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Documenting two emerging sociomathematical norms for examining functions in mathematics teachers' online asynchronous discussions

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Abstract

This study investigated novice mathematics teachers participating in an online teacher education course focused on covariational reasoning and understanding the behavior of functions. The analysis centered on documenting the emergence of participants' sociomathematical norms for engaging in online asynchronous discussions. In this paper, we characterized participants' initial mathematical discourse and documented two emergent sociomathematical norms, namely *explaining why* and *emergent shape discourse*. When participants *explained why*, they used specific quantities or symbolic representations of functions to justify why function graphs have particular visual features. When participants engaged in *emergent shape discourse*, they coordinated change between covarying quantities to justify why function graphs behave in certain ways. This study provides evidence that online settings can provide context for mathematics teachers engaging in legitimate collaborative mathematical activity and that activity can be enhanced by participation in discourse featuring specific sociomathematical norms. We discuss conjectures regarding the potential of reflective discussion activities paired with the Notice and Wonder Framework to support the emergence of generative sociomathematical norms. We also discuss potential relationships between characteristics of participants' mathematical discourse and their membership with the core and periphery of a social network.

Keywords Mathematics teacher professional learning · Sociomathematical norms · Online asynchronous discussions · Covariational reasoning · Functions

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Introduction

Students reasoning about quantities and relationships between quantities is important for their success in understanding functions (Adu-Gyamfi, & Bossé, 2014; Ellis et al., 2015, 2016; Ferrari-Escolá et al., 2016; Johnsons et al., 2020; Moore, 2014; Weber & Thompson, 2014). While research has highlighted the importance of teachers' deep understanding of the concept of function (Byerley & Thompson, 2017; Thompson & Carlson, 2017; Thompson et al., 2017), there is little focus on how mathematics teachers can collaboratively develop such understandings in online contexts.

Social and the mathematics specific sociomathematical norms—accepted and expected regularities in mathematical discourse (Cobb et al., 2001)—are powerful social mechanisms that can support generative mathematical activity in various contexts (McClain & Cobb, 2001; Fukawa-Connelly, 2012; Yackel et al., 2000), including in mathematics teacher education settings (Clark et al., 2008; Dean, 2005; Grant et al., 2007; Güven & Dede, 2017; Sánchez & García, 2014; Tatsis & Koleza, 2008; Van Zoest et al., 2012; Whitacre & Rumsey, 2018). The sophistication of emerging sociomathematical norms can evolve over time (Clark et al., 2008; Dean, 2005; Whitacre & Rumsey, 2018), creating context for teachers' mathematical development (Cobb & Yackel, 1996). Further, mathematics teachers participating in sociomathematical norms in professional learning contexts has implications for teachers building similar norms in their classrooms (Clark et al., 2008; Tsai, 2007).

Our work explores the use of online asynchronous collaboration to support mathematics teachers' development of mathematics content knowledge. These efforts are informed by the documented effectiveness of online contexts for mathematics teacher learning (Araujo & Gadanidis, 2020; Beilstein et al., 2020; Engelbrecht et al., 2020; Lafferty & Kopcha, 2016; Martínez et al., 2020; Pape et al., 2015) and the potential of online asynchronous collaboration via threaded discussion forums, hereafter referred to as online discussions, to enhance opportunities for teachers' extended reflection on and discussion of mathematical discourse (Llinares & Valls, 2010; Shumar, 2017). Understanding how online settings can support mathematics teachers' content knowledge development is important because these settings can be scaled, potentially increasing teachers' access to collaborative learning experiences. Despite the importance of participation in sociomathematical norms for generative and collaborative mathematical activity and the increasing need for online contexts to support teacher learning, we are not aware of any studies documenting whether or how sociomathematical norms emerge in online contexts.

The purpose of this study was to better understand mathematics teachers' participation in online mathematical discourse and document the emergence of sociomathematical norms propitious for individual and collective development of skills for examining and reasoning about functions. We conceptualize of collaborative learning as a reflexive relationship between advances in individual mathematical discourse and emerging sociomathematical norms and collective mathematical practices (Cobb & Yackel, 1996). Further, we conceive of advances in the sophistication of teachers' mathematical discourse as *critical moments,* and we argue that these moments can provide insight into how sociomathematical norms emerge. Our research questions are as follows:

- What sociomathematical norms emerge in participants' online discussions about functions?

- What are the critical moments associated with the emergence of the documented sociomathematical norms?

Building on decades of research and practice (e.g., see Clark et al., 2008; Cobb & Yackel, 1996; Fukawa-Connelly, 2012; Whitacre & Rumsey, 2018), this study contributes evidence that online contexts can support legitimate collaborative mathematics learning experiences that include teachers participating in sociomathematical norms for examining functions. An additional contribution of this study is a characterization of the process by which mathematics teachers collectively shift from discussing the shape of function graphs to discussing underlying quantitative relationships to justify the behavior of function graphs. Finally, we contribute a methodology for documenting the emergence of sociomathematical norms in online discussions.

Sociomathematical Norms in Mathematics Teacher Professional Learning

Social and sociomathematical norms can emerge and evolve as individuals interact in collaborative mathematical activity as part of a community (Cobb et al., 2001). Social norms represent accepted and expected discourse that is not specific to mathematical discourse such as explaining one's reasoning or asking clarifying questions. Sociomathematical norms represent accepted and expected mathematical discourse such as sharing explanations that focus on the meaning of mathematical ideas, providing justifications that emphasize why and how methods work, or asking questions that push on the validity of another's mathematical reasoning (Elliot et al., 2009; van Zoest et al., 2012). Participation in communities where specific sociomathematical norms have emerged increases the likelihood that mathematics teachers' ongoing discourse will be consistent with those norms (Clark et al., 2008; Dean, 2005). Sociomathematical norms can also provide insight into characteristics of mathematical discourse that are improper or not accepted (Education Committee of the EMS, 2013). Moreover, as potential sociomathematical norms emerge, teachers can try out new mathematical discourse they access through interactions with colleagues, creating context for both collective and individual development (Cobb & Yackel, 1996).

The sophistication of sociomathematical norms around *how* teachers provide justifications can evolve over time (Clark et al., 2008; Dean, 2005; Whitacre & Rumsey, 2018). A significant shift in the quality of a justification can include moving from describing observable patterns to articulating the *logical necessity* or reasons for why observable patterns cannot be any other way (Simon et al., 2010). As an example, Clark et al. (2008) documented mathematics teachers shifting from procedurally based justifications for the behavior of functions to *speaking with meaning,* a sociomathematical norm where the normative discourse focused on relationships between quantities to justify function behavior. Furthermore, Dean (2005) documented shifts in normative aspects of mathematics teacher discourse from accurately representing data and calculating descriptive statistics to reasoning about part–whole relationships to investigate measures of center.

Learning environments can be designed to create contexts that are conducive for the emergence of increasingly more sophisticated sociomathematical norms. For example, Kynigo and Kalogeria (2012) argued that online discussions elicited differences in teacher educators' reasoning patterns, which resulted in the emergence of more sophisticated norms. Furthermore, Grant et al. (2007) used readings and mathematics tasks that favor non-traditional solution strategies to build norms for explaining and justifying conclusions. These norms can create context for learners negotiating what constitutes acceptable mathematical discourse (Yackel, 2002) and could support mathematics teachers in advancing the quality of their mathematical reasoning.

