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Profiles of teachers' expertise in professional noticing of children's mathematical thinking

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Abstract

This study contributes to the growing body of research that highlights the usefulness of professional noticing of children's mathematical thinking for understanding the complexity and variability in teaching expertise. We explored the noticing expertise of 72 upper elementary school teachers engaged in multi-year professional development focused on children's fraction thinking. Our assessment addressed the three component skills of professional noticing of children's mathematical thinking: (a) attending to children's strategy details, (b) interpreting children's understandings, and (c) deciding how to respond on the basis of children's understandings. We used a latent class analysis to empirically identify three distinct "profiles" of noticing expertise—subgroups of teachers who responded similarly to each other and differently from teachers in other profiles. The profiles differed in their overall noticing expertise as well as their patterns of strengths and areas for growth across the component skills. Thus, the profiles provide a concise, multi-dimensional characterization of noticing expertise that integrates expertise in each of the component skills. The profiles also provide tools for differentiating learning opportunities for teachers in professional development. In addition, our design allowed us to compare teachers' expertise in two common forms of deciding how to respond: deciding on follow-up questions and deciding on next problems. In all three profiles, teachers demonstrated more expertise when deciding on follow-up questions than when deciding on next problems, suggesting not only a starting point for teacher learning but also the need for a line of research focused on different forms of this component skill.

Keywords Teacher noticing · Professional development · Practicing teachers · Responsive teaching · Fractions · Elementary school

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Introduction

As a professional, you are sensitized to notice certain things in professional situations. To develop your professional practice means to increase the range and to decrease the grain size of relevant things you notice, all in order to make informed choices as to how to act in the moment, how to respond to situations as they emerge. (Mason, 2002, p. xi)

Teachers are professionals who regularly *notice* the relevant things children say and do so that they can make informed choices about how to respond during instruction. This type of noticing is central to providing high-quality instruction, but often overlooked because the work is invisible—noticing describes the in-the-moment work teachers do before their observable responses.[1] By naming and studying this invisible instructional practice, researchers have drawn attention to its importance and revealed its complexity.

Mathematics teacher noticing, in particular, has been extensively studied over the past two decades (for compilations, see Schack et al., 2017; Sherin, Jacobs, & Philipp, 2011). Our study contributes to this line of work by identifying profiles of teachers' expertise in mathematics teacher noticing, with the goal of better understanding, capturing, and supporting expertise in this instructional practice. In the following sections, we elaborate on why mathematics teacher noticing is worthy of study and then share our conceptual framework, centered on professional noticing of children's mathematical thinking (Jacobs et al., 2010).

Mathematics teacher noticing as worthy of study

Jacobs and Spangler (2017) identified teacher noticing as a core practice of high-quality mathematics instruction and put forth four main reasons for why teacher noticing is worthy of study. First, teacher noticing expertise has been linked to other outcomes of interest, such as student learning gains (Kersting et al., 2010) and teachers' abilities to adapt mathematical tasks productively (Choppin, 2011). Second, noticing expertise can enhance teachers' abilities to learn from their experiences because they can only reflect on and grow from what they have noticed (Mason, 2002, 2011; Star & Strickland, 2008). Third, teachers usually do not gain noticing expertise solely from years of teaching (Copur-Gencturk & Rodrigues, 2021), but research has shown that expertise can be developed with sustained support (see, e.g., Casey & Amidon, 2020; LaRochelle et al., 2020; Lee, 2019; Roth McDuffie et al., 2014; Schack et al., 2013; Simpson & Haltiwanger, 2017; van Es & Sherin, 2008). Fourth, research outside of mathematics education (e.g., athletics and aviation) has shown how understanding a profession's noticing can enhance the design of supports for developing expertise in that profession (Miller, 2011).

