\tilde{u}||_{L^2}$, we deduce that

$$\inf_{u \in \mathcal{U}} \mathcal{F}^\beta(u) = \inf_{u \in \mathcal{U}_\rho} \mathcal{F}_\rho^\beta(u).$$

Moreover, this choice of ρ guarantees that the Γ-limit $\mathcal{F}_\rho \not\equiv +\infty$, since we have that $\mathcal{F}_\rho(\tilde{u}) < +\infty$.

Proof of Theorem 7.6 We begin with the lim sup condition Eq. 7.9. If $\mathcal{F}_\rho(u) = +\infty$, the inequality is trivially satisfied. Let us assume that $\mathcal{F}_\rho(u) < +\infty$. Then, setting $u_\beta = u$ for every $\beta > 0$, we deduce that $x_u(1) = x_{u_\beta}(1) \in D$, where $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of the Cauchy problem Eq. 2.6 corresponding to the control u. Recalling that $a|_D \equiv 0$, we have that

$$\mathcal{F}_\rho^\beta(u_\beta) = \frac{1}{2}||u||_{L^2}^2 = \mathcal{F}_\rho(u)$$

for every $\beta > 0$. This proves the lim sup condition.

We now prove the lim inf condition Eq. 7.8. Let us consider $(u_\beta)_{\beta \in \mathbb{R}_+} \subset \mathcal{U}_\rho$ such that $u_\beta \rightharpoonup_{L^2} u$ as $\beta \to \infty$, and such that

$$\liminf_{\beta \to +\infty} \mathcal{F}_\rho^\beta(u_\beta) = C. \tag{7.12}$$

We may assume that $C < +\infty$. If this is not the case, then Eq. 7.8 trivially holds. Let us extract $(\beta_m)_{m \geq 0}$ such that $\beta_m \to +\infty$ and

$$\lim_{m \to \infty} \mathcal{F}_\rho^{\beta_m}(u_{\beta_m}) = \liminf_{\beta \to +\infty} \mathcal{F}_\rho^\beta(u_\beta) = C. \tag{7.13}$$

For every $m \geq 0$, let $x_{\beta_m} : [0, 1] \to \mathbb{R}^n$ be the curve defined as the solution of the Cauchy problem Eq. 2.6 corresponding to the control u_{β_m}, and let $x_u : [0, 1] \to \mathbb{R}^n$ be the solution corresponding to u. Using Lemma 7.1, we deduce that $x_{\beta_m} \to_{C^0} x_u$ as $m \to \infty$. In particular, we obtain that $x_{\beta_m}(1) \to x_u(1)$ as $m \to \infty$. On the other hand, the limit in Eq. 7.13 implies that there exists $\bar{m} \in \mathbb{N}$ such that

$$\beta_m a(x_{\beta_m}(1)) \leq \mathcal{F}_\rho^{\beta_m}(u_{\beta_m}) \leq C + 1,$$

for every $m \geq \bar{m}$. Recalling that $\beta_m \to \infty$ as $m \to \infty$, the previous inequality yields

$$a(x_u(1)) = \lim_{m\to\infty} a(x_{\beta_m}(1)) = 0,$$

i.e., that $x_u(1) \in D$. This argument proves that, if $u_\beta \rightharpoonup_{L^2} u$ as $\beta \to \infty$ and if the quantity at the right-hand side of Eq. 7.12 is finite, then the limiting control u steers x_0 to D. In particular, this shows that $\mathcal{F}_\rho(u) < +\infty$, namely $\mathcal{F}_\rho(u) = \frac{1}{2}||u||^2_{L^2}$. Finally, we observe that

$$\mathcal{F}_\rho(u) \leq \liminf_{n\to\infty} \frac{1}{2}||u_{\beta_n}||^2_{L^2} \leq \liminf_{n\to\infty} \mathcal{F}_\rho^{\beta_n}(u_{\beta_n}) = \liminf_{\beta\to+\infty} \mathcal{F}_\rho^{\beta}(u_\beta),$$

and this establishes the lim inf condition Eq. 7.8. □

The next theorem motivates the interest in the Γ-convergence result just established. Indeed, we can investigate the asymptotic the behavior of the sequence $(\inf_{\mathcal{U}_\rho} \mathcal{F}_\rho^\beta)_{\beta\in\mathbb{R}_+}$ as $\beta \to +\infty$. Moreover, it turns out that the minimizers of $\mathcal{F}_\rho^\beta$ provide approximations of the minimizers of the limiting functional $\mathcal{F}_\rho$, with respect to the *strong topology* of L^2. The first part of Theorem 7.8 holds for every Γ-convergent sequence of equi-coercive functionals (see, e.g., [8, Corollary 7.20]). On the other hand, the conclusion of the second part relies on the particular structure of $(\mathcal{F}^\beta)_{\beta\in\mathbb{R}_+}$.

