J Comb Optim (2015) 29:367–372
DOI 10.1007/s10878-013-9595-y

General lower bound on the size of $(H; k)$-stable graphs

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Published online: 15 February 2013

Abstract A graph G is called $(H; k)$-vertex stable if G contains a subgraph isomorphic to H ever after removing any k of its vertices. By $\mathrm{stab}(H; k)$ we denote the minimum size among the sizes of all $(H; k)$-vertex stable graphs. In this paper we present a first (non-trivial) general lower bound for $\mathrm{stab}(H; k)$ with regard to the order, connectivity and minimum degree of H. This bound is nearly sharp for $k = 1$.

Keywords Vertex-stable graph · Minimum degree · Connectivity

1 Introduction

By a word graph we mean a simple graph in which multiple edges (but not loops) are allowed. Given a graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. Furthermore, $|G| := |V(G)|$ is the order of G and $||G|| := |E(G)|$ is the size of G. By $N_G(x)$ we denote the set of vertices adjacent with x in G. For a vertex set X, the set $N_G(X)$ denotes the external neighbourhood of X in G, i.e.

$$N_G(X) = \{y \in V(G) \setminus X : y \text{ is adjacent with some } x \in X\}.$$

Consider the following problem. Suppose that we want to build a construction having certain properties using elements from sets S_1, ..., S_t. Each element from a set S_i has a given cost c_i. Thus, the total cost (depending only on the numbers and costs of used elements) of every construction can be computed.

We will consider this kind of problem in case where the feasible constructions are graphs with certain properties and accesible elements are vertices and edges.

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We require that a feasible (from our point of view) graph G (feasible construction) contains a given subgraph H. In fact, we require more. Some sensors may get damaged, hence, we want that even if some of them are spoiled, the special configuration of sensors and connections is still assured in the net. Clearly, we want to assure this configuration with minimal cost. The following problem has attracted some attention recently. Let H be any graph and k a non-negative integer. A graph G is called $(H; k)-vertexstable$ (in short $(H; k)-stable$) if G contains a subgraph isomorphic to H ever after removing any k of its vertices. Then $\mathrm{stab}(H; k)$ denotes minimum size among the sizes of all $(H; k)$-vertex stable graphs. Note that if H does not have isolated vertices then after adding to or removing from a $(H; k)$-vertex stable graph any number of isolated vertices we still have a $(H; k)$-vertex stable graph with the same size. Therefore, in the sequel we assume that no graph in question has isolated vertices.

The notion of $(H; k)$-vertex stable graphs was introduced in Dudek et al. (2008) (an edge version of this notion was also considered, see Frankl and Katona 2008; Horváth and Katona 2011). So far the exact value of $\mathrm{stab}(H; k)$ for any k is known in the case when $H = C_3, C_4, K_4, K_{1,m}$ (Dudek et al. 2008), $H = K_5$ (Fouquet et al. 2012a), and $H = K_q$ with k sufficiently large (Żak 2011). On the other hand, for small k the value $\mathrm{stab}(H; k)$ is known when $H = K_{m,n}$ and $k = 1$, see Dudek and Zwonek (2009) and Dudek and Żak (2010), and when $H = K_n$ and $k \leq n/2 + 1$, see Fouquet et al. (2012b). In all the above cases minimum vertex stable graphs are characterized. Furthermore, stab $(C_n; 1)$ is known for infinitely many n's and for remaining n's it has one of only two possible values, see Cichacz et al. (2011). An upper and a lower bound on stab $(C_n; k)$ for sufficiently large n is also presented therein.

Our aim in this paper is to prove a more general result. Namely, we give a lower bound for the size of a $(H; k)$ stable graph, where H is an arbitrary graph. The bound depends on the order, connectivity and minimum degree of H. This generalizes a similar lower bound obtained for $k = 1$ only in Cichacz et al. (2012).

