Solving $(k - 1)$ -stable instances of k -terminal cut with isolating cuts

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Abstract

The k -terminal cut problem, also known as the multiterminal cut problem, is defined on an edge-weighted graph with k distinct vertices called “terminals.” The goal is to remove a minimum weight collection of edges from the graph such that there is no path between any pair of terminals. The problem is APX-hard. Isolating cuts are minimum cuts which separate one terminal from the rest. The union of all the isolating cuts, except the largest, is a $(2 - 2/k)$ -approximation to the optimal k -terminal cut. An instance of k -terminal cut is γ -stable if edges in the cut can be multiplied by up to γ without changing the unique optimal solution. In this paper, we show that, in any $(k - 1)$ -stable instance of k -terminal cut, the source sets of the isolating cuts are the source sets of the unique optimal solution to that k -terminal cut instance. We conclude that the $(2 - 2/k)$ -approximation algorithm returns the optimal solution on $(k - 1)$ -stable instances. Ours is the first result showing that this $(2 - 2/k)$ -approximation is an exact optimization algorithm on a special class of graphs. We also show that our $(k - 1)$ -stability result is tight. We construct $(k - 1 - \epsilon)$ -stable instances of the k -terminal cut problem which only have trivial isolating cuts: that is, the source set of the isolating cuts for each terminal is just the terminal itself. Thus, the $(2 - 2/k)$ -approximation does not return an optimal solution.

Keyword stability; isolating cuts; Approximation; k -terminal cut

1 Introduction

The k -terminal cut problem, also known as the multiterminal cut problem, is defined on an edge-weighted graph with k distinct vertices called “terminals.” The

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goal is to remove a minimum weight collection of edges from the graph such that there is no path between any pair of terminals. The k -terminal cut problem is known to be APX-hard (Dahlhaus et al. 1994).

Bilu and Linial introduced the concept of stability for graph cut problems (Bilu and Linial 2012). An instance of k -terminal cut is said to be γ -stable if the optimal cut remains uniquely optimal when the weight of every edge in the cut is multiplied by a factor up to γ . The concepts of sensitivity analysis and robustness in linear programming are closely related (Robinson 1977; Ben-Tal and Nemirovski 2000). Makarychev et al. showed that, for 4-stable instances of k -terminal cut, the solution to the linear programming relaxation of the Calinescu-Karloff-Rabani (CKR) formulation (introduced in Călinescu et al. (1998)) will necessarily be integer (Makarychev et al. 2014). The result was later improved by Angelidakis et al. to $(2 - 2/k)$ -stable instances using the same linear programming technique (Angelidakis et al. 2017a, b).

In an instance of k -terminal cut, minimum isolating cuts are minimum cuts which separate one terminal from the rest of the terminals. They can give useful information about the optimal solution: the source set of a terminal's minimum isolating cut is a subset of that terminal's source set in an optimal solution (Dahlhaus et al. 1994). Furthermore, the union of all the minimum isolating cuts, except for the cut with largest weight, is a $(2 - 2/k)$ -approximation for the k -terminal cut problem (Dahlhaus et al. 1994). This $(2 - 2/k)$ -approximation algorithm is considerably different from the other approximation algorithms for k -terminal cut. All the other currently-known approximation algorithm for k -terminal cut work by probabilistically rounding the linear programming relaxation of the CKR formulation, giving an approximation guarantee in expectation (Călinescu et al. 1998; Karger et al. 2004; Buchbinder et al. 2013; Sharma and Vondrák 2014). It is natural to wonder how the $(2 - 2/k)$ -approximation performs on non worst-case instances.

In this paper, we establish a connection between minimum isolating cuts and stability. We show that in $(k - 1)$ -stable instances of k -terminal cut, the source sets of the minimum isolating cuts equal the source sets of the unique optimal solution to that k -terminal cut instance. It follows that the simple $(2 - 2/k)$ -approximation of Dahlhaus et al. (1994) returns the optimal solution on $(k - 1)$ -stable instances. Ours is the first result showing that this $(2 - 2/k)$ -approximation is an exact optimization algorithm on a special class of graphs.

We also show that our result is tight. For $\epsilon > 0$, we construct $(k - 1 - \epsilon)$ -stable instances of the k -terminal cut problem which only have trivial minimum isolating cuts: that is, the source set of the minimum isolating cut for each terminal is just the terminal itself. In these $(k - 1 - \epsilon)$ -stable instances, the $(2 - 2/k)$ -approximation does not return an optimal solution.

