On the Adjoint of the Eulerian Idempotent in an Analytic Context

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Received: 21 February 2018 / Revised: 5 September 2018 / Published online: 20 November 2018
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Abstract

We generalize Gehrig-Kawski theorem connecting the adjoint of the Eulerian idempotent with the logarithm of identity operator in the convolution product algebra $\text{End}(\mathbb{K}\langle A \rangle)$. This has application in dynamical systems, control theory, coordinates of the first kind, generalized BCH-formula, Magnus expansion, etc., and is connected with iterated integrals and the signature of a path. We also show certain algebraic identities, which are meaningful in context of control and path-signature theory.

Keywords Eulerian idempotent · BCH-formula · Signature of a path

Mathematics Subject Classification (2010) 17B01 · 17B80

1 Introduction

The Eulerian idempotent $\pi_1 : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle$ can be explicitly defined as a linear endomorphism satisfying

$$\pi_1(w) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in A_+^*} (w|u_1 \sqcup \dots \sqcup u_k) u_1 \cdots u_k$$

for all $w \in A^*$. Its importance comes, however, from a more implicate (but natural) definition $\pi_1 = \log \text{Id}$, where the logarithm is taken in the algebra $\text{End}(\mathbb{K}\langle A \rangle)$ with the convolution product $*$. The name “Eulerian idempotent” emanates from the Eulerian numbers, which are related with coefficients (in symmetric group algebras) of higher order Eulerian idempotents $\pi_n = \frac{1}{n!} \pi_1^{*n}$ [1]. π_1 is interesting from many points of view. Purely mathematical aspects concern free Lie algebras, symmetric algebras, Solomon algebras, preLie algebras, etc. In this article, we focus our attention on algebraic applications in dynamical systems, control theory, coordinates of the first kind, generalized BCH-formula, Magnus expansion, etc., which are connected with iterated integrals, the signature of a path and so

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on. In detail, Sussmann [22] showed a product expansion for the solution of a non-linear control-affine system in terms of Lyndon basis of the free Lie algebra on words assigned to the controls. Then Melançon and Reutenauer [16] discovered the same expansion in purely algebraic context, and Reutenauer [20] generalized it to a Hall basis setting. This was rewritten in control-theoretic setting by Kawski and Sussmann [11]. The generalized algebraic version of these results is as follows:

$$\sum_{w \in A^*} w \otimes w = \prod_{h \in \mathcal{H}} \exp(\mathcal{S}_h \otimes \mathcal{P}_h).$$

In this formula, it is crucial that $\exp$ is taken with respect to a product $\text{sh} \otimes \text{conc}$ (see Section 2.2 for the definitions), and $\mathcal{H}$ is a Hall set (see Section 2.4 for the definitions of $\mathcal{H}$, $\mathcal{P}_h$, $\mathcal{S}_h$). After a decade, Gehrig [8] and Kawski with Gehrig [7] proved that the adjoint homomorphism $\pi'_1 : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle$ gives rise to another formula as follows:

$$\sum_{w \in A^*} w \otimes w = \exp \left(\sum_{h \in \mathcal{H}} \pi'_1(\mathcal{S}_h) \otimes \mathcal{P}_h \right)$$

with the same data as previously. They work in a context of control theory. Namely, for a control system with the following:

$$\dot{x}(t) = \sum_{i=1}^m u_i(t) f_i(x(t)) \quad x \in \mathbb{R}^n, (u_1, \dots, u_m)^T \in U \subset \mathbb{R}^m$$

the above formula gives solution as follows:

$$x(t) = \exp \left(\sum_{h \in \mathcal{H}} \phi^t \circ \pi'_1(\mathcal{S}_h) X_h \right) (x(0)),$$

where ϕ^t is a linear mapping defined for $a_1 \cdots a_k \in A_k^*$ by

$$\phi^t(a_1 \cdots a_k) = \int_0^t \cdots \int_0^{t_2} u_{a_1}(t_1) \cdots u_{a_k}(t_k) dt_1 \cdots dt_k,$$

and X_h are appropriate vector fields. In particular, for fixed controls u_i and a fixed $t > 0$, the solution $x(t)$ is an image of $x(0)$ under the t -time flow of a certain vector field. For varying t , the vector field is also t -varying. This is also connected with the theory of rough paths [12–14]. Namely, for a basis (e_i) of $\mathbb{R}^m$, take an alphabet $A = \{e_i\}$; define a linear isomorphism $\iota_\otimes : \mathbb{K}\langle A \rangle \rightarrow T^\otimes \mathbb{R}^m$ by $\iota_\otimes(e_{i_1} \cdots e_{i_k}) = e_{i_1} \otimes \cdots \otimes e_{i_k}$. The signature of a path $\gamma : [0, T] \rightarrow \mathbb{R}^m$ is as follows:

$$T^{\bar{\otimes}} \mathbb{R}^m \ni X(\gamma) = 1 + \sum_{k=1}^{\infty} \int_0^T \cdots \int_0^{t_2} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_k}$$

and Gehrig-Kawski formula gives the logarithm of the signature as follows:

$$\log X(\gamma) = \sum_{h \in \mathcal{H}} \phi_\gamma \circ \pi'_1(\mathcal{S}_h) \iota_\otimes(\mathcal{P}_h),$$

where $\phi_\gamma : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}$ is a homomorphism defined by $\phi_\gamma(P) = (X(\gamma)|_{\iota_\otimes(P)})$.

In this article, we firstly generalize Gehrig-Kawski's result to a non-Hall basis (which means to all basis) of free Lie algebra $\text{Lie}(A)$ (this is a more general answer to the Problem 1 stated in [10]). We prove that for a basis $\{P_h \mid h \in B\}$ in $\text{Lie}(A)$ and a projection

$\rho : \mathbb{K}\langle A \rangle \rightarrow \text{Lie}(A) \subset \mathbb{K}\langle A \rangle$, there exist the canonical subspace $\text{Lie}_\rho^*(A) \subset \mathbb{K}\langle\langle A \rangle\rangle$ and its basis $\{S_h \mid h \in B\}$ for which is as follows:

$$\sum_{w \in A^*} w \otimes w = \exp \left(\sum_{h \in B} \pi'_1(S_h) \otimes P_h \right).$$

We state this result in Theorem 1. After this, we give examples for this theorem choosing different basis in $\text{Lie}(A)$ and a projection ρ . In particular, we obtain the Gehrig-Kawski formula for a Hall basis.

