Journal of Mathematics Teacher Education
https://doi.org/10.1007/s10857-023-09570-x

The use of a scriptwriting task as a window into how prospective teachers envision teacher moves for supporting student reasoning

Victoria Shure[1] · **Peter Liljedahl**[2]

Accepted: 18 January 2023

Abstract
The development of mathematical reasoning skills has increasingly been of focus for the teaching and learning of mathematics. This research utilizes a teaching simulation using the methodology of scriptwriting, in which prospective teachers are asked to complete a script of a dialogue from a classroom simulation involving fraction multiplication and division with justification, assisting fictional students to work through their difficulties and helping them to justify their reasoning. Such tasks allow for the examination of the prospective teacher moves to support student reasoning through their imagined action and choice of words. Scripts from forty-one prospective primary teachers were examined for the study, and five clusters based on the type of teacher move for supporting student reasoning were found. Overall, the prospective teachers emphasized the elicitation and facilitation of students' ideas. The cluster analysis, however, provided a nuanced examination of the cohort's teacher moves. While cluster one saw the highest incident of *eliciting* teacher moves, albeit only in the low potential category, clusters two and three mostly used *facilitating* teacher moves, but varied in their use of high and low potential moves. Cluster four concentrated moves on *facilitating, eliciting,* and *responding to student reasoning*. Cluster five employed teacher moves from all main categories, with some instances of high potential moves in all categories except *extending student reasoning*, which can better support reasoning. The prospective mathematics teachers' scripts and the five clusters that were found during analysis are discussed with implications for future teacher education and the support of building mathematical reasoning.

Keywords Scriptwriting tasks · Mathematical reasoning · Teacher moves · Teacher education · Fraction multiplication · Fraction division

✉ Victoria Shure
victoria.shure@hu-berlin.de

1 Primary Mathematics Education, Institute of Education, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

2 Faculty of Education, Simon Fraser University, 8888 University Drive, Burnaby, BC V5A 1S6, Canada

Published online: 28 February 2023

Introduction

In recent years, the importance of developing students' mathematical reasoning has become a fundamental component of the everyday teaching and learning of mathematics (Boaler, 2010; Mata-Pereira & Da Ponte, 2017). At the same time, research has concentrated on understanding the type of teacher pedagogical activity or practices that support this development, with some studies focusing on discourse patterns and others on the types of teacher questioning that support student thinking, justification, or classroom participation (cf. Ellis et al., 2018; Herbel-Eisenmann et al., 2013; Stein et al., 2008). While in-service teachers' practices for supporting student reasoning can be more readily investigated in classroom settings, the examination of how prospective teachers envision support is less accessible, as prospective teachers often have few opportunities during their teacher education to actually engage in the practice of teaching (Grossman et al., 2009). Providing prospective teachers with simulated-based approaches gives us a window into how they conceptualize supporting student reasoning through the lens of how they plan classroom interactions. In recent years, in order to examine how prospective teaches envision interacting with students, simulation-based approaches such as scriptwriting tasks (Zazkis et al., 2013) have been utilized. The use of scriptwriting as a tool represents an opportunity to consider how prospective teachers envision supporting students' reasoning. By better understanding how prospective teachers envision supporting students to reason mathematically, more targeted support for prospective mathematics teachers can be developed.

This development of reasoning must also occur in the context of mathematical content. For the primary level, or grades one to six, the construction of rational number and fraction knowledge make up an integral part of mathematics and are foundational for helping build proportional reasoning, which are essential for future mathematical learning (Clarke et al., 2008). Although fraction knowledge represents an important part of the curriculum, students' as well as prospective teachers' conceptual understanding has, however, often been shown to be limited (Ball, 1990; Castro-Rodríguez et al., 2016; Marmur et al., 2019). Prospective teacher education thus needs to focus on strengthening conceptual understanding of fractions, thereby enabling prospective teachers to support students to engage in reasoning. Prospective teachers need to not only recognize when students are struggling with conceptual understanding of fractions, but also what teacher moves can support better understanding and reasoning in this content area. In turn, it is important to investigate the teacher moves prospective teachers envision for this support. In this paper we use scriptwriting as a research tool to examine how prospective primary mathematics teachers plan to support students reason mathematically by engaging in mathematical reasoning processes, and particularly in justification, with a simulation-based approach concerning fraction multiplication and division. The application of this research tool provides a window into prospective teachers' envisioned teacher moves while the use of a framework for supporting student reasoning (Ellis et al., 2018), with theoretically and empirically-grounded teacher moves, provides the lens of analysis for considering prospective teachers' moves.

Theoretical framework

In what follows we will first explore conceptions of student mathematical reasoning and the role teachers play in developing this competency. This will be followed by an examination of how teacher moves can support the development of the processes associated with reasoning. In the last section of the theoretical framework, we will discuss characteristics of scriptwriting tasks and their ability to provide prospective teachers with the opportunity to plan classroom interactions, including how they have been previously used to examine prospective teacher pedagogical choices concerning supporting reasoning.

Developing student reasoning and the role teachers play

In order to examine how prospective teachers support the development of student reasoning, it is important to determine what exactly mathematical reasoning entails, the role teachers play in developing this competency, and, lastly, how it fits into the theoretical framework of this study. Mathematical reasoning encompasses justifying a claim or result and understanding what validates an argument, in particular, is something that students begin to learn early on in their schooling and is important as they advance to later grades and learn more formal justifications, going well beyond the routine use of procedures (Mata-Pereira & Da Ponte, 2017). Mathematical reasoning processes are closely connected to the classroom discourse or the social practices that take place in the whole classroom discussion, with discursive moves such "as asking for fulfilling discursive demands (explaining, arguing, reporting, describing)" (Wessel & Erath, 2018, p. 1059). Within each mathematical reasoning process, there are discursive demands that both students and teachers must utilize to convey their meaning. As these processes take place in classroom interactions, and thus as a social endeavor, another central element that is closely related to mathematical reasoning is the sociocultural perspective in which discussions and learning are closely tied to one another (McCrone, 2005). As a part of supporting individual learning, the sociocultural perspective emphasizes inquiry-based instruction, highlighting active problem solving and student interaction through discussion mediated by the mathematics teacher (Bauersfeld, 1995; Elliott, 1996; McCrone, 2005).

These processes, however, and enabling students to make their mathematical thinking more explicit in particular require appropriate support from teachers (Franke et al., 2009, 2015), with teacher actions having an important effect on student learning (Webb et al., 2014). Consequently, mathematics education research has seen a continuing trend of examining how prospective teachers "probe more deeply fundamental mathematical ideas from the school curriculum linked to the learner's activity and related mathematical understanding" (Da Ponte & Chapman, 2015, p. 281), which has also been paralleled with reforms to various national standards concerning mathematical reasoning (e.g., National Council of Teachers of Mathematics, 2000; Kultusministerkonferenz, 2004). Research in the last years has also accordingly highlighted the need to further develop classroom communities that contribute to the development of reasoning (Mueller et al., 2014). The specific teacher actions that teachers can employ, and those which impact student reasoning in particular, will be discussed in the following section.

Teacher moves for supporting student reasoning

In order for such inquiry-based, student-oriented classroom environments to exist, teachers must be able to promote student inquiry and engagement in discussions, and to foster student reasoning. “The teacher has to support both the content of the discussion and its management” (Mata-Pereira & Da Ponte, 2017, p. 172). To support these aims, teachers’ pedagogical activities or practices are operationalized as teacher actions, also known as “teacher moves,” representing an essential part of professional practice (Mata-Periera & Da Ponte, 2017) and originating from teachers’ goals for a desired classroom activity (Christiansen & Walther, 1986).

Accordingly, various researchers have examined how teachers structure and manage classroom discussion (e.g., Franke et al., 2015; Henning et al., 2012), including the actions or teacher moves employed. In a comparative study by Kawanaka and Stigler (1999), the differences in how teachers in different international contexts managed discussions were highlighted: German middle grade mathematics teachers were found to provide information and elicit answers from students and provide students with more opportunities to speak in discussions, whereas US and Japanese teachers tended to focus more on providing information. Moreover, in contrast to US teachers, German teachers and students “engaged in complex problem-solving activities in which students had to explore the solution methods rather than apply the prescribed methods” (Kawanaka & Stigler, 1999, p. 276). In the U.S. context, Franke et al. (2009) identified four types of teacher questioning teachers used to elicit student thinking, including probing sequences with specific questions, general questions, specific questions, leading questions, and other questions, with probing sequences leading to further student elaboration and opportunity for students to express correct and complete explanations. Franke et al. (2015) expanded upon this research examining teachers’ initial actions in a discussion, highlighting the complex role of teachers’ decisions in what they asked of students and how they followed up with varying types of support.

In work concerning in-service teachers’ moves for facilitating mathematical argumentation in discussions, Kosko et al. (2014) examined the types of questions teachers plan to utilize when envisioning classroom situations following Franke et al. (2009) and then the specific questions teachers asked (Boaler & Brodie, 2004), drawing attention to the often passive facilitation of argumentation, meaning that teachers often remained silent and did not provide any facilitation or tended to ask students for other contributions and provide a teacher statement. These findings, in particular the finding that teachers often envisioned starting the discussion with a teacher statement, are in line with Kawanaka and Stigler’s (1999) findings that U.S. teachers frequently provided input or information.

Further studies have aimed to identify which teacher pedagogical activities can support the development of content, including mathematical reasoning processes (Ellis et al., 2018; Herbel-Eisenmann et al., 2013). In efforts to support teachers to foster student discussions and support the growth of student reasoning, a number of frameworks have been conceptualized. The results of a year-long case study of a teacher by Hufferd-Ackles et al. (2004) generated the development of a math-talk learning community framework, which focuses on ways to shift a classroom community to student-centered learning and a discourse orientation. Hufferd-Ackles et al. (2004) describe four dimensions in which a classroom can achieve this shift: (1) questioning, (2) explaining mathematical thinking, (3) source of mathematical ideas, and (4) responsibility for learning. The math-talk learning community framework, however, with its emphasis on student-centered action, did not focus specifically on teacher moves as a part of this shift.