In Sect. 2, we introduce definitions and notation. In Sect. 3, we prove the main structural result, that in $(k - 1)$ -stable instances of k -terminal cut the source sets of the minimum isolating cuts equal the source sets of the optimal k -terminal cut. In Sect. 4, we construct a $(k - 1 - \epsilon)$ -stable graph in which the source set of the minimum isolating cut for each terminal is just the terminal itself. In Sect. 5, we present further analysis of two implications of our results. In Sect. 6, we conclude.

2 Preliminaries

The notation $\{G = (V, E, w), T\}$ refers to an instance of the k -terminal cut problem, where $G = (V, E, w)$ is an undirected graph with vertices V and edges E . $T = \{t_1, \dots, t_k\} \subseteq V$ is a set of k terminals. The weight function w is a function from E to $\mathbb{R}^+$.

For a subset of edges $E' \subseteq E$, the notation $w(E')$ is the total weight of edges in E' :

$$w(E') = \sum_{e \in E'} w(e).$$

For two disjoint subsets of vertices $V_1 \subseteq V$, $V_2 \subseteq V$, the notation $w(V_1, V_2)$ is the total weight of edges between V_1 and V_2 :

$$w(V_1, V_2) = \sum_{\substack{\{v_1, v_2\} \in E \\ v_1 \in V_1 \\ v_2 \in V_2}} w(\{v_1, v_2\}).$$

We can further generalize this notation to allow for several disjoint subsets of vertices $V_1, \dots, V_m \subseteq V$. In this case, we calculate the total weight of edges that go between two distinct subsets:

$$w(V_1, \dots, V_m) = \sum_i \sum_{j>i} w(V_i, V_j).$$

For an instance $\{G, T\}$ of the k -terminal cut problem, we can refer to the optimal solution in two equivalent ways. The first is in terms of the edges that are cut and the second is in terms of the source sets.

Referring to the optimal cut in terms of edges, we use the notation E_{OPT} : the subset of E of minimum total weight whose removal ensures that there is no path between any pair of terminals.

Source sets are a partition of V into $S_1, S_2, \dots, S_k$ such that $t_i \in S_i$. We say that S_i is the source set corresponding to t_i . We denote the optimal source sets $S_1^*, S_2^*, \dots, S_k^*$.

The set of edges in the optimal cut is precisely the set of edges which go between distinct elements of the optimal partition $(S_1^*, \dots, S_k^*)$. Combining the notation introduced in this section,

$$w(E_{\text{OPT}}) = w(S_1^*, \dots, S_k^*).$$

2.1 Stability

Definition 1 (γ -Perturbation) Let $G = (V, E)$ be a weighted graph with edge weights w . Let $G' = (V, E)$ be a weighted graph with the same set of vertices V and edges E

and a new set of edge weights w' such that, for every $e \in E$ and some $\gamma > 1$,

$$w(e) \leq w'(e) \leq \gamma w(e).$$

Then G' is a γ -perturbation of G .

Stable instances are instances where the optimal solution remains uniquely optimal for any γ -perturbation of the weighted graph.

Definition 2 (γ -Stability) An instance $\{G = (V, E, w), T\}$ of k -terminal cut is γ -stable ($\gamma > 1$) if there is an optimal solution E_{OPT} which is uniquely optimal for k -terminal cut for every γ -perturbation of G .

Note that the optimal solution need not be γ times as good as *any* other solution, since two solutions may share many edges. Given an alternative feasible solution, E_{FEAS} , to the optimal cut, E_{OPT} , in a γ -stable instance, we can make a statement about the relative weights of the edges where the cuts differ. The following equivalence was first noted in Makarychev et al. (2014):

Lemma 1 (γ -Stability) Let $\{G = (V, E, w), T\}$ be an instance of k -terminal cut with optimal cut E_{OPT} . Let $\gamma > 1$. G is γ -stable if and only if, for every alternative feasible k -terminal cut $E_{\text{FEAS}} \neq E_{\text{OPT}}$, we have

$$w(E_{\text{FEAS}} \setminus E_{\text{OPT}}) > \gamma w(E_{\text{OPT}} \setminus E_{\text{FEAS}}).$$

Now, we present a new characterization of γ -stable instances. A key realization is that we do not need to consider every γ -perturbation of G in order to check for γ -stability. In fact, there is one γ -perturbation which is, in a sense, the “worst” one: the γ -perturbation in which we multiply the edges in E_{OPT} by γ and leave the rest unchanged. If we find a set of edges with this property, then our instance is γ -stable. This idea is summarized in Lemma 2.