A teacher educator can contribute to supporting the emergence of more sophisticated sociomathematical norms by, for example, challenging mathematics teachers to explain and justify their mathematical reasoning (Clark et al., 2008; Dean, 2005; Van Zoest et al., 2012; Whitacre & Rumsey, 2018). Further, Whitacre and Rumsey (2018) documented how a teacher educator's emphasis on comparing details of mental computation strategies supported the emergence of norms for distinguishing and communicating the details of strategies (Whitacre & Rumsey, 2018). Clark et al. (2008) also reported effective facilitation strategies for building norms such as modeling speaking with meaning, asking teachers to unpack pronouns when sharing reasoning, and encouraging teachers to critique colleagues' reasoning. Lastly, there is potential that the influence of an instructor modeling specific mathematical discourse is greater in online discussions because discourse is permanent in discussion boards and learners pay particular attention to instructors' posts (An et al., 2009).

Online discussions

Collaboration is a feature of effective professional learning (Darling-Hammond et al., 2017) and is, by definition, necessary for building sociomathematical norms in online settings. Online collaboration has supported mathematics teachers in developing mathematics content knowledge (Pape et al., 2015), enhancing professional noticing skills (Fernández et al., 2020; Llinares & Valls, 2010), learning from videos of classroom practice (Beilstein et al., 2020), and developing mathematical knowledge for teaching (Martínez et al., 2020).

Online discussions have affordances that create context for the emergence of sociomathematical norms. Specifically, online discussions afford text-based communication and archival of text in a public space. These affordances both elicit mathematics teachers' discourse and enable records of teachers' discourse to become objects for extended reflection, discussion, and improvement (Llinares & Valls, 2010; Shumar, 2017). In addition, online discussions can bridge individual and collaborative mathematical activity by supporting teachers in privately working on mathematics tasks, sharing mathematical discourse, as well as accessing, reflecting on, discussing, and revising one another's mathematical discourse. Thus, online discussions can enhance access to and engagement with colleagues' mathematical discourse, which can create context for the emergence of sociomathematical norms (Matranga & Silverman, 2021).

Further, there is evidence that the affordances of online discussions can support mathematics teachers' collaborative learning. For example, Llinares & Valls (2010) argued that online discussions supported mathematics teachers in making meaning of their practices when they used text to reify mathematics teaching practices into objects (i.e., equitable practices) for reflection and discussion. Clay et al. (2012) found that the use of screencasting software and online discussions supported teachers in reflecting on one another's discourse, which scaffolded them in conceiving of algebraic expressions as quantities and relationships between quantities rather than sentences that are read from left to right. Pape et al. (2015) documented mathematics teachers developing skills in operating with rational numbers and part of this growth was contributed to their online discussions about problem-solving strategies. We expand on this research by documenting the emergence of sociomathematical norms in online discussions that provided teachers with a context to participate in generative mathematical discourse and potentially increase the sophistication of their reasoning about functions.

Covariational reasoning and shape thinking

Covariational reasoning is an important reasoning process for understanding functions (Carlson et al., 2002), and we define covariational reasoning as learners *coordinating change between varying quantities* (Confrey & Smith, 1995; Saldanha & Thompson, 1998). Covariational reasoning is foundational for students' success with many topics in the K-12 mathematics curricula (Adu-Gyamfi & Bossé, 2014; Ellis et al., 2016; Ferrari-Escolá et al., 2016; Johnsons et al., 2020; Moore, 2014; Weber & Thompson, 2014). Nevertheless, students and teachers tend not to engage in covariational reasoning when examining functions (Thompson & Carlson, 2017). Thompson et al. (2017) argued that covariational reasoning is "nontrivial," noting that most of the nearly 500 mathematics teachers in their study did not demonstrate covariational reasoning in their work. Thus, there is a need to create scalable mechanisms that support mathematics teachers in developing covariational reasoning skills, which can help disrupt the cycle of mathematics students leaving schools with underdeveloped reasoning skills for understanding functions.

Shape thinking is a framework that can be used to characterize the extent to which learners engage in covariational reasoning when constructing or interpreting function graphs (Moore & Thompson, 2015). There are two types of shape thinking, namely static shape thinking and emergent shape thinking. *Static shape thinking* includes focusing on perceptual objects and associating "equations, names, or analytic rules as facts of shape" (Moore & Thompson, 2015, p. 785). Vishnubhotla and Paoletti (2020) documented a pre-service teacher engaging in static shape thinking when using language such as "half of parabola" and "it started…close to zero and then increases very fast" to describe a quadratic relationship graphed in a coordinate plane (p. 1700). This explanation foregrounded a shape-based description without addressing quantitative relationships to unpack why the graph has these visual features. Another example of static shape thinking is a case where mathematics learners associated "moving up" on a graph with corresponding positive changes in a quantity's value even though the coordinate system was oriented so that "moving up" represented a negative change (Lee et al., 2019). Lee et al. (2019) also argued that reasoning that relies on perceptual objects (i.e., static shape thinking) can constrain learners' potential to engage with novel quantitative scenarios or non-canonical coordinate systems.

In contrast to static shape thinking, Moore and Thompson (2015) posit *emergent shape thinking*, or "understanding a graph simultaneously as what is made (a trace) and how it is made (covariation)" (p. 785), as a more sophisticated way of reasoning about functions. Emergent shape thinking includes the logical necessity for why functions behave in particular ways by examining quantitative relationships. Further, there is evidence that emergent shape thinking can generalize to novel functional scenarios (e.g., see Moore et al., 2019). While emergent shape thinking can include features of static shape thinking, it also includes conceiving of quantities, how quantities' magnitudes vary individually, and how they covary to reason about function graphs (see Paoletti & Moore, 2017 for an example of emergent shape thinking).

We argue that static and emergent shape thinking represent two contrasting points along a spectrum from less to more sophisticated ways of reasoning about functions and to support their students' movement along this spectrum, teachers need experiences at the emergent shape thinking end of the spectrum. Sociomathematical norms are powerful social mechanisms and supporting mathematics teachers' participation in communities with discourse characterized by particular sociomathematical norms is one way to scaffold teachers' movement along this spectrum. The current study documents the emergence of sociomathematical norms in an online setting that represent mathematics teachers' collective shift from examining visual features of graphs to the covariation of quantitative relationships that underlie them.

Emerging sociomathematical norms in online discussions

The emergent perspective (Cobb & Yackel, 1996) frames our conceptualization of mathematics teacher learning. From this perspective, mathematics learning can be conceived as a reflexive relationship between individual advances in mathematical reasoning and the emerging social and sociomathematical norms that create context for these advances. Studies framed by the emergent perspective often focus on the "social side" and document the emergence of norms to better understand the social context within which individual mathematics learning occurs (e.g., see Clark et al., 2008; Cobb et al., 2001; Whitacre & Nickerson, 2016).

Following Cobb, Yackel, and colleagues, we argue that teachers participating in an online course can be considered a community that engages in collaborative mathematical activity. In online asynchronous discussions, where text is the primary mode of communication, teachers use *mathematical discourse* to share aspects of their mathematical reasoning with colleagues. Mathematical discourse is a special kind of discourse because in it, everyday words take on meanings specific to the discipline of mathematics (Sfard, 2008). Mathematical discourse can include words associated with mathematical objects such as "quantity," "function," and "graph" (Sfard, 2007). Mathematical discourse that reflects static and emergent shape thinking includes words such as "straight," "curved," and "shape" or "varies," "as x increases/decrease," and "as x increases by," respectively.

In designed learning environments, certain tools, task structures, collaboration structures, and discourse scaffolds can come together to support specific learning outcomes (Sandoval, 2014). These design features can increase the likelihood that teachers' contributions to the mathematical discourse enable and/or support shifts in the sophistication of their mathematical discourse (Pea, 2004). We borrow from Schwarz et al. (2018) and refer to these (spontaneous) shifts as *critical moments*—initial advances in the sophistication of teachers' mathematical discourse. Critical moments can provide insight into both how teachers begin engaging in more sophisticated mathematical discourse and learning environment design features that create context for emerging norms.