In the second part of the article, we focus on the aforementioned mappings $\phi^t \circ \pi'_1$ and $\phi_\gamma \circ \pi'_1$ as particular cases of a mapping $\phi \circ \pi'_1$, with $\phi : \mathbb{K}_{\text{Lie}}\langle\langle A \rangle\rangle \rightarrow R$ an algebra homomorphism (R is a $\mathbb{K}$ -algebra). We show in Theorem 3 that for $w \in A_m^*$, it follows:

$$\phi \circ \pi'_1(w) = \sum_{n=1}^m \frac{(-1)^{n+1}}{n} \binom{m}{n} \phi_n(w),$$

where $\phi_n(w) = \sum_{w_1 \dots w_k = w} \phi(w_1) \dots \phi(w_k)$. From this theorem, we deduce that for a path γ , the homogeneous components of the logarithmic signature of γ can be expressed through the following:

$$\phi_\gamma \circ \pi'_1(w) = \sum_{n=1}^m \frac{(-1)^{n+1}}{n} \binom{m}{n} \phi_{\gamma^{*n}}(w),$$

defined for $w \in A_m^*$. This formula explicitly connects the m -homogeneous component of $\log X(\gamma)$ with the signatures $X(\gamma^{*n})$ ($n = 1, \dots, m$) of the n th powers concatenation of γ .

2 Preliminary

In the article, we assume $\mathbb{N}$ to be the set of natural numbers beginning from 1, $\mathbb{K}$ to be a field of characteristic 0. For a unitary associative $\mathbb{K}$ -algebra $(\text{Alg}, \bullet)$ with unit 1, we use standard notations as follows:

$$\exp(Q) = \sum_{k=0}^{\infty} \frac{1}{k!} Q^{\bullet k}, \quad \log(1 + Q) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} Q^{\bullet k}$$

for any $Q \in \text{Alg}$, for which it makes sense. Here, $Q^{\bullet k}$ is defined recursively by $Q^{\bullet 0} = 1$, and $Q^{\bullet k} = Q \bullet Q^{\bullet k-1}$ for $k \in \mathbb{N}$.

2.1 Tensor Algebra

Let V be a finite dimensional linear space over $\mathbb{K}$. For $k \in \mathbb{N}$, denote by $V^{\otimes k}$ the k th tensor product of V ; $V^{\otimes 0} = \mathbb{K}$. Let $T^{\otimes} V = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ and $T^{\bar{\otimes}} V$ be its algebraic closure. Consider $\otimes : T^{\bar{\otimes}} V \times T^{\bar{\otimes}} V \rightarrow T^{\bar{\otimes}} V$ as a product, i.e., $V^{\otimes k} \times V^{\otimes l} \ni (x, y) \mapsto x \otimes y \in V^{\otimes k+l}$. Then $(T^{\otimes} V, \otimes)$ and $(T^{\bar{\otimes}} V, \otimes)$ are associative K -algebras with $1 \in \mathbb{K} = V^{\otimes 0}$ as the unit.

2.2 Shuffle Algebra

Let A be a certain finite set of cardinality ≥ 2 , called the alphabet. Denote by A_n^* , A^* , $\mathbb{K}\langle A \rangle$, $\mathbb{K}\langle\langle A \rangle\rangle$, the set of words of length n , the set of all words, the K -algebras of non-commutative polynomials, and series in the letters A , respectively; by $1 \in A_0^*$, we denote the empty word; by $A_+^* \subset A^*$, $\mathbb{K}_+\langle A \rangle \subset \mathbb{K}\langle A \rangle$, $\mathbb{K}_+\langle\langle A \rangle\rangle \subset \mathbb{K}\langle\langle A \rangle\rangle$, the subset of non-trivial (non-empty) words, the submodule spanned on non-trivial words, and its algebraic closure, respectively. The module $\mathbb{K}\langle\langle A \rangle\rangle$ is dual to $\mathbb{K}\langle A \rangle$, and we identify $P \in \mathbb{K}\langle\langle A \rangle\rangle$ with the $\mathbb{K}$ -linear functional $(P|\cdot) : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}$ by writing $P = \sum_{w \in A^*} (P|w)w$. The product of two series $P, Q \in \mathbb{K}\langle\langle A \rangle\rangle$ is therefore defined by $(PQ|w) := \sum_{uv=w} (P|u)(Q|v)$. Since $\mathbb{K}\langle A \rangle$ is a finite-free generated algebra, the module $\mathbb{K}\langle A \rangle$ is dual to $\mathbb{K}\langle\langle A \rangle\rangle$, and we also identify $Q \in \mathbb{K}\langle A \rangle$ with $(Q|\cdot) : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}$ by $Q = \sum_{w \in A^*} (Q|w)w$. Clearly, $(P|Q) = (Q|P)$, and it also plays a role of a scalar product in $\mathbb{K}\langle A \rangle$. For a $\mathbb{K}$ -submodule $Y \subset \mathbb{K}\langle A \rangle$, we denote by $Y^\perp = \{S \in \mathbb{K}\langle\langle A \rangle\rangle \mid (S|y) = 0 \forall y \in Y\} \subset \mathbb{K}\langle\langle A \rangle\rangle$ the module of series representing functionals vanishing on Y . If $X, Y \subset \mathbb{K}\langle A \rangle$ are $\mathbb{K}$ -submodules then for an endomorphism $\rho : X \rightarrow Y$, we denote $\ker^\perp \rho := (\ker \rho)^\perp$.

Since $\mathbb{K}\langle A \rangle \subset \mathbb{K}\langle\langle A \rangle\rangle$, and $\mathbb{K}\langle\langle A \rangle\rangle$ is the algebraic closure of $\mathbb{K}\langle A \rangle$, we define all objects in the larger algebra. The concatenation product in $\mathbb{K}\langle\langle A \rangle\rangle$ has its tensorial version which we denote by $\text{conc} : \mathbb{K}\langle\langle A \rangle\rangle \otimes \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle$, i.e., $\text{conc}(P \otimes Q) = PQ$. Introduce another—shuffle—product $\sqcup : \mathbb{K}\langle\langle A \rangle\rangle \otimes \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle$ defined recursively for words by $1 \sqcup w = w \sqcup 1 = w$ for any $w \in A^*$, and

$$(w_1 a_1) \sqcup (w_2 a_2) = (w_1 \sqcup (w_2 a_2))a_1 + ((w_1 a_1) \sqcup w_2)a_2 \quad (1)$$

for all $a_1, a_2 \in A$ and $w_1, w_2 \in A^*$. In what follows, we will use another standard notation for the shuffle product $\text{sh} : \mathbb{K}\langle\langle A \rangle\rangle \otimes \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle$, i.e., $\text{sh}(w_1 \otimes w_2) = w_1 \sqcup w_2$, and its generalized version $\text{sh}_k : \mathbb{K}\langle\langle A \rangle\rangle^{\otimes k} \rightarrow \mathbb{K}\langle\langle A \rangle\rangle$ given by the following:

$$\text{sh}_k(w_1 \otimes \cdots \otimes w_k) := w_1 \sqcup \cdots \sqcup w_k \quad (2)$$

for $w_1, \dots, w_k \in A^*$. We denote by $\mathbb{K}_\sqcup\langle A \rangle$ (resp. $\mathbb{K}_\sqcup\langle\langle A \rangle\rangle$) the commutative $\mathbb{K}$ -algebras of non-commutative polynomials (resp. series) in the letters A with $\sqcup$ as the product. The adjoint coproduct of the shuffle product $\delta = \delta_2 : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle \otimes \mathbb{K}\langle\langle A \rangle\rangle$ and its generalized version $\delta_k : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle^{\otimes k}$ are defined by the following:

$$\delta_k(w) := \sum_{w_1, \dots, w_k \in A^*} (w|w_1 \sqcup \cdots \sqcup w_k) w_1 \otimes \cdots \otimes w_k.$$

