Remarks on Barnette's conjecture

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Abstract

Let P be a cubic 3-connected bipartite plane graph having a 2-factor which consists only of facial 4-cycles, and suppose that P^* is the dual graph. We show that P has at least $3 \left\lceil \frac{2|P^*|}{\Delta^2(P^*)} \right\rceil$ different Hamilton cycles.

Keyword Barnette's conjecture · Hamilton cycle · Induced tree

Mathematics Subject Classification 05C45 · 05C10

1 Introduction

Let $\mathcal{P}$ denote the family of cubic 3-connected bipartite plane graphs. In 1969, Barnette (Tutte 1969, Problem 5) conjectured that every graph in $\mathcal{P}$ has a Hamilton cycle. In Goodey (1975), Goodey proved that if a graph in $\mathcal{P}$ has only faces with 4 or 6 sides, then it is hamiltonian (see also Feder and Subi (2006), and Bagheri et al. (2018)). Holton et al. (1985) have used computer search to confirm Barnette's conjecture for graphs up to 64 vertices. The problem whether a cubic bipartite planar graph has a Hamilton cycle (without the assumption of 3-connectivity) is NP-complete, as shown by Takanori et al. (1980).

All graphs considered in this paper are finite and simple. We use Diestel (2005) as reference for undefined terms. In particular, if P is a plane graph, then $V(P)$ is its vertex set and $E(P)$ is its edge set. The set of neighbours of a vertex v in P is denoted by $N(v)$. $|P|$ is the number of vertices of P and $\Delta(P)$ is the maximum degree of P . Hence, if P^* is the dual graph of P , then $|P^*|$ is the number of faces in P , and $\Delta(P^*)$ is the maximum size of faces in P .

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Let $\mathcal{P}(4)$ be the family of all graphs in $\mathcal{P}$ which possess a 2-factor consisting only of facial 4-cycles. The following theorem yields a lower bound on the number of Hamilton cycles in every graph belonging to $\mathcal{P}(4)$.

Theorem 1 *Every graph $P \in \mathcal{P}(4)$ has at least $3^{\left\lceil \frac{2|P^*|}{\Delta^2(P^*)} \right\rceil}$ different Hamilton cycles.*

A *sequence of faces* in a plane graph P is a sequence of different faces in P such that two faces from the sequence are adjacent if and only if they are consecutive in the sequence. A sequence of faces $f_0 f_1 \dots f_k$ in P is called a sequence from f_0 to f_k of *length* k . The distance of two faces f, h in P is the length of the shortest sequence of faces from f to h . We prove the following theorem which generalizes some results obtained by Florek in Florek (2010).

Theorem 2 *Let $P \in \mathcal{P}(4)$ and suppose that M is a set of faces in P such that the distance of any two different faces in M is greater than 4. If for every face in M any its edge is chosen, then there is a Hamiltonian cycle in P containing all the remaining edges of faces in M .*

We will use a result that expresses hamiltonicity in terms of the dual graph. Let $\mathcal{E}$ be the dual family to $\mathcal{P}$, i.e., $\mathcal{E}$ is the family of all Eulerian plane triangulations. Let $G \in \mathcal{E}$. Stein (1971) proved that G^* is hamiltonian if and only if G possesses two disjoint induced acyclic subgraphs which together contain all vertices of G (see also Florek 2016).

Let $\mathcal{E}(4)$ be the dual family to $\mathcal{P}(4)$. Our aim is to prove the following two theorems which express the theorems formulated above in the dual form:

Theorem 3 *Every graph $G \in \mathcal{E}(4)$ has at least $3^{\left\lceil \frac{2|G|}{\Delta^2(G)} \right\rceil}$ different pairs of disjoint induced acyclic subgraphs which together contain all vertices of G .*

Theorem 4 *Let $G \in \mathcal{E}(4)$ and suppose that $L \subset V(G)$ is a set of vertices such that the distance of every two different vertices of L is greater than 4. If for every $v \in L$ any vertex $n(v) \in N(v)$ is chosen, then there are two induced acyclic subgraphs which together contain all vertices of G and one of them contains the set $N(v) \setminus \{n(v)\}$, for every $v \in L$.*

In the proof of Theorem 3 we will use the following *greedy algorithm* (see Diestel 2005): starting from a fixed vertex enumeration $v_1, \dots, v_m$ of G , we consider the vertices in turn and colour each v_i with the first available colour—e.g., with the smallest positive integer not already used to colour any neighbour of v_i among $v_1, \dots, v_{i-1}$. In this way, we never use more than $\Delta(G) + 1$ colours, even for unfavourable choices of the enumeration $v_1, \dots, v_m$. Notice that greedy algorithm gives a constructive method to find an independent set in G which has at least $\frac{|G|}{\chi(G)} \geq \frac{|G|}{\Delta(G)+1}$ vertices.