Theorem 1 *Let H be a graph of order n, minimal degree $\delta \geq 1$ and connectivity $\kappa \geq 1$. If $\frac{\delta}{2}(n + 1 - \kappa) \geq \sqrt{k\delta\kappa n} + \frac{k(1+\delta-\delta\kappa)}{2}$ then*

$$\mathrm{stab}(H; k) \geq \frac{\delta}{2}n + \sqrt{k\delta\kappa n} - \frac{k(\delta\kappa - \delta - 1)}{2} \tag{1}$$

In particular, Theorem 1 leads to a new lower bound for stab $(C_n; k)$ which, for $k \geq 2$, is signifficantly better than the one obtained in Cichacz et al. (2011). Namely,

Corollary 2 *If $n \geq 3k + \sqrt{k^2 + k} + 1$ then*

$$\mathrm{stab}(C_n; k) \geq n + 2\sqrt{kn} - \frac{k}{2}.$$

In Sect. 3 we present a family of graphs for which our new lower bound gives reasonable estimates.

On the other hand, for regular graphs our bound gives

$$\mathrm{stab}(H; k) \geq ||H|| + \sqrt{k\delta\kappa|H|} - \frac{k(\delta\kappa + \delta + 1)}{2},$$

which in many cases is signifficantly better than the trivial bound

$$\mathrm{stab}(H; k) \geq ||H|| + k\Delta(H).$$

2 Proof of Theorem 1

We start with the following inequality which will be used later

Proposition 3 *If k, l, m are positive integers with $m \geq k + l$, then*

$$\frac{m(m-1)...(m-k+1)}{(m-l)(m-l-1)...(m-l-k+1)} \geq \frac{m+k(l-1)}{m-k}. \tag{2}$$

Proof The proof is by induction on k. For $k = 1$ the assertion is easy to check. Assume that $k > 1$ and the inequality is true for $k - 1$. Then

$$\begin{aligned} &\tfrac{m(m-1)...(m-k+2)(m-k+1)}{(m-l)(m-l-1)...(m-l-k+2)(m-l-k+1)} \\ &\geq \tfrac{m+(k-1)(l-1)}{m-k+1}\,\tfrac{m-k+1}{m-l-k+1} \quad \text{by the induction hypothesis} \\ &= \tfrac{m+(k-1)(l-1)}{m-l-k+1} = 1 + \tfrac{kl}{m-k-l+1} \geq 1 + \tfrac{kl}{m-k} = \tfrac{m+k(l-1)}{m-k}. \end{aligned}$$

□

Recall the following observation.

Proposition 4 (Dudek et al. (2008)) *Let δ_H be a minimal degree of a graph H. Then in any $(H; k)$-vertex stable graph G with minimum size, $\deg_G v \geq \delta_H$ for each vertex $v \in G$.*

Proof of Theorem 1 Let G be a $(H; k)$ stable graph with minimum size and let $|G| = v$. Let $S = \{x_1, ..., x_m\} \subset V(G)$ be a set of vertices of degree greater than or equal to $\delta + 1$ in G. By Proposition 4 all other vertices of G have degree δ. Let $C_1, ..., C_q$ be components of $G - S$. Let $\tilde{G}$ be a graph that arises from G by contracting every edge of each C_i, $i = 1, ..., q$.

Suppose first that $\tilde{G}$ contains a vertex $u \notin S$ with at most $\kappa - 1$ neighbors in $\tilde{G}$. Consider a component C in G that corresponds to u. Since G is minimal $(H; k)$-stable, every vertex of C is contained in some copy of H. So, consider a copy of H that contains at least one vertex of C. Note, that this copy of H may contain only vertices from C and the vertices which are neighbors of u in $\tilde{G}$, because, otherwise H contains a cutting set of cardinality less than κ. Thus, C contains at least $n + 1 - \kappa$ vertices,

$$|C| \geq n + 1 - \kappa. \tag{3}$$

Note that after removing from G any vertex $x_i \in N_G(C)$ each vertex of C is not any longer a vertex of H. Indeed, after removing x_i, its neighbors in $C \cap G$ have degree less than δ. Thus, they cannot be in H. Hence, their neighbors in $C \cap G$ would have degrees less than δ in H. Thus, the latter vertices connot be in H neither, and so on. Therefore, since G is $(H; k)$ stable, $G - C$ contains a copy of H. Thus, $||G - C|| \geq ||H|| \geq \frac{n\delta}{2}$. Hence, by (3) and by the assumption on n,

$$||G|| \geq \frac{n\delta}{2} + \frac{|C|\delta}{2} \geq \frac{n\delta}{2} + \frac{\delta}{2}(n + 1 - \kappa) \geq \frac{\delta}{2}n + \sqrt{k\delta\kappa n} - \frac{k(\delta\kappa - \delta - 1)}{2}. \quad (4)$$