As the field of teacher noticing has grown, we have come to view the synergistic nature of the research on teacher noticing as a fifth benefit—different types of noticing research foreground different aspects of instruction that teachers can notice. For instance, some

[1] In this paper, we focus on teacher noticing during instruction, which is consistent with most of the research on mathematics teacher noticing. However, Sherin (2017) noted that researchers have recently begun to question this boundary. Under question is whether the teacher noticing construct should include not only when teachers interact with children, but also when they prepare for and reflect on their teaching (see, e.g., work on curricular noticing [Amador et al., 2017; Dietiker et al., 2018]).

noticing research foregrounds the variety in what teachers notice during instruction (Huang & Li, 2009; Kaiser et al., 2015; Males, 2017; Star & Strickland, 2008). Other noticing research foregrounds the noticing of children's mathematical thinking. This noticing is often related to specific mathematical content, such as early arithmetic (Schack et al., 2013), fractions (Coskun et al., 2021; Ivars et al., 2020), algebraic thinking (Jong et al., 2021; Walkoe, 2015), slope (Styers et al., 2020), mean and variability (Shin, 2020), and length and its measurement (Moreno et al., 2021). Still other noticing research foregrounds the noticing of equity issues during instruction, such as anti-deficit noticing (Louie et al., 2021), racial noticing (Shah & Coles, 2020), noticing of students' language resources (Crespo et al., 2021), and noticing of participation and status (Baldinger, 2017; Jilk, 2016; Kalinec-Craig, 2017; Wager, 2014). These examples provide a glimpse into the diversity of what has been foregrounded in teacher noticing research. We see this diversity of what teachers can notice as complementary, rather than as mutually exclusive (see also Turner & Drake, 2016). Researchers (and teachers) can embrace noticing that foregrounds one aspect of instruction as a starting point, which can then lead to noticing other aspects. In this way, the richness of the work on teacher noticing can become synergistic and provide a more complete picture of mathematics teaching and learning.

Conceptual framework: professional noticing of children's mathematical thinking

Our vision of high-quality mathematics instruction centers teaching that is responsive to children's mathematical thinking and is informed by extensive research on children's mathematical thinking and numerous policy recommendations (Cai, 2017; National Council of Teachers of Mathematics, 2014; National Research Council, 2001). In this type of responsive teaching, teachers pursue the substance of children's ideas and mathematical connections within those ideas (Bishop, 2021; Richards & Robertson, 2016). Essential for enactment is mathematics teacher noticing that includes a foregrounding of children's mathematical thinking, and thus we adopted professional noticing of children's mathematical thinking as our conceptualization of mathematics teacher noticing (Jacobs et al., 2010).

Professional noticing of children's mathematical thinking includes three interrelated component skills. The first component skill, *attending to children's strategy details*, refers to how teachers recognize mathematically noteworthy aspects of children's strategies. Teachers can gain a more nuanced view of children's thinking by going beyond the answer and focusing on multiple strategy details. The second component skill, *interpreting children's understandings*, refers to how teachers reason about strategy details to discern children's mathematical understandings. A single strategy cannot reflect a complete picture of a child's understandings, but each strategy provides valuable clues so that, over time, children's strategies become windows into their understandings. The third component skill, *deciding how to respond on the basis of children's understandings*, describes how teachers use what they have learned from children's strategy details and understandings to determine their next instructional steps. Because teacher noticing is invisible, this final component skill refers to teachers' *intended* next steps, which occur prior to teachers' observable responses (Jacobs et al., 2010, 2011).

The three component skills were proposed as a set because they are conceptually and temporally linked, occurring almost simultaneously in the midst of instruction. In the ideal enactment, the set has a nested relationship, such that attending to children's strategy details informs the interpretation of children's understandings which informs decisions about how to respond (Jacobs et al., 2010). However, teacher noticing expertise exists on a continuum, and we need to better understand common patterns of teachers' strengths and areas for growth across the component skills. This study explores these patterns as "profiles" of teacher noticing expertise across the component skills.

We situated our work in multi-year professional development (PD) connected to the long-standing research and PD project, Cognitively Guided Instruction (CGI; Carpenter et al., 2003, 2015, 2017; Empson & Levi, 2011). Centered on providing teachers opportunities to engage with research-based knowledge of children's mathematical thinking and how instruction can build on that thinking, CGI is one of the few projects that has consistently documented gains in teacher learning and student achievement (Carpenter et al., 1989; Fennema et al., 1996; Jacobs et al., 2007; Kennedy, 2016; Schoen et al., 2018; Villaseñor & Kepner, 1993; Wilson & Berne, 1999). We used this context of multi-year PD to purposefully assess teachers with varying amounts of PD as a way to maximize the range of noticing expertise in our sample. However, this study was not a study of teacher learning but instead designed to deepen our understanding of the instructional practice of mathematics teacher noticing. We explored the following research question: *What profiles of teachers' expertise in professional noticing of children's mathematical thinking exist among teachers engaged in multi-year PD?*