2 Proofs

Let $G \in \mathcal{E}(4)$. It is well known that every Eulerian plane triangulation has a proper 3-colouring of vertices. Let $\{B, W, R\}$ be the corresponding vertex partition of $V(G)$

Fig. 1 A graph G . If $C = G[a, b, c, d]$ and $D = G[f, g, h, i, j]$, then the pair (C, D) is black-white closed. Proof of Theorem 4: Let $L = \{d\}$ and $n(d) = b$. Then $(C, D) = (S(n(d), P(d, n(d)))$

Fig. 2 Extension of (C, D) to a pair of disjoint induced acyclic subgraphs of G which together contain all vertices of G

into independent black, white and red colour classes. Certainly, one of them (say R) contains only vertices of degree 4. In Figs. 1, 2, 3 the vertices of B , W and R are indicated by black circles, white circles and squares, respectively.

Florek (2010) proved that every graph in $\mathcal{E}(4)$ is 4-connected. Hence, the set of neighbours of any vertex in G induces a cycle of G . For a path svt , which is disjoint with R , we define an operation $\alpha = \alpha(s, v, t)$ in the following way: let $v_1 \dots v_n v_1$ be a cycle induced by neighbours of v ($s = v_1$ and $t = v_k$). Replace the vertex v with a path xuy and join the vertices of this path with the former neighbours of

Fig. 3 A graph G . If $C = G[a, k, c, e]$ and $D = G[b, f, g, h, i]$, then the pair (C, D) is black-white closed. Proof of Theorem 3: Let $K = \{d, k\}$ (notice that $d_G(d, k) = 3$) and $n(d) = j$. Suppose that $W_1 = \emptyset$, $W_2 = \{k\}$, $B_1 = \{d\}$ and $B_2 = \emptyset$. Then $(C, D) = (S(k), P(d, n(d)))$

v in such a way that x is adjacent to $v_1, \dots, v_k$, u is adjacent to v_1 and v_k , and y is adjacent to $v_k, \dots, v_n, v_1$. Let $\alpha(G)$ be the graph constructed from G by α . We put $\alpha(R) = R \cup \{u\}$, $\alpha(B) = B$ and $\alpha(W) = (W \setminus \{v\}) \cup \{x, y\}$, for $v \in W$ (similarly, $\alpha(B) = (B \setminus \{v\}) \cup \{x, y\}$ and $\alpha(W) = W$, for $v \in B$, respectively). Notice that $\alpha(G) \in \mathcal{E}(4)$, because $\{\alpha(B), \alpha(W), \alpha(R)\}$ is a vertex partition of $V(\alpha(G))$ into independent colour classes such that $\alpha(R)$ contains only vertices of degree 4. It was proved in Florek (2010) that every graph $G \in \mathcal{E}(4)$ can be constructed (up to isomorphism) from the octahedron by iterating the operation α . Hence, by induction we obtain:

$$(i) |B \cup W| = \frac{|G|}{2} + 1.$$

Definition 1 We say that a pair (C, D) of disjoint induced subgraphs of G is (black, white)-closed if the following two conditions are satisfied (see Figs. 1 and 3):

- (1) if $v \in B$ ($v \in B \setminus V(D)$) is adjacent with a white (red, respectively) vertex of C , then v is a vertex of C ,
- (2) if $v \in W$ ($v \in W \setminus V(C)$) is adjacent with a black (red, respectively) vertex of D , then v is a vertex of D .

Lemma 1 Let $G \in \mathcal{E}(4)$ and suppose that $V(G)$ has a vertex partition into independent red, black and white colour classes such that the red colour class contains only vertices of degree 4. Every pair of disjoint (black, white)-closed induced acyclic subgraphs of G can be extended to such a pair of disjoint induced acyclic subgraphs which together contain all vertices of G .

Proof Let (C, D) be a pair of disjoint (black, white)-closed induced acyclic subgraphs of G . First we add to C (or D) all black (white, respectively) vertices which not belong

to $C \cup D$. Then we obtain a pair (C', D') of disjoint induced acyclic subgraphs of G . Next we consider any red vertex v not belonging to $C \cup D$. Since (C', D') is (black,white)-closed and all neighbours of v belong to $C' \cup D'$ one of the following cases occurs:

- (j) two black neighbours of v belong to C' ,
- (jj) two white neighbours of v belong to D' .

If black neighbours of v belong to a path contained in C' , then we add v to D' . If white neighbours of v belong to a path contained in D' then we add v to C' . Otherwise, we add v to C' or D' . Then we obtain a pair (X, Y) of disjoint induced acyclic subgraphs of G such that $C' \subset X$, $D' \subset Y$ and $V(X) \cup V(Y) = V(G)$ (see Figs. 1 and 2). $\square$

Now we are ready to prove Theorems 3 and 4.