An affordance of online discussions is that critical moments can become objects for reflection and discussion (Shumar, 2017). In contrast to in-person discussions, where utterances are ephemeral, discourse in online discussions is both permanent and public and teachers maintain access to their colleagues' mathematical discourse for the duration of the learning experience. Therefore, activities can be designed to support teacher examination of past discussion board conversations with the Noticing and Wondering Framework (Hogan & Alejandre, 2010). This framework can scaffold teachers' attention to mathematical details of critical moments and development of generative feedback (Matranga et al., 2018). Such access to and engagement with critical moments can increase the likelihood of individual utterances impacting others' discourse and, ultimately, their mathematical development (Borba et al., 2018). Further, teachers continuing to engage with colleagues' mathematical discourse can influence their conception of the community's expected and accepted mathematical discourse (Lave & Wenger, 1991; Valente, 1995) and create context for teachers negotiating these expectations (McClain & Cobb, 2001)—contributing to an evolution in the sophistication of the community's mathematical discourse. Accordingly, seeding novel mathematical discourse into a community can result in the discourse diffusing through the network and becoming normative.

As teachers contribute novel mathematical discourse to an online discussion, the mathematical discourse can reflect a shift in both what teachers attend to and what the community accepts and expects from one another. While teacher participation in sociomathematical norms does not guarantee they will engage in reasoning that reflects the sociomathematical norms independently (Cobb & Yackel, 1996), Sfard (2008) argued that communication and thinking are interrelated. This suggests that as teachers begin to use mathematical discourse that reflects, for example, coordinating change between quantities, there is potential for their individual ways of reasoning to evolve and include covariational reasoning. Thus, documenting the emergence of sociomathematical norms can provide insight into the quality of teachers' collective mathematical discourse and ways of reasoning that individual teachers may engage in when working with mathematics on their own.

Methodology

We conducted an analysis of norms (Cobb et al., 2001; Rasmussen & Stephan, 2008) to document mathematics teachers' collaborative development of more sophisticated mathematical discourse for examining functions in an online teacher education course. This analytical approach comes from a qualitative tradition of research that aims to better understand collaborative mathematical development (Cobb & Whitenack, 1996). The following presents the online course, participants, data sources, and data analysis procedures.

The online course

We studied a 10-week online course part of a mathematics education master's program offered by a private university in the Northeast region of the USA. This study was approved by the second author's Institutional Review Board. Participants were recruited through social media and at local, regional, and national conferences. All participants in the course consented to participate in this study by completing the approved consent process before any data were collected. The mathematical goal of the course was to support participants in reasoning covariationally about quantitative relationships to understand the behavior of functions. We aimed to support the emergence of sociomathematical norms for analyzing quantitative relationships and justifying why function graphs have particular shapes.

We used the online asynchronous collaboration model (Silverman & Clay, 2009) to structure participants' collaboration. This model capitalizes on the affordances of online discussions to promote extended reflection, discussion, and improvement of mathematical discourse and has five phases: (1) privately working on a mathematics task, (2) sharing a response to the task in an online discussion, (3) reviewing and providing feedback on at least two colleagues' work, (4) revising initial responses to a task, and (5) collectively reflecting on and discussing this mathematical activity. This collaboration structure can bridge individual and collaborative mathematical activity, creating opportunities for regularities in mathematical discourse to emerge (see Duval et al. (2020) for an example online discussion).

We incorporated specific scaffolds for building norms into each phase of the online asynchronous collaboration model. Phase one included interactive applets intended to scaffold participants in connecting quantitative relationships to function graph shapes. Phase two included discussion prompts designed to elicit participants' reasoning about quantities, relationships between quantities, and why functions have certain visual features. Phase three supported participants in using the Noticing and Wondering Framework to provide feedback to their colleagues. We used critiquing guidelines to focus participants' feedback on their colleagues' mathematical discourse that was around quantitative relationships and graph shapes (Table 1). Further, the course instructor consistently challenged participants to explain why function graphs have particular visual features. In phase four, we prompted participants to reflect on the feedback they received and incorporate it into a revised post. Phase five included activities that asked participants to return to past week's discussions and reflect on the effectiveness of their mathematical discourse at communicating why function graphs have particular visual features. These phase five activities are designed to support participants in talking about their mathematical activity and, thus, placing mathematical activity in the background while foregrounding discussions about what constitutes accepted and expected mathematical discourse.

Participants and data sources

Participants in this study were 21 practicing mathematics teachers who participated in our online course. Participants had between one and three years of teaching experience at the secondary level and their geographical locations varied across the USA. The racial and ethnic background of participants was unknown, and 62% of the participants identified as female and 38% identified as male. Data for this study included participants' mathematical discourse in the 1014 course discussion board posts. We extracted these posts from the learning management system and organized them into a spreadsheet. The spreadsheet had ten columns with the following labels: week, forum, activity, thread title, post name, author from, author to, and content of post. The spreadsheet was imported into NVivo 11 for the initial analysis.

Table 1 Critiquing Guidelines from the online PD course

• Did the author mention which quantities they are focusing on in their analysis? If they did, highlight this aspect of their work and note anything you may have noticed or wondered.
• Did the author explain why the graphs look the way they do as a result of underlying quantitative relationships? Point out to your partner specific aspects of their explanation that were particularly effective in conveying why the graph looks the way it does.
• Did the author explain WHY certain important points are important? For example, did they note a change in the direction of the variation of quantities? Did they note specific values in which this change occurred?
• Compare your work to your partner's response. Identify at least one aspect that you noticed is the same and one aspect that you noticed is different and tell your partner how/why these parts of the solution were similar/different [for this criterion try to focus on aspects of mathematical thinking/reasoning as opposed to surface level details].

Data analysis procedures

The goal of our analysis was to understand how participants collaboratively developed more sophisticated ways of examining functions by participating in emerging sociomathematical norms. Our research questions focused our investigation on emerging sociomathematical norms and critical moments in participants' mathematical discourse. The analysis included three phases: We (1) documented regularities in participants' mathematical discourse, (2) examined for whether these regularities were socially accepted and expected, and (3) identified initial shifts in the sophistication of participants' mathematical discourse. As such, our analysis focused on participant mathematical discourse and, therefore, does not formally investigate individual mathematical reasoning.

In phase one of the analysis, we employed constant comparative coding procedures (Strauss & Corbin, 1990) to chronologically examine the dataset and develop themes that characterized regularities in participants' mathematical discourse. We used the shape thinking framework (Moore & Thompson, 2015) and Simon et al.'s (2010) notion of logical necessity—justifications for why certain observable patterns that follow logically cannot be any other way—to conceptualize differences in the sophistication of mathematical discourse. Specifically, we attended to features of participant mathematical discourse that reflected "equations, names, or analytic rules as facts of shape" (static shape thinking) or quantities, the variation of quantities, and the covariation between quantities (emergent shape thinking). Further, we focused on whether and how participants justified why certain observable patterns could not by any other way.

This framing allowed us to distinguish mathematical discourse such as "the graph has a parabolic shape" from discourse such as "as the length of the vertical distance from the x-axis to the unit circle's circumference varies…" because the former uses "parabolic shape" as a fact of shape while the latter reflects discussion of how a quantity ("the length of the vertical distance from the x-axis to the unit circle") varies. Attending to participant justifications was also important because, as argued above, discourse that reflects emergent shape thinking could also include features of static shape thinking. Therefore, when we observed discourse such as "the graph has a parabolic shape," we further scrutinized the discourse for whether and how it was justified, and specifically whether the justification reflected emergent or static shape thinking.

