J Comb Optim (2015) 30:668–674
DOI 10.1007/s10878-013-9658-0

The Cartesian product of cycles with small 2-rainbow domination number

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Published online: 19 September 2013

Abstract The concept of 2-rainbow domination of a graph G coincides with the ordinary domination of the prism $G \Box K_2$ (see Brešar et al., Taiwan J Math 12:213–225, 2008). Hence $\gamma_{r2}(C_m \Box C_n) \geq \frac{mn}{3}$. In this paper we give full characterization of graphs $C_m \Box C_n$ with $\gamma_{r2}(C_m \Box C_n) = \frac{mn}{3}$.

Keywords Domination · Rainbow domination · Cartesian product of graphs

Mathematics Subject Classification 05C69

1 Introduction

For notation and graph theory terminology not given here, we follow Diestel (1997) and Haynes et al. (1998). Let $G = (V(G), E(G))$ be a finite, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex v is $N(v) = \{u \in V(G) : uv \in E(G)\}$. If $A \subset V(G)$, then $N(A)$ denotes the union of open neighborhoods of all vertices of A. For two subsets A, B of $V(G)$, $E(A, B) = \{ab \in E(G) : a \in A, b \in B\}$.

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The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other.

We restrict our attention to the Cartesian product of C_n and C_m, $n, m \geq 3$. Let $V(C_n) = \{0, 1, \ldots, n-1\}$, $E(C_n) = \{i(i+1), (n-1)0 : i = 0, 1, \ldots, n-2\}$. Hence we will denote vertices of $V(C_m \Box C_n)$ by (i, j) for $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, n-1$. For the arbitrary integers i and j we will use the following notation

$$[i, j] = (i \bmod m, j \bmod n).$$

A function $f : V(G) \to \mathscr{P}(\{1, \ldots, k\})$ is called a *k-rainbow dominating function of G* (for short $kRDF$ of G) if $\bigcup_{u \in N(v)} f(u) = \{1, \ldots, k\}$, for each vertex $v \in V(G)$ with $f(v) = \emptyset$. By $w(f)$ we mean $\sum_{v \in V(G)} |f(v)|$ and we call it the *weight of a function f* in G. The minimum weight of a $kRDF$ of G is called the *k-rainbow domination number of G* and it is denoted by $\gamma_{rk}(G)$. If f is a $kRDF$ of G and $w(f) = \gamma_{rk}(G)$, then f is called a *γ_{rk}-function*. For more information about rainbow domination we refer the reader to Brešar and Šumenjak (2007), Tong et al. (2009), Hartnell and Rall (1998), Wu and Rad (2013), Wu and Xing (2010), Šumenjak et al. (2013) and Xu (2009), where authors consider, in particular, connections between rainbow domination and Vizing conjecture.

Let f be any $2RDF$ of $C_m \Box C_n$. Define the following sets

$$\begin{aligned}
V_0 &= \{v \in V(C_m \Box C_n) : f(v) = \emptyset\},\\
V_1 &= \{v \in V(C_m \Box C_n) : f(v) = \{1\} \text{ or } f(v) = \{2\}\},\\
V_2 &= \{v \in V(C_m \Box C_n) : f(v) = \{1, 2\}\},\\
V_{i_1 i_2} &= \{v \in V_0 : |N(v) \cap V_t| = i_t, t = 1, 2\},\\
E_1 &= \{uv \in E(C_m \Box C_n) : u, v \in V_1\},\\
E_2 &= \{uv \in E(C_m \Box C_n) : u, v \in V_2\},\\
E_{12} &= \{uv \in E(C_m \Box C_n) : u \in V_1, v \in V_2\}.
\end{aligned}$$

We need the following technical lemma.