Theorem 7.8 *Under the same assumptions of Theorem 7.6, given $\rho > 0$ we have that*

$$\lim_{\beta\to\infty} \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho^\beta = \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho. \tag{7.14}$$

Moreover, under the further assumption that $\mathcal{F}_\rho \not\equiv +\infty$, for every $\beta > 0$ let $\hat{u}_\beta$ be a minimizer of $\mathcal{F}_\rho^\beta$. Then, for every non-decreasing sequence $(\beta_m)_{m\geq1}$ such that $\beta_m \to +\infty$ as $m \to \infty$, $(\hat{u}_{\beta_m})_{m\geq1}$ is pre-compact with respect to the strong topology of $\mathcal{U}_\rho$, and every limiting point of $(\hat{u}_{\beta_m})_{m\geq1}$ is a minimizer of $\mathcal{F}_\rho$.

Proof For every $\beta > 0$ let $\hat{u}_\beta$ be a minimizer of $\mathcal{F}_\rho^\beta$, that exists in virtue of Proposition 7.2. Let us consider a non-decreasing sequence $(\beta_m)_{m\geq1}$ such that $\beta_m \to +\infty$ as $m \to \infty$ and such that

$$\lim_{m\to\infty} \mathcal{F}_\rho^{\beta_m}(\hat{u}_{\beta_m}) = \lim_{m\to\infty} \inf_{U_\rho} \mathcal{F}_\rho^{\beta_m} = \liminf_{\beta\to+\infty} \inf_{U_\rho} \mathcal{F}_\rho^{\beta}. \tag{7.15}$$

Recalling that $(\hat{u}_{\beta_m})_{m\geq1} \subset \mathcal{U}_\rho$, we have that there exists $\hat{u}_\infty \in \mathcal{U}_\rho$ and a sub-sequence $(\beta_{m_j})_{j\geq1}$ such that $\hat{u}_{\beta_{m_j}} \rightharpoonup_{L^2} \hat{u}_\infty$ as $j \to \infty$. Since $\mathcal{F}_\rho^\beta \to_\Gamma \mathcal{F}_\rho$ as $\beta \to +\infty$, the inequality Eq. 7.10 derived in Remark 7.4 implies that

$$\mathcal{F}_\rho(\hat{u}_\infty) \leq \lim_{j\to\infty} \mathcal{F}_\rho^{\beta_{m_j}}(u_{\beta_{m_j}}) = \liminf_{\beta\to+\infty} \inf_{U_\rho} \mathcal{F}_\rho^\beta, \tag{7.16}$$

where we used Eq. 7.15 in the last identity. On the other hand, for every $u \in \mathcal{U}_\rho$ let $(u_\beta)_{\beta\in\mathbb{R}_+}$ be a recovery sequence for u, i.e., a sequence that satisfies the lim sup condition Eq. 7.9. Therefore, we have that

$$\mathcal{F}_\rho(u) \geq \limsup_{\beta\to+\infty} \mathcal{F}_\rho^\beta(u_\beta) \geq \limsup_{\beta\to+\infty} \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho^\beta. \tag{7.17}$$

From Eqs. 7.16 and 7.17, we deduce that

$$\mathcal{F}_\rho(u) \geq \mathcal{F}_\rho(\hat{u}_\infty)$$

for every $u \in \mathcal{U}_\rho$, i.e.,

$$\mathcal{F}_\rho(\hat{u}_\infty) = \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho. \tag{7.18}$$

Finally, setting $u = \hat{u}_\infty$ in Eq. 7.17, we obtain

$$\mathcal{F}_\rho(\hat{u}_\infty) = \lim_{\beta\to\infty} \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho^\beta. \tag{7.19}$$

From Eqs. 7.18 and 7.19, it follows that Eq. 7.14 holds.

We now focus on the second part of the thesis. For every $\beta > 0$ let $\hat{u}_\beta$ be a minimizer of $\mathcal{F}_\rho^\beta$, as before. Let $(\beta_m)_{m\geq 1}$ be a non-decreasing sequence such that $\beta_m \to +\infty$ as $m \to \infty$, and let us consider $(\hat{u}_{\beta_m})_{m\geq 1}$. Since $(\hat{u}_{\beta_m})_{m\geq 1}$ is L^2-weakly pre-compact, there exists $\hat{u} \in \mathcal{U}_\rho$ and a sub-sequence $(\hat{u}_{\beta_{m_j}})_{j\geq 1}$ such that $\hat{u}_{\beta_{m_j}} \rightharpoonup_{L^2} \hat{u}$ as $j \to \infty$. From the first part of the thesis, it descends that $\hat{u}$ is a minimizer of $\mathcal{F}_\rho$. Indeed, in virtue of Eq. 7.10, we have that

$$\mathcal{F}_\rho(\hat{u}) \leq \liminf_{j\to\infty} \mathcal{F}_\rho^{\beta_{m_j}}(\hat{u}_{\beta_{m_j}}) = \lim_{j\to\infty} \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho^{\beta_{m_j}} = \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho,$$

where we used $\mathcal{F}_\rho^{\beta_{m_j}}(\hat{u}_{\beta_{m_j}}) = \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho^{\beta_{m_j}}$ and the identity Eq. 7.14. The previous relation guarantees that

$$\mathcal{F}_\rho(\hat{u}) = \inf_{\mathcal{U}_\rho} \mathcal{F}_\rho, = \lim_{j\to\infty} \mathcal{F}_\rho^{\beta_{m_j}}\left(\hat{u}_{\beta_{m_j}}\right). \tag{7.20}$$