Methods

The teacher noticing data were drawn from a larger study, Responsive Teaching in Elementary Mathematics (RTEM), in which the goals included building a model of teaching that is responsive to children's mathematical thinking (Empson & Jacobs, 2021). In this paper, we focus on one of the instructional practices in the model, professional noticing of children's mathematical thinking.

Participants

We assessed the noticing expertise of 72 upper elementary school teachers (64 females and 8 males) who had voluntarily enrolled in our 3-year PD. This group included 68 classroom teachers in grades 3–5 and 4 teaching specialists, such as resource teachers and instructional facilitators. The teachers' professional experiences and instructional contexts varied in four ways, which increased the likelihood that we would capture the range of noticing expertise for teachers engaged in multi-year PD.

First, teachers varied in their *amount of teaching experience*, ranging from 2 to 36 years ($M = 11.8$ years). Second, teachers varied in their *amount of PD*. Teachers in the RTEM project were divided into three PD cohorts with staggered starts. We collected the noticing data at one point in time from teachers who were at multiple points in the 3-year PD—22 at the end of their first year, 26 at the end of their second year, and 24 at the end of their third year. Third, teachers varied in the *instructional contexts of their districts*. They worked in three neighboring districts in the southern region of the United States. These districts had varied instructional histories in that all administrations had endorsed teaching that was responsive to children's mathematical thinking, but for different amounts of time. Fourth, teachers varied in the *demographic diversity of their students*. To provide a sense of this diversity, we share summary data from the 36 participating schools (11–14 schools per district). Across the schools, students who qualified for free or reduced-cost lunch ranged from 10–98% ($M = 59.7\%$), and students classified as Limited English Proficiency ranged from 2–85% (M=33.3%). Student race and ethnicity classifications also varied in that White students ranged from 6–85% (M=49.6%), Hispanic students ranged from 4–81% (M=34.8%), Black students ranged from 0–20% (M=4.3%), Hawaiian or Pacific Islander students ranged from 0–31% (M=5.4%), and students with classifications of "Other" ranged from 0–14% (M=6.0%).

Professional development

The overall goal of the PD was to help teachers develop expertise in teaching that is responsive to children's mathematical thinking, with special emphasis on the teaching and learning of fractions (Jacobs, Empson, Pynes et al., 2019). Key resources included research-based frameworks of children's mathematical thinking (Carpenter et al., 2015; Empson & Levi, 2011) and research-based frameworks of instructional practices, such as noticing children's mathematical thinking (Jacobs et al., 2010) and questioning children's mathematical thinking (Jacobs & Ambrose, 2008; Jacobs & Empson, 2016).

Teachers engaged in more than 150 hours of face-to-face workshops offered over 3 years. Each year included 4.5 workshop days during the summer and 4 workshop days during the school year (2 consecutive days in the fall and 2 consecutive days in the spring). In addition, the PD included several school-based activities between workshops that teachers enacted without a facilitator present (Pynes et al., 2020).

Workshops provided teachers with opportunities to reflect on their practice, explore new ideas, try new practices, and collaborate with colleagues. Activities focused on working with children, analyzing children's written work, and discussing videos depicting whole-class instruction, small-group instruction, and one-on-one conversations with children. Teachers also engaged in various other activities, including reading about children's thinking and instruction that builds on that thinking, solving mathematics problems using children's strategies, and adapting curriculum materials using a lens of opening spaces for children's thinking (Drake et al., 2015).

Noticing assessment

Because of the invisible nature of teacher noticing, studying teacher noticing during instruction would likely change the instruction, including the teacher noticing (Sherin, Russ, & Colestock, 2011). Therefore, teacher noticing expertise has often been studied with proxies, such as video and written-work artifacts. We followed this precedent by capturing teachers' noticing expertise with a written assessment structured around three instructional scenarios in which teachers had opportunities to notice the thinking of children engaged in fraction problem solving. The scenarios were conveyed via authentic, strategically selected artifacts of practice (see Table 1).