Lemma 1 (Stępień and Zwierzchowski 2012) *Let f be any $2RDF$ of $C_m \Box C_n$. Then*

$$w(f) = \frac{mn}{3} + \frac{\beta}{6},$$

where

$$\begin{aligned}
\beta = {} & 2|V_2| + |V_{11}| + 3|V_{12}| + 5|V_{13}| + 2|V_{21}| + 4|V_{22}| + |V_{30}|\\
& +3|V_{31}| + 2|V_{40}| + 2|V_{02}| + 4|V_{03}| + 6|V_{04}|\\
& +3|E_{12}| + 2|E_1| + 4|E_2|.
\end{aligned}$$

Corollary 1 *Let f be any $2RDF$ of $C_m \Box C_n$. Then $w(f) \geq \frac{mn}{3}$ and equality holds if and only if $\beta = 0$.*

In this paper we will use the following form of the Chinese Reminder Theorem.

Theorem 1 (Chinese Reminder Theorem) *Two simultaneous congruences*

$$\begin{aligned} x &\equiv a \pmod{m}, \\ x &\equiv b \pmod{n} \end{aligned}$$

are solvable if and only if $a \equiv b \pmod{\gcd(m,n)}$. *Moreover the solution is unique modulo* $\mathrm{lcm}(m,n)$.

2 Results

For any integer s, let $L_s = \{[k, k-s] \in V(C_m \square C_n) : k = 0, \pm 1, \pm 2, \ldots\}$. The following theorem is a consequence of the Chinese Reminder Theorem.

Theorem 2 *We have*

$$V(C_m \square C_n) = \bigcup_{s=0}^{\gcd(m,n)-1} L_s.$$

The sum is disjoint and $|L_s| = \mathrm{lcm}(m,n)$.

Proof By definition of L_s we have $\bigcup_{s=0}^{\gcd(m,n)-1} L_s \subseteq V(C_m \square C_n)$. Let $(i, j) \in V(C_m \square C_n)$ and let $s \in \{0, 1, \ldots, \gcd(m,n) - 1\}$ be such that $s \equiv i - j \pmod{\gcd(m,n)}$. By Theorem 1 there exists an integer k such that

$$\begin{aligned} k &\equiv i \pmod{m}, \\ k &\equiv j + s \pmod{n}. \end{aligned}$$

Consequently, $(i, j) = [k, k-s] \in L_s$.

Next if the above system has any solution for a fixed (i, j) and some s, then again by Theorem 1 we have $s \equiv i - j \pmod{\gcd(m,n)}$. Hence $L_{s_1} \cap L_{s_2} = \emptyset$ for $s_1 \neq s_2$ and $s_1, s_2 \in \{0, 1, \ldots, \gcd(m,n) - 1\}$. Finally, observe that cardinality of L_s is the same for each s. Therefore $|L_s| = \mathrm{lcm}(m,n)$. □

For any integer s, let us denote

$$[\![s]\!] = s \bmod \gcd(m,n).$$

Corollary 2 *The following holds:*

1. *for any integers* i, j *we have* $[i, j] \in L_{[\![i-j]\!]}$,
2. *if* $\gcd(m,n) > 1$, *then for any integer* s *we have*

$$N\left(L_{[\![s]\!]}\right) = L_{[\![s-1]\!]} \cup L_{[\![s+1]\!]},$$

3. *if* $\gcd(m,n) = 1$, *then we have*

$$V(C_m \square C_n) = L_0.$$

Now we introduce some definitions. Let f be a 2-rainbow dominating function of $C_m \Box C_n$. We say that f is *positive* if for any $(i, j) \in V(C_m \Box C_n)$ the following implication holds:

$$(i, j) \in V_1 \Rightarrow L_{[\![i-j]\!]} \subset V_1.$$

Let $L_s^- = \{[k, -k+s] \in V(C_m \Box C_n) : k = 0, \pm 1, \pm 2, \ldots\}$. Note that $[i, j] \in L^-_{[\![i+j]\!]}$. We say that f is *negative* if for any $(i, j) \in V(C_m \Box C_n)$ the following implication holds:

$$(i, j) \in V_1 \Rightarrow L^-_{[\![i+j]\!]} \subset V_1.$$

Lemma 2 *Let f be a 2-rainbow dominating function of $C_m \Box C_n$ such that $w(f) = \frac{mn}{3}$. Then f is either positive or negative.*

Proof Let f be a 2-rainbow dominating function of $C_m \Box C_n$ such that $w(f) = \frac{mn}{3}$. By Corollary 1, we get $\beta = 0$. Hence $|V_2| = |E_1| = |V_{30}| = |V_{40}| = 0$.