Similarly, we define the adjoint coproduct of the concatenation $\delta' = \delta'_2 : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle \otimes \mathbb{K}\langle\langle A \rangle\rangle$ and its generalized version $\delta'_k : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle^{\otimes k}$ by the following:

$$\delta'_k(w) := \sum_{w_1 \cdots w_k = w} w_1 \otimes \cdots \otimes w_k, \quad (3)$$

where the sum is taken over all $w_i \in A^*$. Note that $(\mathbb{K}\langle A \rangle, \text{conc}, \delta)$ and $(\mathbb{K}\langle\langle A \rangle\rangle, \text{sh}, \delta')$ are mutually adjoint bialgebras [20, Prop. 1.9].

2.3 Free Lie Algebra on A

Let $[\cdot, \cdot] : \mathbb{K}\langle A \rangle \times \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle$ be the standard Lie bracket, i.e., a bilinear mapping given by $[P, Q] := PQ - QP$. We denote by $\text{Lie}(A)$ the smallest $\mathbb{K}$ -submodule of $\mathbb{K}\langle A \rangle$, which

contains A , and is closed under the Lie bracket, i.e., the free Lie algebra generated by A , and by $\text{Lie}(\langle A \rangle)$ its algebraic closure. By $\text{Lie}^*(A)$, we denote the module of $\mathbb{K}$ -linear functionals $\text{Lie}(A) \rightarrow \mathbb{K}$. Denote by $(S \lfloor X)$ the action of a $\mathbb{K}$ -homomorphism $S : \text{Lie}(A) \rightarrow \mathbb{K}$ on an element $X \in \text{Lie}(A)$. Let B be an totally ordered set, and $\{P_h \mid h \in B\}$ be a basis in $\text{Lie}(A)$. Let $\{S_h \mid h \in B\}$ be the dual basis of $\text{Lie}^*(A)$, i.e., the one defined by the following:

$$X = \sum_{h \in B} (S_h \lfloor X) P_h$$

for any $X \in \text{Lie}(A)$.

2.4 Hall Sets & Basis

Let $\mathcal{M}(A)$ be the set of binary, complete, planar, rooted trees with leaves labeled by A . Each such tree can be naturally identified with the unique expression in the set $\mathcal{E}(A)$ defined by the following two conditions: (i) if $a \in A$, then $a \in \mathcal{E}(A)$, and (ii) if $t, t' \in \mathcal{E}(A)$, then $(t, t') \in \mathcal{E}(A)$. In the sequel, we will not distinguish between these sets, i.e., we assume $\mathcal{M}(A) = \mathcal{E}(A)$. Define the mapping $\mathcal{F}(\cdot) : \mathcal{M}(A) \rightarrow A^*$, which assigns to a tree $t \in \mathcal{M}(A)$, the word given by dropping all brackets in it, i.e., $\mathcal{F}(a) = a$ for all $a \in A$, and $\mathcal{F}((t, t')) = \mathcal{F}(t)\mathcal{F}(t')$ for all $t, t' \in \mathcal{M}(A)$. The word $\mathcal{F}(t)$ is called the foliage of $t \in \mathcal{M}(A)$. Define also the mapping $\mathcal{P} : \mathcal{M}(A) \rightarrow \text{Lie}(A)$, which changes the rounded brackets into the Lie brackets, i.e., $\mathcal{P}_a = a$ for all $a \in A$, and $\mathcal{P}_{(t, t')} := [\mathcal{P}_t, \mathcal{P}_{t'}]$ for all $t, t' \in \mathcal{M}(A)$. We will generalize this definition in the sequel. A Hall set $\mathcal{H}$ on the letters A [9]¹, is a subset of $\mathcal{M}(A)$ totally ordered by $\leq$ and satisfying:

- (I) $A \subset \mathcal{H}$;
- (II) if $h = (h', h'') \in \mathcal{H} \setminus A$, then $h'' \in \mathcal{H}$ and $h' < h''$;
- (III) for all $h = (h', h'') \in \mathcal{M}(A) \setminus A$ we have $h \in \mathcal{H}$ iff
 - $h', h'' \in \mathcal{H}$ and $h' < h''$, and
 - $h' \in A$ or $h' = (x, y)$ such that $y \geq h''$.

Fix a Hall set $\mathcal{H}$ on the letters A totally ordered by $\leq$. Each Hall tree $h \in \mathcal{H}$ corresponds to a word $\mathcal{F}(h) \in A^*$ called a Hall word. Denote by $\mathcal{W}$, the set of Hall words with ordering $\leq$ inherited from the ordering on $\mathcal{H}$ in the natural way. Each word $w \in A^*$, is the unique concatenation of a unique non-increasing series of Hall words, that is, $w = h_1 \cdots h_k$ for some unique $k \in \mathbb{N}$, and $h_i \in \mathcal{W}$ such that $h_1 \geq \cdots \geq h_k$. Let $\mathcal{P} : A^* \rightarrow \mathbb{K}\langle A \rangle$ be the mapping defined by the following

- (i) $\mathcal{P}_1 := 1$;
- (ii) $\mathcal{P}_a := a$ for $a \in A$;
- (iii) $\mathcal{P}_h := \mathcal{P}_t \in \text{Lie}(A)$ for $h \in \mathcal{W}$ such that $h = \mathcal{F}(t)$, $t \in \mathcal{H} \subset \mathcal{M}(A)$;
- (iv) $\mathcal{P}_w := \mathcal{P}_{h_1} \cdots \mathcal{P}_{h_k} \in \mathbb{K}\langle A \rangle$ for $w = h_1 \cdots h_k$, where $k \in \mathbb{N}$ and $h_i \in \mathcal{W}$ such that $h_1 \geq \cdots \geq h_k$.