We purposefully included all three instructional scenarios to capture teachers' noticing expertise throughout multiple facets of their work. When selecting the videos and written work, our overall goal was to depict meaningful fraction content and problems as well as children's strategies that reflected a range of understanding. We chose to focus on children's valid strategies, and most strategies also had correct answers, although some were shared in nontraditional forms (e.g., use of fraction words instead of symbols). We also focused on fraction story problems involving equal sharing because they were introduced early in our PD, thereby ensuring that all teachers in the study—including the subset of teachers who had completed only one year of PD—would have been familiar with them.

Table 1 Three Instructional Scenarios for the Noticing Assessment

Instructional scenarios	Artifact description	Fraction story problem(s) showcased
Classroom interactions	8-min video of a fifth-grade lesson that included multiple interactions and highlighted the thinking of 3 children	There are 5 candy bars. 8 students want to share them so that each person gets the same amount. How much will each person get?
Written work	written work of 3 fourth graders	The teacher has 4 pancakes to share equally among 6 children. How much pancake does each child get?
One-on-one conversation	9-min video of Nicholas, a fourth grader, working individually with a teacher	Divine had 12 giant chocolate bars to share with the kids on her soccer team. She wants to give each person 3/4 of a bar of chocolate. How many kids will she be able to give chocolate to before she runs out? Including Andrew, there are 12 kids on the basketball team. Andrew's mom made 16 small pizzas as a treat for the team to eat after practice. If the kids share the pizza fairly, how much should each person get?

The paper's supplemental materials provide the strategies involved in each scenario

We administered the three instructional scenarios on separate days within about one month. To approximate teacher noticing in the moment, we chose to show videos only one time so the videos could serve as proxies for instructional situations in which children share their ideas verbally, without a rewind option.

For each instructional scenario, teachers were asked to engage with the artifact and then to respond, in writing, to prompts linked to the component skills of professional noticing of children's mathematical thinking (see Table 2). For the component skill of deciding how to respond on the basis of children's understandings, we chose to separate the prompts (and scores) for two common forms that we viewed as conceptually distinct: (a) *deciding on follow-up questions* and (b) *deciding on next problems*. This separation highlights a secondary goal of our study, which was elaboration of this component skill by comparing teachers' expertise with two forms of the skill.

Scoring of the noticing assessment

Each teacher received 12 noticing scores—one score linked to each of 4 prompts for the noticing component skills within the 3 instructional scenarios (see Table 3). Drawing on past noticing research (Jacobs et al., 2010), scoring was done holistically on a 0–2 scale, indicating the extent to which we had evidence for teachers' engagement with children's mathematical thinking: lack of evidence (0), limited evidence (1), or robust evidence (2). Data were blinded so that teachers and their number of years of PD were hidden during scoring, and we scored the 72 teachers' responses to one prompt before moving on to the next prompt. We also made sure to adjust scores when teachers had provided information relevant to a given prompt elsewhere in the assessment. All data were double-scored, and interrater reliability for each of the 12 noticing scores was 80% or more, with discrepancies resolved through discussion. More information about the scoring for each noticing component skill is described below, and examples of responses illustrating the scores are provided in Fig. 3 in the Findings section. (For additional examples, see Jacobs & Empson, 2021.)

Scoring for attending to children's strategy details

Teachers received three scores for the attending component skill, with one score for each instructional scenario (column 1 in Table 3).[2] Scores reflected the extent of evidence teachers demonstrated in attending to children's strategy details. Specifically, we looked for inclusion of mathematically significant details such as how children used drawings to represent and partition quantities, how they combined fraction amounts, or how they described amounts using fraction names or notation.

Responses with scores of robust evidence of engagement with children's mathematical thinking included explicit descriptions of the majority of the mathematically significant details. Responses with scores of limited evidence and lack of evidence included

[2] Teachers' attending scores for each instructional scenario were generated in two steps given that our prompts requested that teachers describe each strategy (see Table 2 for the prompts). First, we scored individual strategy descriptions for a scenario. Second, we averaged the scores for individual strategy descriptions for that scenario, and all further analyses used this average as the attending score for that scenario. Averaging multiple strategy descriptions within a scenario provided a more stable measure of teachers' expertise in attending to children's strategy details given that individual strategies can have both typical and idiosyncratic mathematical details of interest.

