The Stokes Phenomenon for Some Moment Partial Differential Equations

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Abstract

We study the Stokes phenomenon for the solutions of general homogeneous linear moment partial differential equations with constant coefficients in two complex variables under condition that the Cauchy data are holomorphic on the complex plane but finitely many singular or branching points with the appropriate growth condition at the infinity. The main tools are the theory of summability and multisummability, and the theory of hyperfunctions. Using them, we describe Stokes lines, anti-Stokes lines, jumps across Stokes lines, and a maximal family of solutions.

Keywords Linear PDEs with constant coefficients · Moment-PDEs · Borel summability · Multisummability · Maximal family of solutions · Stokes phenomenon · Hyperfunctions

Mathematics Subject Classification (2010) 35C10 · 35C20 · 35E15 · 40G10

1 Introduction

In this article, we generalise our results from [18] concerning summability and Stokes phenomenon for the formal solutions of the Cauchy problem for the complex heat equation. In the present paper, we consider the Cauchy problem for general homogeneous linear moment partial differential equation with constant coefficients in two complex variables (t, z)

$$\begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})u = 0 \\ \partial_{m_1,t}^j u(0, z) = \varphi_j(z) \in \mathcal{O}(D), \quad j = 0, \dots, N-1, \end{cases} \quad (1)$$

where $P(\lambda, \zeta)$ is a polynomial of two variables of degree N with respect to λ . Here, $\partial_{m_1,t}$ and $\partial_{m_2,z}$ denote the formal moment differentiations introduced by W. Balser and M. Yoshino [3], which generalise the usual and fractional differentiations.

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Such type of equations was previously investigated by the first author [15–17] and by A. Lastra, S. Malek, and J. Sanz [11], mainly in the context of multisummability in a given direction.

Now, we use the similar methods as in the abovementioned papers to the study of multisummable normalised formal solution $\widehat{u}$ of Eq. 1. It means that $\widehat{u}$ has to be multisummable in every direction but finitely many singular directions. For this reason, we assume that the Cauchy data have finitely many singular or branching points $z_0, \dots, z_n \in \mathbb{C} \setminus \{0\}$ and are analytically continued to $\mathbb{C} \setminus \bigcup_{j=0}^n \{z_j t : t \geq 1\}$, and that satisfy the appropriate exponential growth condition at the infinity. Observe that by the linearity of Eq. 1, it is sufficient to consider the case when there is exactly one such point, say $z_0 \in \mathbb{C} \setminus \{0\}$. Therefore, we only consider the case when $\varphi_j(z) \in \mathcal{O}(\mathbb{C} \setminus \{z_0\})$.

Using such formal multisummable solution $\widehat{u}$, for any nonsingular admissible multidirection $\mathbf{d}$, we are able to construct its multisum $u^{\mathbf{d}}$. This multisum is an actual solution of Eq. 1 as a holomorphic function in some sectorial neighbourhood of the origin.

The main purpose of this article is the description of these actual solutions and the study of the relations between them. To this end, we introduce the concept of maximal family of solutions. It is defined as the whole family of actual solutions, which can be obtained by the method of multisummability.

The relations between solutions are studied in the context of the Stokes phenomenon. It means that we find the Stokes lines, which separate different actual solutions constructed from the same multisummable formal power series solution. We also calculate the differences between actual solutions on such lines, which are called jumps across the Stokes lines. To study such jumps, we apply the Laplace-type hyperfunctions supported on the Stokes line.

In this way, we get the main result of the paper about the maximal family of solutions and the Stokes phenomenon for Eq. 1, which is given in Theorem 3.

In the special case when $\partial_{m_1, t}$ and $\partial_{m_2, z}$ are replaced by ∂_t and ∂_z , we get the description of the Stokes phenomenon for general linear PDEs with constant coefficients.

In this sense, the paper gives the application of theory of summability for PDEs to the description of maximal family of solutions and to the study of Stokes phenomenon for such equations.

Let us recall that the theory of summability of the formal solutions of PDEs has been recently intensively developed by such authors as M. Hibino [4]; K. Ichinobe and M. Miyake [7]; K. Ichinobe [5, 6]; A. Lastra, S. Malek and J. Sanz [12]; P. Remy [19], H. Tahara and H. Yamazawa [21]; H. Yamazawa and M. Yoshino [23]; M. Yoshino [24, 25]; and others.

The paper is organised as follows. Section 2 consists of basic notations. In Section 3, we recall Balser's theory of moment summability. In particular, we introduce kernel functions and connected with them moment functions, Gevrey order, moment Borel and Laplace transforms, k -summability, and multisummability. In the next section, we recall the concept of moment differential operators and their generalisation to pseudodifferential operators. In Section 5, we recall the notion of Stokes phenomenon. We define Stokes lines and jumps across them for multisummable formal power series. We also introduce Laplace-type hyperfunction on Stokes lines, which allows us to describe these jumps. In Section 6, we introduce the idea of a maximal family of normalised actual solutions of non-Kowalevskian equation. We describe such family of solutions of Eq. 1 in the case when formal solution $\widehat{u}$ is multisummable (Theorem 1). In Section 7, we recall how to reduce the Cauchy problem (1) to a family of the Cauchy problems of simple pseudodifferential equations. Next, using the theory of moment summability, we find the integral representation of actual solutions of these

simple pseudodifferential equations in the case when their formal solutions are summable (Proposition 5). It allows us to describe a maximal family of solutions of simple equations, Stokes lines, and jumps across them (Theorem 2). Finally, we return to the equation (1), and using the theory of multisummability, we get the main result of the paper, i.e. the description of a maximal family of solution, Stokes lines, and jumps across them for the equation (1), which is given in Theorem 3. In the last section, we present a few examples of special cases of moment partial differential equations with constant coefficients, where by using hyperfunctions we derive the form of jumps across obtained Stokes lines.

2 Notation

A sector S in a direction $d \in \mathbb{R}$ with an opening $\alpha > 0$ and a radius $R \in \mathbb{R}_+$ in the universal covering space $\tilde{\mathbb{C}}$ of $\mathbb{C} \setminus \{0\}$ is defined by the following:

$$S = S_d(\alpha, R) = \left\{ z \in \tilde{\mathbb{C}} : z = re^{i\phi}, r \in (0, R), \phi \in (d - \alpha/2, d + \alpha/2) \right\}.$$

This sector is called *unbounded* if $R = +\infty$ and the notation $S = S_d(\alpha)$ will be used. If the opening α is not essential, the sector $S_d(\alpha)$ is denoted briefly by S_d .

A complex disc D_r in $\mathbb{C}$ with a radius $r > 0$ is a set of the form

$$D_r = \{z \in \mathbb{C} : |z| < r\}.$$

In case that the radius r is not essential, the set D_r will be designated briefly by D . We also denote briefly a *disc-sector* $S_d(\alpha) \cup D$ (resp. $S_d \cup D$) by $\hat{S}_d(\alpha)$ (resp. $\hat{S}_d$).

If a function f is holomorphic on a domain $G \subset \mathbb{C}^n$, then it will be denoted by $f \in \mathcal{O}(G)$. Analogously, the space of holomorphic functions of the variable $z^{1/\gamma} = (z_1^{1/\gamma_1}, \dots, z_n^{1/\gamma_n})$ on a domain $G \subset \mathbb{C}^n$ is denoted by $\mathcal{O}_{1/\gamma}(G)$, where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ and $1/\gamma = (1/\gamma_1, \dots, 1/\gamma_n)$. In other words, $f \in \mathcal{O}_{1/\gamma}(G)$ if and only if the function $w \mapsto f(w^\gamma)$ is analytic for every $w^\gamma = (w_1^{\gamma_1}, \dots, w_n^{\gamma_n}) \in G$.

More generally, if $\mathbb{E}$ denotes a complex Banach space with a norm $\|\cdot\|_{\mathbb{E}}$, then by $\mathcal{O}(G, \mathbb{E})$ (resp. $\mathcal{O}_{1/\gamma}(G, \mathbb{E})$), we shall denote the set of all $\mathbb{E}$ -valued holomorphic functions (resp. holomorphic functions of the variables $z^{1/\gamma}$) on a domain $G \subseteq \mathbb{C}^n$. For more information about functions with values in Banach spaces, we refer the reader to [2, Appendix B]. In the paper, as a Banach space $\mathbb{E}$, we will take the space of complex numbers $\mathbb{C}$ (we abbreviate $\mathcal{O}(G, \mathbb{C})$ to $\mathcal{O}(G)$ and $\mathcal{O}_{1/\gamma}(G, \mathbb{C})$ to $\mathcal{O}_{1/\gamma}(G)$) or the space of functions $E_{1/\gamma}(D) := \mathcal{O}_{1/\gamma}(D) \cap C(\overline{D})$ equipped with the norm $\|\varphi\|_{E_{1/\gamma}(D)} := \max_{z \in \overline{D}} |\varphi(z)|$.

The space of formal power series $\sum_{n=0}^{\infty} a_n t^n$ with $a_n \in \mathbb{E}$ is denoted by $\mathbb{E}[[t]]$.

We use the “hat” notation $(\hat{u}, \hat{u}_i, \hat{f})$ to denote the formal power series. If the formal power series $\hat{u}$ (resp. $\hat{u}_i, \hat{f}$) is convergent, we denote its sum by u (resp. u_i, f).

Definition 1 Suppose $k \in \mathbb{R}$, S is an unbounded sector and $u \in \mathcal{O}_{1/\gamma}(S, \mathbb{E})$. The function u is of *exponential growth of order at most k* , if for every proper subsector $S^* \prec S$ (i.e. $\overline{S^*} \setminus \{0\} \subseteq S$) there exist constants $C_1, C_2 > 0$ such that $\|u(x)\|_{\mathbb{E}} \leq C_1 e^{C_2 |x|^k}$ for every $x \in S^*$. If this is so, one can write $u \in \mathcal{O}_{1/\gamma}^k(S, \mathbb{E})$ and $u \in \mathcal{O}_{1/\gamma}^k(\mathbb{C}, \mathbb{E})$ for $S = \mathbb{C}$.

More generally, if G is an unbounded domain in $\mathbb{C}^n$ and $u \in \mathcal{O}_{1/\gamma}(G, \mathbb{E})$, then $u \in \mathcal{O}_{1/\gamma}^k(G, \mathbb{E})$ if for every set G^* satisfying $\overline{G^*} \subset \text{Int } G$ there exist constants $C_1, C_2 > 0$ such that $\|u(x)\|_{\mathbb{E}} \leq C_1 e^{C_2 |x|^k}$ for every $x \in G^*$.

3 Kernel and Moment Functions, K-Summability, and Multisummability

In this section, we recall the notion of moment methods introduced by Balser [2]. It allows us to describe moment Borel transforms, Gevrey order, Borel summability, and multisummability.

Definition 2 (see [2, Section 5.5]) A pair of functions e_m and E_m is said to be *kernel functions of order k* ($k > 1/2$) if they have the following properties:

1. $e_m \in \mathcal{O}(S_0(\pi/k))$, $e_m(z)/z$ is integrable at the origin, $e_m(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and e_m is exponentially flat of order k as $z \rightarrow \infty$ in $S_0(\pi/k)$ (i.e., for every $\varepsilon > 0$ there exist $A, B > 0$ such that $|e_m(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S_0(\pi/k - \varepsilon)$).
2. $E_m \in \mathcal{O}^k(\mathbb{C})$ and $E_m(1/z)/z$ is integrable at the origin in $S_\pi(2\pi - \pi/k)$.
3. The connection between e_m and E_m is given by the *corresponding moment function m of order $1/k$* as follows. The function m is defined by the Mellin transform of e_m

$$m(u) := \int_0^\infty x^{u-1} e_m(x) dx \quad \text{for } \operatorname{Re} u \geq 0 \quad (2)$$

and the kernel function E_m has the power series expansion

$$E_m(z) = \sum_{n=0}^{\infty} \frac{z^n}{m(n)} \quad \text{for } z \in \mathbb{C}. \quad (3)$$

4. Additionally, we assume that the corresponding moment function satisfies the normalisation property $m(0) = 1$.

Remark 1 Observe that by the inverse Mellin transform and by Eq. 3, the moment function m uniquely determines the kernel functions e_m and E_m .

In case $k \leq 1/2$ the set $S_\pi(2\pi - \pi/k)$ is not defined, so the second property in Definition 2 can not be satisfied. It means that we must define the kernel functions of order $k \leq 1/2$ and the corresponding moment functions in another way. To this end, we use the ramification at $z = 0$.

Definition 3 (see [2, Section 5.6]) A function e_m is called a *kernel function of order $k > 0$* if we can find a pair of kernel functions $e_{\tilde{m}}$ and $E_{\tilde{m}}$ of order $pk > 1/2$ (for some $p \in \mathbb{N}$) so that

$$e_m(z) = e_{\tilde{m}}(z^{1/p})/p \quad \text{for } z \in S_0(\pi/k).$$

For a given kernel function e_m of order $k > 0$, we define the *corresponding moment function m of order $1/k > 0$* by Eq. 2 and the *kernel function E_m of order $k > 0$* by Eq. 3.

Remark 2 Observe that by Definitions 2 and 3 we have

$$m(u) = \tilde{m}(pu) \quad \text{and} \quad E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} = \sum_{j=0}^{\infty} \frac{z^j}{\tilde{m}(jp)}.$$

As in [16], we extend the notion of moment functions to real orders.

Definition 4 We say that m is a *moment function of order* $1/k < 0$ if $1/m$ is a moment function of order $-1/k > 0$.

We say that m is a *moment function of order 0* if there exist moment functions m_1 and m_2 of the same order $1/k > 0$ such that $m = m_1/m_2$.

By Definition 4 and by [2, Theorems 31 and 32] we have the following:

Proposition 1 Let m_1, m_2 be moment functions of orders $s_1, s_2 \in \mathbb{R}$ respectively. Then we have as follows:

- $m_1 m_2$ is a moment function of order $s_1 + s_2$.
- m_1/m_2 is a moment function of order $s_1 - s_2$.