Lemma 2 (Checking γ -Stability) Consider an instance of k -terminal cut $\{G = (V, E, w), T\}$. Let $E^* \subseteq E$ be a subset of the edges. For some $\gamma > 1$, consider the instance $\{G' = (V, E, w'), T\}$, where

$$w'(e) = \begin{cases} \gamma w(e) & e \in E^* \\ w(e) & e \notin E^*. \end{cases}$$

If E^* is the unique optimal solution to k -terminal cut in $\{G', T\}$, then the instance $\{G, T\}$ is γ -stable and E^* is the unique optimal solution to k -terminal cut in $\{G, T\}$.

Proof First, we claim that E^* is the unique optimal solution in G . For any alternative feasible cut E_{FEAS} ,

$$w'(E^*) < w'(E_{\text{FEAS}}).$$

By construction, $\gamma w(E^*) = w'(E^*)$ and $w'(E_{\text{FEAS}}) < \gamma w(E_{\text{FEAS}})$. Thus,

$$w(E^*) < w(E_{\text{FEAS}}).$$

Now, consider an alternative feasible cut E_{FEAS} . We express its weight in G' in terms of its weight in G :

$$\begin{aligned} w'(E_{\text{FEAS}}) &= w'(E_{\text{FEAS}} \setminus E^*) + w'(E_{\text{FEAS}} \cap E^*) \\ &= w(E_{\text{FEAS}} \setminus E^*) + \gamma w(E_{\text{FEAS}} \cap E^*). \end{aligned}$$

We do the same for E^* :

$$\begin{aligned} w'(E^*) &= w'(E^* \setminus E_{\text{FEAS}}) + w'(E^* \cap E_{\text{FEAS}}) \\ &= \gamma w(E^* \setminus E_{\text{FEAS}}) + \gamma w(E^* \cap E_{\text{FEAS}}). \end{aligned}$$

Thus,

$$w'(E^*) < w'(E_{\text{FEAS}}) \iff \gamma w(E^* \setminus E_{\text{FEAS}}) < w(E_{\text{FEAS}} \setminus E^*).$$

From Lemma 1, we conclude that $\{G, T\}$ is γ -stable. $\square$

We make a few observations about γ -stability:

Fact 1 Any k -terminal cut instance that is stable with $\gamma > 1$ must have a unique optimal solution.

Proof By Definition 1, any graph is a γ -perturbation of itself. Thus, by Definition 2, the optimal solution must be unique. $\square$

Fact 2 Any k -terminal cut instance that is γ_2 -stable is also γ_1 -stable for any $1 < \gamma_1 < \gamma_2$.

Proof The set of γ_1 -perturbations is a subset of the set of γ_2 -perturbations, since

$$w(e) \leq w'(e) \leq \gamma_1 w(e) \implies w(e) \leq w'(e) \leq \gamma_2 w(e).$$

$\square$

Thus, for example, every instance which is 4-stable is necessarily 2-stable. On the other hand, there exist instances which are 2-stable but not 4-stable.

2.2 Isolating cuts

Definition 3 (Minimum t_i -Isolating Cut) The minimum t_i -isolating cut is a minimum cut which separates source terminal $s = t_i$ from all the other terminals.

The problem of calculating a minimum isolating cut for t_i can be reduced to the problem of computing a minimum (s, t) -cut. This is accomplished by contracting all the terminals $\cup_{j \neq i} t_j$ into a single sink vertex t , then calculating a minimum (s, t) -cut between $s = t_i$ and t .

We will use the notation Q_i to denote the source set of this minimum isolating cut (the set of vertices which remain connected to t_i). We use E_i to denote the set of edges which form the cut (the subset of edges with exactly one endpoint in Q_i). Let E_{ISO} be the union of the E_i except for the E_i with largest weight. The following two lemmas are due to Dahlhaus et al. (1994):

Lemma 3 E_{ISO} is a $(2 - 2/k)$ -approximation for the optimal k -terminal cut.

Lemma 4 Let $\{G = (V, E, w), T\}$ be an instance of k -terminal cut. Let $i \in \{1, \dots, k\}$. Then there exists an optimal solution $(S_1^*, \dots, S_k^*)$ in which

$$Q_i \subseteq S_i^*.$$

The condition that *there exists* an optimal solution can make the implication of Lemma 4 somewhat complicated when there are multiple optimal solutions, since the equation $Q_i \subseteq S_i^*$ need not be simultaneously true for all i . Conveniently, when an instance is γ -stable ($\gamma > 1$), it has a unique optimal solution (fact 1). Thus, in such instances, the condition $Q_i \subseteq S_i^*$ will be simultaneously true for all i .