The set $\{\mathcal{P}_h \in \text{Lie}(A) \mid h \in \mathcal{W}\}$ is the Hall basis of $\text{Lie}(A)$ corresponding to the Hall set $\mathcal{H}$. By the Poincaré-Birkhoff-Witt theorem, the set of ordered products $\mathcal{P}_{h_1} \cdots \mathcal{P}_{h_k}$, where $h_1 \geq \cdots \geq h_k$ are Hall words, creates a basis for the enveloping algebra of $\text{Lie}(A)$, which

¹A Hall set should probably be called a Hall-Shirshov-Viennot set because of an important contribution of the other two authors to this theory [21, 23].

in the free case is isomorphic to $\mathbb{K}_+\langle A \rangle$. Therefore, $\{\mathcal{P}_w \mid w \in A^*\}$ is a basis in $\mathbb{K}\langle A \rangle$. Consider the dual basis $\{\mathcal{S}_w \mid w \in A^*\}$ in $\mathbb{K}\langle\langle A \rangle\rangle$.

Proposition 1 ([20, Theorem 5.3]) (i) $\mathcal{S}_1 = 1$;

(ii) If $h = av \in \mathcal{W}$ is a Hall word, where $a \in A$, $v \in A^*$, then $\mathcal{S}_h = a\mathcal{S}_v$;

(iii) If $w = h_1^{i_1} \cdots h_k^{i_k} \in A^*$ is any word, where $h_1 > \cdots > h_k$ are Hall words and $i_1, \dots, i_k \in \mathbb{N}$, then

$$\mathcal{S}_w = \frac{1}{i_1! \cdots i_k!} \mathcal{S}_{h_1}^{\sqcup i_1} \sqcup \cdots \sqcup \mathcal{S}_{h_k}^{\sqcup i_k}.$$

One of the consequences of this proposition is that $\mathbb{K}_{\sqcup}\langle\langle A \rangle\rangle$ is the algebraic closure of the free commutative algebra over $\{\mathcal{S}_h \mid h \in \mathcal{W}\}$. In particular, $\mathcal{S}_{h_1} \sqcup \cdots \sqcup \mathcal{S}_{h_k}$, where $h_1 \geq \cdots \geq h_k$ are Hall words, creates a basis for $\mathbb{K}_+\langle\langle A \rangle\rangle$.

2.5 Algebra of Endomorphisms $\text{End}(\mathbb{K}\langle A \rangle)$

Consider $\text{End}(\mathbb{K}\langle A \rangle)$ —the $\mathbb{K}$ -module of linear endomorphisms from $\mathbb{K}\langle A \rangle$ to $\mathbb{K}\langle A \rangle$. For $f, g \in \text{End}(\mathbb{K}\langle A \rangle)$ define their convolution product as follows:

$$f * g = \text{conc} \circ (f \otimes g) \circ \delta \in \text{End}(\mathbb{K}\langle A \rangle).$$

Let $\epsilon : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle$ be a projection $\epsilon(Q) = (Q|1)1$. Then, $(\text{End}(\mathbb{K}\langle A \rangle), *)$ is an associative $\mathbb{K}$ -algebra with unit ϵ . Introduce the complete tensor product as follows:

$$\mathcal{A} = \mathbb{K}\langle\langle A \rangle\rangle \bar{\otimes} \mathbb{K}\langle A \rangle$$

and a product $\bar{*} = \text{sh} \otimes \text{conc} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, i.e.,

$$(P_1 \otimes Q_1) \bar{*} (P_2 \otimes Q_2) = \text{sh}(P_1 \otimes P_2) \otimes \text{conc}(Q_1 \otimes Q_2),$$

for $P_1, P_2 \in \mathbb{K}\langle\langle A \rangle\rangle$, $Q_1, Q_2 \in \mathbb{K}\langle A \rangle$. Then, $(\mathcal{A}, \bar{*})$ is an associative $\mathbb{K}$ -algebra with unit $\bar{1} = 1 \otimes 1$. The canonical isomorphism of modules $\text{Im} : \text{End}(\mathbb{K}\langle A \rangle) \rightarrow \mathcal{A}$ given by the following:

$$\text{Im}(f) = \sum_{u \in A^*} u \otimes f(u)$$

is a homomorphism of algebras $(\text{End}(\mathbb{K}\langle A \rangle), *)$ and $(\mathcal{A}, \bar{*})$, i.e.,

$$\sum_{u \in A^*} u \otimes f * g(u) = \left(\sum_{u \in A^*} u \otimes f(u) \right) \bar{*} \left(\sum_{u \in A^*} u \otimes g(u) \right).$$

Note that in the definition of Im , we choose the most natural basis $\{u \mid u \in A^*\}$ in $\mathbb{K}\langle A \rangle$ and its dual basis $\{u \mid u \in A^*\}$ in $\mathbb{K}\langle\langle A \rangle\rangle$, but in general one can take a different basis and its dual.

3 Eulerian Idempotent and its Adjoint

Let $I : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle$ be a projection given by $I(Q) = Q - \epsilon(Q)$. We are particularly interested in an endomorphism $\pi_1 : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle$,

$$\pi_1 = \log \text{Id} = \log(\epsilon + I) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} I^{*k}.$$

called the Eulerian idempotent. A straightforward calculation shows that $\text{Im}(\pi_1)$ equals

$$\sum_{w \in A^*} w \otimes \pi_1(w) = \log \left(\sum_{w \in A^*} w \otimes w \right).$$

It is known that π_1 is a projection, $\pi_1(\mathbb{K}\langle A \rangle) = \text{Lie}(A)$, and $(P \sqcup Q | \pi_1(R)) = 0$ for all $P, Q, R \in \mathbb{K}_+ \langle A \rangle$. The second statement means $\ker^\perp \pi_1 = \text{span} \{ v \sqcup w \mid v, w \in A_+^* \}$. From Proposition 1, we conclude that

$$\left\{ S_{h_1} \sqcup \cdots \sqcup S_{h_k} \mid k \geq 2, h_1 \geq \cdots \geq h_k \in \mathcal{W} \right\}$$

is a basis of this space. The kernel of π_1 have another characterization: $\ker \pi_1 = \bigcup_{k \geq 2} \pi_1^{*k}(\mathbb{K}\langle A \rangle)$, where $\pi_1^{*k}(\mathbb{K}\langle A \rangle) = \{ P^k \mid P \in \text{Lie}(A) \}$ [20, Thm. 3.7].

The main results in this article concern $\pi'_1 : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow \mathbb{K}\langle\langle A \rangle\rangle$ —the adjoint endomorphism to π_1 , i.e., the one defined by the following;

$$(\pi'_1(P) | Q) = (P | \pi_1(Q))$$

for all $P \in \mathbb{K}\langle\langle A \rangle\rangle$, $Q \in \mathbb{K}\langle A \rangle$. A straightforward calculation using adjointness of conc_k and δ'_k , adjointness of sh_k and δ_k , and self-adjointness of I , brings to the following formula

$$\pi'_1 = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \text{sh}_k \circ I^{\otimes k} \circ \delta'_k.$$

Since π_1 is a projection, the same is true for π'_1 . In particular, $\pi'_1(\mathbb{K}\langle\langle A \rangle\rangle) = \ker^\perp \pi_1$ and $\ker \pi'_1 = \pi_1(\mathbb{K}\langle A \rangle)^\perp = (\text{Lie}(A))^\perp$, which we denote by $\text{Lie}^\perp(A)$.