Stein et al. (2008) created, and later further developed by Smith and Stein (2011), the *Five Practices for Facilitating Mathematical Discussions Around Cognitively Demanding Tasks* as a set of tools for teachers to be better prepared for discussions. The model includes five practices that teachers can incorporate into their planning: anticipating likely student responses to tasks, selecting certain student responses to discuss with the class, selecting students to present responses to the class, purposefully sequencing responses that will be discussed, and helping the class to make connections between responses and mathematical ideas. The emphasis of the tool was thus on a way to support teachers in using students' responses to further the mathematical understanding of the whole class, thereby improving their discussion facilitating (Stein et al., 2008).

With a similar focus, Leatham et al. (2015) developed a framework with the goal of identifying *Mathematically Significant Pedagogical Opportunities to Build on Student Thinking* (MOST). These instances are worth building on, as "student thinking [is] worth making the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea (Van Zoest et al., 2017, p. 36). While the MOST framework illuminated aspects of teachers' practice, the original focus of the framework primarily emphasized "students and their mathematics rather than teachers and their" mathematics (Leatham et al., 2015, p. 120).

Based on experience developing professional development practice-based materials, Herbel-Eisenmann et al. (2013) created the *Teacher Discourse Moves* (TDMs) with the goal of supporting mathematics teachers become "purposeful about engaging students in mathematical explanations, argumentation, and justification (p. 181). Furthermore, Herbel-Eisenmann et al. (2013) underscore the aim of helping teachers "thoughtfully plan for and use [these moves] to open up the classroom discourse" (p. 183). The framework includes such teacher discourse moves like: (1) waiting, (2) inviting student participation, (3) revoicing, (4) asking students to revoice, (5) probing students' thinking, and (6) creating opportunities to engage with another's reasoning (Herbel-Eisenmann et al., 2013).

Ellis et al. (2018) further extended the *Teacher Discourse Moves* (TDMs) tool from Herbel-Eisenmann et al. (2013) with a focus on supporting inquiry-oriented learning with students seen as problem solvers engaging in mathematical reasoning processes. In differentiation to the other frameworks, the Ellis et al. (2018) *Teacher Moves for Supporting Student Reasoning* (TMSSR) framework organizes teacher moves into four categories with subcategories across a continuum, based on their potential for supporting student reasoning (see Table 1). The four main categories of the Ellis et al. (2018) TMSSR framework include: (1) eliciting student reasoning, (2) responding to student reasoning, (3) facilitating student reasoning, and (4) extending student reasoning. While these main categories are reported to not be "strictly hierarchical… they represent a continuum of potential for supporting student reasoning, in that the extending moves were typically, but not always, more effective in fostering the processes of searching for similarity or difference, validating, and exemplifying" (Ellis et al., 2018, p. 117). Furthermore, Ellis et al. (2018) found that eliciting moves were often a first step, frequently followed by responding and facilitating teacher moves.

As is visible in Fig. 1, the subcategories of teacher moves within each of the four main categories are grouped based on their high or low potential for supporting student reasoning. Moves that are considered to be high potential give "students more responsibility as doers of mathematics" (Ellis et al., 2018, p. 116), whereas moves with less or low potential give "teachers a more prominent role" (Ellis et al., 2018, p. 116). An example from the framework that highlights this differentiation includes a teacher correcting student errors (low potential) versus a teacher prompting students to correct their own errors

	Eliciting Student Reasoning		Responding to Student Reasoning	
	Low ⟷	High	Low ⟷	High
	Eliciting Answer	Eliciting Ideas	Correcting Student Error	Prompting Error Correction
	Eliciting Facts or Procedures	Eliciting Understanding	Re-voicing	Re-representing
	Asking for Clarification	Pressing for Explanation	Encouraging Student Re-voicing	
	Figuring Out Student Reasoning		Validating a Correct Answer	
	Checking for Understanding			
	Facilitating Student Reasoning		**Extending Student Reasoning**	
	Low ⟷	High	Low ⟷	High
Guiding	Cueing	Providing Guidance	Encouraging Evaluation	Encouraging Reflection
Guiding	Funneling	Encouraging Multiple Solution Strategies	Pressing for Precision	Encouraging Reasoning
Guiding	Topaze Effect	Building	Topaze for Justification	Pressing for Justification
Providing	Providing Information	Providing Alternative Solution Strategies		Pressing for Generalization
Providing	Providing Procedural Explanation	Providing Conceptual Explanation		
Providing	Providing Summary Explanation			

Fig. 1 Ellis et al. (2018, p. 117) Teacher Moves for Supporting Student Reasoning (TMSSR) framework

(high potential). Ellis et al. (2018) explain, "we use the term potential rather than impact in recognition that our analysis focused more on the classroom discussion than on individual students' performance" (p. 116), with the consequence that the same moves could have different outcomes in different circumstances and thus just the potential of a particular move should be considered. While the framework emphasizes the role of the teacher and his or her moves for supporting student reasoning, the differentiation of low and high potential also relates to the student involvement in a discussion, with high potential teacher moves being more open, often pushing students to elaborate their ideas and be more active participants of the discussion. With this focus, the framework does not explicitly address how teachers can promote dialogue between students within the social context of the classroom, which has recently been considered regarding how teachers can orient students to the thinking of others (e.g., Shaughnessy et al., 2021b), but instead places emphasis on teacher moves that support students in becoming active leaners responsible for their learning.

The basis for the grouping of these teacher moves is grounded in the conceptual model of mathematical reasoning for the teaching and learning of school mathematics by Jeannotte and Kieran (2017). In addition to considering structural aspects (related to form) of mathematical reasoning, Jeannotte and Kieran (2017) further considered the processes associated with mathematical reasoning that take place in the school setting, as opposed to the theoretical idea of mathematical reasoning, and categorized processes associated with mathematical reasoning in the school environment into those related to (1) the search for similarities or differences and (2) validating. Furthermore, they highlighted exemplifying as a support for the two processes. With its foundation in the Jeannotte and Kieran (2017) conceptual model, the TMSSR framework (Ellis et al., 2018) represents a current and well-founded tool for examining a variety of teacher moves with varying degrees of potential that specifically support the development of student reasoning in the mathematics classroom.

While the aforementioned frameworks vary in their focuses, all of these frameworks are predicated on the idea that (a) the teacher has a classroom in which to implement the principles of the framework and (b) that the teacher has an experience base that gives him or her the motivation and confidence to implement them within his or her classroom. In essence, all of these frameworks are geared toward helping teachers to change their practice. The examination of prospective teachers' practices or teacher moves presents a complex undertaking, as this group encompasses teachers who do not yet have a practice, who do not have a practice to change, and do not have a classroom in which to implement these practices. The following section will address what is currently understood about prospective teachers' plans for supporting student reasoning and how this can be examined while considering these constraints.

Examining prospective teacher moves for supporting reasoning

One such methodology that works around these constraints to examine what teacher moves prospective teachers plan to employ to support students is the lesson play or scriptwriting task (Zazkis et al., 2009, 2013). Zazkis et al. (2009) designed the scriptwriting task as a tool that allows for the consideration of alternate student conceptions of content and student questions, which generally are not a part of the traditional lesson plan (Zazkis et al., 2009). One such example of a scriptwriting task that Zazkis et al. (2009) designed for prospective teachers to complete includes a conversation between a student and a teacher. The teacher asks the student "why do you say that 462 is divisible by 4?" and the student explains "because the sum of the digits is divisible by 4" (Zazkis et al., 2009, p. 53). The tool thereby allows prospective teachers to envision possible interactions with students in the form of a fictional dialogue or script and thereby to plan how they will address such alternate conceptions and questions with specific language and teacher moves. This exercise provides prospective teachers with a bridge between planning for interactions and the actual course of action in a classroom environment (Zazkis et al., 2013), as the fictional dialogue requires that the prospective teachers think beyond lesson content and desired outcomes to possible misunderstandings that must first be diagnosed and then clarified (Zazkis & Marmur, 2018). Thus, the scriptwriting task has the dual function of being a means of helping prospective teachers prepare for instruction as well as serving as a diagnostic tool in teacher education (Zazkis & Zazkis, 2016).

Recently, Buchbinder and Cook (2018) examined prospective teachers' mathematical knowledge for teaching in the context of scriptwriting involving a proving task in geometry, looking at both mathematical content knowledge and general pedagogical knowledge of twenty-seven prospective teachers, including how prospective teachers managed discussions. They provided examples of productive pedagogical moves related to leading discussions such as assessing agreement of an idea and encouraging discussion and non-productive moves such as providing praise for incorrect responses. Biza and Nardi (2020) incorporated a scripting approach into their research on how prospective teachers reflect on mathematical content and how it is taught, illustrating the importance of a teacher's questioning techniques on student (dis)engagement and, more broadly, the value of scriptwriting in teacher education. In a case study of two prospective teachers, Lim et al. (2018) examined learning trajectories for orchestrating productive mathematics discussions based on the frameworks from Smith and Stein (2011) and Hufferd-Ackles et al. (2004), which do not have an explicit focus on mathematical reasoning.

In addition to scriptwriting studies, other simulation-based research specifically concerning prospective teachers has focused on how prospective teachers elicit students' understanding (Shaughnessy et al., 2021a), as well as respond to students' errors (Campbell & Baldinger, 2021; Hallman-Thrasher, 2017), highlighting, for example, the importance of planning and calling for further examination of these teacher moves. These studies, however, did not consider the further practices identified by Ellis et al. (2018) that can support student reasoning, and instead primarily focused on the questions that the prospective teachers posed, in particular, relating to eliciting student understanding.