Example 1 For any $k > 0$, the classical kernel functions and the corresponding moment function, satisfying Definition 2 or 3, are given by the following:

- $e_m(z) = kz^k e^{-z^k}$,
- $m(u) = \Gamma(1 + u/k)$,
- $E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+j/k)} =: \mathbf{E}_{1/k}(z)$, where $\mathbf{E}_{1/k}$ is the Mittag-Leffler function of index $1/k$.

They are used in the classical theory of k -summability.

Example 2 For any $s \in \mathbb{R}$, we will denote by Γ_s the function as follows:

$$\Gamma_s(u) := \begin{cases} \Gamma(1 + su) & \text{for } s \geq 0 \\ 1/\Gamma(1 - su) & \text{for } s < 0. \end{cases}$$

Observe that by Example 1 and Definition 4, Γ_s is an example of a moment function of order $s \in \mathbb{R}$.

The moment functions Γ_s will be extensively used in the paper, since every moment function m of order s has the same growth as Γ_s . Precisely speaking, we have the following:

Proposition 2 (see [2, Section 5.5]) If m is a moment function of order $s \in \mathbb{R}$, then there exist constants $a, A, c, C > 0$ such that

$$ac^n \Gamma_s(n) \leq m(n) \leq AC^n \Gamma_s(n) \quad \text{for every } n \in \mathbb{N}_0.$$

Using Balser's theory of general moment summability ([2, Section 6.5], in particular [2, Theorem 38]), we apply the moment functions to define moment Borel transforms, the Gevrey order and the Borel summability. We first introduce the following:

Definition 5 Let m be a moment function. Then, the linear operator $\widehat{\mathcal{B}}_m: \mathbb{E}[[t]] \rightarrow \mathbb{E}[[t]]$ defined as follows:

$$\widehat{\mathcal{B}}_m \left(\sum_{j=0}^{\infty} u_j t^j \right) := \sum_{j=0}^{\infty} \frac{u_j}{m(j)} t^j$$

is called an m -moment Borel transform.

We define the Gevrey order of formal power series as follows:

Definition 6 Let $s \in \mathbb{R}$. Then, $\widehat{u} \in \mathbb{E}[[t]]$ is called a *formal power series of Gevrey order s* if there exists a disc $D \subset \mathbb{C}$ with centre at the origin such that $\widehat{\mathcal{B}}_{\Gamma_s} \widehat{u} \in \mathcal{O}(D, \mathbb{E})$. The space of formal power series of Gevrey order s is denoted by $\mathbb{E}[[t]]_s$.

Remark 3 By Proposition 2, we may replace Γ_s in Definition 6 by any moment function m of the same order s .

Remark 4 If $\widehat{u} \in \mathbb{E}[[t]]_s$ and $s \leq 0$ then the formal series $\widehat{u}$ is convergent, so its sum u is well-defined. Moreover, $\widehat{u} \in \mathbb{E}[[t]]_0 \iff u \in \mathcal{O}(D, \mathbb{E})$ and $\widehat{u} \in \mathbb{E}[[t]]_s \iff u \in \mathcal{O}^{-1/s}(\mathbb{C}, \mathbb{E})$ for $s < 0$.

Definition 7 Let e_m, E_m be a pair of kernel functions of order $1/k > 0$ with a moment function m and let $d \in \mathbb{R}$.

- If $v \in \mathcal{O}^k(\widehat{S}_d, \mathbb{E})$, then the integral operator $T_{m,d}$ defined as follows:

$$(T_{m,d}v)(t) := \int_{e^{id}\mathbb{R}_+} e_m(s/t)v(s) \frac{ds}{s}$$

is called an *m -moment Laplace transform in a direction d* .

- If $v \in \mathcal{O}(S_d(\frac{\pi}{k} + \varepsilon, R), \mathbb{E})$ for some $\varepsilon, R > 0$, then the integral operator $T_{m,d}^-$ is defined as follows:

$$(T_{m,d}^-v)(s) := -\frac{1}{2\pi i} \int_{\gamma(d)} E_m(s/t)v(t) \frac{dt}{t}$$

(where a path $\gamma(d)$ is the boundary of a sector contained in $S_d(\frac{\pi}{k} + \varepsilon, R)$ with bisecting direction d , a finite radius, an opening slightly larger than π/k , and the orientation is negative) is called an *inverse m -moment Laplace transform in a direction d* .

Remark 5 Observe, that $T_{m,d}(t^n) = m(n)t^n$ for every $n \in \mathbb{N}_0$. Hence, $T_{m,d}\widehat{\mathcal{B}}_m u = u$ for every $u \in \mathcal{O}(D)$.

Now, we are ready to define the summability of formal power series.

Definition 8 Let $k > 0$ and $d \in \mathbb{R}$. Then, $\widehat{u} \in \mathbb{E}[[t]]$ is called *k -summable in a direction d* if there exist $\varepsilon > 0$ and a disc-sector $\widehat{S}_d = \widehat{S}_d(\varepsilon)$ in a direction d such that $v = \widehat{\mathcal{B}}_{\Gamma_{1/k}} \widehat{u} \in \mathcal{O}^k(\widehat{S}_d, \mathbb{E})$.

Moreover, the *k -sum of $\widehat{u}$ in the direction d* is given by the following:

$$u^d(t) = \mathcal{S}_{k,d}\widehat{u}(t) := (T_{m,\theta}v)(t) = \int_{e^{i\theta}\mathbb{R}_+} e_m(s/t)v(s) \frac{ds}{s} \quad \text{for } \theta \in (d - \varepsilon/2, d + \varepsilon/2). \quad (4)$$

Definition 9 If $\widehat{u} \in \mathbb{E}[[t]]$ is k -summable in all directions d but (after identification modulo 2π) finitely many directions $d_1, \dots, d_n$, then $\widehat{u}$ is called *k -summable* and $d_1, \dots, d_n$ are called *singular directions of $\widehat{u}$* .

Next, we extend the notion of k -summable formal power series to that which are multisummable.

Definition 10 Let $k_1 > \dots > k_n > 0$ and let $\kappa_1, \dots, \kappa_n$ be defined by $\kappa_1 = k_1$, $1/\kappa_j = 1/k_j - 1/k_{j-1}$, $2 \leq j \leq n$. We say that a real vector $\mathbf{d} = (d_1, \dots, d_n)$ is an *admissible multidirection with respect to $\mathbf{k} = (k_1, \dots, k_n)$* if

$$2\kappa_j |d_j - d_{j-1}| \leq \pi \quad \text{for } j = 2, \dots, n.$$

Remark 6 Admissibility of $\mathbf{d}$ with respect to $\mathbf{k}$ is equivalent to the inclusions $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n$, where $I_j := (d_j - \frac{\pi}{2\kappa_j}, d_j + \frac{\pi}{2\kappa_j})$ for $j = 1, \dots, n$.

Definition 11 Let $m_1, \dots, m_n$ be moment functions of positive orders respectively $1/\kappa_1, \dots, 1/\kappa_n$, where $\kappa_1, \dots, \kappa_n$ are constructed in Definition 10. A formal power series $\hat{u}(t) = \sum_{j=0}^{\infty} u_j t^j \in \mathbb{E}[[t]]$ is called *$\mathbf{k}$ -multisummable in the admissible multidirection*, provided by the following:

- $v_n(t) := (\hat{\mathcal{B}}_{m_n} \dots \hat{\mathcal{B}}_{m_1} \hat{u})(t) = \sum_{j=0}^{\infty} \frac{u_j}{m_1(j) \dots m_n(j)} t^j \in \mathcal{O}^{\kappa_n}(\hat{S}_{d_n})$.
- $v_{j-1}(t) := (T_{m_j, d_j} v_j)(t) \in \mathcal{O}^{\kappa_{j-1}}(S_{d_{j-1}})$ for $j = n, n-1, \dots, 2$.

Moreover, the $\mathbf{k}$ -multisum of $\hat{u}$ in the multidirection $\mathbf{d}$ is given by the following:

$$u^{\mathbf{d}}(t) = \mathcal{S}_{\mathbf{k}, \mathbf{d}} \hat{u}(t) := (T_{m_1, d_1} \dots T_{m_n, d_n} v_n)(t).$$

Definition 12 If $(d_1, \dots, d_n)$ is an admissible multidirection and the functions $v_n, \dots, v_j$ all exist, but $v_j \notin \mathcal{O}^{\kappa_j}(S_{d_j})$, then d_j is called a *singular direction of $\hat{u}$ of level k_j* (for $j = 1, \dots, n$).

Definition 13 If $\hat{u}$ has at most (after identification modulo 2π) finitely many singular directions of each level k_j , $1 \leq j \leq n$, then $\hat{u}$ is called *$\mathbf{k}$ -multisummable*.

Remark 7 If $k_1 > \dots > k_n > 0$, $(d_1, \dots, d_n)$ is an admissible multidirection and $\hat{u}_j$ is k_j -summable in a direction d_j for $j = 1, \dots, n$, then, by [2, Lemma 20], $\hat{u} := \hat{u}_1 + \dots + \hat{u}_n$ is $\mathbf{k}$ -multisummable in the multidirection $\mathbf{d}$ and $\mathcal{S}_{\mathbf{k}, \mathbf{d}} \hat{u}(t) = \mathcal{S}_{\kappa_1, d_1} \hat{u}_1(t) + \dots + \mathcal{S}_{\kappa_n, d_n} \hat{u}_n(t)$.

Moreover, if additionally $\hat{u}_j$ is k_j -summable with n_j singular directions $d_{j,1}, \dots, d_{j,n_j}$ (for $j = 1, \dots, n$) then $\hat{u}$ is $\mathbf{k}$ -multisummable and $d_{j,1}, \dots, d_{j,n_j}$ are singular directions of $\hat{u}$ of level k_j .

4 Moment Operators

In this section, we recall the notion of moment differential operators constructed by Balser and Yoshino [3] and the concept of moment pseudodifferential operators introduced in the previous papers of the first author [15, 16].

Definition 14 Let m be a moment function. Then, the linear operator $\partial_{m,x}: \mathbb{E}[[x]] \rightarrow \mathbb{E}[[x]]$ is defined by the following:

$$\partial_{m,x} \left(\sum_{j=0}^{\infty} \frac{u_j}{m(j)} x^j \right) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{m(j)} x^j$$

and is called an *m -moment differential operator $\partial_{m,x}$* .

Below, we present most important examples of moment differential operators. Other examples, including also integro-differential operators, can be found in [16, Example 3].

Example 3 If $m(u) = \Gamma_1(u)$, then the operator $\partial_{m,x}$ coincides with the usual differentiation ∂_x . More generally, if $s > 0$ and $m(u) = \Gamma_s(u)$, then the operator $\partial_{m,x}$ satisfies $(\partial_{m,x}\hat{u})(x^s) = \partial_x^s(\hat{u}(x^s))$, where ∂_x^s denotes the Caputo fractional derivative of order s defined by the following:

$$\partial_x^s \left(\sum_{j=0}^{\infty} \frac{u_j}{\Gamma_s(j)} x^{sj} \right) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{\Gamma_s(j)} x^{sj}.$$

Immediately by the definition, we obtain the following connection between the moment Borel transform and the moment differentiation.

Proposition 3 Let m and m' be two moment functions. Then, the operators $\widehat{\mathcal{B}}_{m'}, \partial_{m,t}: \mathbb{E}[[t]] \rightarrow \mathbb{E}[[t]]$ satisfy the following commutation formulas for every $\widehat{u} \in \mathbb{E}[[t]]$ and for $\overline{m} = mm'$:

- i) $\widehat{\mathcal{B}}_{m'} \partial_{m,t} \widehat{u} = \partial_{\overline{m},t} \widehat{\mathcal{B}}_{m'} \widehat{u}$,
- ii) $\widehat{\mathcal{B}}_{m'} P(\partial_{m,t}) \widehat{u} = P(\partial_{\overline{m},t}) \widehat{\mathcal{B}}_{m'} \widehat{u}$ for any polynomial P with constant coefficients.

Now, following [16], we generalise moment differential operators to a kind of pseudodifferential operators. Namely, we have the following:

Definition 15 ([16, Definition 13]) Let m be a moment function of order $1/k > 0$ and $\lambda(\zeta)$ be an analytic function of the variable $\xi = \zeta^{1/\gamma}$ for $|\zeta| \geq r_0$ (for some $\gamma \in \mathbb{N}$ and $r_0 > 0$) of polynomial growth at infinity. A moment pseudodifferential operator $\lambda(\partial_{m,z}): \mathcal{O}_{1/\gamma}(D) \rightarrow \mathcal{O}_{1/\gamma}(D)$ is defined by the following:

$$\lambda(\partial_{m,z})\varphi(z) := \frac{1}{2\gamma\pi i} \oint_{|w|=\varepsilon}^{\gamma} \varphi(w) \int_{r_0 e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_{\widetilde{m}} \left(\zeta^{1/\gamma} z^{1/\gamma} \right) \frac{e_m(\zeta w)}{\zeta w} d\zeta dw$$

for every $\varphi \in \mathcal{O}_{1/\gamma}(D_r)$ and $|z| < \varepsilon < r$, where $\widetilde{m}(u) := m(u/\gamma)$, $E_{\widetilde{m}}(\zeta^{1/\gamma} z^{1/\gamma}) = \sum_{n=0}^{\infty} \frac{\zeta^{n/\gamma} z^{n/\gamma}}{m(n/\gamma)}$, $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$ and $\oint_{|w|=\varepsilon}^{\gamma}$ means that we integrate γ times along the positively oriented circle of radius ε . Here, the integration in the inner integral is taken over a ray $\{re^{i\theta} : r \geq r_0\}$.