Take any vertex $(i, j) \in C_m \Box C_n$ such that $(i, j) \in V_1$. Assume, without loss of generality, that $f((i, j)) = \{1\}$. Since $|E_1| = 0$, we have

$$N((i, j)) \subset V_0. \tag{1}$$

We claim that also

$$\{[i-2, j], [i+2, j], [i, j-2], [i, j+2]\} \subset V_0. \tag{2}$$

To prove (2) suppose the contrary: assume, without loss of generality, that $[i, j+2] \in V_1$. Then $[i+1, j+1], [i+1, j+2] \in V_0$ (otherwise $|V_{30} \cup V_{40} \cup E_1| \neq 0$). Since the vertex $[i+1, j+1]$ must be dominated in the sense of 2-rainbow domination, we get $|V_2| \neq 0$, a contradiction.

Observe that exactly one of $[i+1, j+1]$, $[i+1, j-1]$ belongs to V_1. Indeed, on the one hand at most one of $[i+1, j+1]$, $[i+1, j-1]$ belongs to V_1, since otherwise $|V_{30}| \neq 0$. On the other hand at least one of $[i+1, j+1]$, $[i+1, j-1]$ belongs to V_1 and $f([i+1, j+1]) = \{2\}$ or $f([i+1, j-1]) = \{2\}$ (otherwise $[i+1, j]$ would not be dominated in the sense of 2-rainbow domination). Thus either $f([i+1, j+1]) = \{2\}$ or $f([i+1, j-1]) = \{2\}$.

Assume that $f([i+1, j+1]) = \{2\}$. This assumption combined with (1) and (2) imply that $[i+2, j+2] \in V_1$. By induction we get $L_{[[i-j]]} \subset V_1$. Similarly, if $f([i+1, j-1]) = \{2\}$, then we get $L^-_{[[i+j]]} \subset V_1$. We have shown that if $(i, j) \in V_1$, then either $L_{[[i-j]]} \subset V_1$ or $L^-_{[[i+j]]} \subset V_1$. Thus f is positive or negative.

Finally, suppose that f is positive and negative. This means that $L_{s_1} \subset V_1$ and $L^-_{s_2} \subset V_1$ for $s_1, s_2 \in \{0, 1, \ldots, \gcd(m, n) - 1\}$. To eliminate this possiblity, we will consider the following two cases.

(a) There exists k such that $s_1 + s_2 \equiv 2k \pmod{\gcd(m, n)}$. Consider the following system of simultaneous congruences

$$l \equiv k \pmod{m},$$
$$l \equiv -k + s_1 + s_2 \pmod{n}.$$

By Theorem 1, there exists a solution l. Hence $[k, k - s_1] = [l, -l + s_2]$, which means that $L_{s_1} \cap L^-_{s_2} \neq \emptyset$. It is easy to see that this contradicts the fact that $|V_{30}| = 0$.

(b) For all k we have $s_1 + s_2 \equiv 2k + 1 \pmod{\gcd(m, n)}$. Consider the following system of simultaneous congruences for some fixed k

$$l \equiv k \pmod{m},$$
$$l \equiv -k + s_1 + s_2 - 1 \pmod{n}.$$

By Theorem 1, there exists a solution l. Hence $[l, -l + s_2 - 1] = [k, k - s_1] \in L_{s_1} \subset V_1$. Since vertices $[l, -l+s_2-1]$ and $[l, -l+s_2] \in L^-_{s_2} \subset V_1$ are adjacent, it contradicts the fact that $|E_1| = 0$.