Table 2 Writing Prompts for the Noticing Assessment

Instructional scenarios	Noticing component skills			
	Attending to children's strategy details[a]	Interpreting children's understandings	Deciding how to respond on the basis of children's understandings	
			Deciding on follow-up questions	Deciding on next problems
Classroom interactions & written work	Describe in detail what you think each child did in response to this problem.	Explain what you learned about these children's understandings.	Imagine that you are the teacher of these children, and you want to have a one-on-one conversation with one of them. Which child would you choose? Describe some ways you might respond to their work on this problem, and explain why you chose those responses.	Imagine that you are the teacher of these children. What problem or problems might you pose next? What is your rationale?
One-on-one conversation	Describe in detail what you think Nicholas did in response to each problem.	Explain what you learned about Nicholas's understandings.	Imagine that you are Nicholas's teacher. Describe some ways you might respond to his work on this problem, and explain why you chose those responses.	Imagine that you are Nicholas's teacher. What problem or problems might you pose next? What is your rationale?

[a] For the Classroom Interactions and Written Work scenarios, the three children's names were listed after the prompt for attending to strategy details to ensure that we captured the teachers' engagement with each strategy. We did not force this precision with prompts for interpreting children's understandings and deciding on next problems, which allowed us to distinguish when teachers focused on the three children as a group or as individuals. (Teachers' decisions about follow-up questions were automatically focused on a single child as they chose one child's strategy as a focus.) Similarly, for the One-on-One Conversation scenario, Nicholas's two problems were listed after the prompt for attending to strategy details

Table 3 Teacher Noticing Scores: Noticing Component Skill by Instructional Scenario

Instructional scenarios	Noticing component skills			
	Attending to children's strategy details	Interpreting children's understandings	Deciding how to respond on the basis of children's understandings	
			Deciding on follow-up questions	Deciding on next problems
Classroom interactions	Score (0–2)	Score (0–2)	Score (0–2)	Score (0–2)
Written work	Score (0–2)	Score (0–2)	Score (0–2)	Score (0–2)
One-on-one conversation	Score (0–2)	Score (0–2)	Score (0–2)	Score (0–2)

Five summary scores, also ranging from 0–2, were computed for each teacher: an *overall noticing score* (mean of all 12 scores) and four *noticing component-skill scores* (column means)—one for each of the noticing component skills

progressively fewer details and less clear descriptions and, at times, descriptions that were mathematically incorrect or inconsistent with the children's strategies.

Scoring for interpreting children's understandings

Teachers received three scores for the interpreting component skill, with one score for each instructional scenario (column 2 in Table 3). Scores reflected the extent of evidence teachers demonstrated in interpreting children's understandings. We did not seek a single best interpretation but instead looked for an emphasis on what children understood (vs. did not understand) and reasoning grounded in children's strategies and consistent with the research on children's mathematical development. When describing the three children's understandings in the Classroom Interactions and Written Work scenarios, teachers often compared strategies and related understandings, and we noted whether they chose to convey children's understandings reflected in those strategies as uniform or individualized.

Responses with scores of robust evidence of engagement with children's mathematical thinking captured the breadth or depth of understandings reflected in children's strategies, and they typically differentiated the understandings of individual children, when appropriate. Responses with scores of limited evidence typically focused on children's understandings in a general fashion, using few strategy details as evidence. Further, these responses sometimes overgeneralized, meaning that they went beyond the evidence provided, such as attributing one child's understandings to all children, even when strategies indicated that the children had different understandings. Overgeneralizations can be problematic because they minimize the importance of *individual* children's strategies and understandings (Jacobs et al., 2010). Responses with scores of lack of evidence were even more general, often unclear, and rarely differentiated children's understandings.