To conclude, we have to show that

$$\lim_{j\to\infty} \left\| \hat{u}_{\beta_{m_j}} - \hat{u} \right\|_{L^2} = 0. \tag{7.21}$$

Using the assumption $\mathcal{F}_\rho \not\equiv +\infty$, from the minimality of $\hat{u}$ we deduce that $\mathcal{F}_\rho(\hat{u}) = \frac{1}{2}||\hat{u}||_{L^2}^2$. Hence, Eq. 7.20 implies that

$$\frac{1}{2}||\hat{u}||_{L^2}^2 = \lim_{j\to\infty} \mathcal{F}_\rho^{\beta_{m_j}}\left(\hat{u}_{\beta_{m_j}}\right) \geq \limsup_{j\to\infty} \frac{1}{2}||u_{\beta_{m_j}}||_{L^2}^2, \tag{7.22}$$

where we used that $\mathcal{F}_\rho^\beta(u) \geq \frac{1}{2}||u||_{L^2}^2$ for every $\beta > 0$ and for every $u \in \mathcal{U}_\rho$. From Eq. 7.22 and from the weak convergence $\hat{u}_{\beta_{m_j}} \rightharpoonup_{L^2} \hat{u}$ as $j \to \infty$, we deduce that Eq. 7.21 holds. □

Conclusions

In this paper, we have considered an optimal control problem in a typical framework of sub-Riemannian geometry. In particular, we have studied the functional given by the weighted sum of the energy of the admissible trajectory (i.e., the squared 2-norm of the control) and of an end-point cost.

We have written the gradient flow induced by the functional on the Hilbert space of admissible controls. We have proved that, when the data of the problem are real-analytic, the gradient flow trajectories converge to stationary points of the functional as soon as the starting point has Sobolev regularity.

The Γ-convergence result bridges the functional considered in the first part of the paper with the problem of joining two assigned points with an admissible length-minimizer path. This fact may be of interest for designing methods to approximate sub-Riemannian length-minimizers. Indeed, a natural approach could be to project the gradient flow onto a proper finite-dimensional subspace of the space of admissible controls, and to minimize the weighted functional restricted to this subspace. We leave further development of these ideas for future work.

Appendix A: Proofs of Subsection 2.2

Proof (Proposition 2.3) Using the fact that x_u and x_{u+v} are solutions of Eq. 2.6, for every $s \in [0, 1]$ we have that

$$|x_{u+v}(s) - x_u(s)|_2 \leq \int_0^s \sum_{i=1}^k \left(\left| F^i(x_{u+v}(\tau)) \right|_2 |v^i(\tau)| \right) d\tau \\ + \int_0^s \sum_{i=1}^k \left(|F^i(x_{u+v}(\tau)) - F^i(x_u(\tau)|_2 |u^i(\tau)| \right) d\tau.$$

Recalling that $||v||_{L^2} \leq R$, in virtue of Lemma 2.2, we obtain that there exists $C_R > 0$ such that

$$\sup_{\tau \in [0,1]} \sup_{i=1,\dots,k} |F^i(x_{u+v}(\tau))|_2 \leq C_R.$$

Hence, using Eq. 2.10, we deduce that

$$\int_0^s \sum_{i=1}^k \left(|F^i(x_{u+v}(\tau))|_2 |v^i(\tau)| \right) d\tau \leq C_R \sqrt{k} ||v||_{L^2}. \tag{A.1}$$

On the other hand, from the Lipschitz-continuity condition Eq. 2.2, it follows that

$$|F^i(x_{u+v}(\tau)) - F^i(x_u(\tau)|_2 \leq L |x_{u+v}(\tau) - x_u(\tau)|_2 \tag{A.2}$$

for every $i = 1, \dots, k$ and for every $\tau \in [0, 1]$. Using Eqs. A.1 and A.2, we deduce that

$$|x_{u+v}(s) - x_u(s)|_2 \leq C_R \sqrt{k} ||v||_{L^2} + L \int_0^s |u(\tau)|_1 |x_{u+v}(\tau) - x_u(\tau)|_2 \, d\tau, \tag{A.3}$$

for every $s \in [0, 1]$. By applying Grönwall inequality to Eq. A.3, we obtain that

$$|x_{u+v}(s) - x_u(s)|_2 \leq e^{L||u||_{L^1}} C_R \sqrt{k} ||v||_{L^2},$$

for every $s \in [0, 1]$. Recalling Eq. 2.10 and setting

$$L_R := e^{L\sqrt{k}R} C_R \sqrt{k},$$

we prove Eq. 2.12. □

Proof (Proposition 2.4) Setting $R := ||u||_{L^2} + ||v||_{L^2}$, we observe that $||u + \varepsilon v||_{L^2} \leq R$ for every $\varepsilon \in (0, 1]$. Owing to Lemma 2.2, we deduce that there exists a compact $K_R \subset \mathbb{R}^n$