Definition 16 [15, Definition 9] Let $\lambda(\zeta)$ be an analytic function of the variable $\xi = \zeta^{1/\gamma}$ for $|\zeta| \geq r_0$ (for some $\gamma \in \mathbb{N}$ and $r_0 > 0$) of polynomial growth at infinity. Then, we define the pole order $q \in \mathbb{Q}$ and the leading term $\lambda_0 \in \mathbb{C} \setminus \{0\}$ of $\lambda(\zeta)$ as the numbers satisfying the formula $\lim_{\zeta \rightarrow \infty} \lambda(\zeta)/\zeta^q = \lambda_0$. We write it also $\lambda(\zeta) \sim \lambda_0 \zeta^q$.

5 Stokes Phenomenon and Hyperfunctions

Now, we extend the concept of the Stokes phenomenon (see [18, Definition 7]) to multisummable formal power series $\widehat{u} \in \mathbb{E}[[t]]$.

Definition 17 Assume that $\widehat{u} \in \mathbb{E}[[t]]$ is $\mathbf{k}$ -multisummable with singular directions $d_{j,1}, \dots, d_{j,n_j}$ of level k_j , $1 \leq j \leq n$. Then, for every $l = 1, \dots, n_j$ and $j = 1, \dots, n$, the set $\mathcal{L}_{d_{j,l}} = \{t \in \mathbb{C}^\times : \arg t = d_{j,l}\}$ is called a *Stokes line of level k_j* for $\widehat{u}$.

Assume now that for fixed $j \in \{1, \dots, n\}$ the vector $\mathbf{d} = (d_1, \dots, d_n)$ is an admissible multidirection with a singular direction d_j of level k_j and with nonsingular directions d_l of level k_l for $l \neq j$, and let $\mathbf{d}_j^\pm := (d_1, \dots, d_j^\pm, \dots, d_n)$ be the admissible multidirections, where d_j^+ (resp. d_j^-) denotes a direction close to d_j and greater (resp. less) than d_j , and let $u^{\mathbf{d}_j^+} := S_{\mathbf{k}, \mathbf{d}_j^+} \widehat{u}$ (resp. $u^{\mathbf{d}_j^-} := S_{\mathbf{k}, \mathbf{d}_j^-} \widehat{u}$) then the difference $J_{\mathcal{L}_{d_j}, k_j} \widehat{u} := u^{\mathbf{d}_j^+} - u^{\mathbf{d}_j^-}$ is called a *jump for $\widehat{u}$ across the Stokes line $\mathcal{L}_{d_j}$ of level k_j* .

Remark 8 Every Stokes line $\mathcal{L}_{d_j}$ of level k_j for $\widehat{u}$ determines also so called *anti-Stokes lines* $\mathcal{L}_{d_j \pm \frac{\pi}{2k_j}}$ of level k_j for $\widehat{u}$.

We will describe jumps across the Stokes lines in terms of hyperfunctions. The similar approach to the Stokes phenomenon one can find in [8, 13, 20]. For more information about the theory of hyperfunctions, we refer the reader to [9].

We will consider the space as follows:

$$\mathcal{H}^k(\mathcal{L}_d) := \mathcal{O}^k(D \cup (S_d \setminus \mathcal{L}_d)) / \mathcal{O}^k(\widehat{S}_d)$$

of Laplace-type hyperfunctions supported by $\mathcal{L}_d$ with exponential growth of order k . It means that every hyperfunction $G \in \mathcal{H}^k(\mathcal{L}_d)$ may be written as follows:

$$G(s) = [g(s)]_d = \left\{ g(s) + h(s) : h(s) \in \mathcal{O}^k(\widehat{S}_d) \right\}$$

for some defining function $g(s) \in \mathcal{O}^k(D \cup (S_d \setminus \mathcal{L}_d))$.

Let γ_d be a path consisting of the half-lines from $e^{id^-} \infty$ to 0 and from 0 to $e^{id^+} \infty$, i.e. $\gamma_d = -\gamma_{d^-} + \gamma_{d^+}$ with $\gamma_{d^\pm} = \mathcal{L}_{d^\pm}$. By the Köthe-type theorem [10], one can treat the hyperfunction $G(s) = [g(s)]_d$ as the analytic functional defined by the following:

$$G(s)[\varphi(s)] := \int_{\gamma_d} g(s)\varphi(s) \, ds, \quad (5)$$

for such small $\varphi \in \mathcal{O}^{-k}(\widehat{S}_d)$ that the function $s \mapsto g(s)\varphi(s)$ belongs to the space $\mathcal{O}^{-k}(D \cup (S_d \setminus \mathcal{L}_d))$.

To describe the jumps across the Stokes lines in terms of hyperfunctions, first assume that $\widehat{f} \in \mathbb{C}[[t]]$ is k -summable, m is a moment function of order $1/k$ and d is a singular direction. By Eq. 4, the jump for $\widehat{f}$ across the Stokes line $\mathcal{L}_d$ is given by the following:

$$J_{\mathcal{L}_d} \widehat{f}(t) = f^{d^+}(t) - f^{d^-}(t) = (T_{m,d^+} - T_{m,d^-}) \widehat{B}_m \widehat{f}(t). \quad (6)$$

Observe that we can treat $g_0(t) := \widehat{B}_m \widehat{f}(t) \in \mathcal{O}^k(D \cup (S_d \setminus \mathcal{L}_d))$ as a defining function of the hyperfunction $G_0(s) := [g_0(s)]_d \in \mathcal{H}^k(\mathcal{L}_d)$. So, combining Eq. 5 with Eq. 6, we conclude that

$$J_{\mathcal{L}_d} \widehat{f}(t) = G_0(s) \left[\frac{e_m(s/t)}{s} \right] \quad \text{for sufficiently small } r > 0 \text{ and } t \in S_d\left(\frac{\pi}{k}, r\right). \quad (7)$$

Moreover, it is natural to define the m -moment Laplace operator $T_{m,d}$ acting on the hyperfunction $G(s)$ as $T_{m,d} G(t) := G(s) \left[\frac{e_m(s/t)}{s} \right]$ for $t \in S_d\left(\frac{\pi}{k}, r\right)$, where $G(s)[\varphi(s)]$ is defined by Eq. 5. So, by Eq. 7, we may describe the jump in terms of the m -moment Laplace operator acting on the hyperfunction as $J_{\mathcal{L}_d} \widehat{f}(t) = T_{m,d} G_0(t)$.

Now, let $\widehat{f} \in \mathbb{C}[[t]]$ be $\mathbf{k}$ -multisummable and $\mathbf{d}$ be as in Definition 17 with $\mathcal{L}_{d_j}$ being the Stokes line of level k_j . We additionally assume as in Remark 7 that $\widehat{f} = \widehat{f}_1 + \cdots + \widehat{f}_n$, where $\widehat{f}_j$ is k_j -summable. Then, by Remark 7, analogously as in the summable case, the jump across $\mathcal{L}_{d_j}$ of level k_j is given by the following:

$$J_{\mathcal{L}_{d_j}, k_j} \widehat{f} = f^{\mathbf{d}_j^+} - f^{\mathbf{d}_j^-} = f_j^{d_j^+} - f_j^{d_j^-} = (T_{m_j, d_j^+} - T_{m_j, d_j^-}) \widehat{\mathcal{B}}_{m_j} \widehat{f}_j,$$

and we may describe this jump in terms of hyperfunctions as in the previous case.

Similarly, if $\mathcal{L}_d$ is a Stokes line for k -summable $\widehat{u} = \widehat{u}(t, z) \in \mathcal{O}(D)[[t]]$, then we are able to describe jumps for $\widehat{u}(t, z)$ at the point $z = 0$ in terms of hyperfunctions. Namely, we have the following:

$$\begin{aligned} J_{\mathcal{L}_d} \widehat{u}(t, 0) &= (T_{m, d} F_0)(t) = F_0(s) \left[\frac{e_m(s/t)}{s} \right], \quad \text{where } F_0(s) \\ &= [\widehat{\mathcal{B}}_m \widehat{u}(s, 0)]_d \in \mathcal{H}^k(\mathcal{L}_d). \end{aligned}$$

Analogously, we calculate jumps across a Stokes line $\mathcal{L}_{d_j}$ of level k_j for $\mathbf{k}$ -multisummable $\widehat{u}$ satisfying $\widehat{u} = \widehat{u}_1 + \cdots + \widehat{u}_n$, where $\widehat{u}_i$ is k_i -summable ($i = 1, \dots, n$).

Remark 9 In some special cases, we are also able to describe jumps for $\widehat{u}(t, z)$ at any point $z \in D$. It is possible in the case when $\widehat{u}$ is a multisummable solution of

$$\begin{cases} P(\partial_{m_1, t}, \partial_z)u = 0 \\ \partial_{m_1, t}^j u(0, z) = \varphi_j(z) \quad \text{for } j = 0, \dots, N-1, \end{cases}$$

instead of Eq. 1. In this case, we are able to reduce the problem of description of jumps for $\widehat{u}(t, z)$ at the fixed point $z \in D$, to the problem of description of jumps for the auxiliary formal power series $\widehat{u}_z(t, s) := \widehat{u}(t, s + z)$ at the point $s = 0$. Since the derivative operator ∂_z is invariant under the translation, i.e. $(\partial_z \widehat{u})(t, s + z) = \partial_s(\widehat{u}(t, s + z))$, we conclude that $\widehat{u}_z(t, s)$ is a multisummable solution of

$$\begin{cases} P(\partial_{m_1, t}, \partial_s)u_z = 0 \\ \partial_{m_1, t}^j u_z(0, s) = \varphi_{z, j}(s) := \varphi_j(s + z) \quad \text{for } j = 0, \dots, N-1. \end{cases}$$

Hence, $J_{\mathcal{L}_d} \widehat{u}(t, z) = J_{\mathcal{L}_{d_z}} \widehat{u}_z(t, 0)$, where $\mathcal{L}_{d_z}$ is a Stokes line of $\widehat{u}_z$, which corresponds to a Stokes line $\mathcal{L}_d$ of $\widehat{u}$.

Since in general the moment differential operators are not invariant under translation, we are not able to use this method to describe the jumps for solutions of $P(\partial_{m_1, t}, \partial_{m_2, z})u = 0$ at any point $z \in D$.

6 A Maximal Family of Solutions

Now, we are ready to describe a family of normalised actual solutions of given non-Kowalevskian equation using sums of multisummable formal power series solution. More precisely, we consider the Cauchy problem

$$\begin{cases} P(\partial_{m_1, t}, \partial_{m_2, z})u = 0 \\ \partial_{m_1, t}^j u(0, z) = \varphi_j(z) \in \mathcal{O}(D), \quad j = 0, \dots, N-1, \end{cases} \quad (8)$$

where m_1, m_2 are moment functions of orders $s_1, s_2 > 0$ respectively and

$$P(\lambda, \zeta) = P_0(\zeta)\lambda^N - \sum_{j=1}^N P_j(\zeta)\lambda^{N-j} \quad (9)$$

is a general polynomial of two variables, which is of order N with respect to λ .

If $P_0(\zeta)$ defined by Eq. 9 is not a constant, then a formal solution of Eq. 8 is not uniquely determined. To avoid this inconvenience, we choose some special solution which is already uniquely determined. To this end, we factorise the polynomial $P(\lambda, \zeta)$ as follows:

$$P(\lambda, \zeta) = P_0(\zeta)(\lambda - \lambda_1(\zeta))^{N_1} \cdots (\lambda - \lambda_l(\zeta))^{N_l} =: P_0(\zeta)\tilde{P}(\lambda, \zeta),$$

where $\lambda_1(\zeta), \dots, \lambda_l(\zeta)$ are the roots of the characteristic equation $P(\lambda, \zeta) = 0$ with multiplicity $N_1, \dots, N_l$ ($N_1 + \cdots + N_l = N$) respectively.

Since $\lambda_\alpha(\zeta)$ are algebraic functions, we may assume that there exist $\gamma \in \mathbb{N}$ and $r_0 < \infty$ such that $\lambda_\alpha(\zeta)$ are holomorphic functions of the variable $\xi = \zeta^{1/\gamma}$ (for $|\zeta| \geq r_0$ and $\alpha = 1, \dots, l$) and, moreover, there exist $\lambda_\alpha \in \mathbb{C} \setminus \{0\}$ and $q_\alpha = \mu_\alpha/\nu_\alpha$ (for some relatively prime numbers $\mu_\alpha \in \mathbb{Z}$ and $\nu_\alpha \in \mathbb{N}$) such that $\lambda_\alpha(\zeta) \sim \lambda_\alpha \zeta^{q_\alpha}$ for $\alpha = 1, \dots, l$. Observe that $\nu_\alpha | \gamma$ for $\alpha = 1, \dots, l$.

Hence, $\lambda_\alpha(\partial_{m_2, z})$ are well-defined moment pseudodifferential operators and consequently also the operator

$$\tilde{P}(\partial_{m_1, t}, \partial_{m_2, z}) = (\partial_{m_1, t} - \lambda_1(\partial_{m_2, z}))^{N_1} \cdots (\partial_{m_1, t} - \lambda_l(\partial_{m_2, z}))^{N_l}$$

is well-defined.

Under the above assumption, by a *normalised formal solution* $\hat{u}$ of Eq. 8, we mean such solution of Eq. 8, which is also a solution of the pseudodifferential equation $\tilde{P}(\partial_{m_1, t}, \partial_{m_2, z})\hat{u} = 0$ (see [15, Definition 10]).

Since the principal part of the pseudodifferential operator $\tilde{P}(\partial_{m_1, t}, \partial_{m_2, z})$ with respect to $\partial_{m_1, t}$ is given by $\partial_{m_1, t}^N$, the Cauchy problem (8) has a unique normalised formal power series solution $\hat{u} \in \mathcal{O}(D)[[t]]$. If we additionally assume that $\hat{u}$ is multisummable, then using the procedure of multisummability in nonsingular directions, we obtain a family of normalised actual solutions of Eq. 8 on some sectors with respect to t . This motivates us to introduce the following definitions.