With this framing, while also maintaining an open stance to allow themes to emerge from the data, we individually reviewed the dataset to gain familiarity with participant mathematical discourse. This was followed by weekly data sessions that included developing short codes describing participant mathematical discourse, comparing data and emerging codes to existing codes and associated data, condensing codes into categories, and then identifying themes with descriptive titles that represented commonalities in the categories.

The goal of the second phase of analysis was to further analyze themes in participants' mathematical discourse identified in phase one to document whether the themes represented sociomathematical norms. Consistent with Dean (2005), we used two forms of evidence to determine whether these regularities in mathematical discourse were accepted and expected and represented sociomathematical norms: (1) evidence of participants *explicitly discussing* expectations for and value of mathematical discourse, and (2) occasions of participants collegially *challenging* one another to engage in particular mathematical discourse. The second form of evidence is grounded in the assumption that norms are valued and participants violating a norm might be pressed to align their mathematical discourse with class expectations (Sfard, 2000).

To prepare for phase two of the analysis, we organized the coded data into a norms and practices chart—an adaptation of Rasmussen and Stephan's (2008) mathematical ideas chart. The chart's columns included the week of the online PD course, the thread number, a link to posts in the thread, participants who contributed to the thread, the mathematics task of focus in the thread, theme(s) identified in that thread (if any), and annotations regarding posts with explicit discussion or challenge in the thread (Table 2). We went row by row through the chart and examined for explicit discussion and challenges related to the identified themes in mathematical discourse during additional data sessions. This included looking at rows in the chart that were and were not labeled with the identified themes to investigate whether participants challenged colleagues when their discourse did not align with the emerging theme. We documented participant interactions identified as explicit discussion and challenge in the chart with annotations in the appropriate column and row.

The purpose of the final phase of analysis was to identify critical moments and the associated mathematical task that provided context for critical moments and emerging sociomathematical norms. We conceptualized characteristics of a critical moment as the first time we observed participants engaging in, challenging a colleague to engage in, or explicitly discussing the mathematical discourse that constituted participation in the class's norms. We went back through the data using the norms and practices chart to identify the first instance of data that was connected to the emergent sociomathematical norms while noting the mathematical activity that created context for the critical moment.

Findings

Overview of themes in mathematical discourse

We identified four broad themes in participant mathematical discourse (Table 3). These themes included *static shape discourse*, *explaining why*, *action orientation*, and *emergent shape discourse*. Table 3 introduces each theme and its frequency in our data.

These themes characterized approximately 50% of the 687 posts in the course discussion boards that included participants engaging in mathematical activity. Posts not included in the table above included participants praising their colleagues' work, making agreement statements, and sharing and comparing information not related to the core mathematics task. While posts of this type are common in teacher education and can have implications for community-building (Zhang et al., 2017), they are not pertinent to the focus of this study.

The second phase of our analysis began with identifying candidates for sociomathematical norms by identifying occurrences of explicit discussion and occasions of challenge related to these themes. In our analysis, we observed both explicit discussion and occasions of challenge for both *explaining why* and *emergent shape thinking*. As we did not observe explicit discussion or occasions of challenge related to *static shape discourse* or *action orientation*, we focus on the themes *explaining why* and *emergent shape discourse* and seek to document how each can be characterized as (emerging) sociomathematical norms.

Before introducing the sociomathematical norms, we briefly discuss aspects of participants' mathematical discourse that were prevalent during the initial weeks of the course. It is against this background that we document both the emerging sophistication of participant mathematical discourse and the critical moments that catalyzed them.

Table 2 Example norms and practices chart

Week	Thread number and link	Participants	Task	Themes	Annotations
Week 3	*5*	Ava, Nina, Paul	Intro to trig task	Explaining why	Nina violates explaining why and Paul challenges Nina to explain why the graph has "hills and valleys"
Week 3	*6*	Cindy, Chloe, Instructor	Intro to trig task	NA	No explicit discussion or challenges related to the themes

Table 3 Overview of themes in discourse

Theme	Description	Posts
Static shape discourse	Participants used names of shapes and familiar function graph shapes (e.g., parabola) to discuss function graphs. They also made broad generalizations about connections between a graph's shape and the problem context	95
Explaining why	Participants justified why function graphs have particular visual features by relating symbolic representations of functions to specific visual features of function graphs and relating quantities to specific visual features of function graphs	87
Action orientation	Participants used the language of input–output to engage with functions and consider specific values of a function's domain and range	60
Emergent shape discourse	Participants used mathematical discourse reflecting covariational reasoning to justify why function graphs behave in particular ways	95

Participants' initial mathematical discourse: static shape discourse

There is consensus that mathematics learners tend to approach functions by associating function names, equations, or rules with shapes of graphs and, as we discussed above, such approaches have been referred to as static shape thinking (Moore & Thompson, 2015). As expected, our initial observations of participants' discourse were consistent with this approach; therefore, we refer to participants initial mathematical discourse as *static shape discourse*. The following presents the initial mathematics task of our online course and features of *static shape discourse* to set the stage for documenting the emergence of participants' sociomathematical norms.

The goal of the initial week's mathematics task (adapted from Saldanha and Thompson (1998)) was to support participants in examining functions by coordinating change between two covarying quantities. The quantitative scenario includes two points (City A and B) that can be placed in a two-dimensional plane and a third point (a car) that moves along a linear path (a road) through this plane (Fig. 1). An interactive applet models this scenario and allows participants to investigate the covarying quantities.

The task scaffolds participants in visualizing how the quantities' magnitudes vary and covary in the coordinate plane. The quantities' magnitudes are superimposed along horizontal and vertical axes. The blue segment represents the distance from the car to City B, and the red segment represents the distance from the car to City A. As the car travels from left to right, for instance, the distance from the car to City B decreases, resulting in an equivalent change in the length of the blue segment along the vertical axis. The correspondence point traces a record of the covariation between quantities. The activity prompts participants to (1) pick locations for the cities, (2) move the car along the road, (3) examine how the distance from the car to each city varies individually and covaries, and (4) explain why the graph looks the way that it does. Participants worked privately on this task, submitted their responses to a discussion board, and then asynchronously discussed their work.

In their online discussions, participants engaged in *static shape discourse* when they described their observations from the car problem with discourse of (1) shapes, (2) familiar function graph shapes, (3) a moving correspondence point, and (4) a connection between the context and graph shape. Table 4 provides representative samples of these features.

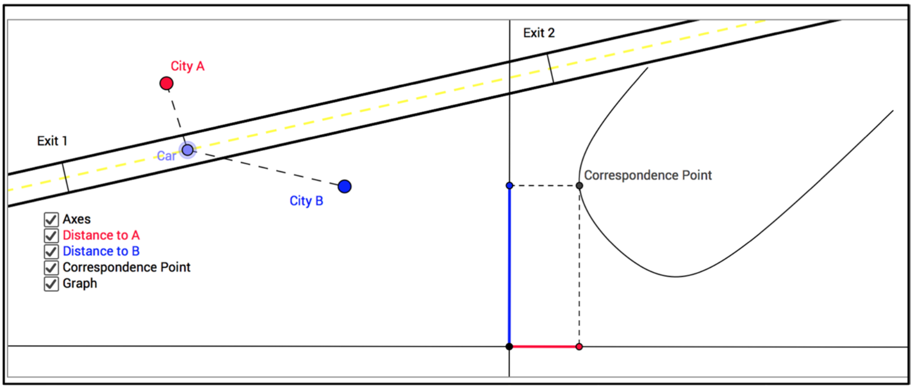

Fig. 1 The car problem interactive applet

Table 4 Features of static shape discourse

Features of mathematical discourse	Representative samples from our data
Shapes	The graph will create a short and wide rectangle which morphs into a tall and skinny rectangle. (Hank)
	The "U shape" or the curve of the graph can be explained because of the way the two lengths of the segments decrease then increase at the same rate. (Nina)
Familiar function graph shapes	When the correspondence point was added, I noticed the graph has parabolic features instead of features of an absolute value ("V") function. Both are similar shapes, so I'm not sure if I am correct. I just saw more of a curve at the bottom as compared to that sharp turn. (Jazmine)
The movement of the correspondence point	Initially, the point moved down and to the left. The correspondence point begins to move more slowly in both directions, but slows down more quickly in the horizontal direction until the graph is only moving down. The graph then begins to move and accelerate to the right as it slows down vertically, thereby moving down and to the right for a brief while. Once the car reaches the minimum distance to city B, the correspondence point is only moving horizontally. Finally, the correspondence point begins to accelerate upward again and moves up and to the right. (Charles)
Connecting the context and graph shape	When two cities are close to each other the graph is more like a parabola. However, when the cities are further apart from each other, the graph becomes a polynomial of a higher even degree because it flattens out. (Taylor)