Previous studies outside the realm of scriptwriting and simulations have focused on both student and prospective mathematics teacher knowledge and pedagogical choices, in particular, in the specific content area of fraction multiplication and division, highlighting the challenges for prospective teachers and the frequent use of algorithms for solving fraction tasks without the construction of conceptual knowledge (Ball, 1990; Olanoff et al., 2014; Siebert, 2002; Tirosh, 2000). Isiksal and Cakiroglu (2011) assessed prospective primary teachers' pedagogical content knowledge of fraction multiplication, highlighting various ways the prospective teachers recognized student misconceptions concerning this area of knowledge and providing general suggestions from the prospective teachers of how to mitigate such student challenges such as using multiple representations, emphasizing practice, focusing on the meaning of the concept, or developing positive attitudes toward mathematics. These suggestions, however, remain general and do not provide an examination of the teacher moves prospective teachers employ to support students in overcoming misconceptions and reason in the context of fraction multiplication and division, which is possible with using the scriptwriting task as a research tool.

Further use and development of scriptwriting tasks as a tool for examining and building upon student reasoning competencies thereby represent an opportunity in teacher education and were selected as the research tool for this study, particularly as scriptwriting tasks have not yet been employed to examine the specific types of high and low potential teacher moves for supporting student reasoning in teacher education, and in particular, utilizing the Ellis et al. (2018) framework of teacher moves for supporting student reasoning going beyond eliciting and responding to students' reasoning. As previous work has indicated cultural differences in how teachers manage discussions (Kawanaka & Stigler, 1999), the application of the TMSSR framework (Ellis et al., 2018) in the German context also represents an opportunity for comparison, and in particular with prospective teachers. Furthermore, with the completion of a scriptwriting task, prospective teachers demonstrate how they plan to support students engage in reasoning without the requirement of a classroom setting.

Present study and research questions

This study utilizes a scriptwriting task as a tool to better understand how prospective teachers employ teacher moves for supporting student reasoning, and justification in particular, in the context of fraction multiplication and division. Fraction knowledge represents an essential foundation for the development of proportional reasoning and future learning (Clarke et al., 2008) and was thereby selected as the content area for this study. This area of research has grown in recent years with findings that prospective teachers' conceptual understanding is often weak (Ball, 1990; Castro-Rodríguez et al., 2016; Marmur et al., 2019). Notably, building conceptual understanding of fraction multiplication and division has proven difficult for both students (Rule & Hallagan, 2006) and in-service and prospective teachers (Alenazi, 2016; Hohensee & Jansen, 2017; Izsák et al., 2019; Ölmez & Izsák, 2021). An aspect of both fraction multiplication and division that proves challenging for students is the "met-before" (Tall, 2007, 2008) that the multiplication of two natural numbers produces a bigger result than the division of the same two natural numbers. This prior experience can lead to misunderstanding, or the "intuitively based mistake" (Tirosh, 2000, p. 7), when students operate on fractions and assume that when dividing fractions, the result must be smaller than the input, and when multiplying fractions, the result must be greater than the input, as they experienced with natural numbers.

To examine how prospective teachers plan to support the students work through their difficulties concerning fraction multiplication and division and envision supporting students reason and particularly justify, we provide prospective teachers with a scriptwriting task to create a simulated environment in which they plan their support of reasoning. Furthermore, we follow the Ellis et al. (2018) framework as a lens of analysis for the prospective teachers' moves for supporting student reasoning, considering the four main categories of teacher moves (eliciting, responding to, facilitating, and extending student reasoning), as well as the teacher moves that have high or low potential for supporting student reasoning. More specifically, we aim to study the clusters of teacher moves that are identifiable within the cohort of prospective teachers to consider frequent patterns of teacher moves. This analysis is situated within the sociocultural perspective by considering the manner in which the prospective teachers mediate the discussion such that students can actively engage in reasoning.

To operationalize these aims, this project draws upon the methodology of the scriptwriting task (Zazkis et al., 2013) as a tool for examining the manner in which prospective primary mathematics teachers plan to support students in developing mathematical reasoning skills, and what patterns of high and low potential teacher moves from the Ellis et al. (2018) teacher moves for supporting student reasoning (TMSSR) framework emerge in the scriptwriting tasks. Prospective primary mathematics teachers received the scriptwriting task, including the hypothetical student dialogue, were asked to identify student difficulties, and then complete the dialogue, attending to the identified student difficulties and supporting the students in completing the task using reasoning, thereby employing teacher moves. The following research questions guided the project:

RQ1: Which teacher moves for supporting student reasoning do prospective primary mathematics teachers employ to support student justification in completing the scriptwriting task?

RQ2: What clusters of prospective primary mathematics teachers emerge when a TMSSR analysis is applied to their responses to the scriptwriting task?

Analyzing the teacher moves and specifically the high and low potential teacher moves that the prospective teachers employ in their scriptwriting tasks provides an understanding of what prospective teachers conceptualize when they plan to support students reason mathematically and justify in particular. The identification of clusters of prospective teachers as based on similarities in their use of teacher moves for supporting student reasoning provides a nuanced examination of the frequently employed teacher moves of the whole cohort by examining patterns of teacher moves.

Materials and methods

Participants and context

This project draws upon data from a cohort of 54 prospective primary mathematics teachers at a Berlin-area university in a master's level course, "Foundations of Mathematics

Instruction in Grades 5 and 6," during the winter semester 2018/2019. The course had both a mathematical subject matter knowledge and mathematical pedagogical knowledge lens, concentrating on the major subjects in fifth and sixth grade in mathematics in Berlin, including operations with rational numbers, with the multiplication and division of fractions. Moreover, the course provided the prospective teachers with opportunities to learn about typical student misconceptions regarding multiplication and division of fractions, including how to recognize, understand, and work through such misconceptions and support students in developing conceptual understanding. This focus on building conceptual understanding was exemplified with strategies for engaging students in inquiry and discourse around these mathematical operations, in particular with ways of expressing and connecting different representations and registers for fraction multiplication and division.

Additionally, the course represents the last mathematical content course within the master's program and thus the last opportunity for new mathematical subject matter knowledge acquisition in the context of university education. The second half of the semester was selected for data collection to reflect the moment in which the prospective teachers had gained the foundations for teaching primary grade mathematics and particularly for supporting conceptual understanding of fraction multiplication and division. The prospective teachers were exposed to scriptwriting in the context of the course, in particular in relation to supporting conceptual understanding. In total, 41 prospective teachers, consisting of seven male and 34 female prospective teachers with a median age of 28 years, completed the scriptwriting task.

Research tool: the scriptwriting task

Part I: the classroom level task

The classroom level task for this project is drawn from a task involving the multiplication and division of fractions from Padberg and Wartha (2017). In the task, the prospective primary mathematics teachers consider a fictional student's claim that the multiplication of two fractions is larger than the division of the same two fractions and are asked to justify their response regarding the truth of the student's claim (Fig. 2). The prospective teachers who participated in the study had all been exposed to this style of task and had discussed the differences and challenges for students regarding natural numbers and fractions in a master's level course concerning upper primary mathematics content. The classroom level task served as the foundation for the second and third parts of the scriptwriting task.

The classroom level task

Textbook task adapted from Padberg and Wartha (2017, p. 148):

Mia claims: The product of 2/5 and 1/3 is bigger than the quotient of the division of 2/5 and 1/3, because division always makes things smaller. Is Mia right? Justify your response.

Fig. 2 The classroom level task (reproduced with permission from Research in Mathematics Education, Shure and Rösken-Winter, 2022)

Part II: the student dialogue based on the task

The second part of the scriptwriting task contains a dialogue (Fig. 3) between three students who are working together on the classroom level task, in which several empirically founded language- and accompanying content-related difficulties were embedded.

The three fictional students encounter the following difficulties concerning both language-related aspects and accompanying conceptual elements as they work through the task:

- Language-related aspects:
  - Azra experiences confusion with the use of the preposition "into" and the accompanying grammatical structure conveying that a divisor divides a dividend, which leads her to struggle to determine, grammatically, which fraction represents the subject and which represents the object. This leads her to not be able to ascertain mathematically, which fraction represents the divisor and which represents the dividend.
  - Leonie utilizes the phrase "cut the pizza more" to refer to splitting it in reference to the operation of multiplication. Azra then associates cutting the pizza with division ("dividing") and because of this confusion, does not understand the mathematical explanation of the multiplication of the two fractions in Leonie's leftover pizza example.

The dialogue beginning about the classroom level task

Ceyda, Azra, and Leonie are working on the textbook task from part I. The dialogue below is a part of their conversation:

Ceyda: Okay, product means times. Two-fifths times one third is two-fifteenths, or one-third times two-fifths is also two-fifteenths.

Azra: But two fifteenths is not as big as the other two numbers… With times, I thought we always get a big number.

Leonie: I'll show you guys. [Leonie draws a fraction circle *Figure 1a*] We still have two-fifths of a pizza from yesterday. There are three of us and I get one-third of it... two-fifteenths. That looks like this. [Leonie further partitions the pizza and shows her proportion of the remaining pizza *Figure 1b*]

Fig. 1a

Fig. 1b

Azra: How did you know the new number? Did you calculate it?

Leonie: No, I cut the pizza more... to split it.

Azra: But that is completely backwards. Timesing makes things smaller...? But you cut the pizza more, so did you divide the pizza? So Mia is right, isn't she?

Leonie: Nope, I multiplied. Now we are going to divide it up... divide it.