Definition 18 Let S be a sector in the universal covering space $\tilde{\mathbb{C}}$. A function $u \in \mathcal{O}(S \times D)$ is called a *normalised actual solution* of Eq. 8 if it satisfies

$$\begin{cases} \tilde{P}(\partial_{m_1, t}, \partial_{m_2, z})u = 0 \\ \lim_{t \rightarrow 0, t \in S} \partial_{m_1, t}^j u(t, z) = \varphi_j(z) \in \mathcal{O}(D), \quad j = 0, \dots, N-1. \end{cases}$$

In [18], we introduced a maximal family of solutions of Eq. 8 in the case when a formal power series solution is k -summable. It is a collection of all actual solutions of Eq. 8 constructed by the procedure of k -summability. Now, we generalise this definition to the multisummable case.

Definition 19 Assume that the normalised formal power series solution $\hat{u}$ of Eq. 8 is $\mathbf{k}$ -multisummable, $\mathcal{J}$ is a finite set of indices, and V is a sector with an opening greater than π/k_n on the Riemann surface of $t^{\frac{1}{q}}$ for some $q \in \mathbb{Q}_+$.

We say that $\{u_i\}_{i \in \mathcal{J}}$ with $u_i \in \mathcal{O}(V_i \times D)$ is a *maximal family of solutions* of Eq. 8 on $V \times D$ if the following conditions hold:

- (a) $V_i \subseteq V$ is a sector of opening greater than π/k_1 for every $i \in \mathcal{J}$.
- (b) $\{V_i\}_{i \in \mathcal{J}}$ is a covering of V .
- (c) $u_i \in \mathcal{O}(V_i \times D)$ is a normalised actual solution of Eq. 8 for every $i \in \mathcal{J}$.
- (d) If $V_i \cap V_j \neq \emptyset$ then $u_i \not\equiv u_j$ on $(V_i \cap V_j) \times D$ for every $i, j \in \mathcal{J}, i \neq j$.
- (e) For every $i \in \mathcal{J}$, there exists an admissible nonsingular multidirection $\mathbf{d}$ such that $u_i = \mathcal{S}_{\mathbf{k}, \mathbf{d}} \widehat{u}$ on $\tilde{V} \times D$ for some non-empty sector $\tilde{V} \subseteq V_i$.
- (f) For every admissible nonsingular multidirection $\mathbf{d}$, there exists $i \in \mathcal{J}$ such that $\mathcal{S}_{\mathbf{k}, \mathbf{d}} \widehat{u} = u_i$ on $\tilde{V} \times D$ for some sector $\tilde{V} \subseteq V_i$.

Now, we are ready to describe a maximal family of solutions of Eq. 8 generalising our previous result [18, Theorem 3] to the multisummable case.

Theorem 1 *Let $\widehat{u}$ be a $\mathbf{k}$ -multisummable normalised formal power series solution of Eq. 8 with a $\mathbf{k}$ -multisum in a nonsingular admissible multidirection $\mathbf{d}$ given by $u^{\mathbf{d}} = \mathcal{S}_{\mathbf{k}, \mathbf{d}} \widehat{u}$ and satisfying $\widehat{u} = \widehat{u}_1 + \dots + \widehat{u}_n$, where $\widehat{u}_j$ is k_j -summable for $j = 1, \dots, n$. Assume that there exists $q \in \mathbb{Q}_+$, which is the smallest positive rational number such that $u^{\mathbf{d}}(t, z) = u^{\mathbf{d}}(te^{2q\pi i}, z)$ for every nonsingular multidirection $\mathbf{d}$. Suppose that the set of singular directions of $\widehat{u}$ of level k_j modulo $2q\pi$ is given by $\{d_{j,1}, \dots, d_{j,n_j}\}$, where $0 \leq d_{j,1} < \dots < d_{j,n_j} < 2q\pi$ ($j = 1, \dots, n$).*

Furthermore, let

$$I_{j,l} := \left(d_{j,l} - \frac{\pi}{2k_j}, d_{j,l+1} + \frac{\pi}{2k_j} \right) \quad \text{for } l = 1, \dots, n_j, \quad j = 1, \dots, n,$$

where $d_{j,n_j+1} := d_{j,1} + 2q\pi$, and let

$$\mathcal{J} := \left\{ \mathbf{l} = (l_1, \dots, l_n) \in \mathbb{N}^n : 1 \leq l_j \leq n_j, |I_{j,l_j} \cap \dots \cap I_{n,l_n}| > \frac{\pi}{k_j}, j = 1, \dots, n \right\},$$

where $|I|$ denotes the length of the interval I .

Then,

- (i) for every $\mathbf{l} \in \mathcal{J}$ there exists an admissible multidirection $\mathbf{d} = (d_1, \dots, d_n)$ satisfying

$$d_j \in (d_{j,l_j}, d_{j,l_j+1}), \quad j = 1, \dots, n, \quad (10)$$

for which the function $u_{\mathbf{l}} := \mathcal{S}_{\mathbf{k}, \mathbf{d}} \widehat{u}$ is well-defined.

- (ii) For every sufficiently small $\varepsilon > 0$, there exists $r > 0$ such that $u_{\mathbf{l}} \in \mathcal{O}(V_{\mathbf{l}}(\varepsilon, r) \times D)$ for every $\mathbf{l} \in \mathcal{J}$, where

$$V_{\mathbf{l}}(\varepsilon, r) := \left\{ t \in W_r : \left(\arg t - \frac{\varepsilon}{2}, \arg t + \frac{\varepsilon}{2} \right) \subseteq I_{1,l_1} \cap \dots \cap I_{n,l_n} \right\}$$

and $W_r = \{t \in W : 0 < |t| < r\}$ with W being the Riemann surface of $t \mapsto t^{\frac{1}{q}}$.

- (iii) $\{u_{\mathbf{l}}\}_{\mathbf{l} \in \mathcal{J}}$ is a maximal family of solutions of Eq. 8 on $W_r \times D$.

Remark 10 Observe that $\mathcal{L}_{d_{j,l}}$ and $\mathcal{L}_{d_{j,l} \pm \frac{\pi}{2k_j}}$, with $d_{j,l}$ satisfying the assumptions of Theorem 1, are respectively Stokes and anti-Stokes lines of level j for $l = 1, \dots, n_j$ and $j = 1, \dots, n$. They play an important role in our description of the maximal family of solutions of Eq. 8.

Proof of Theorem 1 (i) First, observe that condition

$$|I_{j,l_j} \cap \cdots \cap I_{n,l_n}| > \frac{\pi}{k_j} \quad \text{for } j = 1, \dots, n$$

guarantees that there exist $d_j \in (d_{j,l_j}, d_{j,l_j+1})$, $j = 1, \dots, n$, such that

$$\left(d_1 - \frac{\pi}{2k_1}, d_1 + \frac{\pi}{2k_1}\right) \subseteq \left(d_2 - \frac{\pi}{2k_2}, d_2 + \frac{\pi}{2k_2}\right) \subseteq \cdots \subseteq \left(d_n - \frac{\pi}{2k_n}, d_n + \frac{\pi}{2k_n}\right).$$

It means that for every $\mathbf{l} \in \mathcal{J}$, one can find an admissible multidirection $\mathbf{d} = (d_1, \dots, d_n)$ satisfying (10).

Moreover, for every admissible multidirections $\mathbf{d} = (d_1, \dots, d_n)$ and $\tilde{\mathbf{d}} = (\tilde{d}_1, \dots, \tilde{d}_n)$ such that $d_j, \tilde{d}_j \in (d_{j,l_j}, d_{j,l_j+1})$ for $j = 1, \dots, n$ we have as follows:

$$S_{\mathbf{k}, \mathbf{d}} \hat{u} = S_{k_1, d_1} \hat{u}_1 + \cdots + S_{k_n, d_n} \hat{u}_n = S_{k_1, \tilde{d}_1} \hat{u}_1 + \cdots + S_{k_n, \tilde{d}_n} \hat{u}_n = S_{\mathbf{k}, \tilde{\mathbf{d}}} \hat{u},$$

since by [2, Lemma 10]

$$S_{k_j, d_j} \hat{u}_j = S_{k_j, \tilde{d}_j} \hat{u}_j \quad \text{for every } j = 1, \dots, n.$$

It means that for every $\mathbf{l} \in \mathcal{J}$, the function $u_{\mathbf{l}}$ is well-defined.

To show (ii), observe that for every sufficiently small $\varepsilon > 0$ there exists $r > 0$ such that $S_{k_j, d_j} \hat{u}_j$ is analytically continued to the set as follows:

$$\{t \in W : |t| \in (0, r), (\arg t - \frac{\varepsilon}{2}, \arg t + \frac{\varepsilon}{2}) \subseteq I_{j,l_j}\} \times D \quad \text{for every } j = 1, \dots, n.$$

Hence, the whole function $u_{\mathbf{l}}$ is analytically continued to the set $V_{\mathbf{l}}(\varepsilon, r) \times D$.

Finally, we prove (iii). Since the inequality $|I_{1,l_1} \cap \cdots \cap I_{n,l_n}| > \frac{\pi}{k_{\mathbf{l}}}$ holds for every $\mathbf{l} \in \mathcal{J}$, we are able to take such small $\varepsilon > 0$ that the opening of $V_{\mathbf{l}}(\varepsilon, r)$ ($V_{\mathbf{l}}$ for short) is greater than $\frac{\pi}{k_{\mathbf{l}}}$.

We claim that $\{V_{\mathbf{l}}\}_{\mathbf{l} \in \mathcal{J}}$ is a covering of W_r . To this end, we take any $t \in W_r$. Then, we may choose $\mathbf{l} \in \mathbb{N}^n$ such that $1 \leq l_j \leq n_j$ and $\arg t \in [d_{j,l_j}, d_{j,l_j+1})$ for $j = 1, \dots, n$. For such choice of $\mathbf{l}$ there exists $\delta > 0$ such that

$$\left[\arg t - \frac{\pi}{2k_j} + \frac{\delta}{2}, \arg t + \frac{\pi}{2k_j} + \frac{\delta}{2}\right] \subset I_{j,l_j} \cap \cdots \cap I_{n,l_n} \quad \text{for } j = 1, \dots, n.$$

It means that $|I_{j,l_j} \cap \cdots \cap I_{n,l_n}| \geq \frac{\pi}{k_j} + \frac{\delta}{2} > \frac{\pi}{k_j}$ for $j = 1, \dots, n$, so $\mathbf{l} \in \mathcal{J}$ and $t \in V_{\mathbf{l}}$. By the freedom of choice of $t \in W_r$, $\{V_{\mathbf{l}}\}_{\mathbf{l} \in \mathcal{J}}$ is a covering of W_r .

By the moment version of [1, Theorem 6.2], we conclude that the space of $\mathbf{k}$ -multisummable series in a multidirection $\mathbf{d}$ is a moment differential algebra over $\mathbb{C}$. It means that it is a linear space, which is also closed under multiplication and moment differentiations, and which for any $\mathbf{k}$ -multisummable series $\hat{f}$ and $\hat{g}$ satisfies: $S_{\mathbf{k}, \mathbf{d}}(\hat{f} + \hat{g}) = S_{\mathbf{k}, \mathbf{d}}\hat{f} + S_{\mathbf{k}, \mathbf{d}}\hat{g}$, $S_{\mathbf{k}, \mathbf{d}}(\hat{f} \cdot \hat{g}) = S_{\mathbf{k}, \mathbf{d}}\hat{f} \cdot S_{\mathbf{k}, \mathbf{d}}\hat{g}$, $S_{\mathbf{k}, \mathbf{d}}(\partial_{m_1, t}\hat{f}) = \partial_{m_1, t}(S_{\mathbf{k}, \mathbf{d}}\hat{f})$, and $S_{\mathbf{k}, \mathbf{d}}(\partial_{m_2, z}\hat{f}) = \partial_{m_2, z}(S_{\mathbf{k}, \mathbf{d}}\hat{f})$.

Hence,

$$P(\partial_{m_1, t}, \partial_{m_2, z})u_{\mathbf{l}} = P(\partial_{m_1, t}, \partial_{m_2, z})S_{\mathbf{k}, \mathbf{d}}\hat{u} = S_{\mathbf{k}, \mathbf{d}}P(\partial_{m_1, t}, \partial_{m_2, z})\hat{u} = 0 \text{ on } V_{\mathbf{l}} \times D.$$

Additionally, since $\hat{u}(t, z) = \sum_{j=0}^{\infty} u_j(z)t^j$ on $V_{\mathbf{l}} \times D$, by [2, Proposition 8] and by the definition of multisummable series we get the following:

$$\lim_{t \rightarrow 0, t \in V_{\mathbf{l}}} \partial_{m_1, t}^j u_{\mathbf{l}}(t, z) = \frac{m_1(j)}{m_1(0)} u_j(z) = \varphi_j(z) \quad \text{for } j = 0, \dots, N-1.$$

Therefore $u_{\mathbf{l}}$ is an actual solution of Eq. 8 for $\mathbf{l} \in \mathcal{J}$.