The first aspect to notice in these samples is that participants treated the function graph and interactive applet as the primary source of their examination. For example, participants referenced "the graph" and used names of shapes such as "U shape" and "rectangles" or familiar function graph shapes such as "parabolic features" and "absolute value ('V') function" to describe the graph. In addition, participants described the direction ("down and to the left") and/or speed ("begins to move more slowly") of the correspondence point's movement when it traced the graph. Participants also connected the car problem context and function graph by using names of familiar function shapes to describe graphs that correspond to particular orientations of the cities ("When two cities are close the graph is more like a parabola").

These representative samples of participants' discourse focus on perceptual objects from the car problem such as the graph itself, the correspondence point, or the location of the cities and appear to include names as claims about shape such as "parabola" or "absolute value." These features of discourse are similar to explanations reflecting static shape thinking documented in Vishnubhotla and Paoletti (2020). While participants' examination of the car problem was accurate, their discourse did not include focus on covarying quantities to explain why the car problem graphs had particular shapes. Although some participants did connect the context to the graph shape, their explanations foregrounded broad generalizations of the context and names of functions to describe the graph. Thus, we use *static shape discourse* to characterize participants' initial mathematical discourse with the acknowledgment that it may not be completely aligned with Moore and Thompson's (2015) conceptualization of static shape thinking because of these signs of participants connecting the context and graph shape.

It is important to note that our documentation of *static shape discourse* does not indicate that participants could not reason covariationally about quantitative relationships in the car problem. Instead, it is evidence that participants' mathematical discourse did not reflect covariational reasoning. Nevertheless, as *static shape discourse* was an overwhelmingly common theme in participants' initial mathematical discourse, we use it as a starting point for documenting increases in the sophistication of their mathematical discourse.

The following sections document the emergence of two increasingly more sophisticated sociomathematical norms. We provide an overview of our coding and the context within which each norm emerged, critical moments in participant discourse, themes in their discourse, and evidence of participants discussing their engagement in discourse that reflected each theme and challenging one another to engage in such discourse when these details were not included.

The first emerging sociomathematical norm: *explaining why*

Throughout the course, we observed each of the 21 participants in 87 discussion board posts (12.7% of posts) engage in mathematical discourse reflecting the theme explaining why. Furthermore, we found evidence of repeated challenges (48 occasions) and explicit discussion (13 posts) associated with explaining why*,* both of which are examples of evidence of the emergence of sociomathematical norms (Dean, 2005).

It is important to note that initial discourse in the course was not characterized as *explaining why* (see Sect. 4.2); however, as the course progressed, the mathematical discourse evolved to include justifications for why function graphs have particular visual features. Specifically, features of *explaining why* included (1) relating symbolic representations of functions to specific visual features of function graphs, and (2) relating quantities to specific visual features of function graphs. We observed each participant using at least one of these discourse features when engaging in a reflective discussion activity or the introduction to trigonometry activity.

Because *explaining why* included justifying why graphs must have certain visual features by drawing from mathematical objects (i.e., quantities) other than the graph itself, we argue that it became normative for participants to communicate a logical necessity with their mathematical discourse, that is, they were articulating why certain observable patterns must be the case and cannot be any other way. Shifting from describing observable patterns to articulating a logical necessity is evidence of an increase in the sophistication of mathematical discourse (Simon et al., 2010). Thus, *explaining why* represented a collective shift from *static shape discourse* toward a more sophisticated approach to examining the behavior of functions.

A reflective discussion and the trigonometry activity

The reflective discussion activity included reading an article about conceptual conversations (see Thompson et al., 1994), returning to the week one discussion of the car problem, treating past mathematical discourse as objects for reflection, and considering whether this mathematical discourse was conceptual (e.g., by focusing on reasons, relationships, and meanings). Concurrently, participants worked on a trigonometry activity where they extended their work with the car problem by examining the behavior of $y = \sin(n*x)$ using an interactive applet.

The applet was designed to scaffold focus on quantities from the unit circle to understand the behavior of the $y = \sin(n*x)$ graph (Fig. 2). The applet works as follows: The left portion of the applet is a unit circle. Users can vary the openness of the circle's central angle by dragging a point around the circle's circumference. Two quantities of focus include the angle measured in radians and the sine of the angle. The magnitudes of these quantities are mapped onto the coordinate plane. A correspondence point traces a record of how the quantities' magnitudes covary in the coordinate plane when the openness of the central angle varies. The activity prompts teachers to (1) move a point along the circle, (2) examine how the arclength along the circle and the horizontal and vertical components of the point's coordinates vary individually, (3) examine how the arclength and these components covary, and (4) explain why the graph looks the way that it does.

A critical moment

We observed the initial shift in participants' mathematical discourse toward *explaining why* during the reflective discussion activity. Riley was the first participant to initiate a discussion about examining functions by justifying why graphs have particular visual features. Specifically, Riley appeared to consider an alternative approach to examining function graphs that did not include using her knowledge of familiar graph shapes to "describe the graph." Riley described this alternative approach: "I could focus on why the graph was the shape that it was by describing in detail the relationship of the distances between the cities." Her approach included using relationships between quantities (e.g., "distances between the cities") to describe "why the graph was the shape that it was." Thus, Riley's mathematical discourse reflects a shift from *static shape discourse* to considering the use of mathematical objects other than the graph itself to understand the function's behavior.

Features of *explaining why*

During the introduction to trigonometry activity, participants used mathematical discourse indicating that they analyzed the symbolic representation of trigonometric functions and underlying quantities from the unit circle to justify why function graphs have particular

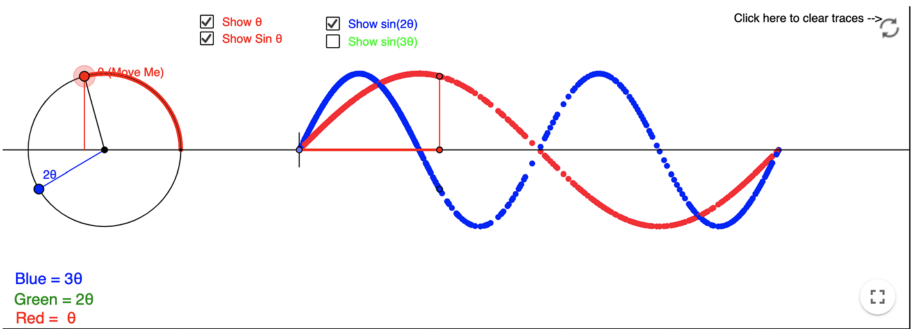

Fig. 2 Trigonometry interactive applet

visual features. Table 5 provides representative samples of discourse features reflecting *explaining why*.