Ceyda: It really doesn't matter, we can start with two-fifths of the pizza or one-third of it, it's timesing. She gets two-fifteenths of the pizza.

Azra: Hmm... okay... with dividing... two-fifths divides into one-third, right?

Fig. 3 The dialogue beginning about the classroom level task (reproduced with permission from Research in Mathematics Education, Shure and Rösken-Winter, 2022)

- Accompanying content-related aspects:
    - The first two difficulties are exacerbated for Azra by the contradictions she experiences when she over-extends her knowledge about the multiplication of two natural or whole numbers and concludes that the same will be true for the multiplication of two fractions.
    - Azra's difficulty in understanding the area model for multiplication relates to her language-related difficulties. When she equates the idea of cutting up the pizza with dividing it, she also misunderstands mathematically that this further portioning of the pizza represents an example of the model for multiplying two fractions.

After reading the dialogue beginning, the prospective teachers are asked to identify up to two language- and accompanying content-related difficulties respectively from the dialogue. This identification serves as the basis for the third part of the scriptwriting task, in which the prospective teachers attend to these difficulties.

Part III: fictional dialogue attending to student difficulties

In the third part of the scriptwriting task, the prospective teachers are given the task to complete the student dialogue, imagining that they are the teacher of the three students. The prospective teachers are given the goal to assist the students in working through language- and accompanying content-related difficulties they identified and to enable the students to justify their answers concerning the truth of the claim that the multiplication of the two fractions is larger than the division of the same fractions. These two elements were combined in the task instructions in Fig. 4.

Your task (cf. Zazkis & Zazkis, 2016):

Complete the dialogue (like a script) between you (the teacher) and the three students:

1. Provide prompts with which you can support the students in independently solving the justification task with reasoning. Include how the students respond to your prompts and how you as the teacher in turn could respond.
2. Work through one language-related and one content-related problem area that you identified (in part II). Highlight the point in your script at which you work through the problem area.

Fig. 4 Your task (reproduced with permission from Research in Mathematics Education, Shure and Rösken-Winter, 2022)

The source of data for this project consisted of the completed scripts (from part 3 of the task) written by the prospective teachers attending to the student difficulties and the justification regarding the original claim.

Data analysis procedure

On the basis of the theoretical framework of teacher moves for supporting student reasoning (Ellis et al., 2018) and a coding system (Kuckartz, 2018), the teacher moves in the scripts written by the prospective teachers in the cohort were categorized into the four main categories of teacher moves: *facilitating student reasoning*, *responding to student reasoning*, *eliciting student reasoning*, and *extending student reasoning* (see Fig. 5).

As a next step, the teacher moves within each category were coded according to the low or high potential of the move (see Fig. 1), thus producing a total of eight codes. It is important to note that during the translation of the completed scripts from German to English, any ambiguous, confusing, or incorrect assertions, both of mathematical and/or linguistic nature, were maintained for the English translation. Thereby, in coding a teacher move, such as an attempt at facilitating student reasoning in which the prospective teacher provides a conceptual explanation that may be ambiguous, a qualitative assessment was not the focus (i.e., mathematical correctness), but rather the representation of a category of teacher moves within the coding framework concerning the low or high potential of the teacher move.

Code	Description	Examples from the scripts
Facilitating student reasoning	Prospective teachers attempt to assist students in developing their reasoning through various forms of guidance and explanation with a focus on making assumptions, identifying patterns, or comparing or classifying ideas.	"Let's look at the whole one more time. Leonie drew a good diagram of this" (Excerpt from script PS_17). / "What multiplication actually means is repeated addition and division is repeated subtraction" (Excerpt from script PS_04).
Responding to student reasoning	Prospective teachers react to students' thinking, including 1) validating students' responses, 2) correcting incomplete or inaccurate reasoning or solution strategies, 3) or by encouraging students to take these roles on themselves.	"Yes, that is true. Division is the reverse operation of multiplication" (Excerpt from script PS_35). / "Exactly, the pieces were made smaller. The diagram was partitioned" (Excerpt from script PS_39).
Eliciting student reasoning	Prospective teachers try to draw out, identify, clarify, and understand students' ideas and contributions, as essential in understanding what students know and understand and are thinking in the moment.	"Can you give two problems where the result will be smaller in one and bigger in the other?" (Excerpt from script PS_08) / "What do you need to pay attention to when dividing?" (Excerpt from script PS_17)
Extending student reasoning	Prospective teachers support students' opportunities to extend their mathematical reasoning, especially in terms of generalizing their strategies or ideas, and developing mathematically appropriate justifications.	"What do you mean by 'cut more?'" (Excerpt from script PS_08) / "Can you explain this in more detail?" (Excerpt from script PS_01)

Fig. 5 Examples from script coding as based on (Ellis et al., 2018) framework

As a part of the data analysis procedure, after a training process and further discussion of the coding system, two researchers utilized the eight aforementioned codes to individually analyze the data. In comparison of the independent analyses, a sufficient inter-rater reliability utilizing Cohen's (1960) kappa (Cohen's $K=0.94$) was attained.

Cluster analysis

In order to determine possible clustering of the prospective teachers based on their employed teacher moves for supporting student reasoning, and justification in particular, the recorded teacher moves for each participant were first put in relation to the total number of teacher moves. This allowed for a comparison of participant teacher moves in instances in which a participant utilized many teacher moves or employed only a few teacher moves (the maximum number of moves seen for low potential facilitating and low potential eliciting was nine; for all other categories the maximum was six moves or fewer). Thus, normalizing the data allowed for a clustering as based on the percentage of employed teacher moves in each category. On this basis, the analysis focused on similarities between individuals in terms of their compositions of employed teacher moves for supporting student reasoning.

Thereafter, a data-driven hierarchical agglomerative cluster analysis was completed using Ward's method (Ward, 1963) in RStudio using the 'hclust' function from the stats package, in which similar types of prospective teacher moves from the four main categories based on high or low potential were grouped together in an accumulating manner to yield clusters of prospective teachers who used similar compositions of teacher moves, and thereby were the nearest observations to one another in the dataset. This methodology attempts to minimize the total within-cluster variance utilizing the sum of squared Euclidean differences between each observation and its group's mean, meaning that two clusters are merged only if they are the most similar in terms of teacher moves used, minimizing the distance between observations and the mean of their cluster. In addition to using the dendrogram as a stopping rule, the elbow method was utilized in order to determine the number of clusters (Clatworthy et al., 2005; Ng et al., 2016), which help to identify a number of clusters in which the variance between the cluster members is minimized. Furthermore, in the consideration of the number of clusters, the relation between different high and low potential teacher moves within a main category were considered and are reflected in the resulting clusters.

Results and discussion

Results of teacher moves for supporting student reasoning

Concerning the first research question and the categories of teacher moves for supporting student reasoning (Ellis et al., 2018) that appeared in the prospective teacher scripts, as a whole, the prospective teachers employed moves mainly across the categories of

Table 1 Instances of teacher moves in percentages for supporting student reasoning based on the Ellis et al. (2018) framework in scripts written by the prospective primary mathematics teachers

	Facilitating student reasoning	Responding to student reasoning	Eliciting student reasoning	Extending student reasoning
High and low potential teacher moves	39.03	23.36	36.18	1.42
Low potential teacher moves	35.04	22.79	31.05	1.42
High potential teacher moves	3.99	0.51	5.17	0

facilitating, *eliciting*, and *responding*, and utilized low-potential moves in particular. Such instances occurred in which a prospective teacher, for example, corrected a student error instead of prompting the student to correct the error herself or a prospective teacher elicited procedures instead of eliciting understanding of a concept. *Facilitating* and *eliciting student reasoning* were found to be the most frequently used categories of teacher moves with 39.03% and 36.18% of teacher moves respectively, followed by moves aimed at *responding to student reasoning* with 23.36% of teacher moves. Moves for *extending student reasoning* were hardly used by the study participants with 1.42% of teacher moves. Table 1 provides an overview of the average occurrences of the teacher moves used in the completed scripts, including the distinction of average instances of low and high potential teacher moves (see Fig. 1 for specific low and high potential examples). In regard to teacher moves used to support justification, which was the goal of the textbook task, none of the prospective teachers attempted to employ moves in the high potential category of *extending student reasoning*, which explicitly focus on encouraging justification, generalization, and other mathematical reasoning processes.

As evident in table one, the prospective primary students most frequently utilized *facilitating* moves, followed by *eliciting* moves. Within the *facilitating* main category, low potential *facilitating* teacher moves such as cueing or providing general information to draw students' attention to specific or more general aspects of the task were overwhelmingly employed. In transcript one (Fig. 6), the students have already discussed the multiplication task and continue the conversation concerning division.

Ceyda:	…We want to know how often 1/3 of a pizza fits in 2/5 of a pizza.
Azra:	Okay, division makes it smaller. But how do we now know, how this looks compared to the division?
Teacher:	Cut out the 2/5 and 1/3 that you drew and place them on one another.
Azra:	They fit almost exactly on one another!
Ceyda:	So this means that the result must be just under 1…
Leonie:	The result of the division must be then bigger than the result of the multiplication!
Azra:	Then in the problem, Mia was wrong!

Fig. 6 Transcript 1: Excerpt from script PS_16

In this situation, the teacher cues the students to cut out the diagrams they made of one-third and two-fifths as a means to consider the division of the two fractions. The students recognize that the two fractions are somewhat similar in size, but misinterpret the size comparison in relation to the division task. Ceyda notes that the result "must be just under 1," which would be correct for the division task one-third divided by two-fifths. The fictional students in the dialogue noted that they needed to consider how many times one-third fits into two-fifths, but instead interpret this for one-third divided by two-fifths. While this interpretation still lends itself for a correct comparison with the result of the multiplication task, the students are able to answer the textbook task correctly in stating that Mia is wrong in her assumption; their reasoning, however, is based on an interpretation of a different task. The teacher does not respond to the students or provide any feedback regarding this incorrect interpretation.