Therefore, assume that every vertex $u \in V(\tilde{G}) \setminus S$ has at least κ neighbors in $\tilde{G}$. We may assume that $m \geq k + \kappa$. Indeed, otherwise by removing k vertices from S, we remove at least one vertex $x \in N_G(C)$ for each component $C \in \{C_1, ..., C_q\}$. Hence, all remaining vertices (possibly except $\kappa - 1 < n$ vertices from S) become useless for H.

We will randomly delete exactly k vertices from S. Hence, each k-tuple is removed with probability $\binom{m}{k}^{-1}$. In the resulting graph some vertices may have degree less than δ and so do not belong to any copy of H. For $u \in V(G) \setminus S$ let X_u denote the indicator random variable with value 1 if u belongs to some copy of H. Since $w \in V(\tilde{G}) \setminus S$ has at least κ neighbors in $\tilde{G}$, the probability that $u \in V(G) \setminus S$ can be used in some copy of H is less than or equal to $\frac{\binom{m-\kappa}{k}}{\binom{m}{k}}$. Hence,

$$E(X_u) \leq \frac{\binom{m-\kappa}{k}}{\binom{m}{k}}.$$

Thus, the expected value of vertices from $V(G) \setminus S$ that can be used in a copy of H is

$$E\left(\sum_{u \in V(G)\setminus S} X_u\right) = \sum_{u \in V(G)\setminus S} E(X_u) \leq \frac{(v - m)\binom{m-\kappa}{k}}{\binom{m}{k}}.$$

Hence,

$$\begin{aligned} E\left(\sum_{u \in V(G)\setminus S} X_u\right) &\leq (v - m)\frac{(m - \kappa)...(m - \kappa - k + 1)}{m...(m - k + 1)} \\ &\leq (v - m)\frac{m - k}{m + k(\kappa - 1)}, \end{aligned} \quad (5)$$

by inequality (2). Thus, there are k vertices in S such that after deleting them we can use in H at most $m - k + (v - m)\frac{m-k}{m+k(\kappa-1)}$ vertices of G. Therefore,

$$m - k + (v - m)\frac{m - k}{m + k(\kappa - 1)} \geq n, \text{ and so}$$

$$v - m \geq (n - m + k)\frac{m + k(\kappa - 1)}{m - k}$$

Hence,

$$||G|| \geq \tfrac{\delta(v-m)+(\delta+1)m}{2} \geq \tfrac{\delta}{2}(n-m+k)\tfrac{m+(\kappa-1)k}{m-k} + \tfrac{\delta+1}{2}m =: f(m)$$

It is not difficult to check (by examining the derivative) that the function $f(x) = \frac{\delta}{2}(n-x+k)\frac{x+(\kappa-1)k}{x-k} + \frac{\delta+1}{2}x$, $x \geq k+l$, has minimum in $x_0 = \sqrt{k\kappa\delta n} + k$. Hence, $||G|| \geq f(x_0) = \frac{\delta}{2}n + \sqrt{k\delta\kappa n} - \frac{k(\delta\kappa-\delta-1)}{2}$ □

3 Question of tightness

For $\kappa, \delta, n, k \in \mathbb{N}$ with $\kappa \leq \delta \leq n+1$ let

$$\begin{aligned}\operatorname{stab}(n,\kappa,\delta;k) := \min\{&\operatorname{stab}(H;k) : H \text{ has order } n,\\ &\text{connectivity } \kappa \text{ and minimum degree } \delta\}.\end{aligned}$$