such that $x_u(s), x_{u+\varepsilon v}(s) \in K_R$ for every $s \in [0,1]$ and for every $\varepsilon \in (0,1]$. Using the fact that $F^1, \ldots, F^k$ are assumed to be C^1-regular, we deduce that their differentials are uniformly continuous on K_R. This is equivalent to say that there exists a non-decreasing function $\delta : [0,+\infty) \to [0,+\infty)$ such that $\delta(0) = \lim_{r\to 0} \delta(r) = 0$ and

$$\left| F^i(x_2) - F^i(x_1) - \frac{\partial F^i(x_1)}{\partial x}(x_2 - x_1) \right|_2 \le C\delta(|x_1 - x_2|)|x_1 - x_2| \tag{A.4}$$

for every $x_1, x_2 \in K_R$ and for every $i = 1, \ldots, k$. Let us consider the non-autonomous affine system Eq. 2.14. Owing to Carathéodory Theorem (see [9, Theorem 5.3]), we deduce that the system Eq. 2.14 admits a unique absolutely continuous solution $y_u^v : [0,1] \to \mathbb{R}^n$. For every $s \in [0,1]$, let us define

$$\xi(s) := x_{u+\varepsilon v}(s) - x_u(s) - \varepsilon y_u^v(s). \tag{A.5}$$

Therefore, in view of Eqs. 2.6 and 2.14, for a.e. $s \in [0,1]$ we compute

$$\begin{aligned} |\dot{\xi}(s)|_2 \le\ & \varepsilon \sum_{i=1}^k |F^i(x_{u+\varepsilon v}(s)) - F^i(x_u(s))|_2 |v^i(s)| \\ & + \sum_{i=1}^k \left| F^i(x_{u+\varepsilon v}(s)) - F^i(x_u(s)) - \varepsilon \frac{\partial F^i(x_u(s))}{\partial x} y_u^v(s) \right|_2 |u^i(s)| \end{aligned}$$

On one hand, using Proposition 2.3 and the Lipschitz-continuity assumption Eq. 2.2, we deduce that there exists $L' > 0$ such that

$$\varepsilon \sum_{i=1}^k |F^i(x_{u+\varepsilon v}(s)) - F^i(x_u(s))|_2 \le L' ||v||_{L^2} \varepsilon^2 \tag{A.6}$$

for every $s \in [0,1]$ and for every $\varepsilon \in (0,1]$. On the other hand, for every $i = 1, \ldots, n$, combining Proposition 2.3, the inequality Eq. A.4 and the estimate of the norm of the Jacobian Eq. 2.4, we obtain that there exists $L'' > 0$ such that

$$\begin{aligned} & \left| F^i(x_{u+\varepsilon v}(s)) - F^i(x_u(s)) - \varepsilon \frac{\partial F^i(x_u(s))}{\partial x} y_u^v(s) \right|_2 \\ & \quad \le \left| F^i(x_{u+\varepsilon v}(s)) - F^i(x_u(s)) - \frac{\partial F^i(x_u(s))}{\partial x} (x_{u+\varepsilon v}(s) - x_u(s)) \right|_2 \\ & \qquad + \left| \frac{\partial F^i(x_u(s))}{\partial x} \big(x_{u+\varepsilon v}(s) - x_u(s) - \varepsilon y_u^v(s) \big) \right|_2 \\ & \quad \le C \left[\delta(L''||v||_{L^2}\varepsilon) L''||v||_{L^2}\varepsilon \right] + L|\xi(s)|_2. \end{aligned}$$

for every $s \in [0,1]$ and for every $\varepsilon \in (0,1]$. Combining the last inequality and Eq. A.6, it follows that

$$|\dot{\xi}(s)|_2 \le L_R \varepsilon^2 + L_R |u(s)|_1 \delta(L_R\varepsilon)\varepsilon + L|u(s)|_1 |\xi(s)|_2 \tag{A.7}$$

for a.e. $s \in [0,1]$ and for every $\varepsilon \in (0,1]$, where $L_R := \max\{L', L''\}||v||_{L^2}$. Finally, recalling that $|\xi(0)|_2 = |x_{u+\varepsilon v}(0) - x_u(0) - \varepsilon y_u^v(0)|_2 = 0$ for every $\varepsilon \in (0,1]$, we have that

$$|\xi(s)|_2 \le \int_0^s |\dot{\xi}(\tau)|_2 \, d\tau \le L_R\varepsilon^2 + L_R ||u||_{L^1} \delta(L_R\varepsilon)\varepsilon + L \int_0^s |u(\tau)|_1 |\xi(\tau)|_2 \, d\tau,$$

for every $s \in [0,1]$ and for every $\varepsilon \in (0,1]$. Using Grönwall inequality and Eq. A.5, we deduce Eq. 2.13. □