Concerning instances of *eliciting student reasoning moves*, the most frequent types of low potential *eliciting* teacher move concerned checking for understanding, asking about facts or procedures, and eliciting the answer to a specific task. Transcript two (Fig. 7) provides an example of this type of situation in the dialogue continuation:

Azra:	But why is Mia wrong and the result with dividing bigger?
Teacher:	We will try to calculate this again. Leonie, please tell us how to divide two fractions.
Leonie:	First you need to write out the task. 1/3 ÷ 2/5 and then you need to flip the second number, then you can multiply both fractions and then it is totally each. 1/3 ÷ 2/5 = 1/3 · 5/2 = 6/5.
Azra:	Oh, so that's how you calculate that. It makes it look somehow so easy.
Teacher:	Leonie, your result is correct, but why is it bigger?
Leonie:	Well when you draw it, then you have more parts than when you divide. So more parts mean a bigger number.
Teacher:	Very good Leonie…

Fig. 7 Transcript 2: Excerpt from script PS_10

In the script excerpt, the teacher elicits a procedural explanation of the division of two fractions from Leonie. This form of questioning was a common type of low potential teacher move that appeared in the scripts, with a 35.01% frequency. The teacher follows up eliciting an explanation as to why the result, six-fifths, was in fact bigger than the result of the multiplication task. At this point, the teacher does not press Leonie further to explain or justify the claim she made regarding "you have more parts than when you divide." While this interaction sets the stage for encouraging student reasoning, the teacher moves employed do not result in a justification of the task at hand.

In examination of the type of *responding* teacher moves utilized by the prospective teachers the most repeatedly used teacher moves were low potential moves like validation of a correct response or result and revoicing a student's thought (Fig. 8).

Teacher:	So is the task 1/3 ÷ 2/5?
Azra:	No, the other way around, 2/5 ÷ 1/3.
Teacher:	Exactly, so the 1/3 divides the 2/5.
Azra:	Ah ya. But I still haven't quite understood it with the getting smaller with multiplication and the pizza. And with division now everything will get bigger?

Fig. 8 Transcript 3: Excerpt from script PS_44

The exchange between the teacher and Azra in transcript three represents one of the most commonly used type of responding teacher move with low potential. The teacher validates Azra's reply as to whether one-third is being divided by two-fifths or two-fifths is being divided by one-third by stating the division task in other words, with the divisor first. While the teacher provides an example of how to state the task in a different manner, there were often instances in the scripts in which the teacher responded with one-word answers, not adding any additional information. Similarly to situations in which *eliciting* moves were utilized, such occasions did not encourage the students to justify their thinking.

In the few instances in which prospective teachers used *extending* moves, in 1.42% of teacher moves, the aim was more geared toward a more precise use of language, but not necessarily focused on a justification of the result of the task or of particular procedures implemented during the calculations. The following excerpt from script PS_08 exemplifies these situations (Fig. 9).

Teacher:	Now we will come to the division task [*2/5 ÷ 1/3 from above*]. What did you calculate for it?
Azra:	Smaller.
Ceyda:	Bigger.
Teacher:	Can you justify this?
Azra:	Because $6 \div 3 = 2$.
Ceyda:	Because we multiply with the reverse and then get 6/5.
Teacher:	Can you calculate this step by step and explain with the "reverse?"
Ceyda:	$2/5 \div 1/3 = 2/5 \cdot 3/1 = 2 \cdot 3 / 5 \cdot 1$ because we have the rule and that equals 6/5.
Teacher:	And what do we call this with this "reverse?"
Leonie:	We take the reciprocal in order to get the result.
Teacher:	And what is the reciprocal?
Leonie:	Well that what was on top goes under the fraction bar and the denominator goes on top.

Fig. 9 Transcript 4: Excerpt from PS_08

In this excerpt, the teacher asks the students to justify their answers as to whether the result of the division problem will be greater or smaller than the multiplication of the same fractions. In the back and forth of questions and responses, the teacher does not press the students to explain their reasoning beyond providing procedural explanations and a description of the reciprocal. The teacher uses an *extending* move, however, when asking the students to be more precise in expressing what occurs in their explanation of the division task, pushing them to express the idea of a reciprocal as opposed to calling it the "reverse." While the students are requested to be more precise in their use of language, they do not provide a claim with backing concerning the teacher's question as to why the result of the division task is greater or smaller than the multiplication task. While the Ellis et al. (2018) framework discusses that combinations of teacher moves from the different categories can work together to foster meaningful student reasoning and mathematical engagement, the authors note that *extending* moves are typically more effective in fostering reasoning and are "on the high end of a continuum for supporting student reasoning precisely because each of the moves reflects an intent to foster more sophisticated mathematical reasoning. Thus, even low-potential moves can still offer significant opportunities for students" (p. 124). The finding that only 1.42% of prospective teacher moves included *extending* moves, was not unexpected, as the use of more sophisticated or complex moves is considered by Ellis et al. (2018) to be higher on the continuum of the potential of teacher moves and could thus prove challenging for prospective teachers.

Results and discussion of clusters of prospective primary mathematics teachers

The previous section outlined overall findings of the types of teacher moves utilized by the prospective teachers in their continuations of the dialogue. This section provides the results of the cluster analysis, examining how the prospective teachers employed high and low potential teacher moves for supporting student reasoning in the completed scriptwriting tasks with findings for each cluster that exemplify differences between groups of prospective teachers not visible in the aforementioned results.

In regard to the second research question, concerning possible clustering of the prospective teachers based on their employed low and high potential teacher moves for supporting student reasoning, five clusters were developed during the cluster analysis based on teacher moves utilized in conjunction with one another, with variance found amongst the clusters. The differentiation of the high and low potential teacher moves for supporting student reasoning based on the Ellis et al. (2018) framework can be found in five clusters (Fig. 10).

As seen in Fig. 10, variance in terms of use of high and low potential teacher moves for supporting student reasoning can be seen amongst the five clusters (see Appendix 1 for further information). Cluster one ($n=15$) saw the highest incident of *eliciting* teacher moves, albeit only in the low potential category. Clusters two ($n=8$) and three ($n=7$) focused on *facilitating student reasoning*, but varied in their use of high and low potential moves. The prospective teachers in cluster four ($n=8$) utilized a more even distribution of *facilitating*, *responding*, and *eliciting* moves, however, in contrast to the other clusters, they attempted the highest relative instances of high potential *eliciting* teacher moves. Lastly, cluster five ($n=3$) was the only group of prospective teachers to attempt all four categories of teacher moves, with some instances of high potential moves in all categories except *extending student reasoning*. The following provides a more detailed examination of the individual clusters based on their patterns of employed teacher moves.

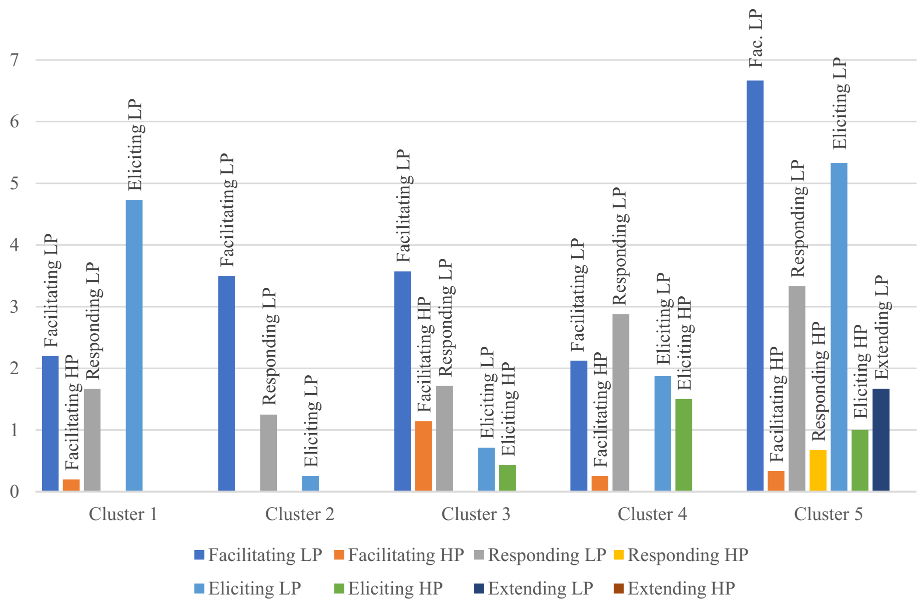


Fig. 10 Average use of high and low potential (HP and LP) teacher moves for supporting student reasoning in the five student clusters

Elicitors—cluster 1

In looking closer at the individual clusters, cluster one most closely followed the identified pattern by Ellis et al. (2018) that *eliciting* moves are often a first step in trying to support student reasoning. The participants in cluster one focused their teacher moves more than half of the time in the category of *eliciting* student reasoning, especially including eliciting an answer, eliciting facts or procedures, and checking for understanding with 55.85% of relative teacher moves in this category. In doing so, all of the teacher moves employed in the *eliciting* category were of low potential. Members of this cluster also employed *facilitating* and *responding* moves to a lesser extent, with 28.73% and 15.41% of relative teacher moves. In considering low and high potential moves for these categories of teacher moves, 26.11% of moves were in the low potential *facilitating* category and 2.62% in the high potential *facilitating* category and no high potential *responding* moves were undertaken. The script from PS_24 (Fig. 11) provides an example of a common pattern of low potential *eliciting* moves found in this cluster, in which a teacher engages in a question and answer dialogue with the students:

Teacher: And now the expert question: When we split up one-fourth a last time, what happens then?