Now, assume that $V_1 \cap V_{\tilde{1}} \neq \emptyset$ and $u_1 \equiv u_{\tilde{1}}$ on $(V_1 \cap V_{\tilde{1}}) \times D$ for some $\mathbf{l}, \tilde{\mathbf{l}} \in \mathcal{J}$ and $\mathbf{l} \neq \tilde{\mathbf{l}}$. It means that there exists admissible multidirections $\mathbf{d} = (d_1, \dots, d_n)$, $d_j \in (d_{j,l_j}, d_{j,l_{j+1}})$ and $\tilde{\mathbf{d}} = (\tilde{d}_1, \dots, \tilde{d}_n)$, $\tilde{d}_j \in (d_{j,\tilde{l}_j}, d_{j,\tilde{l}_{j+1}})$ such that

$$u_1 = \mathcal{S}_{k_1,d_1} \hat{u}_1 + \dots + \mathcal{S}_{k_n,d_n} \hat{u}_n = \mathcal{S}_{k_1,\tilde{d}_1} \hat{u}_1 + \dots + \mathcal{S}_{k_n,\tilde{d}_n} \hat{u}_n = u_{\tilde{1}}.$$

Since $\mathcal{S}_{k_j,d_j} \hat{u}_j$ and $\mathcal{S}_{k_j,\tilde{d}_j} \hat{u}_j$ are both analytic on the non-empty

$$\left\{ t \in W : |t| \in (0, r), \left(\arg t - \frac{\varepsilon}{2}, \arg t + \frac{\varepsilon}{2} \right) \subseteq I_{j,l_j} \cap I_{j,\tilde{l}_j} \right\} \times D,$$

by the Relative Watson's lemma [14, Proposition 2.1], we conclude that $\mathcal{S}_{k_j,d_j} \hat{u}_j = \mathcal{S}_{k_j,\tilde{d}_j} \hat{u}_j$ for $j = 1, \dots, n$. Since $\mathbf{l} \neq \tilde{\mathbf{l}}$, without loss of generality, we may assume that $\tilde{d}_i < d_{i,l_i} < d_i$ for some $i \in \{1, \dots, n\}$. This contradicts the fact that d_{i,l_i} is a singular direction of level k_i . So if $V_1 \cap V_{\tilde{1}} \neq \emptyset$ then $u_1 \equiv u_{\tilde{1}}$ on $(V_1 \cap V_{\tilde{1}}) \times D$ for every $\mathbf{l}, \tilde{\mathbf{l}} \in \mathcal{J}$, $\mathbf{l} \neq \tilde{\mathbf{l}}$.

By the construction of the family $\{u_1\}_{\mathbf{l} \in \mathcal{J}}$, the last two conditions in Definition 19 are also satisfied, which completes the proof. $\square$

7 General Linear Moment Partial Differential Equations with Constant Coefficients

We will study the Stokes phenomenon and the maximal family of solutions for the normalised formal solution $\hat{u}$ of Eq. 8. Let us recall that we may reduce the Cauchy problem (8) of a general linear moment partial differential equation with constant coefficients to a family of the Cauchy problems of simple moment pseudodifferential equations. Namely, we have

Proposition 4 [16, Theorem 1] *Let $\hat{u}$ be the normalised formal solution of Eq. 8. Then, $\hat{u} = \sum_{\alpha=1}^l \sum_{\beta=1}^{N_\alpha} \hat{u}_{\alpha\beta}$ with $\hat{u}_{\alpha\beta}$ being a formal solution of a simple pseudodifferential equation*

$$\begin{cases} (\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^\beta u_{\alpha\beta} = 0 \\ \partial_{m_1,t}^j u_{\alpha\beta}(0, z) = 0 \quad (j = 0, \dots, \beta - 2) \\ \partial_{m_1,t}^{\beta-1} \hat{u}_{\alpha\beta}(0, z) = \lambda_\alpha^{\beta-1}(\partial_{m_2,z}) \varphi_{\alpha\beta}(z), \end{cases}$$

where $\varphi_{\alpha\beta}(z) := \sum_{j=0}^{N-1} d_{\alpha\beta j}(\partial_{m_2,z}) \varphi_j(z) \in \mathcal{O}_{1/\gamma}(D)$ and $d_{\alpha\beta j}(\xi)$ are some holomorphic functions of the variable $\xi = \zeta^{1/\gamma}$ and of polynomial growth.

Moreover, if q_α is a pole order of $\lambda_\alpha(\zeta)$ and $\bar{q}_\alpha = \max\{0, q_\alpha\}$, then $\hat{u}_{\alpha\beta} \in \mathcal{O}_{1/\gamma}(D)[[t]]_{\bar{q}_\alpha s_2 - s_1}$.

For this reason, we will study the following simple moment pseudodifferential equation

$$\begin{cases} (\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^\beta u = 0 \\ \partial_{m_1,t}^j u(0, z) = 0 \quad (j = 0, \dots, \beta - 2) \\ \partial_{m_1,t}^{\beta-1} u(0, z) = \lambda^{\beta-1}(\partial_{m_2,z}) \varphi(z) \in \mathcal{O}_{1/\gamma}(D), \end{cases} \quad (11)$$

where m_1, m_2 are moment functions of orders respectively $s_1, s_2 > 0$ such that $qs_2 > s_1$, $\gamma \in \mathbb{N}$, $\lambda(\zeta) \sim \lambda_0 \zeta^q$ with $q = \mu/\nu$ for some relatively prime $\mu, \nu \in \mathbb{N}$ satisfying $q\gamma \in \mathbb{N}$.

We start from the following representation of summable solutions of Eq. 11.

Proposition 5 Let $d \in \mathbb{R}$, $K = (qs_2 - s_1)^{-1}$, $\varepsilon > 0$ and $m(u)$ be a moment function of order $1/K$. Suppose that $\widehat{u}(t, z) \in \mathcal{O}_{1/\gamma}(D)[[t]]$ is the unique formal power series solution of the Cauchy problem (11) and

$$\varphi(z) \in \mathcal{O}_{1/\gamma}^{qK} \left(\bigcup_{l=0}^{q\gamma-1} \widehat{S}_{(d+\arg \lambda_0+2l\pi)/q}(\varepsilon/q) \right). \quad (12)$$

Then, $\widehat{u}(t, z)$ is K -summable in the direction d and for every $\tilde{d} \in (d - \frac{\varepsilon}{2}, d + \frac{\varepsilon}{2})$ and for every $\tilde{\varepsilon} \in (0, \varepsilon)$, there exists $r > 0$ such that its K -sum $u^{\tilde{d}} \in \mathcal{O}_{1,1/\gamma}(S_{\tilde{d}}(\pi/K - \tilde{\varepsilon}, r) \times D)$ is given by

$$u^{\tilde{d}}(t, z) = \mathcal{S}_{K, \tilde{d}} \widehat{u}(t, z) = (T_{m, \tilde{d}} v)(t, z) = \int_{e^{i\tilde{d}} \mathbb{R}_+} e_m(s/t) v(s, z) \frac{ds}{s}, \quad (13)$$

where $v(t, z) = \widehat{B}_m \widehat{u}(t, z)$ has the integral representation

$$\begin{aligned} v(t, z) &= \frac{t^{\beta-1}}{(\beta-1)!} \partial_t^{\beta-1} \frac{1}{2\gamma\pi i} \oint_{|w|=\varepsilon}^{\gamma} \varphi(w) \int_{r_0 e^{i\theta}}^{\infty e^{i\theta}} \\ &E_{\overline{m}_1}(t\lambda(\zeta)) E_{\tilde{m}_2} \left(\zeta^{1/\gamma} z^{1/\gamma} \right) \frac{e_{m_2}(\zeta w)}{\zeta w} d\zeta dw \end{aligned} \quad (14)$$

with $\overline{m}_1(u) = m_1(u)m(u)$, $\tilde{m}_2(u) = m_2(u/\gamma)$ and $\theta \in (-\arg w - \frac{s_2\pi}{2}, -\arg w + \frac{s_2\pi}{2})$.

Proof First, observe that by Proposition 4 we get $\widehat{u}(t, z) \in \mathcal{O}_{1/\gamma}(D)[[t]]_{qs_2-s_1}$. Hence, the function $v(t, z) := \widehat{B}_m \widehat{u}(t, z)$ belongs to the space $\mathcal{O}_{1,1/\gamma}(D^2)$. Moreover, by Proposition 3 it satisfies the following:

$$\begin{cases} (\partial_{\overline{m}_1, t} - \lambda(\partial_{m_2, z}))^\beta v = 0 \\ \partial_{\overline{m}_1, t}^j v(0, z) = 0 \quad (j = 0, \dots, \beta-2) \\ \partial_{\overline{m}_1, t}^{\beta-1} v(0, z) = \lambda^{\beta-1}(\partial_{m_2, z}) \varphi(z) \in \mathcal{O}_{1/\gamma}(D). \end{cases}$$

Hence by [16, Lemma 3], we get the integral representation (14) of $v(t, z)$.

Since $\varphi(z)$ satisfies (12), by [16, Lemma 4], we conclude that $v(t, z) \in \mathcal{O}_{1,1/\gamma}^K(\widehat{S}_d(\varepsilon) \times D)$. So, for every $\tilde{d} \in (d - \frac{\varepsilon}{2}, d + \frac{\varepsilon}{2})$, the function $u^{\tilde{d}}(t, z) := T_{m, \tilde{d}} v(t, z)$ is well-defined and by the definitions of kernel functions (Definitions 2 and 3) for every $\tilde{\varepsilon} \in (0, \varepsilon)$, there exists $r > 0$ such that $u^{\tilde{d}} \in \mathcal{O}_{1,1/\gamma}(S_{\tilde{d}}(\pi/K - \tilde{\varepsilon}, r) \times D)$. $\square$

Now, we are ready to describe the Stokes phenomenon and the maximal family of solutions of the simple moment pseudodifferential equation (11) with the Cauchy data having the separate singular point at $z_0 \in \mathbb{C} \setminus \{0\}$.

Theorem 2 Let $\widehat{u}$ be a formal solution of Eq. 11 with $\varphi \in \mathcal{O}_{1/\gamma}^{qK}(\widetilde{\mathbb{C} \setminus \{z_0\}})$ for some $z_0 \in \mathbb{C} \setminus \{0\}$. Set $K := (qs_2 - s_1)^{-1}$, $\delta_l := q \arg z_0 + \frac{2l\pi}{\nu} - \arg \lambda_0$ and $u_l := u^{\tilde{d}}$ for $\tilde{d} \in (\delta_l, \delta_{l+1}) \bmod 2q\pi$ (for $l = 0, \dots, \mu-1$ with $\delta_\mu := \delta_0 + 2q\pi$), where $u^{\tilde{d}}$ is given by Eq. 13. Finally, let $W_r = \{t \in W : 0 < |t| < r\}$ for $r > 0$, where W is the Riemann surface of the function $t \mapsto t^{\frac{1}{q}}$.

Then, for every $\tilde{\varepsilon} > 0$, there exists $r > 0$ such that $u_l \in \mathcal{O}_{1,1/\gamma}(S_{\delta_l + \frac{\pi}{\nu}}((K^{-1} + \frac{2}{\nu})\pi - \tilde{\varepsilon}, r) \times D)$ ($l = 0, \dots, \mu-1$) and $\{u_0, \dots, u_{\mu-1}\}$ is a maximal family of solutions of Eq. 11 on $W_r \times D$.

Moreover, the sets $\mathcal{L}_{\delta_l}$ and $\mathcal{L}_{\delta_l \pm \frac{\pi}{2K}}$ ($l = 0, \dots, \mu - 1$) are respectively Stokes lines and anti-Stokes lines for $\widehat{u}$. The jump across the Stokes line $\mathcal{L}_{\delta_l}$ is given by the following:

$$J_{\mathcal{L}_{\delta_l}} \widehat{u}(t, 0) = u_l(t, 0) - u_{l-1}(t, 0) = u^{\delta_l^+}(t, 0) - u^{\delta_l^-}(t, 0) = F_l(s, 0) \left[\frac{e_m(s/t)}{s} \right],$$

where $F_l(s, 0) \in \mathcal{H}^q(\mathcal{L}_{\delta_l})$ is defined by $F_l(s, 0) := [v(s, 0)]_{\delta_l}$ and $v(s, z) = \widehat{\mathcal{B}}_m \widehat{u}(s, z)$ has the representation (14).

Proof First observe that, if $d \neq \delta_l \bmod 2q\pi$ for $l = 0, \dots, \mu - 1$ then φ satisfies the assumption (12) for sufficiently small $\varepsilon > 0$. Hence, by Proposition 5, $\widehat{u}$ is K -summable in a direction $\tilde{d} \in \mathbb{R}$, $\tilde{d} \neq \delta_l \bmod 2q\pi$ for $l = 0, \dots, \mu - 1$ and its K -sum $u^{\tilde{d}}(t, z)$ satisfies (13).

Observe that $u^{\tilde{d}}(t, z) = u^{\tilde{d}}(te^{2q\pi i}, z)$ and q is the smallest positive rational number for which this equality holds. Moreover, the set of singular directions of $\widehat{u}(t, z)$ modulo $2q\pi$ is given by $\{\delta_l \bmod 2q\pi : l = 0, \dots, \mu - 1\}$. Hence by [18, Theorem 3], for every $\tilde{\varepsilon} > 0$, there exists $r > 0$ such that $\{u_0, \dots, u_{\mu-1}\}$ with $u_l \in \mathcal{O}_{1,1/\gamma}(S_{\delta_l + \frac{\pi}{v}}((K^{-1} + \frac{2}{v})\pi - \tilde{\varepsilon}, r) \times D)$ ($l = 0, \dots, \mu - 1$) is a maximal family of solutions of Eq. 11. Moreover, Stokes lines for $\widehat{u}$ are the sets $\mathcal{L}_{\delta_l}$ and anti-Stokes lines for $\widehat{u}$ are the sets $\mathcal{L}_{\delta_l \pm \frac{\pi}{2K}}$.