3 Proof of main result

Theorem 1 Let $\{G = (V, E, w), T\}$ be a $(k - 1)$ -stable instance of k -terminal cut. Then, for all i , $Q_i = S_i^*$.

Proof We will primarily be working with the k vertex sets $Q_1, \dots, Q_k$ and the k vertex sets $S_1^* \setminus Q_1, \dots, S_k^* \setminus Q_k$. For convenience, we will use the notation $R_i = S_i^* \setminus Q_i$. As a consequence of Lemma 4, $S_i^* = Q_i \cup R_i$. We will assume, for the sake of contradiction, that at least one R_i is non-empty.

Since Q_i is the source set for the minimum isolating cut for terminal t_i :

$$\begin{aligned} & w(Q_i, V \setminus Q_i) \leq w(S_i^*, V \setminus S_i^*) \\ \iff & w(Q_i, V \setminus Q_i) \leq w(R_i, V \setminus S_i^*) + w(Q_i, V \setminus S_i^*) \\ \iff & -w(Q_i, V \setminus S_i^*) + w(Q_i, V \setminus Q_i) \leq w(R_i, V \setminus S_i^*) \\ \iff & w(Q_i, R_i) \leq w(R_i, V \setminus S_i^*) \\ \iff & w(Q_i, R_i) \leq \sum_{\{j|j \neq i\}} w(R_i, R_j) + \sum_{\{j|j \neq i\}} w(R_i, Q_j) \end{aligned}$$

Summing these inequalities over all the i :

$$\begin{aligned} & \sum_i w(Q_i, R_i) \leq \sum_i \sum_{\{j|j \neq i\}} w(R_i, R_j) + \sum_i \sum_{\{j|j \neq i\}} w(R_i, Q_j) \\ \iff & \sum_i w(Q_i, R_i) \leq 2w(R_1, \dots, R_k) + \sum_i \sum_{\{j|j \neq i\}} w(R_i, Q_j) \end{aligned} \quad (1)$$

(a) The optimal partition, in which each R_i is in a source set with its respective Q_i . **(b)** The alternative partition used in theorem 1 when $i = 2$, where all of the R are in a source set with Q_2 .

Fig. 1 The sets Q_1, Q_2, Q_3 and R_1, R_2, R_3 defined in Theorem 1 when $k = 3$. Solid lines represent edges which are in the cut. Dashed lines represent edges which are not in the cut

Next, we will consider alternatives to the optimal cut $(S_1^*, \dots, S_k^*)$ and apply Lemma 1. The optimal cut can be written as

$$(S_1^*, \dots, S_k^*) = (Q_1 \cup R_1, \dots, Q_k \cup R_k).$$

We will consider alternative cuts $E_{\text{FEAS}}^{(i)}$ where all the R_j are in the same set of the partition, associated with Q_i . That is, we will consider

$$\begin{aligned} & (S_1, \dots, S_{i-1}, S_i, S_{i+1}, \dots, S_k) \\ &= (Q_1, \dots, Q_{i-1}, Q_i \cup (R_1 \cup \dots \cup R_k), Q_{i+1}, \dots, Q_k). \end{aligned}$$

See Fig. 1 for an illustration. We assumed that at least one of the R_i is non-empty, so at least $k - 1$ of these alternative cuts are distinct from the optimal one¹. In order to apply Lemma 1, we need to calculate $w(E_{\text{OPT}} \setminus E_{\text{FEAS}}^{(i)})$ and $w(E_{\text{FEAS}}^{(i)} \setminus E_{\text{OPT}})$.

To calculate $w(E_{\text{FEAS}}^{(i)} \setminus E_{\text{OPT}})$, consider the edges in $E_{\text{FEAS}}^{(i)}$ with one endpoint in Q_j ($j \neq i$). The only edges which are *not* counted in E_{OPT} are those which go to R_j . Thus,

$$w(E_{\text{FEAS}}^{(i)} \setminus E_{\text{OPT}}) = \sum_{\{j \mid j \neq i\}} w(R_j, Q_j).$$

Next, we calculate $w(E_{\text{OPT}} \setminus E_{\text{FEAS}}^{(i)})$. For an edge not to be in $E_{\text{FEAS}}^{(i)}$, it must be internal to one of the Q_j ($j \neq i$) or internal to $Q_i \cup (R_1 \cup \dots \cup R_k)$. None of the