Since π_1 is onto $\text{Lie}(A)$, we also consider its adjoint as an epimorphism. Namely, for $\bar{\pi}_1 : \mathbb{K}\langle A \rangle \rightarrow \text{Lie}(A)$, $\bar{\pi}_1(P) := \pi_1(P)$, we introduce its adjoint endomorphism $\bar{\pi}'_1 : \text{Lie}^*(A) \rightarrow \mathbb{K}\langle\langle A \rangle\rangle$. It is a $\mathbb{K}$ -linear monomorphism onto $\ker^\perp \pi_1$ satisfying the following:

$$(\alpha | \pi_1(Q)) = (\bar{\pi}'_1(\alpha) | Q)$$

for all $\alpha \in \text{Lie}^*(A)$, $Q \in \mathbb{K}\langle A \rangle$.

Proposition 2 *Let $\{P_h \mid h \in B\}$ and $\{S_h \mid h \in B\}$ be bases in $\text{Lie}(A)$ and its dual in $\text{Lie}^*(A)$, respectively. It follows that*

$$\mathcal{A} \ni \text{Im}(\text{Id}) = \exp \left(\sum_{h \in B} \bar{\pi}'_1(S_h) \otimes P_h \right).$$

Proof We prove that

$$\log \left(\sum_{w \in A^*} w \otimes w \right) = \sum_{h \in B} \bar{\pi}'_1(S_h) \otimes P_h,$$

Since $\pi_1(\mathbb{K}\langle A \rangle) \subset \text{Lie}(A)$ it follows that

$$\begin{aligned} \log \left(\sum_{w \in A^*} w \otimes w \right) &= \sum_{w \in A^*} w \otimes \pi_1(w) = \sum_{w \in A^*} w \otimes \sum_{h \in B} (S_h \lfloor \pi_1(w)) P_h \\ &= \sum_{w \in A^*} w \otimes \sum_{h \in B} (\tilde{\pi}'_1(S_h) | w) P_h = \sum_{h \in B} \left(\sum_{w \in A^*} (\tilde{\pi}'_1(S_h) | w) w \right) \otimes P_h \\ &= \sum_{h \in A^*} \tilde{\pi}'_1(S_h) \otimes P_h. \end{aligned}$$

This ends the proof. $\square$

Choose a basis $\{P_h \mid h \in B\}$ in $\text{Lie}(A)$, and a projection $\rho : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle$ on $\text{Lie}(A)$. Then, $\text{Lie}^*(A)$ is naturally identified with $\text{Lie}^*_\rho(A) := \ker^\perp \rho \subset \mathbb{K}\langle\langle A \rangle\rangle$ by the isomorphism $\iota_\rho : \text{Lie}^*_\rho(A) \rightarrow \text{Lie}^*(A)$ given by $(\iota_\rho(S) \rfloor P) = (S \rfloor P)$. In $\text{Lie}^*_\rho(A)$, there exists the dual basis $\{S_h \mid h \in B\}$ to the one in $\text{Lie}(A)$ given by the following:

$$\begin{aligned} (S_h \rfloor Q) &= (S_h \rfloor \rho(Q)) & \forall Q \in K\langle A \rangle, \\ P &= \sum_{h \in B} (S_h \rfloor P) P_h & \forall P \in \text{Lie}(A). \end{aligned} \quad (4)$$

Theorem 1 *Let $\{P_h \mid h \in B\}$ a basis in $\text{Lie}(A)$, $\rho : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle$ a projection on $\text{Lie}(A)$, and $\{S_h \mid h \in B\}$ the basis in $\text{Lie}^*_\rho(A)$ given by (4). It follows that*

$$\mathcal{A} \ni \text{Im}(\text{Id}) = \exp \left(\sum_{h \in B} \pi'_1(S_h) \otimes P_h \right).$$

Proof Using Proposition 2 for $\{P_h \mid h \in B\}$ and $\{\iota_\rho(S_h) \mid h \in B\}$ we get

$$\text{Im}(\text{Id}) = \exp \left(\sum_{h \in B} \tilde{\pi}'_1(\iota_\rho(S_h)) \otimes P_h \right).$$

Since $(\iota_\rho(\alpha) \rfloor \pi_1(Q)) = (\alpha \rfloor \pi_1(Q))$ for all $\alpha \in \text{Lie}^*_\rho(A)$, $Q \in \mathbb{K}\langle A \rangle$, we have $\tilde{\pi}'_1 \circ \iota_\rho = \pi'_1$ on $\text{Lie}^*_\rho(A)$. This ends the proof. $\square$

A few remarks are in order. First of all, we recall that in the algebra $\mathcal{A}$ the product is $\text{sh} \otimes \text{conc}$, so the shuffle product is used to compute the left side of the tensor product. It means that good properties of $\pi'_1(S_h)$ with respect to sh are welcome. Secondly, the proved formula is similar to the quite clear formula (6.2.1) in Reutenauer's book [20] (in which the sum is taken over all words). In our case, however, the sum is taken over the basis of the Lie algebra, which contains essential information about the logarithm of a series [4, 18]. Therefore, this theorem generalizes the Gehrig-Kawski theorem (see Theorem 2 beneath) to its most extent. The advantage of the theorem is that it can be used to compute BCH-formula, Magnus expansion, logarithm of the signature, coordinates of the second kind, etc., both in general case, as well as for particular situations. In each

case, one can try to choose a basis in $\text{Lie}(A)$ and ρ to utilize specific features of a given problem.

There are several natural choices for ρ . The first one is $\rho = \pi_1$. Recall that $\ker \pi_1 = \{P^k | k \geq 2, P \in \text{Lie}(A)\} \oplus \mathbb{K}1$ and $\text{Lie}_{\pi_1}^*(A) = \ker^\perp \pi_1 = \text{span} \{v \sqcup w | v, w \in A_+^*\}$ in which there are natural basis

$$\left\{ S_{h_1} \sqcup \cdots \sqcup S_{h_k} \mid k \geq 2, h_1 \geq \cdots \geq h_k \in \mathcal{W} \right\}.$$

written in terms of the dual elements to a Hall basis in $\text{Lie}(A)$ [17, Th. 3.1.1]. The description of its dual in $\text{Lie}(A)$ is unknown.