Now, we are ready to calculate the jump across Stokes line $\mathcal{L}_{\delta_l}$.

$$\begin{aligned} J_{\mathcal{L}_{\delta_l}} \widehat{u}(t, z) &= u^{\delta_l^+}(t, z) - u^{\delta_l^-}(t, z) \stackrel{(13)}{=} \int_{e^{i\delta_l^+} \mathbb{R}_+} e_m(s/t) v(s, z) \frac{ds}{s} \\ &\quad - \int_{e^{i\delta_l^-} \mathbb{R}_+} e_m(s/t) v(s, z) \frac{ds}{s}. \end{aligned}$$

Hence,

$$J_{\mathcal{L}_{\delta_l}} \widehat{u}(t, 0) = F_l(s, 0) \left[\frac{e_m(s/t)}{s} \right],$$

where $F_l(s, 0) \in \mathcal{H}^q(\mathcal{L}_{\delta_l})$ is a hyperfunction on $\mathcal{L}_{\delta_l}$ defined by $F_l(s, 0) := [v(s, 0)]_{\delta_l}$. $\square$

Now, we return to the general equation (8). For convenience, we assume that

$$P(\lambda, \zeta) = P_0(\zeta) \tilde{P}(\lambda, \zeta) = P_0(\zeta) \prod_{i=1}^{\tilde{n}} \prod_{\alpha=1}^{l_i} (\lambda - \lambda_{i\alpha}(\zeta))^{N_{i\alpha}}, \quad (15)$$

where $\lambda_{i\alpha}(\zeta) \sim \lambda_{i\alpha} \zeta^{q_i}$ is the root of the characteristic equation with $q_i \in \mathbb{Q}$ and $\lambda_{i\alpha} \in \mathbb{C} \setminus \{0\}$ for $i = 1, \dots, \tilde{n}$ and $\alpha = 1, \dots, l_i$. Without loss of generality, we may assume that there exist exactly n , $n \leq \tilde{n}$, pole orders q_i , which are greater than s_1/s_2 , where s_1, s_2 are orders of moment functions m_1, m_2 respectively. We also assume that $s_1/s_2 < q_1 < \dots < q_n < \infty$ and let $K_i := (q_i s_2 - s_1)^{-1}$ for $i = 1, \dots, n$. Under the above conditions, we have

Theorem 3 Let $\widehat{u}$ be a normalised formal solution of Eq. 8, $z_0 \in \mathbb{C} \setminus \{0\}$ and $\varphi_j(z) \in \mathcal{O}^{q_n K_n}(\mathbb{C} \setminus \{z_0\})$ for $j = 0, \dots, N - 1$. Let $Q := \frac{LCM(\mu_1, \dots, \mu_n)}{GCD(v_1, \dots, v_n)}$ and let

$$\Lambda_i := \left\{ \delta : \delta = q_i \arg z_0 + \frac{2j\pi}{v_i} - \arg \lambda_{i\alpha} \bmod 2Q\pi, 0 \leq j \leq Qv_i - 1, 1 \leq \alpha \leq l_i \right\}$$

for $i = 1, \dots, n$. It means that we may assume that there exist $0 \leq \delta_{i,1} < \dots < \delta_{i,n_i} < 2Q\pi$ such that $\Lambda_i = \{\delta_{i,1}, \dots, \delta_{i,n_i}\}$. Moreover, let

$$I_{i,j} := \left(\delta_{i,j} - \frac{\pi}{2K_i}, \delta_{i,j+1} + \frac{\pi}{2K_i} \right) \quad \text{for } j = 1, \dots, n_i, \quad i = 1, \dots, n,$$

with $\delta_{i,n_i+1} := \delta_{i,1} + 2Q\pi$,

$$\mathcal{J} := \left\{ \mathbf{l} = (l_1, \dots, l_n) \in \mathbb{N}^n : 1 \leq l_i \leq n_i, |I_{i,l_i} \cap \dots \cap I_{n,l_n}| > \frac{\pi}{K_i} \text{ for } i = 1, \dots, n \right\},$$

$\mathbf{K} := (K_1, \dots, K_n)$ and $W_r = \{t \in W : 0 < |t| < r\}$ for $r > 0$, where W is the Riemann surface of the function $t \mapsto t^{1/Q}$.

Then, the following conditions holds:

- (a) The formal solution $\widehat{u}$ is $\mathbf{K}$ -multisummable, $\widehat{u} = \widehat{u}_0 + \widehat{u}_1 + \dots + \widehat{u}_n$, where $\widehat{u}_0$ is a convergent power series solution of

$$\left(\prod_{i=n+1}^{\tilde{n}} \prod_{\alpha=1}^{l_i} (\partial_{m_1,t} - \lambda_{i\alpha}(\partial_{m_2,z}))^{N_{i\alpha}} \right) u_0 = 0 \quad (16)$$

and $\widehat{u}_i$ is a K_i -summable power series solution of

$$\left(\prod_{\alpha=1}^{l_i} (\partial_{m_1,t} - \lambda_{i\alpha}(\partial_{m_2,z}))^{N_{i\alpha}} \right) u_i = 0 \quad \text{for } i = 1, \dots, n. \quad (17)$$

Moreover,

$$\mathcal{S}_{\mathbf{K},\mathbf{d}}\widehat{u} = u_0 + \mathcal{S}_{K_1,d_1}\widehat{u}_1 + \dots + \mathcal{S}_{K_n,d_n}\widehat{u}_n \quad (18)$$

for any admissible nonsingular multidirection $\mathbf{d} = (d_1, \dots, d_n)$.

- (b) For every $\mathbf{l} \in \mathcal{J}$, the function $u_{\mathbf{l}}(t, z) := \mathcal{S}_{\mathbf{K},\mathbf{d}}\widehat{u}$ is a well-defined actual solution of Eq. 8, where $\mathbf{d}$ is an admissible nonsingular multidirection satisfying $d_i \in (\delta_{i,l_i}, \delta_{i,l_i+1})$ for $i = 1, \dots, n$.
- (c) For every $\varepsilon > 0$, there exists $r > 0$ such that $u_{\mathbf{l}} \in \mathcal{O}_{1,1/\gamma}(V_1(\varepsilon, r) \times D)$, where

$$V_1(\varepsilon, r) := \left\{ t \in W_r : \left(\arg t - \frac{\varepsilon}{2}, \arg t + \frac{\varepsilon}{2} \right) \subseteq I_{1,l_1} \cap \dots \cap I_{n,l_n} \right\}.$$

- (d) $\{u_{\mathbf{l}}\}_{\mathbf{l} \in \mathcal{J}}$ is a maximal family of solutions of Eq. 8.
- (e) For every $i \in \{1, \dots, n\}$, the sets $\mathcal{L}_{\delta_{i,j}}$ (resp. $\mathcal{L}_{\delta_{i,j} \pm \frac{\pi}{2K_i}}$), $j = 1, \dots, n_i$ are Stokes lines (resp. anti-Stokes lines) of level K_i .
- (f) For every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n_i\}$, the jump across the Stokes line $\mathcal{L}_{\delta_{i,j}}$ of level K_i is given by

$$J_{\mathcal{L}_{\delta_{i,j},K_i}}\widehat{u}(t, 0) = u_{\mathbf{l}}(t, 0) - u_{\mathbf{l}'}(t, 0) = u_i^{\delta_{i,j}^+}(t, 0) - u_i^{\delta_{i,j}^-}(t, 0) = F_{i,j}(s, 0) \left[\frac{e_{\overline{m}_i}(s/t)}{s} \right],$$

where $u_i^{\delta_{i,j}^{\pm}} = \mathcal{S}_{K_i, \delta_{i,j}^{\pm}}\widehat{u}_i$, $\overline{m}_i$ is a moment function of order $1/K_i$, $F_{i,j}(s, 0)$ is a hyperfunction on $\mathcal{L}_{\delta_{i,j}}$ defined by $F_{i,j}(s, 0) := [v_i(s, 0)]_{\delta_{i,j}}$, $v_i(s, z) := \widehat{\mathcal{B}}_{\overline{m}_i}\widehat{u}_i(s, z)$ and $\mathbf{l}, \mathbf{l}' \in \mathcal{J}$ satisfy $l'_i = j - 1$ in the case when $l_i = j$ and $j > 1$, $l'_i = n_i$ in the case when $l_i = 1$, and $l'_\alpha = l'_\alpha$ for $\alpha \neq i$.

Moreover, under the additional condition that $\widehat{u}$ is a normalised formal solution of Eq. 8 with $m_2(u) = \Gamma(1+u)$ (i.e. when $\widehat{u}$ is a normalised formal solution of the Cauchy problem $P(\partial_{m_1,t}, \partial_z)u = 0$, $\partial_{m_1,t}^j u(0, z) = \varphi_j(z)$ for $j = 0, \dots, N-1$), we may replace the assertion (f) by

(f') For every $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n_i\}$ and $z \in D$, the jump across the Stokes line $\mathcal{L}_{\delta_{i,j}}$ of level K_i is given by the following:

$$J_{\mathcal{L}_{\delta_{i,j}}, K_i} \widehat{u}(t, z) = u_1(t, z) - u_I(t, z) = u_i^{\delta_{i,j}^+}(t, z) - u_i^{\delta_{i,j}^-}(t, z) = F_{i,j}(s, z) \left[\frac{e_{\overline{m}_i}(s/t)}{s} \right],$$

where $u_i^{\delta_{i,j}^\pm} = \mathcal{S}_{K_i, \delta_{i,j}^\pm} \widehat{u}_i$, $\overline{m}_i$ is a moment function of order $1/K_i$, $F_{i,j}(s, z)$ is a hyperfunction on $\mathcal{L}_{\delta_{i,j}}(z)$ defined by $F_{i,j}(s, z) := [v_i(s, z)]_{\delta_{i,j}(z)}$, $\delta_{i,j}(z) := \delta_{i,j} + q_i(\arg(z_0 - z) - \arg z_0)$, $v_i(s, z) := \widehat{\mathcal{B}}_{\overline{m}_i} \widehat{u}_i(s, z)$ and $\mathbf{l}, \mathbf{l}' \in \mathcal{J}$ satisfy the same conditions as in (f).

Proof Since $P(\lambda, \zeta)$ is given by Eq. 15, by Proposition 4, a normalised formal solution $\widehat{u}$ of Eq. 8 may be written as $\widehat{u} = \widehat{u}_0 + \widehat{u}_1 + \dots + \widehat{u}_n$, where $\widehat{u}_0$ is a convergent power series solution of the pseudodifferential equation (16) and $\widehat{u}_i$ is a $1/K_i$ -Gevrey power series solution of Eq. 17 with the initial data having the same holomorphic properties as $\varphi_j(z)$. More precisely, by Proposition 4 we conclude that $\widehat{u}_i = \sum_{\alpha=1}^{l_i} \sum_{\beta=1}^{N_{i\alpha}} \widehat{u}_{i\alpha\beta}$, where $\widehat{u}_{i\alpha\beta}$ is a formal solution of a simple pseudodifferential equation as follows:

$$\begin{cases} (\partial_{m_1,t} - \lambda_{i\alpha}(\partial_{m_2,z}))^\beta u_{i\alpha\beta} = 0 \\ \partial_{m_1,t}^j u_{i\alpha\beta}(0, z) = 0 \quad (j = 0, \dots, \beta - 2) \\ \partial_{m_1,t}^{\beta-1} \widehat{u}_{i\alpha\beta}(0, z) = \lambda_{i\alpha}^{\beta-1}(\partial_{m_2,z}) \varphi_{i\alpha\beta}(z), \end{cases}$$

where $\varphi_{i\alpha\beta}(z) := \sum_{j=0}^{N-1} d_{i\alpha\beta j}(\partial_{m_2,z}) \varphi_j(z) \in \mathcal{O}_{1/\gamma}(D)$ and $d_{i\alpha\beta j}(\zeta)$ are some holomorphic functions of the variable $\xi = \zeta^{1/\gamma}$ and of polynomial growth. Since $\varphi_j(z) \in \mathcal{O}^{q_n K_n}(\mathbb{C} \setminus \{z_0\})$ and $q_i K_i \leq q_n K_n$, we see that $\varphi_{i\alpha\beta}(z) \in \mathcal{O}_{1/\gamma}^{q_i K_i} \left(\bigcup_{l=0}^{q_i \gamma - 1} \widehat{S}_{(d+\arg \lambda_{i\alpha} + 2l\pi)/q_i} \right)$ for every direction $d \neq q_i \arg z_0 + \frac{2j\pi}{v_i} - \arg \lambda_{i\alpha} \bmod 2\pi q_i$, $j = 0, \dots, \mu - 1$.

Hence, by Proposition 5 and Theorem 2, we see that $\widehat{u}_{i\alpha\beta}$ is K_i -summable with the singular directions given by $q_i \arg z_0 + \frac{2j\pi}{v_i} - \arg \lambda_{i\alpha} \bmod 2\pi q_i$ for $j = 0, \dots, \mu_i - 1$. Consequently, $\widehat{u}$ is $\mathbf{K}$ -multisummable in any nonsingular admissible multidirection $\mathbf{d} = (d_1, \dots, d_n)$. Since a formal power series $\widehat{u}_0$ is convergent, its sum u_0 is well-defined and by Remark 7, we conclude that $\mathbf{K}$ -multisum $\mathcal{S}_{\mathbf{K}, \mathbf{d}} \widehat{u}$ of $\widehat{u}$ is given by Eq. 18, so (a) holds.

Since $\widehat{u}$ is K -multisummable, using Theorem 1 we conclude that (b), (c), and (d) hold.

Since the set of singular directions of order K_i is given by Λ_i , we get the description of Stokes lines $\mathcal{L}_{\delta_{i,j}}$ and anti-Stokes lines $\mathcal{L}_{\delta_{i,j} \pm \frac{\pi}{2K_i}}$ of level K_i for $\delta_{i,j} \in \Lambda_i$ and $i = 1, \dots, n$, so (e) is also satisfied.

Finally, to obtain (f) by Theorem 2, we calculate the jumps for $\widehat{u}$ across the Stokes lines $\mathcal{L}_{d_i}$ of level K_i . Using Remark 9, we get (f'). $\square$

Let us illustrate our theory on the following simple example.