Azra: Can I try it?

Teacher: Of course!

[Azra goes to the board and halves each of the fourths and then has a circle with eighths]

Ceyda: Azra made eighths!

Teacher: What is this result called in our scientist language?

Leonie: An eighth!

Teacher: It would be best to now summarize the whole task. Azra, would you want to dictate the task and result for everyone?

Azra: Sure, well: one-fourth times one-half equals one-eighth.

Teacher: Super, that was it for today. You earned a break, off with you!

Fig. 11 Transcript 5: Excerpt from script PS_24 (Cluster 1)

In the exchange, the teacher primarily focuses on eliciting responses from the students. The teacher asks the students to demonstrate how partitioning using the fraction circle works, the justification of the original task, however, does not play a part in the exchange, nor is the action of halving each fourth discussed in relation to the multiplication of fractions and why the resulting fraction is smaller. The division task is not considered altogether and thus the aim to justify whether Mia's claim that the multiplication of the two fractions is bigger than the division of the two fractions is not supported by the conversation in the script. This focus on procedure as opposed to a conceptual discussion of the division of fractions was a commonality of the majority of members of this cluster.

Low potential facilitators—cluster 2

In cluster two, the group mostly employed teacher moves geared overwhelmingly toward *facilitating student reasoning* with 72.56% of relative moves, with some moves aimed at *responding to student reasoning* (23.55%), and a few instances of *eliciting student reasoning* (3.89%). In this cluster there were no instances of *extending* moves. In all three categories of teacher moves, only low potential moves were attempted. Examples of common *facilitating* teacher moves included low potential moves like cueing students to focus on particular aspects of the task, asking leading questions, providing procedural information, or providing general information. An excerpt from a script (Fig. 12) from this cluster exemplifies this tendency with the use of low potential *facilitating* teacher moves aimed at cueing the students' attention to particular aspects of the task:

Teacher:	To begin, consider a whole pizza and split it into 5 pieces. Of these 5 pieces, we want to next look at just 2 pieces. The rest of the pizza we won't consider further.
Azra:	And now we will times with the one number?
Teacher:	Do you mean we will multiply with the other variable?
Azra:	Yes exactly, I meant that other number.
Teacher:	The second variable is called factor. We are only considering the portion of the pizza and we will in turn determine a portion of it.
Leonie:	So we multiply 2/5 with 1/3?
Teacher:	Exactly, that is the right way to do it.

Fig. 12 Transcript 6: Excerpt from PS_03 script (Cluster 2)

In the dialogue in transcript six, the teacher guides the students by cueing the students to what they should consider when examining the multiplication task, by first providing general information regarding the part-whole concept. Next, the teacher guides Azra by asking a leading question, prompting Azra with what the teacher considers to be the correct technical term. In a subsequent teacher move, the teacher provides procedural assistance in prompting the students with the necessary steps for multiplication, which Leonie correctly translates into the context of the problem. The teacher responds to Leonie without adding additional information and without a conceptual explanation. These predominantly low potential facilitation moves, however, do not focus on conceptual aspects of multiplying two fractions and do not support the students in constructing reasoning concerning the task. This focus on procedure was seen in scripts written by other members of this cluster, in particular concerning the conceptual discussion of fraction division.

High potential facilitators—cluster 3

The participants in cluster three applied teacher moves for *facilitating* with 69.38% of relative moves, followed by 16.10% of relatives moves in the *responding to student reasoning* category and 14.41% of moves focused on *eliciting student reasoning*. In contrast to cluster two, this cluster attempted to employ high potential teacher moves in both the *facilitating* and *eliciting student reasoning* categories, with 18.76% of relative moves in the high potential *facilitating* moves category and 6.80% of relative moves in the *eliciting* high potential category. Examples of typical moves in the high potential *facilitating* category in this cluster included encouraging multiple solution strategies and providing conceptual explanations. In a similar fashion to all clusters except cluster one, this group of participants did not employ any teacher moves for *extending* reasoning. Transcript seven (Fig. 13) provides an example of the type of teacher moves utilized by this cluster:

…	…
Azra:	…I think I got it: With division, we need to ask the question, "how many times does 1/3 fit in 2/5?"
Teacher:	Ya, you are all on the right path, try to draw it exactly like Azra just said. Maybe you can use a different representation than a circle.
Leonie:	Ok, I will try it, I will use the fifteenths that we just used to calculate. [Leonie draws two number lines that are split into thirds and fifths]

Ceyda:	Look, now you can see that 1/3 is smaller than 2/5… then it fits into it a whole time.
Leonie:	Ya, but then there is something remaining… two-fifths is then one fifteenth more than one-third… so one-third fits one time, plus one-fifteenth in two-fifths?
Azra:	Well, when it fits one time, then the remainder is actually one-sixth, right? It's one-sixth from the next one-third…
Ceyda:	Ok, then that is the answer for the division 1 1/6… so one-third fits 1 1/6 into two-fifths…
Teacher:	And do you have an idea now, how we could imagine this for multiplication?

Fig. 13 Transcript 7: Excerpt from script PS_29 (Cluster 3)

In the preceding excerpt, the teacher engages with the students by encouraging them to explore the task using an alternative representation, an attempt at a high potential *facilitating* teacher move. The three students continue the dialogue discussing the issue of the remainder in the fraction division task and what it represents, coming to an agreement of its meaning without the assistance of the teacher. To guide the students to complete the task in comparing the results of the fraction multiplication and division tasks, the teacher elicits the students' ideas for how the multiplication task could be imagined, a second attempt at a high potential teacher move, this time *eliciting student reasoning*. The remainder of the script contains a conceptual discussion of the fraction multiplication task with a similar pictorial representation made by the students. This dialogue continuation represents one of the few instances in which the discussion encompasses both a conceptual understanding of both fraction multiplication and division, and includes attempts at high potential teacher moves that guide the students toward the justification of the task, which were representative of this cluster. Overall, cluster three saw the highest instances of prospective teachers discussing conceptual aspects of fraction multiplication and/or division.

Facilitate, respond, and elicit combiners—cluster 4

The group of prospective teachers in cluster four concentrated most of their relative teacher moves either in the categories of *facilitating student reasoning* (27.36%), *eliciting student reasoning* (38.84%) and *responding to student reasoning* (33.78%). The participants in this group did not utilize moves for *extending student reasoning*. While the all of the *responding* moves and a majority of *facilitating* moves (25.09% of relative teacher moves) encompassed low potential teacher moves, 17.73% of the *eliciting* teacher moves were of high potential. Transcript eight (Fig. 14) provides a glimpse into the types of teacher moves employed by cluster four, with a combination

Teacher:	Let's take another look at this. Leonie already has drawn a good diagram here. Leonie, can you explain again what you did with your first drawing?
Leonie:	I drew a pizza and split it into five equal parts, two of those were colored, since we don't have a whole pizza but only two-fifths.
Teacher:	Exactly, you thus partitioned the pizza, the pizza is the whole thing. It doesn't matter how we divide up the pizza, the pizza will not become more or less, only the partitioning changes.
Azra:	Yea, that makes sense. I can split the pizza into 5 pieces or in 10 or 7, it will always be a whole pizza. But what does that have to do with multiplication?
Teacher:	Let's get to that now. If I have a fraction, then it always describes the part of the whole that it has. So, in our example two-fifths. So, we have two out of five parts of a pizza, like in Leonie's picture. Who can explain to me how multiplication generally works?
…	…
Leonie:	Exactly, that's what it's called. I further partitioned the parts. In doing so, I split each fifth into three parts. I needed to do that because I can't split two-fifths into three so easily. When I split each fifth into three parts, then I can take a part from each and then only need to put them together, to see how many I have. With two-fifths I have two parts. I then times each numerator with each other, so two times one.

Fig. 14 Transcript 8: Excerpt from script PS_17 (Cluster 4)

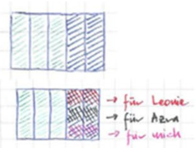
of predominantly low potential moves from the *facilitating* and *responding* categories, with some high potential *eliciting* moves.

The teacher first prompts Leonie to explain again what she did with her pictorial representation of the pizza. In a following move, the teacher responds to Leonie's answer to her request, revoicing Leonie's explanation of partitioning for the other students. In response to Azra's question concerning how the pizza representation and partitioning are related to fraction multiplication, the teacher elicits a general explanation of how multiplication functions. In an ensuing discussion, the teacher eventually elicits an explanation of fraction multiplication from Leonie. While Leonie attempts to explain why further partitioning is necessary, a conceptual explanation of fraction multiplication is not discussed. As was seen in most of the previously discussed clusters, justification of the claims in the textbook task is not a focus of the discussion.

All-category attempters—cluster 5

The prospective teachers in cluster five were the only individuals to utilize all main categories of teacher moves for supporting student reasoning, including some attempts at *extending student reasoning*, and are thus in a cluster of their own. In this group, the participants employed 36.28% of relative teacher moves in the *facilitating student reasoning* category with 32.58% of relative moves categorized as low potential and 3.70% high potential. In the category of *responding*, 26.24% of relative teacher moves appeared within this type of move, with 23.21% low potential and 3.03% high potential moves. Furthermore, 28.65% of relative teacher moves were attempts to *elicit student reasoning* with 24.57% categorized as low potential and 4.08% as high potential. Lastly, 8.83% of relative teacher moves were employed in the category of *extending student reasoning*, with all low potential teacher moves. While all four categories of moves were attempted, the majority of moves appeared as low potential teacher moves. Transcript nine (Fig. 15) provides examples of teacher moves from all four categories.