The second one is to take $\rho = \rho_\perp$ as the orthogonal projection with respect to the scalar product $(\cdot | \cdot)$ in $\mathbb{K}\langle A \rangle$ (for more details about this projection see [5]). In this case, $\text{Lie}_{\rho_\perp}^*(A) = \ker^\perp \rho_\perp = \text{Lie}(\langle A \rangle)$. In particular, $\rho_\perp \neq \pi_1$. Take a basis $\{P_h \mid h \in B\}$ in $\text{Lie}(A)$ on a well-ordered set B . Using Gram-Schmidt process obtain the orthonormal basis $\{\hat{P}_h \mid h \in B\}$. Then, this is also the dual basis in $\text{Lie}(A)$. Therefore, from the above

theorem, we get $\text{Im}(\text{Id}) = \exp \left(\sum_{h \in B} \pi_1'(\hat{P}_h) \otimes \hat{P}_h \right)$. The same is clearly true for any orthonormal basis in $\text{Lie}(A)$.

The third one is to take $\rho = \rho_{\text{PBW}}$ as the projection with kernel derived from Poincaré-Birkhoff-Witt theorem. More precisely, universal enveloping algebra of the free Lie algebra $\text{Lie}(A)$ is $\mathbb{K}\langle A \rangle$. If $\{P_h \mid h \in B\}$ is a basis in $\text{Lie}(A)$ then, using Poincaré-Birkhoff-Witt theorem, the set of ordered products $P_{h_1} \cdots P_{h_k}$, where $h_1 \geq \cdots \geq h_k \in B$, is a basis for the enveloping algebra of $\text{Lie}(A)$, i.e., in $\mathbb{K}_+ \langle \langle A \rangle \rangle$. Denote by $S_{h_1, \dots, h_k}$, $h_1 \geq \cdots \geq h_k \in B$, the elements of the dual basis in $\mathbb{K}_+ \langle \langle A \rangle \rangle$. We define ρ_{PBW} as a projection on $\text{Lie}(A)$ with the kernel $\ker \rho_{\text{PBW}} = \text{span} \{P_{h_1} \cdots P_{h_k} \mid k \geq 2, h_1 \geq \cdots \geq h_k \in B\} \oplus \mathbb{K}1$. From PBW theorem, the elements written in this formula are linearly independent, hence creates a basis of $\ker \rho_{\text{PBW}}$. On the dual side, $\ker^\perp \rho_{\text{PBW}} = \text{span} \{S_h \mid h \in B\}$, but in general case, there is not known explicit formulas for S_h . It follows that $\ker \rho_{\text{PBW}} \neq \ker \pi_1$, because for $h > h' \in B$, we have $\ker \pi_1 \ni (P_h + P_{h'})^2 = P_h^2 + P_{h'}^2 + 2P_h P_{h'} - [P_h, P_{h'}]$, but $\rho_{\text{PBW}}((P_h + P_{h'})^2) = -[P_h, P_{h'}]$. Therefore, $\rho_{\text{PBW}} \neq \pi_1$. It also follows that $\ker^\perp \rho_{\text{PBW}} \neq \ker^\perp \rho_\perp$, because for $h > h' \in B$, we have $(P_h P_{h'} | [P_h, P_{h'}]) = (P_h P_{h'} | P_h P_{h'}) - (P_h P_{h'} | P_{h'} P_h) > 0$, since $P_h P_{h'}, P_{h'} P_h$ are linearly independent and have the same norm. Therefore, $\rho_{\text{PBW}} \neq \rho_\perp$. Note that each choice of the basis in $\text{Lie}(A)$ gives a different projection ρ_{PBW} , so in this case, we actually define a class of examples.

The fourth one, a subclass of the previous one, is to take $\rho = \rho_{\text{Hall}}$, the projection derived as above taking a Hall basis in $\text{Lie}(A)$. Recall that we described these basis in Section 2.4. In this case, Theorem 1 is equivalent to the following Gehrig-Kawski theorem.

Theorem 2 ([8] Theorem 28) *Let $\mathcal{W}$ be a set of Hall words on the letters A . It follows that*

$$\sum_{w \in A^*} w \otimes w = \exp \left(\sum_{h \in \mathcal{W}} \pi_1'(S_h) \otimes P_h \right) \in \mathcal{A}.$$

In Tables 1 and 2, we gather known informations about $\ker \rho$, $\ker^\perp \rho = \text{Lie}_\rho^*(A)$ and basis in $\text{Lie}(A)$ and its dual in $\text{Lie}_\rho^*(A)$ associated with projections ρ just described.

Table 1 Kernel and dual space associated with a projection ρ on $\text{Lie}(A)$

ρ	$\ker \rho$	$\ker^\perp \rho = \text{Lie}_\rho^*(A)$
π_1	$\{P^k \mid k \geq 2, P \in \text{Lie}(A)\} \oplus \mathbb{K}1$	$\text{span} \{v \sqcup w \mid v, w \in A_+^*\}$
$\rho_\perp$	?	$\text{Lie}(A)$
ρ_{PBW}	$\text{span} \left\{ P_{h_1} \cdots P_{h_k} \mid \begin{matrix} k \geq 2, \\ h_1 \geq \cdots \geq h_k \in B \end{matrix} \right\} \oplus \mathbb{K}1$	$\text{span} \{S_h \mid h \in B\}$
ρ_{Hall}	$\text{span} \{\mathcal{P}_w \mid w \in A^* \setminus \mathcal{W}\}$	$\text{span} \{S_h \mid h \in \mathcal{W}\}$

4 Computing $\phi \circ \pi'_1$

Let (R, μ) be a $\mathbb{K}$ -algebra with a multiplication $\mu : R \otimes R \rightarrow R$. If there is no confusion, we will use standard notation $ab = \mu(a \otimes b)$. We also recursively introduce $\mu_k : R^{\otimes k} \rightarrow R$ by $\mu_1(a) = a$, $\mu_2 = \mu$, $\mu_{k+1} = \mu \circ (\mu_k \otimes \mu_1)$. Let $(R', \cdot)$ be an K -algebra with a multiplication $\cdot : R' \times R' \rightarrow R'$ (we use simpler definition of this product, because we will not use its tensorial properties).

Let $\phi : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow R$ and $\psi : \mathbb{K}\langle A \rangle \rightarrow R'$ be algebra homomorphisms. Then, it is easy to see that

$$\phi \otimes \psi \left(\exp \left(\sum_{h \in B} \pi'_1(S_h) \otimes P_h \right) \right) = \exp \left(\sum_{h \in B} \phi \circ \pi'_1(S_h) \otimes \psi(P_h) \right).$$

From Theorem 1, we conclude that

$$\phi \otimes \psi(\text{Im}(\text{Id})) = \exp \left(\sum_{h \in B} \phi \circ \pi'_1(S_h) \otimes \psi(P_h) \right).$$

Our aim is to give expression for $\phi \circ \pi'_1$. For $k \in \mathbb{N}$, let us define a linear mapping $\phi_k : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow R$, for $v \in A^*$ given by the following:

$$\phi_k(v) = \sum_{v_1 \cdots v_k = v} \phi(v_1) \cdots \phi(v_k), \quad (5)$$

where the sum is taken over all $v_i \in A^*$. We emphasize the existence of the empty word $1 \in A^*$. In particular $\phi_1(v) = \phi(v)$.