The first aspect of participants' mathematical discourse to notice is their explicit intention to explain why function graphs have certain visual features. For example, Rose noted, "we can predict the behavior of the sine graph" and then discussed the "waves." Ava also stated, "the graph looks the way it does because…." The second aspect to notice is how participants related mathematical objects to particular visual features of the function graph. Rose described a relationship between a parameter of the symbolic representation of the function ("the number in front of x") and a specific visual feature of the sine graph ("the amount of waves"). Similarly, Ava described a relationship between quantities in the unit circle ("the radius of the circle") and a specific visual feature of the sine graph ("constant height in [the graph's] waves"). This provides evidence that participants' attention shifted from the function graph itself toward reasons for why the function graph has a certain shape.

Occasions of challenge

We observed 48 occasions where participants collegially challenged their colleagues to explain why graphs have particular visual features when these details were not included. In their challenges, participants referenced a specific visual feature of a function graph and requested explanation of why the function has this visual feature. Further, participants used the discourse of noticing and wondering to frame a large proportion of these challenges. Consider the following representative interaction between Nina and Paul.

> **Nina:** This graph appears as it does because of the Unit Circle. Essentially as the values of sin (x) make their way around the circle, they start again at zero.

Table 5 Features of explaining why

Features of mathematical discourse	Representative samples from our data
Relating the symbolic representation of functions to specific visual features	I believe that we can predict the behavior of the sine graph…The amount of waves for one period is equivalent to twice the number in front of x. So, w = 2n (where w represents the amount of waves per period). (Rose)
Relating underlying quantities to specific visual features	I believe the graph looks the way it does because the range is restricted by the radius of the circle. I was looking at a unit circle and I knew that the range for sin(x) is -1 < x < 1. It matches the radius, so I thought it might have something to do with that. Also, the radius is the hypotenuse of the triangle, which is in the ratio of sine, which relates the sides together (opposite/*hypotenuse*). That is why the graph stays at a constant height in its waves. (Ava)

> **Paul:** I noticed that you wrote: "This graph appears as it does because of the Unit Circle. Essentially as the values of sin(x) make their way around the circle, they start again at zero." I wonder... if you could elaborate on this concept more. Why do the values start again at zero? Why does the graph have hills and valleys?

Nina reasoned about why the sine graph "appears as it does" by referencing the unit circle and the values of sine. However, Nina did not relate specific quantities from the unit circle or the symbolic representation of the function to specific visual features of the sine graph. Instead, Nina's mathematical discourse reflects *static shape discourse* because the unit circle, in this case, is a perceptual object from the interactive applet (Fig. 2) that scaffolded participants in reasoning about the sine function. Nina's discourse of "make their way around" suggests she was describing how the intersection point of the unit circle's arc length and vertical distance from this arc length to the x-axis moved around the circle, without explicitly connecting this to particular visual features of the sine graph. Thus, from our perspective, Nina's mathematical discourse appeared to align more with *static shape discourse* than the emerging regularity in participants' discourse for explaining why function graphs have certain visual features.

In his response, Paul noticed a specific aspect of Nina's mathematical discourse and then used an "I wonder" statement to challenge her to explain why. Paul's wondering appeared to press Nina to focus on quantities in the unit circle to explain why the "values start again at zero" and then to relate this discourse to specific visual features ("hills and valleys"). Thus, Paul's challenge possibly scaffolded by the Noticing and Wondering Framework intended to push Nina to explain why the sine graph looks a particular way. This suggests that Paul may have interpreted Nina's mathematical discourse as violating what he perceived as accepted and expected mathematical discourse.

Explicit discussion

We observed three online discussions where participants explicitly discussed their focus on what they referred to as "the why." The following representative sample conversation was initiated by Nina and included responses from Ava and Cindy.

> **Nina:** Looking at the function of sin(x) is something I have done many times. Instead of telling you that it crosses 0 at these points, and equals 1 at these points, I had to look at WHY [authors did not add emphasis] it did this…I found that for so long, I knew how to answer questions on sin(x) but I really did not have a full understanding as to why.
>
> **Ava to Nina:** I know about sine functions, but never considered the why or reasoning before. Now I want to understand and be able to explain not just do.
>
> **Cindy to Ava:** It was very difficult for me to see the whys of this week's activity initially. The last time I encountered trigonometry was in high school, and I only remembered...SOH CAH TOA. After discovering what it is that we were trying to find, I realized how much more valuable my learning experience was this time around.

In this conversation, each participant referenced a focus on why or "the why." Nina explained that she was beginning to examine why the sine graph has particular features such as crossing zero and one at specific points. Ava and Cindy referenced "the why" and a value for this approach to examining trigonometric functions. Given Nina's initial post in the thread, our observations of participants' mathematical discourse, and participants' challenges of one another to explain why, we argue that "the why" emerged as participants' short hand for the expectation to explain why function graphs have certain visual features.

In sum, participants' explicit discussions about "the why" and the occasions of challenge provide evidence supporting our conjecture that *explaining why* was a sociomathematical norm in the online course. Central to this sociomathematical norm was participants articulating a logical necessity for why function graphs have certain shapes and their reasons included relating symbolic representations of functions and underlying quantities to visual features of function graphs. González (2021) documented similar features in mathematics teachers' reasoning when they examined a quantitative scenario around global warming (p. 12), while DeJarnette (2018) showed that students' "empirical conception" included adjusting parameters of the symbolic representation of a sine/cosine function to produce specific graph shapes. Because participants did not coordinate change between varying quantities when they *explained why*, our analysis suggests that participants were collectively moving toward the course goal of examining the covariation between quantities to justify the behavior of functions.

The second emerging sociomathematical norm: *emergent shape discourse*

Throughout the course, we observed 19 of the 21 participants in 95 discussion board posts (13.8% of posts) engaging mathematical discourse reflecting the theme emergent shape discourse. Furthermore, we found evidence of repeated challenges (29 occasions) and explicit discussion (26 posts) associated with emergent shape discourse*,* both of which are examples of evidence of an emerging sociomathematical norm (Dean, 2005).

It is important to note that initial discourse in the course was not characterized by *emergent shape discourse* (see Sect. 4.2); however, as the course progressed, the mathematical discourse evolved from static shape discourse to include *explaining why* (see Sect. 4.3) and then further evolved to include mathematical discourse reflecting covariational reasoning to justify why function graphs behave in particular ways—a key feature of emergent shape thinking (Moore & Thompson, 2015). The features of *emergent shape discourse* included approaches to coordinating change between varying quantities. This mathematical discourse emerged as participants engaged in a discussion about their quiz and composite trigonometric functions.

Consistent with *explaining why*, *emergent shape discourse* included using mathematical objects other than the function graph itself to justify why graphs behave in certain ways—indicating that participants continued to articulate a logical necessity. Because *emergent shape discourse* included coordinating change between covarying quantities to justify why function graphs behave in certain ways, participants articulated a logical necessity that is more aligned with the broader mathematical community's approach (e.g., see Carlson et al., 2002; Moore & Thompson, 2015). Thus, we argue that *emergent shape discourse* represented a collective shift in the sophistication of participants' mathematical discourse beyond *explaining why*.

The quiz and composite functions

The quiz included participants discussing the behavior of the functions $y = \cos(2x)$ and $y = \cos(4x)$. The composite functions activity included participants examining the graph of

$y = \sin(x^2)$ or $y = \sin(2/7x)$ in Desmos and participating in an online discussion about why these functions behave in certain ways.