Scoring for deciding on follow-up questions

Teachers received three scores for the first form of the component skill of deciding how to respond—deciding on follow-up questions—with one for each instructional scenario (column 3 in Table 3). We use the term *follow-up questions* broadly to mean questions, comments, and other instructional moves that the teacher proposed as a follow-up to a child's work. Teachers proposed follow-up questions for only one child in each scenario—Nicholas in the One-on-One Conversation scenario and a child of their choosing in the Classroom Interactions and Written Work scenarios (see Table 2 for the prompts). Scores reflected the extent of evidence teachers demonstrated in deciding on follow-up questions on the basis of the child's understandings. We did not seek a single best set of follow-up questions but instead looked for questions that aligned with teachers' rationales, linked to the child's strategies and understandings, and left space for the child's thinking.

Responses with scores of robust evidence of engagement with children's mathematical thinking included questions and rationales that were aligned and explored or built on the child's thinking. These questions used details from the child's existing work and left space for the child's ways of thinking versus taking over or funneling that thinking toward a particular strategy or answer (Jacobs, Empson, Jessup, & Baker, 2019; Wood, 1998). Responses with scores of limited evidence also generally explored or built on the child's thinking, but with less specificity and clarity. In addition, some responses had a hybrid nature—teachers began with the child's thinking but ended by funneling or taking over that thinking to move it in a particular direction. Responses with scores of lack of evidence did not provide evidence of exploring or building on the child's thinking, often because they were unclear or consistently focused on funneling or taking over that thinking.

Scoring for deciding on next problems

Teachers received three scores for the second form of the component skill of deciding how to respond—deciding on next problems—with one for each instructional scenario (column 4 in Table 3). Scores reflected the extent of evidence teachers demonstrated in deciding on next problems on the basis of children's understandings. We did not seek a best next problem (or set of problems) but instead looked for problems that aligned with teachers' rationales, linked to children's strategies and understandings, and left space for children's thinking.

Responses with scores of robust evidence of engagement with children's mathematical thinking included problems and rationales that were aligned and explored or built on the children's thinking. Often showcased were anticipation of the children's potential strategies on the proposed problems, purposeful number selection, and, for the Classroom Interactions and Written Work scenarios, differentiation (by problem type, number selection, or instructional goal) for the three children (see also Jacobs et al., 2010). Responses with scores of limited evidence and lack of evidence included progressively less specificity, less clarity, and less alignment among the problems, rationales, and children's thinking.

Analysis

We identified profiles of teacher noticing expertise across the component skills. We began by determining that the internal consistency for the noticing assessment was acceptable, as indicated by Cronbach's alpha of 0.77. We then conducted a latent class analysis, which can be used to identify individuals with similarities across a set of variables. We used the latent class analysis to identify subgroups of the 72 teachers displaying similar patterns of responses across their 12 scores on the noticing assessment. The final number of subgroups was determined using a combination of information criteria, theory, and researcher expertise. (See the findings section and the supplemental materials for more information on our selection of number of subgroups.) We considered these subgroups to be "profiles" of teachers' expertise in professional noticing of children's mathematical thinking, each with different response patterns.

Findings

We identified profiles of teachers' noticing expertise on the basis of teachers' responses on the noticing assessment. Each noticing profile included a subgroup of teachers who responded similarly to each other and differently from teachers in other profiles. Our goal in empirically identifying profiles was not to "label" teachers, but to better understand the variety of ways teachers were taking up the PD focused on professional noticing of children's mathematical thinking. Ultimately, this information can both inform future research on teacher noticing and support related PD efforts (see also Halpin & Kieffer, 2015). In the following sections, we introduce the noticing profiles and then provide illustrations using excerpts from the teachers' responses on the noticing assessment.

Identifying profiles of noticing expertise

We used a latent class analysis that involved consideration of teachers' 12 scores, which reflected their expertise on each of the noticing component skills in each of the instructional scenarios. We considered a 3, 4, and 5-profile solution and ultimately chose the 3-profile solution based on (a) the lowest Bayesian Information Criterion (BIC) goodness-of-fit statistic (Schwarz, 1978), (b) appropriate average classification probabilities, (c) conceptually interpretable profile patterns, and (d) sufficient sample sizes for comparisons among profiles. We then assigned each teacher the profile for which they had the highest probability based on their response pattern across the noticing assessment. (See the supplemental materials for additional details on our selection of 3 profiles.)