¹ If only one R_i is non-empty, then $E_{\text{OPT}} = E_{\text{FEAS}}^{(i)}$ for this i . The corresponding inequality in Eq. 2 is not strict (both sides are 0), but the other $k - 1$ inequalities are strict and so the average (Eq. 3) is still a strict inequality.

internal edges of the Q_j are in E_{OPT} , so we only need to consider the internal edges of $Q_i \cup (R_1 \cup \dots \cup R_k)$:

$$w(E_{\text{OPT}} \setminus E_{\text{FEAS}}^{(i)}) = w(R_1, \dots, R_k) + \sum_{\{j|j \neq i\}} w(R_j, Q_i).$$

We apply Lemma 1, with $\gamma = k - 1$:

$$(k - 1) \cdot w(E_{\text{OPT}} \setminus E_{\text{FEAS}}^{(i)}) < w(E_{\text{FEAS}}^{(i)} \setminus E_{\text{OPT}}) \quad (2)$$

Substituting in the formulas derived earlier for $w(E_{\text{OPT}} \setminus E_{\text{FEAS}}^{(i)})$ and $w(E_{\text{FEAS}}^{(i)} \setminus E_{\text{OPT}})$:

$$(k - 1) \cdot w(R_1, \dots, R_k) + (k - 1) \cdot \sum_{\{j|j \neq i\}} w(R_j, Q_i) < \sum_{\{j|j \neq i\}} w(R_j, Q_j).$$

Averaging over the k inequalities (one for each i)¹:

$$(k - 1) \cdot w(R_1, \dots, R_k) + \frac{k - 1}{k} \sum_i \sum_{\{j|j \neq i\}} w(R_j, Q_i) < \frac{k - 1}{k} \sum_i w(R_i, Q_i). \quad (3)$$

We combine this with the inequality derived in Eq. 1:

$$\begin{aligned} & (k - 1) \cdot w(R_1, \dots, R_k) + \frac{k - 1}{k} \sum_i \sum_{\{j|j \neq i\}} w(R_j, Q_i) \\ & < 2 \frac{k - 1}{k} w(R_1, \dots, R_k) + \frac{k - 1}{k} \sum_i \sum_{\{j|j \neq i\}} w(R_i, Q_j). \end{aligned}$$

Notice that

$$\sum_i \sum_{\{j|j \neq i\}} w(R_j, Q_i) = \sum_i \sum_{\{j|j \neq i\}} w(R_i, Q_j).$$

$$\text{Therefore, } (k - 1) \cdot w(R_1, \dots, R_k) < 2 \frac{k - 1}{k} w(R_1, \dots, R_k).$$

This is a contradiction, so it must be the case that $R_i = \emptyset$ for all i . Thus, $Q_i = S_i^*$ for all i . $\square$

Corollary 1 *Let $\{G = (V, E, w), T\}$ be a $(k - 1)$ -stable instance of k -terminal cut. Then E_{ISO} is the unique optimal solution to k -terminal cut on $\{G, T\}$.*

Proof Assume, without loss of generality, that E_k is the largest among the k isolating cuts in G . Then E_{ISO} is the set of edges between $Q_1, \dots, Q_{k-1}, V \setminus (Q_1 \cup \dots \cup Q_{k-1})$. From Theorem 1, E_{ISO} is the set of edges between $S_1^*, \dots, S_{k-1}^*, V \setminus (S_1^* \cup \dots \cup S_{k-1}^*) = S_1^*, \dots, S_{k-1}^*, S_k^*$. Thus, $E_{\text{ISO}} = E_{\text{OPT}}$. $\square$ $\square$

Fig. 2 The construction used in Theorem 2 when $k = 5$

4 Tightness of main result

To prove that the factor of $(k - 1)$ is tight, it would be sufficient to find an instance which is $(k - 1 - \epsilon)$ -stable, for arbitrarily small $\epsilon > 0$, for which $Q_i \neq S_i^*$ for some i . In fact, we will exhibit an instance with a stronger property: $Q_i \neq S_i^*$ for all i .

Theorem 2 For all $0 < \epsilon < k - 2$, there exists a $(k - 1 - \epsilon)$ -stable instance of k -terminal cut for which $Q_i = \{t_i\} \neq S_i^*$ for all $i \in \{1, \dots, k\}$.