Recall $I = \text{Id} - \epsilon : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle$, and

$$\pi'_1 = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \text{sh}_k \circ I^{\otimes k} \circ \delta'_k,$$

Table 2 Basis in $\text{Lie}(A)$ and its dual in $\text{Lie}_\rho^*(A)$ associated with a projection ρ on $\text{Lie}(A)$

ρ	Basis in $\text{Lie}(A)$	Dual basis in $\text{Lie}_\rho^*(A)$
π_1	?	$\{S_{h_1} \sqcup \cdots \sqcup S_{h_k} \mid k \geq 2, h_1 \geq \cdots \geq h_k \in \mathcal{W}\}$
$\rho_\perp$	Any orthonormal one	The same one
ρ_{PBW}	$\{P_h \mid h \in B\}$	$\{S_h \mid h \in B\} = ?$
ρ_{Hall}	$\{\mathcal{P}_h \mid h \in \mathcal{W}\}$	$\{S_h \mid h \in \mathcal{W}\}$

where sh_k and δ'_k are defined by (2) and (3), respectively. Therefore,

$$\phi \circ \pi'_1 = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \mu_k \circ (\phi \circ I)^{\otimes k} \circ \delta'_k.$$

For $k \in \mathbb{N}$, let us define linear mappings $\tilde{\phi}_k : \mathbb{K}\langle\langle A \rangle\rangle \rightarrow R$, for $v \in A^*$ given by the following:

$$\tilde{\phi}_k(v) = \sum_{u_1 \cdots u_k = v} \phi(u_1) \cdots \phi(u_k),$$

where the sum is taken over all $u_i \in A_+^*$ —the set of non-trivial words. Since I annihilates empty word, it is easy to see that

$$\mu_k \circ (\phi \circ I)^{\otimes k} \circ \delta'_k(v) = \tilde{\phi}_k(v).$$

We conclude that

$$\phi \circ \pi'_1 = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \tilde{\phi}_k(v).$$

In the following lemma, we derive $\tilde{\phi}_k$'s in terms of ϕ_k 's.

Lemma 4.1 *It follows that*

$$\tilde{\phi}_k = \sum_{n=1}^k (-1)^{k-n} \binom{k}{n} \phi_n.$$

Proof Since in the definition of ϕ_k 's, we sum over all words and in the definition of $\tilde{\phi}_k$'s, we sum over all non-empty words, we see that

$$\phi_k = \tilde{\phi}_k + \binom{k}{1} \tilde{\phi}_{k-1} + \cdots + \binom{k}{k-1} \tilde{\phi}_1. \quad (6)$$

Now, we use induction on k to prove the hypothesis. For $k = 1$, this is clear. Then, using (6) and then induction hypothesis, we have the following:

$$\begin{aligned} \tilde{\phi}_{k+1} &= \phi_{k+1} - \sum_{m=1}^k \binom{k+1}{n} \tilde{\phi}_{k+1-m} \\ &= \phi_{k+1} - \sum_{m=1}^k \binom{k+1}{n} \sum_{n=1}^{k+1-m} (-1)^{k+1-m-n} \binom{k+1-m}{n} \phi_n. \end{aligned}$$

Since $\binom{k+1}{m} \binom{k+1-m}{n} = \binom{k+1}{n} \binom{k+1-n}{m}$, and changing the order of summation, we have the following:

$$\tilde{\phi}_{k+1} = \phi_{k+1} + \sum_{n=1}^k (-1)^{k-n} \binom{k+1}{n} \left[\sum_{m=1}^{k+1-n} (-1)^m \binom{k+1-n}{m} \right] \phi_n.$$

The expression in the square brackets equals -1 , and we are done. $\square$

Using this lemma, we see that

$$\phi \circ \pi'_1 = \sum_{k \geq 1} \sum_{n=1}^k \frac{(-1)^{n+1}}{k} \binom{k}{n} \phi_n.$$

If $v_m \in A_m^*$, then clearly $\tilde{\phi}_n(v_m) = 0$ for all $n > m$ (since we can not divide m -letter word on n non-trivial words). It means that

$$\phi \circ \pi'_1(v_m) = \sum_{k=1}^m \frac{(-1)^{k+1}}{k} \tilde{\phi}_k(v_m) = \sum_{k=1}^m \sum_{n=1}^k \frac{(-1)^{n+1}}{k} \binom{k}{n} \phi_n(v_m).$$

After changing the order of summation, we get

$$\phi \circ \pi'_1(v_m) = \sum_{n=1}^m (-1)^{n+1} \sum_{k=n}^m \frac{1}{k} \binom{k}{n} \phi_n(v_m).$$

A simple induction on m shows that

$$\sum_{k=n}^m \frac{1}{k} \binom{k}{n} = \frac{1}{n} \binom{m}{n},$$

and therefore

$$\phi \circ \pi'_1(v_m) = \sum_{n=1}^m \frac{(-1)^{n+1}}{n} \binom{m}{n} \phi_n(v_m).$$

The above reasoning brings us to the following theorem.

Theorem 3 Let $S \in \mathbb{K}_+(\langle A \rangle)$, and assume $S = \sum_{m \geq 1} S_m$, where S_m is the homogeneous part spanned by words of length m , i.e., $S_m = \sum_{v \in A_m^*} (S)v$. Then,

$$\begin{aligned} \phi \circ \pi'_1(S) &= \sum_{m \geq 1} \sum_{n=1}^m \frac{(-1)^{n+1}}{n} \binom{m}{n} \phi_n(S_m) \\ &= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{(-1)^{n+1}}{n} \binom{m}{n} \phi_n(S_m). \end{aligned}$$

5 Signature of a Path

Let γ be a continuous path with finite variation in a finite dimensional linear space V over $\mathbb{K} = \mathbb{R}$ equipped with the metric d_V . More precisely, take $T > 0$ and let $t = (t_1, \dots, t_r)$, such that $0 = t_0 < t_1 < \dots < t_r \leq T$ and $\#t := r \in \mathbb{N}$. Denote the set of all such tuples by $\mathcal{P}$. For a continuous mapping $\gamma : [0, T] \rightarrow V$ its length is defined by the following:

$$|\gamma| = \sup_{\mathbf{a} \in \mathcal{P}} \sum_{i=1}^{\#\mathbf{a}} d_V(\gamma_{t_i}, \gamma_{t_{i-1}}).$$

If $|\gamma| < +\infty$, then it is of finite variation. In particular, this implies that γ is differentiable almost everywhere and

$$\gamma_t = \int_0^t d\gamma_\tau,$$

for $t \in [0, T]$. Recursively define $X_k^t(\gamma) \in V^{\otimes k}$ for $k \in \mathbb{N}$, $t \in [0, T]$, by

$$X_1^t(\gamma) = \int_0^t d\gamma_\tau, \quad X_{k+1}^t(\gamma) = \int_0^t X_k^\tau(\gamma) \otimes d\gamma_\tau.$$

Denote $X_k(\gamma) = X_k^T(\gamma)$ for $k > 0$, and $X_0(\gamma) = 1$ the neutral element in $T^{\otimes}V$. The signature of the path [13] γ is

$$X(\gamma) = 1 + X_1(\gamma) + X_2(\gamma) + X_3(\gamma) + \dots = \sum_{k \geq 0} X_k(\gamma)$$

in the tensor algebra $T^{\otimes}V$.