The 3-profile solution generated profiles that we labeled *Accomplished Noticing* ($N=14$), *Mixed Noticing* ($N=33$), and *Emerging Noticing* ($N=25$). To characterize the profiles in terms of their overall expertise, we noted the ordered nature of the profile means of teachers' overall noticing scores: 1.42, 0.98, and 0.60 for the *Accomplished Noticing*, *Mixed Noticing*, and *Emerging Noticing* profiles, respectively. To characterize the profiles in terms of their patterns of strengths and areas for growth across the component skills, we used the profile means of teachers' noticing component-skill scores (see Fig. 1). All scores ranged from 0–2, reflecting lack of evidence (0), limited evidence (1), or robust evidence (2) of engagement with children's mathematical thinking. Thus, we were especially interested in whether mean scores were above or below a score of 1—limited evidence of engagement with children's mathematical thinking—as it was the mid-point in our scale.

The *Accomplished Noticing* profile was characterized by consistently strong expertise across the component skills, with all mean scores above 1. For these teachers, their expertise in attending to children's strategy details was their strongest skill, and deciding on next problems—while still strong—was the skill that showed the most room for growth.

The *Mixed Noticing* profile showed overall less expertise than the *Accomplished Noticing* profile and was characterized by a split performance, with mean scores above 1 for attending to children's strategy details and interpreting children's understandings, and mean scores below 1 for the two deciding how-to-respond skills (deciding on follow-up questions and deciding on next problems). In other words, these teachers demonstrated

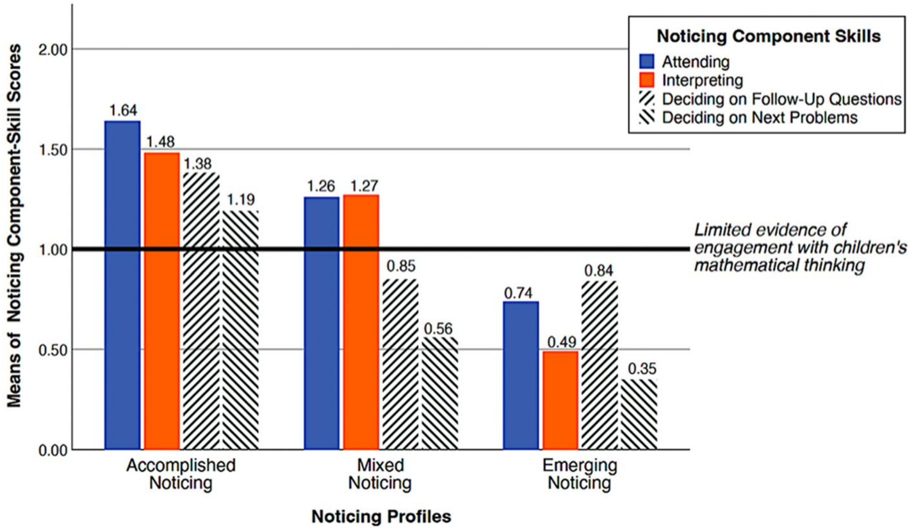

Fig. 1 Profile Means of Teachers' Noticing Component-Skill Scores (Across Instructional Scenarios)

relative strengths in attending to and interpreting children's strategy details, but they still had room to grow in knowing what to do with that information in terms of an instructional response.

Finally, the *Emerging Noticing* profile showed overall less expertise than the other two profiles and was characterized by consistently weak expertise, with all mean scores below 1. A closer look showed that these teachers especially needed support in interpreting children's understandings reflected in strategy details and deciding on next problems on the basis of those understandings. However, they had relative strengths in attending to children's strategy details and deciding on follow-up questions. This profile's relative strength in deciding on follow-up questions was striking given that in much of the noticing literature—as well as in the other two profiles—teachers have typically demonstrated less expertise in deciding how to respond than in attending and interpreting (see, e.g., Haltiwanger & Simpson, 2014; Jacobs et al., 2010; Krupa et al., 2017; LaRochelle et al., 2019; Santagata et al., 2021). We return to this finding in the Discussion section.

By looking across the profiles, we were able to address our secondary study goal of comparing different forms of the component skill of deciding how to respond. Teachers demonstrated more expertise when deciding on follow-up questions than when deciding on next problems in all three profiles, which provided empirical support for our design decision to elaborate this component skill by separating the two forms. In the next section, we provide an illustrative case for each of the profiles.