A Critical moment

We observed a shift in participants' expectations for how to examine functions when discussing their quiz. Specifically, the instructor pressed a participant who was *explaining why* to examine the covariation between quantities to explain why. Consider the following interaction between Ava and the instructor:

> **Ava**: Please let me know what you think on my thoughts: cos(2x) has 2 graphs in the period from 0-2pi and the period of 1 graph is ½ as long as cos(x). cos(4x) has 4 graphs in period 0-2(pi).
> **Instructor to Ava**: … we want to push for you to be able to explain WHY "cos(2x) has 2 graphs in the period 0-2(pi)." You might think: (1) I remember if its cos(ax) and a > 1, the period will be 1/a as long as cosx; (2) I looked at the graph and saw there were 2 graphs in the period 0 to 2pi; (3) I examined the covariation. Either of these perspectives provides insight into WHY the period was 1/2 as long….I'm pushing you to describe why from the third perspective.

In this post, Ava described the relationship between the symbolic representation of the function and the "number of graphs," which we interpreted as the number of times the function cycled through its range of values, to explain why the function has particular visual features. The instructor challenged Ava to explain why, outlined three possible ways to do so, and one of those ways included examining the covariation. While the instructor challenged participants to refine their mathematical discourse multiple times, this critical moment was the first occasion where "examining covariation" emerged in participants' online discussions as an approach to justifying why functions have particular visual features.

Features of *emergent shape discourse*

We observed participants engage in *emergent shape discourse* during online discussions around the composite trig functions task. Features of this discourse included descriptions of (1) changes in the value of one quantity with respect to changes in the other quantity, (2)

Table 6 Features of emergent shape discourse

Features of mathematical discourse	Representative samples from our data
Describing changes in the value of one quantity with respect to change in the other quantity	As x varies, x^2 also varies, and $\sin(x^2)$ varies until x^2 reaches a value of pi/2 (when x = (pi/2)^0.5). (Chloe)
Describing the direction of change between quantities	As x increases from 21pi/4 radians to 7pi radians, 2x/7 increases from 3pi/2 radians to 2pi radians and sin(2x/7) increases from -1 rad to 0 radians. (Cindy)
Describing the amount of change in one quantity with respect to the other quantity	As x increases by 1 rad from 0 to 1, 2x/7 increases by 2/7 radians from o to 2/7, sin(2x/7) increases by .28 rad from 0 to .28. (Hank)

the direction of change between quantities, and (3) the amount of change in one quantity with respect to the other quantity. Table 6 includes representative data samples of these features.

One import aspect to notice in this discourse is how participants described the quantities as dynamic using discourse such as "as x varies" and "as x increases." Participants also described the change of at least two different quantities in proximity to one another ("as x varies, x^2 also varies," "as x increases…$2x/7$ increases…and $\sin(2x/7)$ increases…," and "as x increases by 1 rad…$2x/7$ increases by 2/7 radians…$\sin(2x/7)$ increases by 0.28 radians…"), illustrating their focus on relationships between changing quantities. What distinguishes these features of participants' mathematical discourse is the specificity with which they coordinated change between quantities by using discourse such as "varies," "increases," and "increases by." "Varies" indicates change, "increases" indicates the direction of change, while "increases by" indicates the direction and amount of change. These features of mathematical discourse and associated distinction between them are consistent with what is outlined in Carlson et al. (2002) as indicators of coordinating change between quantities and, thus, mathematical discourse that reflects covariational reasoning.

Each of the 19 participants who used mathematical discourse reflecting covariational reasoning also stated that their examination of the covariation between quantities was a justification for why the function graph has a particular look. For example, Chloe connected the covariation between quantities (x, x^2, and $\sin(x^2)$) to a visual feature of the function's graph: "this is shown in the graph; the graph travels up and down at an increasing rate." Thus, coordinating change between quantities emerged as participants' approach to justifying why functions behave in certain ways.

Occasions of challenge

We observed 28 occasions where participants challenged their colleagues to examine the covariation between quantities when these details were not included in their discussion post. Although there were three different features of *emergent shape discourse*, the observed challenges requested a focus on the covariation between quantities. Further, participants used the discourse of noticing and wondering to frame a large proportion of these challenges. The following presents a representative sample where Summer examined the function $y = \sin(x^2)$ by *explaining why* and Cindy pressed her to examine the covariation between quantities.

> **Summer:** The graph looks like an "m" shape because of the ratio of the opposite side over the hypotenuse (radius), because on the positive side both signs are positive so the graph will stay above the x-axis, but then on the negative side because both signs will be negative so the result will be positive therefore staying above the x-axis. Also as the x-values get farther away from the origin the graph waves get closer and closer together, and I believe this is because of the x^2 part, the inputs are getting so large so fast that the graph needs to "move" faster.

Summer began by mentioning the shape of the graph ("*m* shape") and then justified why this is the case by referencing the unit circle ("the ratio of the opposite side over the hypotenuse"). Then, she further discussed the behavior of the "waves:" "as the x-values get farther away from the origin the graph waves get closer together…" and justified this behavior by relating it to the function's symbolic representation ("x^2 part"). Summer *explained why* because she used the symbolic representation of the function to justify why the graph has a particular shape. Thus, her mathematical discourse was not consistent with the emerging regularity in the class's mathematical discourse because she did not coordinate change between varying quantities to formulate justifications.

Cindy provided Summer with feedback by noticing a portion of Summer's explanation and then wondering about how Summer could examine covariation between quantities to explain why the graph has a particular look.

> **Cindy's feedback:** I noticed that you wrote "As the x-values get farther away from the origin the graph waves get closer and closer together" …and I wondered if you can explain this using the relationship and covariation between the underlying quantities instead of just describing what the graph looks like.

Cindy's noticing referenced Summer's description of a visual feature of the function's graph: "waves get closer together." Then, Cindy used a wondering to press Summer to examine "covariation between the underlying quantities" to provide justification for this visual feature. This challenge possibly scaffolded by the Noticing and Wondering Framework suggests Cindy perceived Summer's mathematical discourse as violating what she was beginning to understand as an accepted and expected approach for examining the behavior of functions.

Explicit discussion

Participants also discussed the importance of examining covariation between quantities to reason about why functions behave in certain ways. In a representative example conversation, Paul raised covariation to explain why as a topic for discussion:

> **Paul (initial):** I am gaining a greater appreciation as to why we are focusing so heavily on covariation - the fact that we need to talk about covariation in order to get at the 'why.'
> **Chloe to Paul:** I am also gaining a greater appreciation for these ideas of covariation because prior to these activities, I would only be able to explain the different properties of the sine function and not why these properties are true.

Paul suggested that covariation is one way to explain "the why." Chloe agreed and implied that covariational reasoning can be used to explain why when examining the sine function. Our interpretation of Paul's use of "the why" is that he was referring to *explaining why*. This suggests that Paul was arguing that one needs to examine covarying quantities to *explain why*. We observed repeated occasions where participants referenced and discussed covariation as a way to explain "the why."

In summary, these explicit discussions about using covariation to explain "the why" and occasions of challenge supported our conjecture that *emergent shape discourse* was a sociomathematical norm. A central feature of *emergent shape discourse* was analyzing covariation between quantities to justify why graphs have certain visual features. While we documented several features of participants' mathematical discourse reflecting covariational reasoning, there was no evidence of participants making similar distinctions in their discussions or challenges. Further, it was unclear whether participants imagined the trace of the function graph emerging while examining the covariation between quantities (a feature of emergent shape thinking Moore & Thompson, 2015). This suggests that participants may have experienced *emergent shape discourse* as an expectation to coordinate change between quantities to explain why functions behave in certain ways, which suggests movement along the continuum toward emergent shape thinking.