Teacher:	Thanks for your explanation Leonie. I found a lot of good starting points in your conversation that we will consider individually. Could you pictorially represent the two fractions, 2/5 and 1/3 and the multiplication of the two differently?
Ceyda:	Ya, for example with a rectangular chocolate bar.
Teacher:	That would be a possibility. Can you explain that in more detail?
Ceyda:	Ya, so in the picture we have 5 pieces of chocolate [Draws rectangular chocolate bar with 5 pieces]. 3 pieces of the chocolate bar go to Mrs. Beauty [marks these in green]. 2 pieces go to Leonie, Azra, and me [marks these pieces blue]. This means that we now get 2/5 of the chocolate bar, so 2 from 5 pieces of chocolate.



Azra:	Ya, this I get.
Ceyda:	Because we are three, Leonie, you, and me, we each get 1/3 of the 2/5 of the chocolate bar. You can see this well with the colors [red for Leonie, black for Azra, pink for Ceyda]. Each of us gets 2 boxes of the chocolate bar. In order to know how much this is in total from the whole bar of chocolate, we need to count all of the boxes.
…	…
Teacher:	Very good explanation, Ceyda. Ceyda just showed the multiplication (multiplying) of 2/5 and 1/3 in a drawing.

Fig. 15 Transcript 9: Excerpt from PS_01 (Cluster 5)

The teacher starts the dialogue by asking the students to provide another representation of the multiplication problem, as a means of facilitating, by encouraging another manner of solving the task pictorially. When the teacher asks Ceyda to expand and explain her idea in more detail, Ceyda provides a precise answer based on her pictorial representation based on a bar of chocolate. This exchange offers an example of an attempt at *extending student reasoning*. In pushing the student to provide a detailed explanation, Ceyda provides a conceptual explanation of her means of solving. In the following teacher move, the teacher provides validation of Ceyda's pictorial explanation of the multiplication task, without providing additional information. This script provides an example of a situation in which a teacher move pushing the student to expand an idea in a more precise manner results in a pictorial representation and corresponding explanation of the multiplication of the two fractions. While the dialogue in the remainder of the script does not result in the justification of the original task, the students are encouraged by the teacher in the aforementioned situation, to think about the multiplication part of the task more conceptually, building understanding of the multiplication of fractions interpretation. This focus on the conceptual aspects only of fraction multiplication was seen by all members of this cluster.

Conclusion

This study highlighted the variation of teacher moves for supporting student reasoning, with different strategies and foci and a progression of moves often divergent from the ideal progression as suggested by Ellis et al. (2018). In terms of the types of eliciting moves employed by the prospective teachers, these results align with previous studies (e.g., Shaughnessy et al., 2021a) concerning how prospective teachers plan their eliciting moves, with prospective teachers placing more of an emphasis on low potential moves geared toward eliciting students' understanding of facts and procedural aspects of the steps they used to complete the task. As this study was conducted toward the end of the teacher education program, while the Shaughnessy et al., (2021a) study, for example, was conducted during the first week of a teacher education program, this suggests the need to place more emphasis on understanding what it means to elicit (or facilitate, respond to, or extend) student reasoning in a high potential manner throughout all phases of teacher education.

The in large part major emphasis on low potential moves for facilitating student reasoning led to more teacher-centered interactions with the teacher providing procedural explanations, providing general information, cuing students to certain aspects of the task, and funneling students down a specific path reflects previous research that in-service teachers often lead students down a specific path with leading questions (e.g., Franke et al., 2009) and highlights the need to discuss ways of making interactions less teacher dominated and more focused on conceptual understanding. This is especially important as research highlights the domination of teacher-centered classrooms that is often still prevalent (Cazden, 2001; Ellis et al., 2018; Franke et al., 2009) and could be more explicitly targeted as prospective teachers plan future interactions.

The overall focus of the prospective teachers on both eliciting and facilitating student reasoning aligns with Kawanaka and Stigler's (1999) results of German mathematics teachers in which in-service teachers almost equally employed moves that elicited and facilitated student thought. The cluster analysis for this study, on the other hand, demonstrates the differences in the actions between different groups of prospective teachers, with the nuance afforded by conducting such an analysis and considering different clusters of teacher moves. While examining the frequently employed teacher moves of the whole cohort of prospective teachers indicated a tendency to elicit and facilitate student thinking, the cluster analysis showed a variety of combinations of moves.

The finding that only three prospective teachers (cluster 5) in the study attempted to incorporate all categories of moves (albeit mostly low potential moves), including *extending* moves that can better foster reasoning, suggests the importance of promoting a diversity of teacher moves and assisting prospective teachers in considering what will best support students in a particular situation in building understanding and supporting reasoning. Furthermore, as this group also predominantly utilized low potential moves, the discussion of the uses of different types of moves should also focus on what high potential moves can mean for instruction and learning. Together, this could lead to more critical consideration of which moves foster reasoning best in a particular situation, which may ultimately reflect a closer version of the TMSSR framework progression of teacher moves, with a focus on strategically using different categories of moves, but decisively extending student reasoning.

The situated examination provided by this scriptwriting task acts as a window into how prospective teachers envision acting in a specific simulated classroom situation utilizing a theoretically and empirically-grounded framework of teacher moves for supporting student reasoning represents an extended use of scriptwriting approaches. As prospective teachers do not yet have their own classroom in which to practice or change their practice, this tool represents an opportunity methodologically to consider the combinations of moves that prospective teachers envision for classroom interaction without being in an actual classroom. While this research considered one mathematical content area and cannot be generalized to other mathematical content, this methodology and form of analysis, however, can guide future scriptwriting approaches to explore the types of teacher moves prospective teachers implement to support reasoning in different settings and with other mathematical content. Furthermore, these results can help inform how prospective teachers are educated about particular teacher moves, as a means of discussing which moves are pertinent in which situations, thus making the utility of these moves more explicit for prospective teachers as they develop their discussion-mediation skills.

We suggest further research to examine the impact of scriptwriting tasks as simulation-based tools for supporting prospective teachers understand the goals of specific teacher moves, and extending moves in particular, as a means of helping them to adjust and plan their moves to guide students in extending their reasoning as well as in other situations as a means of interactive practice, which often is not the focus of teacher preparation programs (Grossman et al., 2009). Furthermore, different lenses of analysis such as sociocultural aspects of classroom interaction beyond students being active and independent learners can also be examined with this tool. Beyond the scope of prospective teacher education, the use of scriptwriting tasks to help in-service teachers who may still struggle with employing such teacher moves to support high potential reasoning can be incorporated into professional development programs, with reflection after scriptwriting and teaching sequences, as well as for facilitators of professional development programs, as a tool for enabling mathematics teachers to (further) develop their skills in supporting student reasoning in an interactive manner.

Appendix 1

Average use of high and low potential teacher moves for supporting student reasoning in the five prospective teacher clusters.

	Cluster 1	Cluster 2	Cluster 3	Cluster 4	Cluster 5
Facilitating student reasoning—low potential	2.20	3.60	2.13	3.5	6.67
Facilitating student reasoning—high potential	0.20	1.14	0.25	0	0.33
Responding to student reasoning—low potential	1.67	1.71	2.88	1.25	3.33
Responding to student reasoning—high potential	0	0	0	0	0.67
Eliciting student reasoning—low potential	4.73	0.71	1.88	0.25	5.33
Eliciting student reasoning—high potential	0	0.43	1.5	0	1
Extending student reasoning—low potential	0	0	0	0	1.67
Extending student reasoning—high potential	0	0	0	0	0

Acknowledgements We would like to thank all the study participants without whom this research would not have been possible.

Author contributions Victoria Shure contributed to conceptualization, methodology, software Victoria Shure, formal analysis, investigation, and writing—original draft and revisions; Peter Liljedahl performed writing—reviewing and editing.

Funding Open Access funding enabled and organized by Projekt DEAL. No funding was received for conducting this study.

Declarations

Conflict of interest One of the authors of this paper is an Editorial Board Member. Otherwise, the other author has no competing interests to declare that are relevant to the content of this article.

Consent for participants All study participants provided consent for the collection, use, and storage of their data in accordance with University requirements.

References

Alenazi, A. (2016). Examining middle school pre-service teachers' knowledge of fraction division interpretations. *International Journal of Mathematical Education in Science and Technology, 47*(5), 696–716. https://doi.org/10.1080/0020739X.2015.1083127

Ball, D. (1990). Preservice elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*(2), 132–144.

Bauersfeld, H. (1995). The structuring of the structures: Development and function of mathematizing as a social practice. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 137–158). Lawrence Erlbaum Associates Inc.