In the space of continuous paths with finite variation, we introduce a natural concatenation product. Namely, for two paths $\gamma : [0, T] \rightarrow V$, $\tilde{\gamma} : [0, \tilde{T}] \rightarrow V$, we define the concatenation of these paths $\gamma \star \tilde{\gamma} : [0, T + \tilde{T}] \rightarrow V$ by

$$\gamma \star \tilde{\gamma} = \begin{cases} \gamma(t) & t \in [0, T] \\ \tilde{\gamma}(t - T) - \tilde{\gamma}(0) + \gamma(T) & t \in [T, T + \tilde{T}] \end{cases}.$$

It follows from [2] that

$$X(\gamma \star \tilde{\gamma}) = X(\gamma) \otimes X(\tilde{\gamma}).$$

for any paths $\gamma, \tilde{\gamma}$. In particular, this means that for each $k \in \mathbb{N}$

$$X_k(\gamma \star \tilde{\gamma}) = X_k(\gamma) + X_{k-1}(\gamma) \otimes X_1(\tilde{\gamma}) + \dots + X_1(\gamma) \otimes X_{k-1}(\tilde{\gamma}) + X_k(\tilde{\gamma})$$

Clearly, this can be generalized to $X(\gamma \star \dots \star \tilde{\gamma}) = X(\gamma) \otimes \dots \otimes X(\tilde{\gamma})$. We are particularly interested in a case $\gamma = \dots = \tilde{\gamma}$, in which we use a notation $\gamma \star \dots \star \gamma = \gamma^{\star k}$ for the k -times concatenation of γ .

Take a basis (e_i) of V and take an alphabet A consisting of the basis elements, i.e., $A = \{e_i\}$. Define a natural algebra homomorphism $\iota_{\otimes} : \mathbb{K}\langle A \rangle \rightarrow T^{\otimes}V$ given by $\iota_{\otimes}(e_{i_1} \dots e_{i_k}) = e_{i_1} \otimes \dots \otimes e_{i_k}$. Introduce a scalar product $(\cdot | \cdot) : T^{\otimes}V \times T^{\otimes}V \rightarrow \mathbb{K}$ for which $\{\iota_{\otimes}(w) \mid w \in A^*\}$ is an orthonormal basis, and define a linear homomorphism $\phi_{\gamma} : \mathbb{K}_{\sqcup} \langle \langle A \rangle \rangle \rightarrow \mathbb{K}$ given by $\phi_{\gamma}(e_{i_1} \dots e_{i_k}) = (X(\gamma) | e_{i_1} \otimes \dots \otimes e_{i_k})$ and $\phi_{\gamma}(1) = 1$.

Assuming $\gamma_t = \sum_i \gamma_t^i e_i$ we see that

$$\phi_{\gamma}(e_{i_1} \dots e_{i_k}) = \int_0^T \int_0^{\tau_k} \dots \int_0^{\tau_2} d\gamma_{\tau_1}^{i_1} \dots d\gamma_{\tau_k}^{i_k}.$$

Chen [3] proved that ϕ_{γ} is a shuffle algebra homomorphism, i.e., $\phi_{\gamma}(v \sqcup w) = \phi_{\gamma}(v)\phi_{\gamma}(w)$ for $v, w \in A^*$. This means we can apply Theorem 2 in which we express $\phi_{\gamma} \circ \pi'$ in terms of $(\phi_{\gamma})_k$ (defined in (5)). Now observe that

$$\begin{aligned} (\phi_{\gamma})_k(v) &= \sum_{v_1 \dots v_k = v} \phi_{\gamma}(v_1) \dots \phi_{\gamma}(v_k) \\ &= \sum_{v_1 \dots v_k = v} (X(\gamma) | \iota_{\otimes}(v_1)) \dots (X(\gamma) | \iota_{\otimes}(v_k)) \\ &= (X(\gamma) \otimes \dots \otimes X(\gamma) | \iota_{\otimes}(v)) \\ &= (X(\gamma^{\star k}) | \iota_{\otimes}(v)) = \phi_{\gamma^{\star k}}(v). \end{aligned}$$

Finally, Theorem 3 states that

$$\begin{aligned} \phi_{\gamma} \circ \pi'_1(S) &= \sum_{m \geq 1} \sum_{n=1}^m \frac{(-1)^{n+1}}{n} \binom{m}{n} \phi_{\gamma^{\star n}}(S_m) \\ &= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{(-1)^{n+1}}{n} \binom{m}{n} \phi_{\gamma^{\star n}}(S_m), \end{aligned}$$

where $S_m = \sum_{v \in A_m^*} (S|v)v$.

Let us also look what is the meaning of Theorem 2 in this context. Using canonical identification $\mathbb{K} \otimes T^{\otimes} V \simeq T^{\otimes} V$ and the definition of ϕ_γ we see that

$$X(\gamma) = \phi_\gamma \otimes \iota_\otimes \left(\sum_{v \in A^*} v \otimes v \right).$$

Since ϕ_γ is a shuffle algebra homomorphism, $\iota_\otimes$ a concatenation algebra homomorphism, and using Theorem 2, we conclude that

$$\log X(\gamma) = \sum_{h \in B} \phi_\gamma \circ \pi'_1(S_h) \iota_\otimes(P_h).$$

If for all $h \in B$, S_h is homogeneous of order $\#h$, i.e., $S_h \in \text{span} \{ w \in A_{\#h}^* \}$ (this is the case for Hall bases), then

$$\phi_\gamma \circ \pi'_1(S_h) = \sum_{n=1}^{\#h} \frac{(-1)^{n+1}}{n} \binom{\#h}{n} \phi_{\gamma^{*n}}(S_h).$$

This means that having got the signature of a path, the logarithm of this signature can be computed in terms of the signatures of concatenations of the path. For more information about log signature (see [6, 15, 19]).

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