Proof (Lemma 2.5) Let us consider the absolutely continuous curve $N_u : [0,1] \to \mathbb{R}^{n\times n}$ that solves

$$\begin{cases} \dot{N}_u(s) = -N_u(s)A_u(s) \text{ for a.e. } s \in [0,1], \\ N_u(0) = \mathrm{Id}. \end{cases} \tag{A.8}$$

The existence and uniqueness of the solution of Eq. A.8 is guaranteed by Carathéodory Theorem. Recalling the Leibniz rule for Sobolev functions (see, e.g., [6, Corollary 8.10]), a simple computation shows that the identity $N_u(s)M_u(s) = \mathrm{Id}$ holds for every $s \in [0,1]$. This proves that $M_u(s)$ is invertible and that $N_u(s) = M_u^{-1}(s)$ for every $s \in [0,1]$. In order to prove the bound on the norm of the matrix $M_u(s)$, we shall study $|M_u(s)z|_2$, for $z \in \mathbb{R}^n$. Using Eq. 2.16, we deduce that

$$\begin{aligned} |M_u(s)z|_2 &\leq |z|_2 + \textstyle\int_0^s |A_u(\tau)|_2 |M_u(\tau)z|_2 \, d\tau \\ &\leq |z|_2 + L \textstyle\int_0^s |u(s)|_1 |M_u(\tau)z|_2 \, d\tau, \end{aligned}$$

where we used Eq. 2.4. Using Grönwall inequality and Eq. 2.10, we obtain that the inequality Eq. 2.17 holds for $M_u(s)$, for every $s \in [0,1]$. Using Eq. A.8 and applying the same argument, it is possible to prove that Eq. 2.17 holds as well for $N_u(s) = M_u^{-1}(s)$, for every $s \in [0,1]$. □

Proof (Lemma 2.10) Let us consider $R > 0$, and let $u, w \in \mathcal{U}$ be such that $||u||_{L^2}, ||w||_{L^2} \leq R$. We observe that Lemma 2.2 implies that there exists a compact set $K_R \subset \mathbb{R}^n$ such that $x_u(s), x_{u+w}(s) \in K_R$ for every $s \in [0,1]$. The hypothesis that $F^1, \dots, F^2$ are C^2-regular implies that there exists $L'_R > 0$ such that the differentials $\frac{\partial F^1}{\partial x}, \dots, \frac{\partial F^k}{\partial x}$ are Lipschitz-continuous in K_R with constant L'_R. From Eq. 2.16, we have that

$$|\dot{M}_{u+w}(s) - \dot{M}_u(s)|_2 = |A_{u+w}(s)M_{u+w}(s) - A_u(s)M_u(s)|_2, \tag{A.9}$$

for a.e. $s \in [0,1]$. In particular, for a.e. $s \in [0,1]$, we can compute

$$\begin{aligned} |A_{u+w}(s) - A_u(s)|_2 \leq & \sum_{i=1}^k \left| \frac{\partial F^i(x_{u+w}(s))}{\partial x} - \frac{\partial F^i(x_u(s))}{\partial x} \right|_2 |u^i(s)| \\ & + \sum_{i=1}^k \left| \frac{\partial F^i(x_{u+w}(s))}{\partial x} \right|_2 |w^i(s)|, \end{aligned}$$

and using Proposition 2.3, the Lipschitz continuity of $\frac{\partial F^1}{\partial x}, \dots, \frac{\partial F^k}{\partial x}$ and Eq. 2.4, we obtain that there exists $L''_R > 0$ such that

$$|A_{u+w}(s) - A_u(s)|_2 \leq L''_R ||w||_{L^2} |u(s)|_1 + L|w(s)|_1, \tag{A.10}$$

for a.e. $s \in [0,1]$. Using once again Eq. 2.4, we have that

$$|A_u(s)|_2 \leq L|u(s)|_1, \tag{A.11}$$

for a.e. $s \in [0,1]$. Combining Eqs. A.10–A.11 with the triangular inequality at the right-hand side of Eq. A.9, we deduce that

$$\begin{aligned} |\dot{M}_{u+w}(s) - \dot{M}_u(s)|_2 \leq & C'_R \left(L''_R ||w||_{L^2} |u(s)|_1 + L|w(s)|_1 \right) \\ & + L|u(s)|_1 |M_{u+w}(s) - M_u(s)|_2, \end{aligned}$$

for a.e. $s \in [0,1]$, where we used Lemma 2.5 to deduce that there exists $C'_R > 0$ such that $|M_{u+w}(s)| \leq C'_R$ for every $s \in [0,1]$. Recalling that the Cauchy datum of Eq. 2.16 prescribes $M_{u+w}(0) = M_u(0) = \mathrm{Id}$, the last inequality yields

$$|M_{u+w}(s) - M_u(s)|_2 \leq \int_0^s |\dot{M}_{u+w}(\tau) - \dot{M}_u(\tau)|_2 \, d\tau$$
$$\leq C''_R ||w||_{L^2} + L \int_0^s |u(s)|_1 |M_{u+w}(\tau) - M_u(\tau)|_2 \, d\tau,$$

for every $s \in [0,1]$, where we used Eq. 2.10 and where $C''_R > 0$ is a constant depending only on R. Finally, Grönwall Lemma implies the first inequality of the thesis. Recalling that $s \mapsto M_u^{-1}(s)$ and $s \mapsto M_{u+w}^{-1}(s)$ are absolutely continuous curves that solve Eq. A.8, repeating *verbatim* the same argument as above, we deduce the second inequality of the thesis. □