Example 4 Let $\widehat{u}$ be a formal solution of the Cauchy problem

$$\begin{cases} (\partial_t - \partial_z^2)(\partial_t - \partial_z^3)\widehat{u} = 0, \\ u(0, z) = \varphi_1(z) \in \mathcal{O}^{3/2}(\widetilde{\mathbb{C} \setminus \{z_0\}}), \\ \partial_t u(0, z) = \varphi_2(z) \in \mathcal{O}^{3/2}(\widetilde{\mathbb{C} \setminus \{z_0\}}), \end{cases} \quad (19)$$

where $\arg z_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Then by Proposition 4 (see also [16, Theorem 1]), $\widehat{u} = \widehat{u}_1 + \widehat{u}_2$, where $\widehat{u}_1, \widehat{u}_2$ are formal solutions of the following:

$$(\partial_t - \partial_z^2)u_1 = 0, \quad u_1(0, z) = c_{11}(\partial_z)\varphi_1(z) + c_{12}(\partial_z)\varphi_2(z) =: \tilde{\varphi}_1(z),$$

$$(\partial_t - \partial_z^3)u_2 = 0, \quad u_2(0, z) = c_{21}(\partial_z)\varphi_1(z) + c_{22}(\partial_z)\varphi_2(z) =: \tilde{\varphi}_2(z),$$

where $c_{ij}(\partial_z)$, $i, j = 1, 2$, are pseudodifferential operators defined by $c_{11}(\zeta) := \frac{\zeta}{\zeta-1}$, $c_{12}(\zeta) := \frac{1}{\zeta^2-\zeta^3}$, $c_{21}(\zeta) := \frac{1}{1-\zeta}$ and $c_{22}(\zeta) := \frac{1}{\zeta^3-\zeta^2}$.

Now, we are ready to describe the Stokes phenomenon and the maximal family of solutions of Eq. 19. By Theorem 2, $\widehat{u}_1$ is 1-summable with singular directions $d_{1,l} := 2 \arg z_0 + 2l\pi$ and $\widehat{u}_2$ is $1/2$ -summable with singular directions $d_{2,l} := 3 \arg z_0 + 2l\pi$ for $l \in \mathbb{Z}$. Hence $\widehat{u}$ is $(1, 1/2)$ -summable, the set of singular directions (modulo 12π) of level 1 is given by $\{d_{1,0}, \dots, d_{1,5}\}$, and the set of singular directions (modulo 12π) of level $1/2$ is given by $\{d_{2,0}, \dots, d_{2,5}\}$. It means that $\mathcal{L}_{d_{1,l}}$ and $\mathcal{L}_{d_{1,l} \pm \frac{\pi}{2}}$ are respectively Stokes and anti-Stokes lines of level 1, and analogously $\mathcal{L}_{d_{2,l}}$ and $\mathcal{L}_{d_{2,l} \pm \pi}$ are respectively Stokes and anti-Stokes lines of level $1/2$ ($l = 0, \dots, 5$).

Next, let $I_{1,j} := (-\frac{\pi}{2} + 2\pi j + 2 \arg z_0, \frac{\pi}{2} + 2\pi(j+1) + 2 \arg z_0)$ for $j = 0, \dots, 5$ and $I_{2,k} := (-\pi + 2\pi k + 3 \arg z_0, \pi + 2\pi(k+1) + 3 \arg z_0)$ for $k = 0, \dots, 5$. Since $\arg z_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we conclude that

$$|I_{1,j} \cap I_{2,k}| = \begin{cases} 3\pi & |j-k| = 0 \\ \frac{3}{2}\pi \pm \arg z_0 & |j-k| = \pm 1 \\ 0 & |j-k| > 1. \end{cases}$$

Hence, $\mathcal{J} = \{(j, k): 0 \leq j, k \leq 5, |j-k| \leq 1\}$ and $\{u_{(j,k)}\}_{(j,k) \in \mathcal{J}}$ is a maximal family of solutions of Eq. 19 on the Riemann surface of $t \mapsto t^{\frac{1}{6}}$, where $u_{(j,k)} := u_{1,j} + u_{2,k}$, $u_{1,j} := u_1^d = \mathcal{S}_{1,d}\widehat{u}_1$ for $d \in (d_{1,j}, d_{1,j+1}) = (2 \arg z_0 + 2\pi j, 2 \arg z_0 + 2\pi(j+1))$ and $u_{2,k} := u_2^d = \mathcal{S}_{2,k}\widehat{u}_2$ for $d \in (d_{2,j}, d_{2,j+1}) = (3 \arg z_0 + 2\pi k, 3 \arg z_0 + 2\pi(k+1))$.

Using [18] and [22], we are also able to calculate the jumps across the Stokes lines. Namely,

$$\begin{aligned} J_{\mathcal{L}_{d_{1,j,1}}}\widehat{u}(t, z) &= u_{(j,j)}(t, z) - u_{(j-1,j)}(t, z) = u_{1,j}(t, z) - u_{1,j-1}(t, z) \\ &= J_{\mathcal{L}_{d_{1,j}}}\widehat{u}_1 = u_1^{d_{1,j}^+}(t, z) - u_1^{d_{1,j}^-}(t, z) = F_{1,z}(s) \left[\frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}} \right], \end{aligned}$$

where $F_{1,z}(s) := [\tilde{\varphi}_1(z+s)]_{\arg(z_0-z)}$ is a hyperfunction on $\{s \in \tilde{\mathbb{C}}: \arg s = \arg(z_0 - z)\}$.

Analogously,

$$\begin{aligned} J_{\mathcal{L}_{d_{2,j,1/2}}}\widehat{u}(t, z) &= u_{(j,j)}(t, z) - u_{(j,j-1)}(t, z) = u_{2,j}(t, z) - u_{2,j-1}(t, z) \\ &= J_{\mathcal{L}_{d_{1,j}}}\widehat{u}_2 = u_2^{d_{2,j}^+}(t, z) - u_2^{d_{2,j}^-}(t, z) = F_{2,z}(s) \left[\frac{1}{3\sqrt[3]{t}} C_3 \left(s/\sqrt[3]{t} \right) \right], \end{aligned}$$

where $F_{2,z}(s) := [\tilde{\varphi}_2(z+s)]_{\arg(z_0-z)}$ is a hyperfunction on $\{s \in \tilde{\mathbb{C}} : \arg s = \arg(z_0 - z)\}$ and $C_3(\tau)$ is the Ecalle kernel defined by $C_3(\tau) := \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n! \Gamma(1 - \frac{n+1}{3})}$.

8 Moment Partial Differential Equations—Special Cases

In this section, we will consider certain special cases of moment partial differential equations. We derive Stokes lines and jumps across these Stokes lines in terms of hyperfunctions.

Case 1. Let us consider the following equation as follows:

$$\begin{cases} \partial_{m_1,t}^p u(t, z) = \partial_z^q u(t, z) & \text{with } 0 < \text{ps}_1 < q, \\ u(0, z) = \varphi(z), \\ \partial_{m_1,t}^j u(0, z) = 0, & \text{for } j = 1, 2, \dots, p-1 \end{cases}$$

with $\varphi(z) \in \mathcal{O}^{\frac{q}{q-\text{ps}_1}}(\mathbb{C} \setminus \{z_0\})$ for some $z_0 \in \mathbb{C} \setminus \{0\}$; where m_1 is a moment function of order $s_1 > 0$ corresponding to a kernel function $e_{m_1}(z)$ of order $1/s_1$.

The above Cauchy problem has a unique formal solution as follows:

$$\widehat{u}(t, z) = \sum_{n=0}^{\infty} \frac{\varphi^{(\text{qn})}(z)}{m_1(\text{pn})} t^{pn},$$

to which we first apply the m -moment Borel transform. We obtain the following:

$$\begin{aligned} (\widehat{\mathcal{B}}_m \widehat{u})(t, z) &= \sum_{n=0}^{\infty} \frac{\varphi^{(\text{qn})}(z)}{m_1(\text{pn})} \cdot \frac{m_1(\text{pn})}{\Gamma(1 + \frac{q}{p} \cdot \text{pn})} t^{(\frac{q}{p})\text{qn}} = \sum_{n=0}^{\infty} \frac{\varphi^{(\text{qn})}(z)}{(\text{qn})} t^{(\frac{q}{p})\text{qn}} \\ &= \frac{1}{q} \left(\varphi \left(z + \sqrt[q]{t^p} \right) + \varphi \left(z + e^{\frac{2\pi i}{q}} \sqrt[q]{t^p} \right) + \dots + \varphi \left(z + e^{\frac{2(q-1)\pi i}{q}} \sqrt[q]{t^p} \right) \right), \end{aligned}$$

where $m(n) := \frac{\Gamma(1 + \frac{q}{p}n)}{m_1(n)} = \frac{\Gamma \frac{q}{p}(n)}{m_1(n)}$ is a moment function of order $\frac{q}{p} - s_1$ corresponding to a kernel function $e_m(z)$ of order $\frac{p}{q-\text{ps}_1}$.

Let $f(s, z) := (\widehat{\mathcal{B}}_m \widehat{u})(s, z)$, then by using m -moment Laplace transform in a nonsingular direction d we get the following:

$$\begin{aligned} (T_{m,d} f)(t, z) &= \int_{e^{id}\mathbb{R}_+} e_m(s/t) f(s, z) \frac{ds}{s} \\ &= \frac{1}{q} \int_{e^{id}\mathbb{R}_+} e_m(s/t) \left(\varphi \left(z + \sqrt[q]{s^p} \right) + \varphi \left(z + e^{\frac{2\pi i}{q}} \sqrt[q]{s^p} \right) + \dots \right. \\ &\quad \left. + \varphi \left(z + e^{\frac{2(q-1)\pi i}{q}} \sqrt[q]{s^p} \right) \right) \frac{ds}{s}. \end{aligned}$$

Thus, by Proposition 5, the unique formal solution $\widehat{u}(t, z)$ of this Cauchy problem is $\frac{p}{q-ps_1}$ -summable in the direction d , and for every $\varepsilon > 0$, there exists $r > 0$ such that its $\frac{p}{q-ps_1}$ -sum $u \in \mathcal{O}(S_d(\frac{\pi(q-ps_1)}{p} - \varepsilon, r) \times D)$ is given by the following:

$$\begin{aligned} u(t, z) &= u^d(t, z) \\ &= \frac{1}{q} \int_{e^{id}\mathbb{R}_+} e_m(s/t) \left(\varphi \left(z + \sqrt[q]{s^p} \right) + \varphi \left(z + e^{\frac{2\pi i}{q}} \sqrt[q]{s^p} \right) + \dots \right. \\ &\quad \left. + \varphi \left(z + e^{\frac{2(q-1)\pi i}{q}} \sqrt[q]{s^p} \right) \right) \frac{ds}{s}. \end{aligned}$$

Let $\theta := \arg z_0$, $\delta := \frac{q\theta}{p}$. Then, $\mathcal{L}_{\delta + \frac{2\pi j}{p}}$ ($j = 0, 1, \dots, p-1$) are Stokes lines for $\widehat{u}$. For every sufficiently small $\varepsilon > 0$, there exists $r > 0$ such that for every fixed $z \in D_r$ the jump is given by the following:

$$\begin{aligned} J_{\mathcal{L}_\delta} \widehat{u}(t, z) &= u^{\delta+\varepsilon}(t, z) - u^{\delta-\varepsilon}(t, z) = F_z(s) \left[\frac{e_m(s/t)}{s} \right] \\ &= \left[\varphi \left(z + \sqrt[q]{s^p} \right) + \varphi \left(z + e^{\frac{2\pi i}{q}} \sqrt[q]{s^p} \right) + \dots \right. \\ &\quad \left. + \varphi \left(z + e^{\frac{2(q-1)\pi i}{q}} \sqrt[q]{s^p} \right) \right]_{q\frac{\theta_z}{p}} \left[\frac{e_m(s/t)}{qs} \right] \\ &= \left[\varphi(z + \sqrt[q]{s^p}) \right]_{q\frac{\theta_z}{p}} \left[\frac{e_m(s/t)}{qs} \right] \end{aligned}$$

with $\theta_z = \arg(z_0 - z)$. The last equality arising from the fact that in this case all singular points appear in the function $s \mapsto \varphi(z + \sqrt[q]{s^p})$.

Observe that from [2, Theorem 32] one can derive the function

$$e_m(u) = T_{m_1, d}^- (e_{m_2}(1/z)) (1/u) = -\frac{1}{2\pi i} \int_{\gamma(d)} E_{m_1} \left(\frac{1}{uz} \right) \frac{p}{q} \left(\frac{1}{z} \right)^{\frac{p}{q}} e^{-\left(\frac{1}{z}\right)^{\frac{p}{q}}} \frac{dz}{z},$$

where $E_{m_1} \left(\frac{1}{uz} \right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{uz}\right)^n}{m_1(n)}$, $m_2(n) = \Gamma(1 + \frac{q}{p}n)$ and, by Example 1, $e_{m_2}(z) = \frac{p}{q} z^{\frac{p}{q}} e^{-z^{\frac{p}{q}}}$.

Case 2. Let us now study the formal solution

$$\widehat{u}(t, z) = \sum_{n=0}^{\infty} \frac{\partial_{m_2, z}^n \varphi(z)}{m_1(n)} t^n$$

of the following equation

$$\begin{cases} \partial_{m_1, t} u(t, z) = \partial_{m_2, z} u(t, z), \\ u(0, z) = \varphi(z) \end{cases}$$

with $\varphi(z) \in \mathcal{O}_{s_2-s_1}^{\frac{1}{s_2-s_1}}(\mathbb{C} \setminus \{z_0\})$ for some $z_0 \in \mathbb{C} \setminus \{0\}$; where m_1 is a moment function of order $s_1 > 0$ corresponding to a kernel function $e_{m_1}(z)$ of order $1/s_1$, m_2 is a moment function of order $s_2 > 0$ corresponding to a kernel function $e_{m_2}(z)$ of order $1/s_2$ and $s_2 > s_1$. First, we apply to $\widehat{u}(t, z)$ the m -moment Borel transform

$$(\widehat{\mathcal{B}}_m \widehat{u})(t, z) = \widehat{\mathcal{B}}_m \left(\sum_{n=0}^{\infty} \frac{\partial_{m_2, z}^n \varphi(z)}{m_1(n)} t^n \right) = \sum_{n=0}^{\infty} \frac{\partial_{m_2, z}^n \varphi(z)}{m_2(n)} t^n,$$

where $m(n) := m_2(n)/m_1(n)$ is a moment function of order $s_2 - s_1$ corresponding to a kernel function $e_m(z)$ of order $k := \frac{1}{s_2 - s_1}$.