Theorem 5 *Let $\kappa, \delta, k \in \mathbb{N}$ where κ is even, $\kappa \leq \delta$. Then for each $n = p^2\delta\kappa$, where $p \geq 2$ is an arbitrary integer,*

$$\operatorname{stab}(n,\kappa,\delta;k) \leq \frac{\delta}{2}n + k\sqrt{\delta\kappa n} + k^2\frac{\kappa}{2}. \tag{6}$$

Proof Let κ, δ be fixed and $t = p^2\delta$ for some integer $p \geq 2$. In order to show the upper bound, we construct the following multigraph $H(t)$. Let $V_i = \{v_{i,1}, ..., v_{i,\kappa/2}\}$ and $V_i' = \{v_{i,1}', ..., v_{i,\kappa/2}'\}$ for $i = 0, ..., t-1$. Then

$$V(H(t)) := V_0 \cup V_0' \cup V_1 \cup V_1' \cup \ldots \cup V_{t-1} \cup V_{t-1}'$$

and

$$E(H(t)) = \binom{V_i \cup V_i'}{2} \cup \{v_{i,j}'v_{i+1,j} : j = 1, ..., \kappa/2\}, \quad i = 0, ..., t-1,$$

where edges $v_{i,j}v_{i,j}'$, $j = 1, ..., \kappa/2$, have multiplicity $\delta - \kappa + 1$. Therefore, there is a clique built on $V_i \cup V_i'$, where some edges are multiple (such that $G[V_i \cup V_i']$ is $\delta - 1$ regular), and there is a perfect matching between V_i' and V_{i+1} ($i+1$ taken modulo t). It is easy to see that $H(t)$ is δ-regular and has connectivity κ. Furthermore, $|H(t)| = t\kappa$ and $||H(t)|| = \frac{t\kappa\delta}{2}$.

We will construct a $(H(t);k)$ stable graph with as few as possible number of edges. Let $G_k(t)$, $0 \leq k \leq t/p$, be a graph which arises from $H(t)$ by adding edges $\{v_{ip,j}'v_{ip+p+1,j}, ..., v_{ip,j}'v_{ip+kp+1,j} : j = 1, ..., \kappa/2\}$ for all $i = 0, \ldots, \frac{t}{p}$ (with indices taken modulo t).

We will show, by induction on k, that $G_k(t+pk)$ is $(H(t);k)$-stable. This is obvious for $k = 0$. Assume that $k \geq 1$ and the statement is true for $G_{k-1}(t+p(k-1))$. It is suficient to prove that for each $x \in V(G_k(t+pk))$, $G_k(t+pk)-x$ is $(H(t); k-1)$-stable. Without loss of generality we may assume that $x \in V_i \cup V_i'$, $i \in \{1, \ldots, p\}$. It can be seen that

$$G_k(t+pk)\left[V_0 \cup V_0' \cup V_{p+1} \cup V_{p+1}' \cup \ldots \cup V_{t+pk-1} \cup V_{t+pk-1}'\right] \\ \supset G_{k-1}(t+p(k-1)).$$

Hence, $G_{k-1}(t+p(k-1))$ is a subgraph of $G_k(t+pk)-x$. Thus, by the induction hypothesis, $G_k(t+pk)-x$ is $(H(t); k-1)$-stable. Therefore, $G_k(t+pk)$ is $(H(t); k)$-stable.

Finally,

$$||G_k(t+pk)|| = ||H(t+pk)|| + \frac{t+pk}{p}\frac{\kappa}{2}k = \frac{(t+pk)\kappa\delta}{2} + \frac{t+pk}{p}\frac{\kappa}{2}k$$

Since $p = \sqrt{\frac{t}{\delta}}$ and $n = t\kappa$, we obtain

$$||G_k(t+pk)|| = \frac{n\delta}{2} + k\sqrt{n\delta\kappa} + k^2\frac{\kappa}{2}.$$

□

Note that Theorem 5 implies that for $k = 1$ the bound (1) is nearly best possible. Namely, the gap between (1) and (6) depends only on δ, κ and does not depend on n.

Acknowledgments The author was partially supported by the Polish Ministry of Science and Higher Education.

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