Proof (Proposition 2.1) In virtue of Proposition 2.7, it is sufficient to prove that there exists $L_R > 0$ such that

$$\left\| g^j_{s,u+w} - g^j_{s,u} \right\|_{L^2} \leq L_R \|w\|_{L^2} \tag{A.12}$$

for every $j = 1, \ldots, n$ and for every $u, w \in \mathcal{U}$ such that $||u||_{L^2}, ||w||_{L^2} \leq R$, where $g^j_{s,u+w}, g^j_{s,u}$ are defined as in Eq. 2.22. Let us consider $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. The inequality Eq. A.12 will in turn follow if we show that there exists a constant $L_R > 0$ such that

$$\left| M_{u+w}(s) M^{-1}_{u+w}(\tau) F(x_{u+w}(\tau)) - M_u(s) M_u^{-1}(\tau) F(x_u(\tau)) \right|_2 \leq L_R ||w||_{L^2}, \tag{A.13}$$

for every $s \in [0,1]$, for every $\tau \in [0,s]$ and for every $u, w \in \mathcal{U}$ that satisfy $||u||_{L^2}, ||w||_{L^2} \leq R$. Owing to Proposition 2.3 and Eq. 2.2, it follows that there exists $L'_R > 0$ such that

$$|F(x_{u+w}(s)) - F(x_u(s))|_2 \leq L'_R ||w||_{L^2}, \tag{A.14}$$

for every $s \in [0,1]$ and for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. Using the triangular inequality in Eq. A.13, we compute

$$\left| M_{u+w}(s) M^{-1}_{u+w}(\tau) \; F \; (x_{u+w}(\tau)) - M_u(s) M_u^{-1}(\tau) F(x_u(\tau)) \right|_2$$
$$\leq |M_{u+w}(s) - M_u(s)|_2 \left| M^{-1}_{u+w}(\tau) \right|_2 |F(x_{u+w}(\tau))|_2$$
$$+ |M_u(s)|_2 \left| M^{-1}_{u+w}(\tau) - M_u^{-1}(\tau) \right|_2 |F(x_{u+w}(\tau))|_2$$
$$+ |M_u(s)|_2 \left| M_u^{-1}(\tau)|_2 \right| F(x_{u+w}(\tau)) - F(x_u(\tau))|_2$$

for every $s \in [0,1]$ and for every $\tau \in [0,s]$. Using Eq. A.14, Lemma 2.5 and Lemma 2.10 in the last inequality, we deduce that Eq. A.13 holds. This concludes the proof. □

Appendix B: Proofs of Subsection 2.3

Proof (Lemma 2.13) It is sufficient to prove the statement for the operator $\mathcal{L}_u : \mathcal{U} \to \mathcal{V}$. Indeed, if $\mathcal{L}_u$ is bounded and compact, then $\mathcal{L}_u^* : \mathcal{V} \to \mathcal{U}$ is as well Indeed, the boundedness of the adjoint descends from Remark 2.12, while the compactness from [6, Theorem 6.4]).

Using Lemma 2.6, we obtain that, for every $u \in \mathcal{U}$, there exists $C > 0$ such that the following inequality holds

$$||\mathcal{L}_u[v]||_{C^0} \le C||v||_{\mathcal{U}}, \tag{B.1}$$

for every $v \in \mathcal{U}$. Recalling the continuous inclusion $C^0([0,1],\mathbb{R}^n) \hookrightarrow \mathcal{V}$, we deduce that $\mathcal{L}_u$ is a continuous linear operator. In view of Theorem 2.1, in order to prove that $\mathcal{L}_u$ is compact, it is sufficient to prove that, for every $u \in \mathcal{U}$, there exists $C' > 0$ such that

$$||\mathcal{L}_u[v]||_{H^1} \le C'||v||_{\mathcal{U}} \tag{B.2}$$

for every $v \in \mathcal{U}$. However, from the definition of $\mathcal{L}_u[v]$ given in Eq. 2.29, it follows that

$$\frac{d}{ds}\mathcal{L}_u[v](s) = \dot{y}_u^v(s)$$

for a.e. $s \in [0,1]$. Therefore, from Eq. 2.14 and Lemma 2.6, we deduce that Eq. B.2 holds. □