Proof Consider a graph with $2k$ vertices. There are k terminals, $T = \{t_1, \dots, t_k\}$, and k other vertices $S = \{s_1, \dots, s_k\}$. The $\binom{k}{2}$ edges between s_i and s_j ($i \neq j$) have weight $a \in \mathbb{R}^+$. The k edges from t_i to s_i have weight $b \in \mathbb{R}^+$. The $k(k - 1)$ edges from t_i to s_j ($i \neq j$) have weight $c \in \mathbb{R}^+$. No other edges exist. Call this graph G_k . See Fig. 2a for a drawing of G_k when $k = 5$. We will show that this graph has the property $Q_i = \{t_i\} \neq \{t_i, s_i\} = S_i^* \forall i$ for the following choices of $a, b, c \in \mathbb{R}^+$:

$$\begin{aligned} a &= 2\epsilon \\ b &= k(k - 1)(k - 1 - \epsilon) \\ c &= k(k - 1) - (k + 1)\epsilon. \end{aligned}$$

For $0 < \epsilon < k - 2$, these values of a , b , and c are all well-defined.

We will be using the idea presented in Lemma 2. Let E_{OPT} be the union of all the edges except those between corresponding t_i and s_i . In our construction, these are all the edges with weights a and c .

Consider the graph G'_k , in which the edges of weight b have their weight divided by $(k - 1 - \epsilon)$ (which is equivalent to multiplying the edges of weight a and c by $(k - 1 - \epsilon)$ and then re-scaling):

$$\begin{aligned} a' &= 2\epsilon \\ b' &= k(k - 1) \\ c' &= k(k - 1) - (k + 1)\epsilon. \end{aligned}$$

We will show that E_{OPT} is the unique optimal solution in G'_k . From Lemma 2, it will follow that the instance $\{G_k, T\}$ is $(k - 1 - \epsilon)$ -stable with optimal cut E_{OPT} .

First, we establish the weight of E_{OPT} in G'_k :

$$w'(E_{\text{OPT}}) = \binom{k}{2}a' + k(k - 1)c'.$$

To verify that E_{OPT} is the optimal cut in G'_k , we need to prove that it has smaller weight than every other feasible cut. Fortunately, we are helped by the symmetry of the construction. We will conduct casework based on the number of $\{t_i, s_i\}$ which are in the feasible cut. Given an alternative feasible cut, let p be the number of edges $\{t_i, s_i\}$ which are in the feasible cut. Without loss of generality, we only need to consider the feasible cut $E_{\text{FEAS}}^{(p)}$ in which $\{t_i, s_i\} \in E_{\text{FEAS}}^{(p)}$ for $i = \{1, \dots, p\}$ and $\{t_i, s_i\} \notin E_{\text{FEAS}}^{(p)}$ for $i = \{p + 1, \dots, k\}$. See Fig. 2b for an illustration. The cut E_{OPT} corresponds to the case $p = 0$: $E_{\text{OPT}} = E_{\text{FEAS}}^{(0)}$.

For fixed s_i , consider the set of edges $T \times \{s_i\} = \{\{t_j, s_i\} \mid \forall j \in \{1, \dots, k\}\}$. Of the k edges in this set, $k - 1$ must be in $E_{\text{FEAS}}^{(p)}$. If $\{t_i, s_i\} \notin E_{\text{FEAS}}^{(p)}$, then $\{t_j, s_i\} \in E_{\text{FEAS}}^{(p)}$ for all $j \neq i$. Otherwise, if $\{t_i, s_i\} \in E_{\text{FEAS}}^{(p)}$, then $k - 2$ edges of weight c' are in $E_{\text{FEAS}}^{(p)}$. In total, for $i \in \{1, \dots, p\}$,

$$w'(E_{\text{FEAS}}^{(p)} \cap (T \times \{s_i\})) = (k - 2)c' + b' = (k - 1)c' + (k + 1)\epsilon.$$

For $i \in \{p + 1, \dots, k\}$,

$$w'(E_{\text{FEAS}}^{(p)} \cap (T \times \{s_i\})) = (k - 1)c'.$$

Thus, we can compute

$$w'(E_{\text{FEAS}}^{(p)} \cap (T \times S)) = k(k - 1)c' + p(k + 1)\epsilon.$$

Next, we will consider the number of edges of weight a' in $E_{\text{FEAS}}^{(p)}$. In other words, the edges $S \times S$. The vertices in $\{s_{p+1}, \dots, s_k\}$ are each connected to the correspond-

ing terminal in $\{t_{p+1}, \dots, t_k\}$. Thus, all the edges between them must be in $E_{\text{FEAS}}^{(p)}$. Formally,

$$|(\{s_{p+1}, \dots, s_k\} \times \{s_{p+1}, \dots, s_k\}) \cap E_{\text{FEAS}}^{(p)}| = \binom{k-p}{2}.$$