Biza, I., & Nardi, E. (2020). Scripting the experience of mathematics teaching: The value of student teacher participation in identifying and reflecting on critical classroom incidents. *International Journal for Lesson and Learning Studies, 9*(1), 43–56. https://doi.org/10.1108/IJLLS-02-2019-0017
Boaler, J. (2010). The road to reasoning. In K. Brodie (Ed.), *Teaching mathematical reasoning in secondary school classrooms* (pp. v–vii). Springer.
Boaler, J., & Brodie, K. (2004). The importance, nature and impact of teacher questions. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the 26th conference of the psychology of mathematics education North America* (pp. 773–781). OISE/UT.
Buchbinder, O., & Cook, A. (2018). Examining the mathematical knowledge for teaching of proving in scenarios written by pre-service teachers. In O. Buchbinder & S. Kuntze (Eds.), *Mathematics teachers engaging with representations of practice. ICME-13 monographs* (pp. 131–154). Springer. https://doi.org/10.1007/978-3-319-70594-1_8
Campbell, M., & Baldinger, E. (2021). Using scripting tasks to reveal mathematics teacher candidates' resources for responding to student errors. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-021-09505-4
Castro-Rodríguez, E., Pitta-Pantazi, D., Rico, L., & Gómez, P. (2016). Prospective teachers' understanding of the multiplicative part-whole relationship of fraction. *Educational Studies in Mathematics, 92*(1), 129–146. https://doi.org/10.1007/s10649-015-9673-4
Cazden, C. B. (2001). *Classroom discourse* (2nd ed.). Heinemann.
Christiansen, B., & Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 243–307). D. Reidel.
Clarke, D. M., Roche, A., & Mitchell, A. (2008). 10 practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School, 13*(1), 373–380.
Clatworthy, J., Buick, D., Hankins, M., Weinman, J., & Horne, R. (2005). The use and reporting of cluster analysis in health psychology: A review. *British Journal of Health Psychology, 10*(3), 329–358. https://doi.org/10.1348/135910705X25697
Cohen, J. (1960). A coefficient of agreement for nominal scales. *Educational and Psychological Measurement, 20*(1), 37–46.
Da Ponte, J., & Chapman, O. (2015). Prospective mathematics teachers' learning and knowledge for teaching. In D. Kirshner & L. D. English (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 275–296). Routledge.
Elliott, P. C. (Ed.). (1996). *Communication in mathematics, K-12 and beyond*. National Council of Teachers of Mathematics.
Ellis, A. B., Özgür, Z., & Reiten, L. (2018). Teacher moves for supporting student reasoning. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-018-0246-6
Franke, M., Webb, N., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education, 60*(4), 380–392. https://doi.org/10.1177/0022487109339906
Franke, M., Turrou, A., Webb, M., Ing, M., Wong, J., Shin, N., & Fernandez, C. (2015). Student engagement with others' mathematical ideas: The role of teacher invitation and support moves. *The Elementary School Journal, 116*(1), 126–148. https://doi.org/10.1086/683174
Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record, 111*, 2055–2100.
Hallman-Thrasher, A. (2017). Prospective elementary teachers' responses to unanticipated incorrect solutions to problem-solving tasks. *Journal of Mathematics Teacher Education, 20*, 519–555. https://doi.org/10.1007/s10857-015-9330-y
Henning, J. E., McKeny, T., Foley, G. D., & Balong, M. (2012). Mathematics discussions by design: Creating opportunities for purposeful participation. *Journal of Mathematics Teacher Education, 15*, 453–479. https://doi.org/10.1007/s10857-012-9224-1
Herbel-Eisenmann, B. A., Steele, M. D., & Cirillo, M. (2013). (Developing) teacher discourse moves: A framework for professional development. *Mathematics Teacher Educator, 1*(2), 181–196.
Hohensee, C., & Jansen, A. (2017). Elementary preservice teachers' transitional conceptions of partitive division with proper-fraction divisors. *Mathematical Thinking and Learning, 19*(4), 210–236. https://doi.org/10.1080/10986065.2017.1346452
Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 35*(2), 81–116. https://doi.org/10.2307/30034933
Isiksal, M., & Cakiroglu, E. (2011). The nature of prospective mathematics teachers' pedagogical content knowledge: The case of multiplication of fractions. *Journal of Mathematics Teacher Education, 14*, 213–230. https://doi.org/10.1007/s10857-010-9160-x

Izsák, A., Jacobson, E., & Bradshaw, L. (2019). Surveying middle-grades teachers' reasoning about fraction arithmetic in terms of measured quantities. *Journal for Research in Mathematics Education, 50*(2), 156–209. https://doi.org/10.5951/jresematheduc.50.2.0156

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1–12. https://doi.org/10.1007/s10649-017-9761-8

Kawanaka, T., & Stigler, J. W. (1999). Teachers' use of questions in eighth-grade mathematics classrooms in Germany, Japan, and the United States. *Mathematical Thinking and Learning, 1*, 255–278. https://doi.org/10.1207/s15327833mtl0104_1

Kosko, K., Rougee, A., & Herbst, P. (2014). What actions do teachers envision when asked to facilitate mathematical argumentation in the classroom? *Mathematics Education Research Journal, 26*, 459–476. https://doi.org/10.1007/s13394-013-0116-1

Kuckartz, U. (2018). *Qualitative Inhaltsanalyse. Methoden, Praxis, Computerunterstützung*. [Qualitative content analysis. Methods, praxis, and computer support]. Juventa.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*(1), 88–124. https://doi.org/10.5951/jresematheduc.46.1.0088

Lim, W., Roberts-Harris, D., & Kim, H.-J. (2018). Preservice teachers' learning paths of classroom discourse through scripting. In R. Zazkis & P. Herbst (Eds.), *Scripting approaches in mathematics education: Mathematical dialogues in research and practice* (pp. 293–319). Springer International Publishing.

Marmur, O., Yan, X., & Zazkis, R. (2019). Fraction images: The case of six and a half. *Research in Mathematics Education, 22*(1), 22–47. https://doi.org/10.1080/14794802.2019.1627239

Mata-Pereira, J., & Da Ponte, J. (2017). Enhancing students' mathematical reasoning in the classroom: Teacher actions facilitating generalization and justification. *Educational Studies in Mathematics, 96*(2), 169–186. https://doi.org/10.1007/s10649-017-9773-4

McCrone, S. (2005). The development of mathematical discussions: An investigation in a fifth-grade classroom. *Mathematical Thinking and Learning, 7*(2), 111–133. https://doi.org/10.1207/s15327833mtl0702_2

Mueller, M., Yankelewitz, D., & Maher, C. (2014). Teachers promoting student mathematical reasoning. *Investigations in Mathematics Learning, 7*(2), 1–20.

Ng, B., Liu, L., & Wang, W. (2016). Student motivation and learning in mathematics and science: A cluster analysis. *International Journal of Science and Mathematics Education, 14*(7), 1359–1376. https://doi.org/10.1007/s10763-015-9654-1

Olanoff, D., Lo, J.-J., & Tobias, J. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast, 11*(2), 267–310.

Ölmez, I., & Izsák, A. (2021). Characterizing reasoning about fraction arithmetic of middle grades teachers in three latent classes. *Mathematical Thinking and Learning, 23*(3), 225–253. https://doi.org/10.1080/10986065.2020.1780368

Padberg, F., & Wartha, S. (2017). *Didaktik der Bruchrechnung [Didactics of rational numbers]*. Springer Spektrum.

Rule, A. C., & Hallagan, J. E. (2006). Preservice elementary teachers use drawings and make sets of materials to explain multiplication and division of fractions. *Open Educational Resources*, *307*.

Shaughnessy, M., DeFino, R., Pfaff, E., & Blunk, M. (2021a). I think I made a mistake: How do prospective teachers elicit the thinking of a student who has made a mistake? *Journal of Mathematics Teacher Education, 24*, 335–359. https://doi.org/10.1007/s10857-020-09461-5

Shaughnessy, M., Garcia, N. M., O'Neill, M. K., Selling, S. K., Willis, A. T., Wilkes, C. E., II., Salazar, S. B., & Ball, D. (2021b). Formatively assessing prospective teachers' skills in leading mathematics discussions. *Educational Studies in Mathematics, 108*, 451–472. https://doi.org/10.1007/s10649-021-10070-z

Shure, V., & Rösken-Winter, B. (2022). Developing and validating a scriptwriting task in the context of student difficulties with fraction multiplication and division. *Research in Mathematics Education, 24*(3), 267–290. https://doi.org/10.1080/14794802.2021.1988689

Siebert, D. (2002). Connecting informal thinking and algorithms: The case of division of fractions. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions* (pp. 247–256). National Council of Teachers of Mathematics.

Smith, M., & Stein, M. K. (2011). *Five practices for orchestrating productive mathematics discussions*. NCTM.

Stein, M., Engle, R., Smith, M. K., & Hughes, E. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313–340. https://doi.org/10.1080/10986060802229675

Tall, D. (2007). Developing a theory of mathematical growth. *ZDM Mathematics Education, 39*(1–2), 145–154.
Tall, D. (2008). *The historical & individual development of mathematical thinking: Ideas that are set-before and met-before* [Plenary presentation]. Colóquio de Histório e Tecnologia no Ensino Da Mathemática UFRJ, Rio de Janeiro, Brazil.
Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's misconceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5–25. https://doi.org/10.2307/749817
Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student mathematical thinking that are worth building on in whole-class discussion. *Mathematical Thinking and Learning, 19*(1), 33–54. https://doi.org/10.1080/10986065.2017.1259786
Ward, J. H. J. (1963). Hierarchical grouping to optimize an objective function. *Journal of the American Statistical Association, 58*, 236–244.
Webb, N. M., Franke, M. L., Ing, M., Wong, J., Fernandez, C. H., Shin, M., & Turrou, A. C. (2014). Engaging with others' mathematical ideas: Interrelationships among student participation, teachers' instructional practices, and learning. *International Journal of Educational Research, 63*, 79–93. https://doi.org/10.1016/j.ijer.2013.02.001
Wessel, L., & Erath, K. (2018). Theoretical frameworks for designing and analyzing language-responsive mathematics teaching–learning arrangements. *ZDM Mathematics Education, 50*(6), 1053–1064. https://doi.org/10.1007/s11858-018-0980-y
Zazkis, R., Liljedahl, P., & Sinclair, N. (2009). Lesson plays: Planning teaching vs. teaching planning. *For the Learning of Mathematics, 29*(1), 40–47.
Zazkis, R., & Marmur, O. (2018). Scripting tasks as a springboard for extending prospective teachers' example spaces: A case of generating functions. *Canadian Journal of Science, Mathematics and Technology Education, 18*(3), 291–312. https://doi.org/10.1007/s42330-018-0019-y
Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. Springer.
Zazkis, D., & Zazkis, R. (2016). Prospective teachers' conceptions of proof comprehension: Revisiting a proof of the Pythagorean theorem. *International Journal of Science and Mathematics Education, 14*(4), 777–803. https://doi.org/10.1007/s10763-014-9595-0