Proof of Theorem 3

Proof Suppose that $G[B \cup W]$ is a subgraph of G induced by all black and white vertices of G . For every different vertices a, b of $G[B \cup W]$ which have the same colour and $d_G(a, b) = 2$ we add the edge ab to $G[B \cup W]$. Then we obtain a graph J such that

$$\Delta(J) \leq \frac{\Delta(G)}{2} + \frac{\Delta(G)}{2} \left(\frac{\Delta(G)}{2} - 1 \right) - 1 < \frac{\Delta^2(G)}{4}.$$

Thus, by the greedy algorithm, $\chi(J) \leq \frac{\Delta^2(G)}{4}$. Hence, J has an independent set of vertices K which, by (i), has at least

$$|K| \geq \frac{|J|}{\chi(J)} = \frac{|B \cup W|}{\chi(J)} > \frac{2|G|}{\Delta^2(G)}$$

vertices.

Notice that $K \subset V(J) = B \cup W \subset V(G)$. Suppose that for every $v \in K$ any red vertex $n(v) \in N(v)$ is chosen. Since G is 4-connected, $N(v) \setminus \{n(v)\}$ induces a path-subgraph in G which is denoted as $P(v, n(v))$. If v is black (white), then $(N(v) \cap W) \cup \{v\}$ ($(N(v) \cap B) \cup \{v\}$, respectively) induces a star-subgraph in G , denoted as $S(v)$.

Suppose that $W_1 \subset K \cap W$ and $B_1 \subset K \cap B$ is chosen. Let $W_2 = K \cap W \setminus W_1$ and suppose that $B_2 = K \cap B \setminus B_1$. Notice that

$$C = \bigcup_{v \in W_1} P(v, n(v)) \cup \bigcup_{v \in W_2} S(v) \text{ and } D = \bigcup_{v \in B_1} P(v, n(v)) \cup \bigcup_{v \in B_2} S(v)$$

are disjoint induced acyclic subgraphs of G because $d_G(a, b) \geq 4$ (≥ 3), for every different vertices $a, b \in K$ of the same colour class (of different colour classes, respectively). Remark that every black (white) vertex, which is adjacent with a vertex belonging to C (or D) belongs to C (D , respectively). Thus, (C, D) is (black,white)-closed (see Fig. 3). Hence follows that G has at least

$$\prod_{v \in K} \left(\frac{\deg_G(v)}{2} + 1 \right) \geq 3^{|K|} \geq 3^{\left\lceil \frac{2|G|}{\Delta^2(G)} \right\rceil}$$

different pairs of disjoint (black,white)-closed induced acyclic subgraphs. Thus, by Lemma 1, G has at least $3^{\left\lceil \frac{2|G|}{\Delta^2(G)} \right\rceil}$ different pairs of disjoint induced acyclic subgraphs which together contain all vertices of G . $\square$

Proof of Theorem 4

Proof Let $L \subset V(G)$ be a set of vertices such that $d(a, b) \geq 5$ for every different vertices $a, b \in L$.

Suppose that for every $v \in V(G)$ any vertex $n(v) \in N(v)$ is chosen. Since G is 4-connected, $N(v) \setminus \{n(v)\}$ induces a path-subgraph in G which is denoted as $P(v, n(v))$. Let R_b (or R_w) be the set of all $v \in R$ such that $n(v)$ is black (white, respectively). If $v \in W \cup R_b$ ($v \in B \cup R_w$), then $(N(n(v)) \cap W) \cup \{n(v)\}$ ($(N(n(v)) \cap B) \cup \{n(v)\}$, respectively) induces a star-subgraph in G , denoted as $S(n(v))$. Notice that if v is white (black), then the pair $(P(v, n(v)), S(n(v)))$ ($(S(n(v)), P(v, n(v)))$, respectively) is (black,white)-closed (see Fig. 1). Hence,

$$\left(\bigcup_{v \in W \cap L} P(v, n(v)) \cup \bigcup_{v \in (B \cup R_w) \cap L} S(n(v)), \bigcup_{v \in B \cap L} P(v, n(v)) \cup \bigcup_{v \in (W \cup R_b) \cap L} S(n(v)) \right)$$

is a pair of disjoint (black,white)-closed induced acyclic subgraphs of G , because $d(a, b) \geq 5$, for every different $a, b \in L$.

Hence, by Lemma 1, G has a pair (X, Y) of disjoint induced acyclic subgraphs which together contain all vertices of G and such that $P(v, n(v)) \subset X$, for $v \in W \cap L$, $S(n(v)) \subset X$, for $v \in R_w \cap L$, $P(v, n(v)) \subset Y$, for $v \in B \cap L$ and $S(n(v)) \subset Y$, for $v \in R_b \cap L$. $\square$

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