Furthermore, each s_i in $\{s_1, \dots, s_p\}$ can remain connected to at most one of the vertices in $\{s_{p+1}, \dots, s_k\}$. This accounts for an additional $p(k-p-1)$ edges which must be in $E_{\text{FEAS}}^{(p)}$. Formally,

$$|(\{s_1, \dots, s_p\} \times \{s_{p+1}, \dots, s_k\}) \cap E_{\text{FEAS}}^{(p)}| \geq p(k-p-1).$$

In total,

$$|(S \times S) \cap E_{\text{FEAS}}^{(p)}| \geq \binom{k-p}{2} + p(k-p-1).$$

Equality is obtained when the vertices $s_1, \dots, s_p$ are all assigned to the source set containing t_k .

Combining the arguments from the previous two paragraphs, we conclude that, for $p \in \{0, \dots, k-1\}$,

$$w'(E_{\text{FEAS}}^{(p)}) = \left(\binom{k-p}{2} + p(k-p-1) \right) a' + k(k-1)c' + p(k+1)\epsilon.$$

Writing this as a function of p , we see that, for all ϵ , it is a quadratic function with negative leading coefficient, thus it is minimized at either $p = 0$ or $p = k-1$. The case $p = k$ clearly leads to a larger feasible cut than $p = k-1$, so we omit it:

$$p = 0 \implies w'(E_{\text{FEAS}}^{(p)}) = \binom{k}{2} a' + k(k-1)c' = k(k-1)c' + k(k-1)\epsilon$$

$$p = k-1 \implies w'(E_{\text{FEAS}}^{(p)}) = k(k-1)c' + (k-1)(k+1)\epsilon.$$

The unique minimizer is $p = 0$, which means that $E_{\text{FEAS}}^{(0)} = E_{\text{OPT}}$ is the unique optimal solution in G'_k .

Last but not least, we need to check that, as claimed, the minimum isolating cuts in G_k have trivial source sets. Knowing that $S_i^* = \{t_i, s_i\}$, thanks to Lemma 4 there are only two possibilities for Q_i . Either $Q_i = \{t_i\}$ or $Q_i = \{t_i, s_i\}$. $Q_i = \{t_i\}$ if and only if

$$\begin{aligned} & b + (k-1)c < (k-1)a + 2(k-1)c \\ \iff & b < (k-1)(a+c) \\ \iff & k(k-1-\epsilon) < k(k-1) - (k+1)\epsilon + 2\epsilon \\ \iff & 0 < \epsilon \end{aligned}$$

Thus, our construction has the property $Q_i = \{t_i\} \neq \{t_i, s_i\} = S_i^* \forall i$ for any ϵ in $(0, k - 2)$. $\square$

5 Further analysis

5.1 Approximation revisited

From Theorem 2, it follows that the $(2 - 2/k)$ -approximation algorithm of Dahlhaus et al. does *not* deliver an optimal solution on the constructed $(k - 1 - \epsilon)$ -stable instance. Still, one might conjecture that as an instance becomes more stable it becomes “easier” to approximate the optimal solution. We ask the following question about the interplay between stability and approximation: as an instance becomes more stable, does the approximation algorithm of Dahlhaus et al. deliver an approximation with better (smaller) approximation ratio? We will answer this question in the negative. We calculate, for $\epsilon \in (0, k - 2)$:

$$\begin{aligned} \frac{w(E_{\text{ISO}})}{w(E_{\text{OPT}})} &= \frac{(k-1)(b + (k-1)c)}{\binom{k}{2}a + k(k-1)c} \\ &= \frac{(k-1)(2k(k-1)(k-1-\epsilon) - (k-1)\epsilon)}{k(k-1)\epsilon + k(k-1)k(k-1-\epsilon) - k(k-1)\epsilon} \\ &= \frac{2k(k-1)(k-1-\epsilon) - (k-1)\epsilon}{k^2(k-1-\epsilon)} \\ &= (2 - 2/k) - \frac{(k-1)\epsilon}{k^2(k-1-\epsilon)}. \end{aligned}$$

This result is somewhat surprising. While one might expect the approximation ratio to improve on more stable instances, we find the opposite: as $\epsilon \rightarrow 0$, the $(k - 1 - \epsilon)$ -stable instance becomes more stable, but the approximation ratio gets worse (larger).