Proof (Lemma 2.14) Recalling the continuous inclusion $C^0([0,1],\mathbb{R}^n) \hookrightarrow \mathcal{V}$, it is sufficient to prove that for every $R > 0$ there exists $L_R > 0$ such that, for every $s \in [0,1]$, the following inequality is satisfied

$$|\mathcal{L}_{u+w}[v](s) - \mathcal{L}_u[v](s)|_2 \le L_R||w||_{\mathcal{U}}||v||_{\mathcal{U}} \tag{B.3}$$

for every $v \in \mathcal{U}$ and for every $u, w \in \mathcal{U}$ such that $||u||_{\mathcal{U}}, ||w||_{\mathcal{U}} \le R$. On the other hand, Eq. 2.30 implies that

$$\begin{aligned}
&|\mathcal{L}_{u+w}[v](s) - \mathcal{L}_u[v](s)|_2 \\
&\le \int_0^s |M_{u+w}(s)M_{u+w}^{-1}(\tau)F(x_{u+w}(\tau)) - M_u(s)M_u^{-1}(\tau)F(x_u(\tau))|_2|v(\tau)|_2\,d\tau.
\end{aligned}$$

However, using Proposition 2.3, Lemma 2.5 and Lemma 2.10, we obtain that there exists $L'_R > 0$ such that

$$\left|M_{u+w}(s)M_{u+w}^{-1}(\tau)F(x_{u+w}(\tau)) - M_u(s)M_u^{-1}(\tau)F(x_u(\tau))\right|_2 \le L'_R||w||_{\mathcal{U}}$$

for every $s,\tau \in [0,1]$ and for every $u, w \in \mathcal{U}$ such that $||u||_{\mathcal{U}}, ||w||_{\mathcal{U}} \le R$. Combining the last two inequalities, we deduce that Eq. B.3 holds. □

Proof (Lemma 2.20) Let us start with $\mathcal{M}_u^\nu : \mathcal{U} \to \mathcal{V}$. Using Lemma 2.5 and Eq. 2.4, we immediately deduce that there exists $C_1 > 0$ such that

$$||\mathcal{M}_u^\nu[v]||_{\mathcal{V}} \le C_1||v||_{\mathcal{U}}$$

for every $v \in \mathcal{U}$. As regards $\mathcal{S}^\nu : C^0([0,1],\mathbb{R}^n) \to \mathcal{V}$, from Eq. 2.45 we deduce that

$$\left|\mathcal{S}_u^\nu[v](\tau)\right|_2 \le \left(\sum_{i=1}^k |u^i(\tau)||S_u^{\nu,i}(\tau)|_2\right)||v||_{C^0}$$

for every $v \in \mathcal{U}$ and for a.e. $\tau \in [0,1]$. Moreover, from Lemma 2.5, from Lemma 2.2 and the regularity of $F^1,\ldots,F^k$, we deduce that there exists $C' > 0$ such that

$$\left|S_u^{\nu,i}(\tau)\right|_2 \le C'$$

for every $\tau \in [0, 1]$. Combining the last two inequalities and recalling that $u \in \mathcal{U} = L^2([0, 1], \mathbb{R}^k)$, we deduce that the linear operator $\mathcal{S}_u^\nu : C^0([0, 1], \mathbb{R}^n) \to \mathcal{V}$ is continuous. □

Appendix C: Proofs of Section 3

Proof (Lemma 3.1) We observe that the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$ is defined as the composition

$$\mathcal{E} = a \circ P_1,$$

where $P_1 : \mathcal{U} \to \mathbb{R}^n$ is the end-point map defined in Eq. 2.20. Proposition 2.4 guarantees that the end-point map P_1 is Gateaux differentiable at every $u \in \mathcal{U}$. Recalling that $a : \mathbb{R}^n \to \mathbb{R}_+$ is assumed to be C^1, we deduce that, for every $u \in \mathcal{U}$, $\mathcal{E}$ is Gateaux differentiable at u and that, for every $v \in \mathcal{U}$, the following identity holds:

$$d_u\mathcal{E}(v) = \sum_{j=1}^{n} \frac{\partial a(x_u(1))}{\partial x^j} D_u P_1^j(v), \tag{C.1}$$

where $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of Eq. 2.6 corresponding to the control $u \in \mathcal{U}$. Recalling that $D_u P_1^1, \dots, D_u P_1^n : \mathcal{U} \to \mathbb{R}$ are linear and continuous functionals for every $u \in \mathcal{U}$ (see Proposition 2.7), from Eq. C.1 we deduce that $d_u\mathcal{E} : \mathcal{U} \to \mathbb{R}$ is as well. Finally, from Eq. 2.21 we obtain Eq. 3.6. □