Thus f is either positive or negative. □

Lemma 3 *Let f be a 2-rainbow dominating function of $C_m \Box C_n$ such that $w(f) = \frac{mn}{3}$ then*

1. $\mathrm{lcm}(m, n) \equiv 0 \pmod{2}$,
2. $\gcd(m, n) \equiv 0 \pmod{3}$.

Proof Let f be a 2-rainbow dominating function of $C_m \Box C_n$ such that $w(f) = \frac{mn}{3}$. By Lemma 2, f is either positive or negative. Assume, without loss of generality, that f is positive. Take any vertex $(i, j) \in C_m \Box C_n$ such that $(i, j) \in V_1$. Hence $L_{[\![i-j]\!]} \subset V_1$.

The same argument as in the proof of Lemma 2 shows that

$$f((i, j)) = f([i + 2, j + 2]) \quad \text{and} \quad f((i, j)) \neq f([i + 1, j + 1]).$$

Hence by induction we have $|L_{[[i-j]]}| \equiv 0 \pmod{2}$. This together with Theorem 2 proves (1).

Now we will prove (2). If $\gcd(m, n) = 1$, then by Corollary 2(3) we have $L_0 = V(C_m \Box C_n)$. Since f is positive, we have $L_0 = V_1$. Therefore, $w(f) = mn$. This contradicts our assumption that $w(f) = \frac{mn}{3}$. Consequently $\gcd(m, n) > 1$. Suppose that $\gcd(m, n) = 2$. Then (i, j), $[i + 1, j + 1]$, $[i + 2, j] \in L_{[\![i-j]\!]} \subset V_1$. This implies that $[i + 1, j] \in V_{30}$. However, this contradicts the fact that $|V_{30}| = 0$. Hence $\gcd(m, n) \geq 3$. Assume now that $\gcd(m, n) > 3$.

By Corollary 2 and inclusions (1), (2) we get

$$L_{[\![i-j]\!]} \subset V_1, \quad L_{[\![i-j+1]\!]} \subset V_0, \quad L_{[\![i-j+2]\!]} \subset V_0.$$

In particular $[i + 1, j]$, $[i + 2, j + 1]$, $[i + 2, j] \in V_0$. To dominate $[i + 2, j]$ we must have $[i + 3, j] \in V_1$, and consequently $L_{[\![i-j+3]\!]} \subset V_1$. Continuing in this way we get that for any $l \geq 1$ we have

$$L_{[\![i-j+3l]\!]} \subset V_1, \quad L_{[\![i-j+3l+1]\!]} \subset V_0, \quad L_{[\![i-j+3l+2]\!]} \subset V_0. \tag{3}$$

To prove (2) we must eliminate the following two possibilities.

(a) Let $\gcd(m,n) = 3k+1$ for some $k \geq 1$. Now $L_{[\![i-j+3k]\!]} \subset V_1$ and $L_{[\![i-j+3k+1]\!]} = L_{[\![i-j]\!]} \subset V_1$. This contradicts (3).
(b) Let $\gcd(m,n) = 3k+2$ for some $k \geq 1$. Now $L_{[\![i-j+3k]\!]} \subset V_1$, $L_{[\![i-j+3k+1]\!]} \subset V_0$ and $L_{[\![i-j+3k+2]\!]} = L_{[\![i-j]\!]} \subset V_1$. This contradicts (3). □

Lemma 4 *For $k, l \geq 1$*

$$\gamma_{r2}(C_{6k} \Box C_{3l}) = \gamma_{r2}(C_{3l} \Box C_{6k}) = \frac{6k \cdot 3l}{3}.$$