5.2 Running time

We proved that $(k - 1)$ -stable instances of k -terminal cut can be solved to optimality by calculating k minimum isolating cuts. Let $C(n, m)$ be the complexity of solving a minimum (s, t) -cut problem on a graph with n vertices and m edges. The complexity of computing k minimum isolating cuts is $O(kC(n, m))$. Using the push-relabel maximum flow algorithm, we have $C(n, m) = O(mn \log \frac{n^2}{m})$. If we ignore logarithmic factors and assume $m > n$, then the complexity of calculating k minimum isolating cuts is $\tilde{O}(km^2)$.

Earlier, A. is et al. proved that $(2 - 2/k)$ -stable instances of k -terminal cut can be solved to optimality by solving the linear programming relaxation of the CKR formulation. They showed that, in these stable instances, the solution to the relaxation will necessarily be integer (Angelidakis et al. 2017a,b). Ignoring redundant constraints, the CKR relaxation is as follows Călinescu et al. (1998):

$$\begin{aligned}
& \min \sum_{\{i,j\} \in E, t \in T} \frac{1}{2} w_{ij} z_{ij}^t \\
& s.t. \quad z_{ij}^t \geq x_i^t - x_j^t \quad \forall \{i,j\} \in E, t \in T \\
& \quad \quad z_{ij}^t \geq x_j^t - x_i^t \quad \forall \{i,j\} \in E, t \in T \\
& \quad \quad \sum_{t \in T} x_i^t = 1 \quad \forall i \in V \\
& \quad \quad x_i^t = 1 \quad \forall t \in T \\
& \quad \quad x_i^t \geq 0 \quad \forall i \in V, t \in T
\end{aligned} \tag{CKR}$$

The number of variables in this formulation is $k(n + m)$ and the number of constraints is $n + 2km$.

Because this linear program has a binary constraint matrix, it is known that it can be solved in strongly polynomial time (Schrijver 1998). That said, even if we knew exactly which variables were in the basis in the optimal solution, determining their values would require solving a system of linear equations with $\min\{k(n + m), n + 2km\}$ variables and equations. Such an operation would be at least quadratic in the number of variables. This suggests that solving the CRK relaxation has complexity $\Omega(k^2 m^2)$. In fact, using the current best-known algorithms for matrix inversion, the complexity of solving the relaxation would be $\Omega(k^{2.37} m^{2.37})$ (Cohen et al. 2019; Le Gall 2014).

In summary, while our proposed algorithm only works on more stable instances, under reasonable assumptions it is at least k times faster than solving the CKR relaxation. In practical instances, the speed-up may be several orders of magnitude (Velednitsky and Hochbaum 2018). This analysis does not account for the possibility of solving the CKR relaxation by a means that avoids solving a systems of equations or inverting a matrix. Determining if such an alternative exists would be an interesting avenue for future research.

5.3 Practical usability

The k minimum isolating cuts are only guaranteed to return the optimal cut in instances which are $(k - 1)$ -stable. Determining the stability of an instance is, itself, an NP-hard problem. This would seem to preclude the practical usage of isolating cuts. However, we can side-step this problem because our algorithm includes a certificate of optimality: the isolating cuts return the optimal solution if and only if their source sets form a partition of the vertex set V .

The proof of this fact follows from Lemma 4. We know from Lemma 4 that the source set of isolating cut for terminal t_i , Q_i , is a subset of the optimal source set, S_i^* . Since $(S_1^*, \dots, S_k^*)$ is a partition of V and $Q_i \subseteq S_i^*$, it follows that the Q_i form a partition if and only if $Q_i = S_i^*$.

To summarize, we side-step the issue of pre-determining stability as follows: rather than computing if an instance is stable and then solving it with isolating cuts, we simply run the isolating cut algorithm. If the algorithm returns a partition of V , then we know that it is the optimal solution. If the algorithm does not return a partition, then we know that the instance was not $(k - 1)$ -stable to begin with.

6 Conclusions

In this paper, we proved that, in $(k - 1)$ -stable instances of `k-terminal cut`, the source sets of the minimum isolating cuts are the source sets of the unique optimal solution to that `k-terminal cut` instance. As an immediate corollary, we concluded that the $(2 - 2/k)$ -approximation algorithm of Dahlhaus et al. is optimal for $(k - 1)$ -stable instances.

We also showed that the factor of $k - 1$ is tight. We constructed $(k - 1 - \epsilon)$ -stable instances of `k-terminal cut` in which the source set of the minimum isolating cut for a terminal is just the terminal itself. In those instances, the $(2 - 2/k)$ -approximation algorithm does not return an optimal solution.

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