Discussion

This study investigated mathematics teachers' participation in online mathematical discourse and documented the emergence of two sociomathematical norms. We provided evidence that participants engaged in *static shape discourse* at the onset of the online course. Whitacre and Nickerson (2016) argued that teachers' existing approaches to participating in mathematical activity can shape commonalities in their initial mathematical practice during a collaborative learning experience. While this initial mathematical practice or discourse in our case can be a resource for teachers' collaborative learning (Smith et al., 1994), it suggests that *static shape discourse* may have been part of participants' existing ways of engaging with functions. We argued above that mathematical activity that relies on perceptual objects and shape-based associations can constrain learners' potential to engage with novel mathematics tasks. Thus, this study adds to research indicating the need for mathematics teacher professional learning opportunities focused on engaging in emergent shape thinking when working with function graphs (e.g., see Thompson et al., 2017).

A contribution of this study is evidence of an online course creating context for shifts in teachers' collective mathematical discourse. We documented our participants' mathematical discourse evolve from emphasizing visual features of function graphs to understand the behavior of functions (*statics shape discourse*) to providing a logical necessity for why function graphs have certain visual features (*explaining why*) and, finally to coordinating change between covarying quantities to explain why functions behave in certain ways (*emergent shape discourse*). *Static shape discourse*, *explaining why*, and *emergent shape discourse* characterize a process by which mathematics teachers can collectively develop more sophisticated approaches to examining functions. This characterization of participants' learning process (Table 7) can function as an instructional tool that supports teacher educators in noticing and capitalizing on critical moments in mathematics teachers' discourse about functions in collaborative learning contexts. Future work is needed to test, revise, and refine this documented collaborative learning process.

The second contribution of this study is that it extends research on norms in general (e.g., see Cobb et al., 2001; Fukawa-Connelly, 2012) and in mathematics teacher professional learning in particular (e.g., see Clark et al., 2008; Dean, 2005; Whitacre & Rumsey, 2018) by providing evidence of the emergence of sociomathematical norms in an online asynchronous course for teachers. Our study suggests that online courses can increase mathematics teachers' access to legitimate collaborative mathematics learning experiences that are so important for effective professional learning (Darling-Hammond et al., 2017). Additionally, the online nature of online courses increases the potential for scalability of these learning experiences. This increased access to professional learning opportunities and community support may be important for supporting novice teachers who are leaving the profession at high rates (Carver-Thomas & Darling-Hammond, 2017). Novice teachers participating in sociomathematical norms in online courses can have implications for their individual mathematical development (Cobb & Yackel, 1996) and their potential to support the emergence of similar norms in their classes (Clark et al., 2008; Tsai, 2007), which can potentially lead to improvements in their instruction and potential to persist in the career.

Table 7 Summary of study results

	Initial Mathematical Discourse: *Static shape discourse*	SM Norm #1: *Explaining why*	SM norm #2: *Emergent shape discourse*
Characterization of the discourse or norm	Describing perceptual objects primarily with shape names	Explaining why function graphs have certain visual features	Using discourse reflecting covariational reasoning to explain why functions behave in certain ways
Mathematics task creating context for emergence of norm	The car problem	Intro to trigonometry task	Composite trig functions—examining $y = \sin(x^2)$ and $y = \sin(2/7x)$
Critical moment	NA	Explicit discussion of explaining why graphs have certain visible features as a reasonable approach to examining functions	Challenge to *explain why* by examining covariation between quantities
			Coordinated
Features of participants' mathematical discourse	Used (1) shapes, (2) familiar function graph shapes, (3) the movement of the correspondence point, and (4) connections between the context and graph shape to examine the car problem	(1) Related the symbolic representation of the function to specific visual features of function graphs and/or	(1) Changes in the value of one quantity with respect to change in the other quantity,
		(2) Related the underlying quantities to specific visual features of function graphs	(2) The direction of change between quantities
			(3) The amount of change in one quantity with respect to changes in the other quantity

We also extend the work of Cobb et al. (2001), Dean (2005), and Rasmussen and Stephan (2008) by contributing a methodology for analyzing mathematical discussions in online settings, which includes identifying themes in teachers' mathematical discourse and providing evidence that these themes represent norms by documenting whether the discourse was accepted and expected. It is important to note that this methodology focuses on teachers' mathematical discourse in online discussions. Thus, an application of this methodology does not allow one to make claims about individual teacher's reasoning or understandings. While Sfard (2008) argues convincingly that discourse and reasoning are interrelated, it will be important to investigate this interrelationship in online contexts to better understand features of learning environments and sociomathematical norms that can support teacher collaborative learning.

To conclude, we offer conjectures to the following question: How did certain course factors, events, and conditions come together to support the emergence of sociomathematical norms? We conjecture that the reflective discussion activity and Noticing and Wondering Framework were important design features that potentially contributed to the emergence of *explaining why* and *emergent shape discourse*. In our theoretical framework, we argued that seeding discourse into online discussions has potential to result in the emergence of sociomathematical norms and we showed, for example, that Riley's critical moment seeded discourse characteristic of *explaining why* into the community. This critical moment emerged as a result of the reflective discussion activity that asked participants to reflect on their mathematical discourse from the car problem and consider whether it was conceptual. Thus, similar to past research (e.g., see Llinares & Valls, 2010), we found that supporting teachers' engagement with colleagues' previous contributions to the discussion boards contributed to shifts in their discussions.

Our past work has shown that the Noticing and Wondering Framework can support mathematics teachers in focusing on the details of their colleague's mathematical discourse and providing generative feedback (Matranga et al., 2018). The current study documented participants noticing specific aspects of their colleagues' mathematical discourse and using "wonderings" to challenge colleagues to justify why function graphs have particular visual features. Thus, scaffolding collaboration with the Noticing and Wondering Framework may have supported participants in attending to colleagues' discourse that violated their expectations for how to examine functions, pressing one another to refine this discourse, and developing emerging regularities in their mathematical discourse. Our findings my also have implications for other modalities of collaboration as, for example, the Noticing and Wondering Framework and reflective discussion activity could be applied to synchronous courses by having teachers notice and wonder about records of colleagues' reasoning from online discussions during a synchronous meeting.

We also conjecture that the core of a course social network may create conditions propitious for the emergence of sociomathematical norms in online discussions. In this study, we documented critical moments that included Riley, Ava, and the instructor. In Matranga and Silverman (2022), we examined these same 21 participants' interactional patterns and found that their social network had a core-periphery structure and resembled a community. Further, we found that Riley, Ava, and the instructor were three of the five participants who remained in the network's core for the entire course. This suggests that the mathematical discourse in these critical moments and subsequent interactions associated with the emergence of *explaining why* and *emergent shape discourse* had increased visibility to the class, which increases the potential of the mathematical discourse to influence what the community perceived as accepted and expected (Lave & Wenger, 1991; Valente, 1995).

We also note that participants' positioning within the community may have an impact on their participation in mathematical discourse and the extent to which they encounter discourse consistent with emerging sociomathematical norms. In looking across this study and Matranga and Silverman (2022), we noticed that the three participants who did not participate in *emergent shape discourse* were members of the periphery. We are currently exploring the use of social network analysis to identify the core and periphery of a network and then scaffold interaction between teachers who are members of these groups to increase access to participation in sociomathematical norms. Further research is needed to understand "where" mathematical discourse reflected in critical moments originates, how it diffuses through teachers' networks, and begins to constitute emerging sociomathematical norms.

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Data availability The data that support the findings of this study are not openly available in order to protect the identity of the participants. The data are available from the corresponding author upon reasonable request.

Code availability Relevant codes are largely presented and discussed in this manuscript; however, additional codes are available upon request.

Declarations

Conflicts of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

Consent to participate This study was approved by the second author's Institutional Review Board and all participants in this study consented to participate.

Consent for publication Both authors approved this version of the manuscript to be considered for publication in the Journal of Mathematics Teacher Education.

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