Using [15, Proposition 3], we see that for $|z| < \varepsilon < r$ and $n \in \mathbb{N}$, we have

$$\partial_{m_2, z}^n \varphi(z) = \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\psi)} \zeta^n E_{m_2}(z\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta dw,$$

where $\psi \in (-\arg w - \frac{\pi s_2}{2}, -\arg w + \frac{\pi s_2}{2})$. Thus,

$$\begin{aligned} (\widehat{\mathcal{B}}_m \widehat{u})(t, z) &= \sum_{n=0}^{\infty} \frac{\partial_{m_2, z}^n \varphi(z) t^n}{m_2(n)} = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\psi)} \frac{\zeta^n t^n}{m_2(n)} \\ &\quad E_{m_2}(z\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta dw \\ &= \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\psi)} E_{m_2}(t\zeta) E_{m_2}(z\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta dw. \end{aligned}$$

Let $f(s, z) := (\widehat{\mathcal{B}}_m \widehat{u})(s, z)$, then by using m -moment Laplace transform in a nonsingular direction d , we get the following:

$$\begin{aligned} (T_{m, d} f)(t, z) &= \int_{e^{\text{id}} \mathbb{R}_+} e_m(s/t) f(s, z) \frac{ds}{s} \\ &= \int_{e^{\text{id}} \mathbb{R}_+} e_m(s/t) \left(\frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\psi)} \right. \\ &\quad \left. E_{m_2}(s\zeta) E_{m_2}(z\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta dw \right) \frac{ds}{s}. \end{aligned}$$

Notice that, by [2, Theorem 32], the function $e_m(u)$ is of the form

$$e_m(u) = T_{m_1, d}^- (e_{m_2}(1/z)) (1/u) = -\frac{1}{2\pi i} \int_{\gamma(d)} E_{m_1} \left(\frac{1}{uz} \right) e_{m_2}(1/z) \frac{dz}{z}, \quad (20)$$

where $E_{m_1} \left(\frac{1}{uz} \right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{uz} \right)^n}{m_1(n)}$.

Thus, by Proposition 5, the unique formal solution $\widehat{u}(t, z)$ of this Cauchy problem is k -summable in the direction d and for every $\varepsilon > 0$, there exists $r > 0$ such that its k -sum $u \in \mathcal{O}(S_d(\frac{\pi}{k} - \varepsilon, r) \times D)$ is given by the following:

$$\begin{aligned} u(t, z) &= u^d(t, z) \\ &= \int_{e^{\text{id}} \mathbb{R}_+} e_m(s/t) \left(\frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\psi)} E_{m_2}(s\zeta) E_{m_2}(z\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta dw \right) \frac{ds}{s}. \end{aligned}$$

Then, $\mathcal{L}_\delta$, with $\delta = \theta := \arg z_0$, is a Stokes line for $\widehat{u}$. For $z = 0$, the jump is given by the following:

$$\begin{aligned} J_{\mathcal{L}_\delta} \widehat{u}(t, 0) &= u^{\delta^+}(t, 0) - u^{\delta^-}(t, 0) = F_0(s) \left[\frac{e_m(s/t)}{s} \right] \\ &= \left[\frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\psi)} E_{m_2}(s\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta dw \right]_\theta \left[\frac{e_m(s/t)}{s} \right]. \end{aligned}$$

Using [2, formula (5.15)], one can derive

$$\int_0^{\infty(\psi)} E_{m_2}(s\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta = \frac{1}{w-s},$$

hence,

$$J_{\mathcal{L}_\delta} \widehat{u}(t, 0) = \left[\frac{1}{2\pi i} \oint_{|w|=\varepsilon} \frac{\varphi(w)}{w-s} dw \right]_\theta \left[\frac{e_m(s/t)}{s} \right] = [\varphi(s)]_\theta \left[\frac{e_m(s/t)}{s} \right],$$

where the last equality follows from the Cauchy integral formula.

Case 3. Now, we take the following equation under consideration

$$\begin{cases} \partial_{m_1,t}^q u(t, z) = \partial_{m_2,z}^q u(t, z), \\ u(0, z) = \varphi(z), \\ \partial_{m_1,t}^j u(0, z) = 0, \text{ for } j = 1, 2, \dots, q-1 \end{cases}$$

with $\varphi(z) \in \mathcal{O}_{\frac{1}{s_2-s_1}}(\mathbb{C} \setminus \{z_0\})$ for some $z_0 \in \mathbb{C} \setminus \{0\}$, where m_1 is a moment function of order $s_1 > 0$ corresponding to a kernel function $e_{m_1}(z)$ of order $1/s_1$, m_2 is a moment function of order $s_2 > 0$ corresponding to a kernel function $e_{m_2}(z)$ of order $1/s_2$ and $s_2 > s_1$. Observe that since

$$\partial_{m_1,t}^q - \partial_{m_2,z}^q = (\partial_{m_1,t} - \partial_{m_2,z}) \left(\partial_{m_1,t} - e^{\frac{2\pi i}{q}} \partial_{m_2,z} \right) \cdot \dots \cdot \left(\partial_{m_1,t} - e^{\frac{2\pi i(q-1)}{q}} \partial_{m_2,z} \right),$$

then, we can write the following:

$$\widehat{u}(t, z) = \widehat{u}_0(t, z) + \widehat{u}_1(t, z) + \dots + \widehat{u}_{q-1}(t, z),$$

where, for $j = 0, 1, \dots, q-1$

$$\widehat{u}_j(t, z) = \frac{1}{q} \sum_{n=0}^{\infty} \frac{\partial_{m_2,z}^n \varphi(z)}{m_1(n)} \left(e^{\frac{2\pi i j}{q}} \right)^n t^n,$$

is a formal solution of the equation

$$\begin{cases} \partial_{m_1,t} u_j(t, z) = e^{\frac{2\pi i j}{q}} \partial_{m_2,z} u_j(t, z), \\ u_j(0, z) = \frac{1}{q} \varphi(z) \in \mathcal{O}_{\frac{1}{s_2-s_1}}(\mathbb{C} \setminus \{z_0\}). \end{cases}$$

Notice that, based on reasoning of the case 2, for each $\widehat{u}_j(t, z)$, we obtain that

$$u_j(t, z) = \int_{e^{\text{id}} \mathbb{R}_+} e_m(s/t) \left(\frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\psi)} E_{m_2}(s e^{\frac{2\pi i j}{q}} \zeta) E_{m_2}(z\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta dw \right) \frac{ds}{s}.$$

So, $\mathcal{L}_{\delta + \frac{2\pi j}{q}}$ are Stokes lines for $\widehat{u}(t, z)$, where $\delta = \theta = \arg z_0$ and $j = 0, \dots, q-1$. Moreover,

$$\begin{aligned} J_{\mathcal{L}_\delta} \widehat{u}_j(t, 0) &= \left[\frac{1}{2\pi i} \oint_{|w|=\varepsilon} \frac{\varphi(w)}{w - \text{se}^{\frac{2\pi i j}{q}}} dw \right]_\theta \left[\frac{e_m(s/t)}{\text{qs}} \right] = \left[\varphi \left(\text{se}^{\frac{2\pi i j}{q}} \right) \right]_\theta \left[\frac{e_m(s/t)}{\text{qs}} \right] \\ &= \begin{cases} [\varphi(s)]_\theta \left[\frac{e_m(s/t)}{\text{qs}} \right], & \text{for } j = 0 \\ 0, & \text{for } j = 1, 2, \dots, q-1, \end{cases} \end{aligned}$$

where e_m is given by Eq. 20. Thus,

$$J_{\mathcal{L}_\delta} \widehat{u}(t, 0) = \left[\sum_{j=0}^{q-1} \varphi \left(\text{se}^{\frac{2\pi i j}{q}} \right) \right]_\theta \left[\frac{e_m(s/t)}{\text{qs}} \right] = [\varphi(s)]_\theta \left[\frac{e_m(s/t)}{\text{qs}} \right].$$

Case 4. In this part, we will study more general case, i.e.

$$\begin{cases} \partial_{m_1, t}^p u(t, z) = \partial_{m_2, z}^q u(t, z), & \text{with } 0 < ps_1 < qs_2, \\ u(0, z) = \varphi(z), \\ \partial_{m_1, t}^j u(0, z) = 0, & \text{for } j = 1, 2, \dots, p-1 \end{cases} \quad (21)$$

with $\varphi(z) \in \mathcal{O}_{\frac{q}{qs_2 - ps_1}}(\mathbb{C} \setminus \{z_0\})$ for some $z_0 \in \mathbb{C} \setminus \{0\}$, where m_1 is a moment function of order $s_1 > 0$ corresponding to a kernel function $e_{m_1}(z)$ of order $1/s_1$, m_2 is a moment function of order $s_2 > 0$ corresponding to a kernel function $e_{m_2}(z)$ of order $1/s_2$. The above Cauchy problem has a formal solution

$$\widehat{u}(t, z) = \sum_{n=0}^{\infty} \frac{\partial_{m_2, z}^{qn} \varphi(z)}{m_1(pn)} t^{pn}.$$

By Theorem 2, $\mathcal{L}_{\delta + \frac{2\pi j}{p}}$ are Stokes lines for $\widehat{u}(t, z)$, where $\delta = \frac{q}{p}\theta$, $\theta := \arg z_0$ and $j = 0, \dots, p-1$. To calculate the jumps across Stokes lines assume that $v(t, z) := u(t^{\frac{q}{p}}, z)$ and $\tilde{m}_1(n) := m_1\left(\frac{pn}{q}\right)$ is a moment function of order $\frac{s_1 p}{q} > 0$ corresponding to a kernel function $e_{\tilde{m}_1}(z) = \frac{q}{p} e_{m_1}\left(z^{\frac{q}{p}}\right)$ of order $\frac{q}{s_1 p}$. Then,

$$\widehat{v}(t, z) = \sum_{n=0}^{\infty} \frac{\partial_{m_2, z}^{qn} \varphi(z)}{\tilde{m}_1(qn)} t^{qn}$$

is a formal solution of the equation

$$\begin{cases} \partial_{m_1, t}^q v(t, z) = \partial_{m_2, z}^q v(t, z), \\ v(0, z) = \varphi(z) \in \mathcal{O}_{\frac{q}{qs_2 - ps_1}}(\mathbb{C} \setminus \{z_0\}), \\ \partial_{m_1, t}^j v(0, z) = 0, & \text{for } j = 1, 2, \dots, q-1. \end{cases} \quad (22)$$

Observe that $\widehat{u}(t, z)$ is a formal solution of the equation (21) if and only if $\widehat{v}(t, z)$ is a formal solution of the equation (22) (see also [15, Lemma 3]).

In this case, we reduce our problem to the one we considered in the case 3. Thus, based on the obtained results, we have the following:

$$v_j(t, z) = \int_{e^{\text{id}} \mathbb{R}_+} e_m(s/t) \left(\frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\psi)} E_{m_2}(\text{se}^{\frac{2\pi ij}{q}} \zeta) E_{m_2}(z\zeta) \frac{e_{m_2}(w\zeta)}{w\zeta} d\zeta dw \right) \frac{ds}{s},$$

so,

$$\begin{aligned} J_{\mathcal{L}_\theta} \widehat{v}_j(t, 0) &= \left[\frac{1}{2\pi i} \oint_{|w|=\varepsilon} \frac{\varphi(w)}{w - \text{se}^{\frac{2\pi ij}{q}}} dw \right]_\theta \left[\frac{e_m(s/t)}{qs} \right] = \left[\varphi \left(s e^{\frac{2\pi ij}{q}} \right) \right]_\theta \left[\frac{e_m(s/t)}{qs} \right] \\ &= \begin{cases} [\varphi(s)]_\theta \left[\frac{e_m(s/t)}{qs} \right], & \text{for } j = 0 \\ 0, & \text{for } j = 1, 2, \dots, q-1, \end{cases} \end{aligned}$$

where e_m is given by the following:

$$e_m(u) = T_{\tilde{m}_1, d}^- (e_{m_2}(1/z))(1/u) = -\frac{1}{2\pi i} \int_{\gamma(d)} E_{\tilde{m}_1} \left(\frac{1}{uz} \right) e_{m_2}(1/z) \frac{dz}{z},$$

with

$$E_{\tilde{m}_1} \left(\frac{1}{uz} \right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{uz} \right)^n}{\tilde{m}_1(n)} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{uz} \right)^n}{m_1\left(\frac{pn}{q}\right)}.$$

Thus

$$J_{\mathcal{L}_\theta} \widehat{v}(t, 0) = \left[\sum_{j=0}^{q-1} \varphi \left(\operatorname{se}^{\frac{2\pi ij}{q}} \right) \right]_\theta \left[\frac{e_m(s/t)}{qs} \right] = [\varphi(s)]_\theta \left[\frac{e_m(s/t)}{qs} \right].$$

Hence

$$J_{\mathcal{L}_\delta} \widehat{u}(t, 0) = J_{\mathcal{L}_\theta} \widehat{v}(t^{\frac{p}{q}}, 0) = [\varphi(s)]_\theta \left[\frac{e_m(s/t^{\frac{p}{q}})}{qs} \right].$$

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