Lemma 3.4 Let us consider $R > 0$. In virtue of Eq. 3.6, it is sufficient to prove that there exists $L_R > 0$ such that

$$\left|\left| \frac{\partial a(x_{u+w}(1))}{\partial x^j} g_{1,u+w}^j - \frac{\partial a(x_u(1))}{\partial x^j} g_{1,u}^j \right|\right|_{L^2} \leq L_R ||w||_{L^2} \tag{C.2}$$

for every $j = 1, \dots, n$ and for every $u, w \in \mathcal{U}$ such that $||u||_{L^2}, ||w||_{L^2} \leq R$. Lemma 2.2 implies that there exists a compact set $K_R \subset \mathbb{R}^n$ depending only on R such that $x_u(1), x_{u+w}(1) \in K_R$ for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. Recalling that $a : \mathbb{R}^n \to \mathbb{R}_+$ is assumed to be C^2-regular, we deduce that there exists $L'_R > 0$ such that

$$\left| \frac{\partial a(y_1)}{\partial x^j} - \frac{\partial a(y_2)}{\partial x^j} \right|_2 \leq L'_R |y_1 - y_2|_2$$

for every $y_1, y_2 \in K_R$. Moreover, combining the previous inequality with Eq. 2.12, we deduce that there exists $L_R^1 > 0$ such that

$$\left| \frac{\partial a(x_{u+w}(1))}{\partial x^j} - \frac{\partial a(x_u(1))}{\partial x^j} \right|_2 \leq L_R^1 ||w||_{L^2} \tag{C.3}$$

for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. On the other hand, using Eq. A.12, we have that there exists $L_R^2 > 0$ such that

$$\left|\left| g_{1,u+w}^j - g_{1,u}^j \right|\right|_{L^2} \leq L_R^2 ||w||_{L^2} \tag{C.4}$$

for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. Combining Eqs. C.3 and C.4, and recalling Eq. 2.25, the triangular inequality yields Eq. C.2. □

Acknowledgements The author acknowledges partial support from INDAM–GNAMPA. The author wants to thank Prof. A. Agrachev and Prof. A. Sarychev for encouraging and for the helpful discussions. Finally, the author is grateful to an anonymous referee for the invaluable comments that contributed to a substantial improvement of the overall quality of the paper.

Author Contribution In the typical framework of sub-Riemannian geometry, we consider the problem of minimizing the weighted sum of an end-point cost and of the energy of the controlled trajectory, whose starting point is fixed. The functional induces a gradient flow on the Hilbert space of admissible controls, and we study its convergence properties via the Lojasiewicz-Simon inequality. The Γ-convergence result implies that minimizers of the weighted functional can be used to approximate horizontal length-minimizer paths that connect the starting point with a minimizer of the end-point cost.

Funding Open access funding provided by Scuola Internazionale Superiore di Studi Avanzati - SISSA within the CRUI-CARE Agreement.

Declarations

Conflict of Interest The author declares no competing interests.

References

1. Attouch H., Buttazzo G., Michaille G. Variational analysis in Sobolev and BV spaces. Series on Optimization, SIAM-MPS, https://doi.org/10.1137/1.9781611973488. 2005.
2. Agrachev A. A., Gamkrelidze R. V. The exponential representation of flows and the chronological calculus. USSR Sbornik. 1975;35:727–785.
3. Agrachev A. A., Sachkov Y. L. Control theory from the geometric viewpoint, Encyclopaedia of Mathematical Sciences. Berlin Heidelberg: Springer-Verlag; 2004, https://doi.org/10.1007/978-3-662-06404-7.
4. Agrachev A. A., Barilari D., Boscain U. A comprehensive introduction to sub-Riemannian geometry. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press; 2019, https://doi.org/10.1017/9781108677325.
5. Ambrosetti A., Prodi G. A primer of nonlinear analysis, Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press; 1995.
6. Brezis H. Functional analysis, Sobolev spaces and partial differential equations. New York: Springer; 2011, https://doi.org/10.1007/978-0-387-70914-7.
7. Chill R. On the Łojasiewicz–Simon gradient inequality. J. Funct. Anal. 2003;201(2):572–601. https://doi.org/10.1016/S0022-1236(02)00102-7.
8. Dal Maso G. An Introduction to Γ-convergence. Progress in nonlinear differential equations and their applications, Birkhäuser Boston MA. 1993.
9. Hale J. Ordinary differential equations. Malabar, Florida: Robert E. Krieger Publishing Company, Inc.; 1980.
10. Ladas G. E., Lakshmikantham V. Differential equations in abstract spaces, Mathematics in Science and Engineering. Amsterdam: Elsevier; 1972.
11. Lojasiewicz S. Une propriété topologique des sous-ensembles analitiques réels. Colloques internationaux du C.N.R.S.: Les equations aux dérivées partielles. 1963;117:87–89.
12. Polyak B. T. Gradient method for the minimization of functionals. USSR Comput. Math. Math. Phys. 1963;3(4):864–878.

13. Simon L. Asymptotics for a class of non-linear evolution equations, with applications to geometric problems. Ann. Mat. 1983;118:525–571. https://doi.org/10.2307/2006981.
14. Simon L. Theorems on regularity and singularity of energy minimizing maps. Lecture notes in Mathematics ETH Zurich, Birkhäuser Basel. https://doi.org/10.1007/978-3-0348-9193-6; 1996.
15. Whittlesey E. F. Analytic Functions in Banach Spaces. Proc. Amer. Math. Soc. 1965;16(5):1077–1083. https://doi.org/10.2307/2035620.