Proof By Lemma 1, we have $\gamma_{r2}(C_m \Box C_n) \geq \frac{mn}{3}$. Hence for the proof it suffices to find a 2RDF of $C_{6k} \Box C_{3l}$ of weight $\frac{6k3l}{3}$. First we define $f : V(C_6 \Box C_3) \to \mathscr{P}(\{1,2\})$ as follows

$$\begin{array}{cccccc} \varnothing & \varnothing & \{1\} & \varnothing & \varnothing & \{2\} \\ \varnothing & \{2\} & \varnothing & \varnothing & \{1\} & \varnothing \\ \{1\} & \varnothing & \varnothing & \{2\} & \varnothing & \varnothing \end{array}.$$

It is easy to see that f is a 2RDF of $C_6 \Box C_3$ of weight 6. The required function on $C_{6k} \Box C_{3l}$ one can construct using this segment. Finally, the equality $\gamma_{r2}(C_{6k} \Box C_{3l}) = \gamma_{r2}(C_{3l} \Box C_{6k})$ follows by the symmetry. □

We are ready to prove our main result.

Theorem 3 $\gamma_{r2}(C_m \Box C_n) = \frac{mn}{3}$ *if and only if* $m = 6k$ *and* $n = 3l$ *or* $m = 3k$ *and* $n = 6l$, $k, l \geq 1$.

Proof Let $\gamma_{r2}(C_m \Box C_n) = \frac{mn}{3}$ and f be a $2RDF$ of $C_m \Box C_n$ such that $w(f) = \frac{mn}{3}$. By Lemma 3, we have $\gcd(m,n) \equiv 0 \pmod 3$ and $\operatorname{lcm}(m,n) \equiv 0 \pmod 2$. Hence $m \equiv 0 \pmod 3$, $n \equiv 0 \pmod 3$ and at least one of m and n is even. This together with Lemma 4 proves our theorem. □

The following theorem is the consequence of our considerations.

Theorem 4 *Let $m = 6k$, $n = 3l$. There are $6 \cdot 2^{\frac{\gcd(m,n)}{3}}$ γ_{r2}-functions of $C_m \Box C_n$ and $2^{\frac{\gcd(m,n)}{3}}$ γ_{r2}-functions up to translations.*

References

Brešar B, Šumenjak TK (2007) Note on the 2-rainbow domination in graphs. Discret Appl Math 155:2394–2400
Brešar B, Henning MA, Rall DF (2008) Rainbow domination in graphs. Taiwan J Math 12(1):213–225

Diestel R (1997) Graph theory. Springer-Verlag, New York

Hartnell B, Rall DF (1998) Domination in Cartesian products: Vizing's conjecture. In: Haynes TW, Hedetniemi ST, Slater PJ (eds) Domination in graphs: advanced topics. Marcel Dekker, New York, pp 163–189

Haynes TW, Hedetniemi ST, Slater PJ (1998) Fundamentals of domination in graphs. Marcel Dekker, New York

Stępień Z, Zwierzchowski M (2012) 2-Rainbow domination number of Cartesian products: $C_n \Box C_3$ and $C_n \Box C_5$, J Comb Optim. doi:10.1007/s10878-012-9582-8

Šumenjak TK, Rall DF, Tepeh A (2013) Rainbow domination in the lexicographic product of graphs. Discret Appl Math 161:2133–2141. arXiv:1210.0514v2 [math.CO]

Tong C, Lin X, Yang Y, Luo M (2009) 2-Rainbow domination of generalized Petersen graphs $P(n, 2)$. Discret Appl Math 157:1932–1937

Wu Y, Rad NJ (2013) Bounds on the 2-rainbow domination number of graphs. Graphs Comb 29:1125–1133

Wu Y, Xing H (2010) Note on 2-rainbow domination and roman domination in graphs. Appl Math Lett 23:706–709

Xu G (2009) 2-Rainbow domination in generalized Petersen graphs $P(n, 3)$,. Discret Appl Math 157:2570–2573