Illustrating profiles of noticing expertise

We selected one teacher in each profile as an illustrative case, and the case teachers' pseudonyms begin with the first letter of the profile name for ease of memory: Alicia (*Accomplished Noticing* profile), Monica (*Mixed Noticing* profile), and Erica (*Emerging Noticing* profile). We caution that no single teacher can ever fully represent a profile because variability exists within each profile. However, the profiles provide a starting point for making sense of the range of teacher noticing expertise, and we use these cases to convey how the profiles differed in their overall noticing expertise as well as their patterns of strengths and areas for growth across the component skills.

We selected teachers as illustrative cases on the basis of how well their responses conveyed the noticing expertise typical for their profile in the broader sample. Table 4 provides the case teachers' overall noticing scores, which reflected the same ordering as the profile means of the overall noticing scores in the broader sample. Table 4 also provides the case teachers' noticing component-skill scores, which reflected the same patterns of strengths and areas for growth for their profiles in the broader sample. To provide a sense of the case teachers' responses, we share sample responses linked to each component skill for one of the scenarios—the Classroom Interactions scenario. We drew all sample responses from the same instructional scenario to facilitate comparison of responses within and across case teachers (and profiles). Thus, we begin by describing the Classroom Interactions scenario before turning to the case teachers' responses.

Description of the classroom interactions scenario

Teachers watched an 8-min video depicting multiple interactions in a fifth-grade lesson involving this fraction story problem: *There are 5 candy bars. 8 students want to share them so that each person gets the same amount. How much will each person get?* The video showed the problem launch in which the teacher posed the problem and then asked the children to show their thinking on paper, write an equation that represented how they were thinking, and solve the problem a second way. The video also included excerpts of the teacher circulating when children were independently solving the problem and excerpts of the subsequent whole-group discussion in which a few children presented their strategies. Interactions with three children—Chase, Aiden, and Emilia—were showcased throughout the video. Figure 2 provides screenshots of these children's strategies, all of which were valid strategies with a correct answer of 5/8. Descriptions of each strategy are then provided, in the order in which teachers viewed the strategies on video.

Chase's strategy was shared via the teacher's one-on-one interaction with Chase during independent problem solving. When the teacher arrived at Chase's desk, he had drawn 8 students and 5 candy bars (as rectangles) and had written the answer "5/8." None of the rectangles showed any partitioning. He explained that he had mentally partitioned the candy bars into eighths and combined 1/8 from each candy bar to arrive at 5/8. During this explanation, Chase added to his drawing to illustrate his reasoning—he partially partitioned each candy bar by drawing one vertical line to represent 1/8 and shaded that amount on each rectangle. The teacher then asked Chase to write an equation for his strategy, but Chase was initially unsure what to write. After some questioning, Chase shared that he had added 1/8 five times and wrote "$1/8 \times 5 = 5/8$."

Aiden's strategy was shared twice—first via the teacher's one-on-one interaction with Aiden during independent problem solving and second via Aiden's presentation of his strategy during the whole-group discussion. Aiden shared that he had drawn 8 students and 5 candy bars (as rectangles). He then described his trial-and-error strategy in which he had initially partitioned the candy bars in half. However, because Aiden wanted to partition all candy bars into pieces of only one size, his halving strategy did not work because the number of pieces could not be shared equally with the number of sharers. Therefore, he "kept going up" until he got to eighths. Aiden's paper showed some initial partitioning attempts

Table 4 Case Teachers' Summary Scores

Case teacher	Noticing profile	Overall noticing score	Noticing component-skills scores (across instructional scenarios)			
			Attending to children's strategy details	Interpreting children's understandings	Deciding how to respond on the basis of children's understandings	
					Deciding on follow-up questions	Deciding on next problems
Alicia	Accomplished noticing	1.56	1.89	1.67	1.67	1.00
Monica	Mixed noticing	1.01	1.39	1.33	0.67	0.67
Erica	Emerging noticing	0.50	0.67	0.33	1.00	0.00

Problem: *There are 5 candy bars. 8 students want to share them so that each person gets the same amount. How much will each person get?*

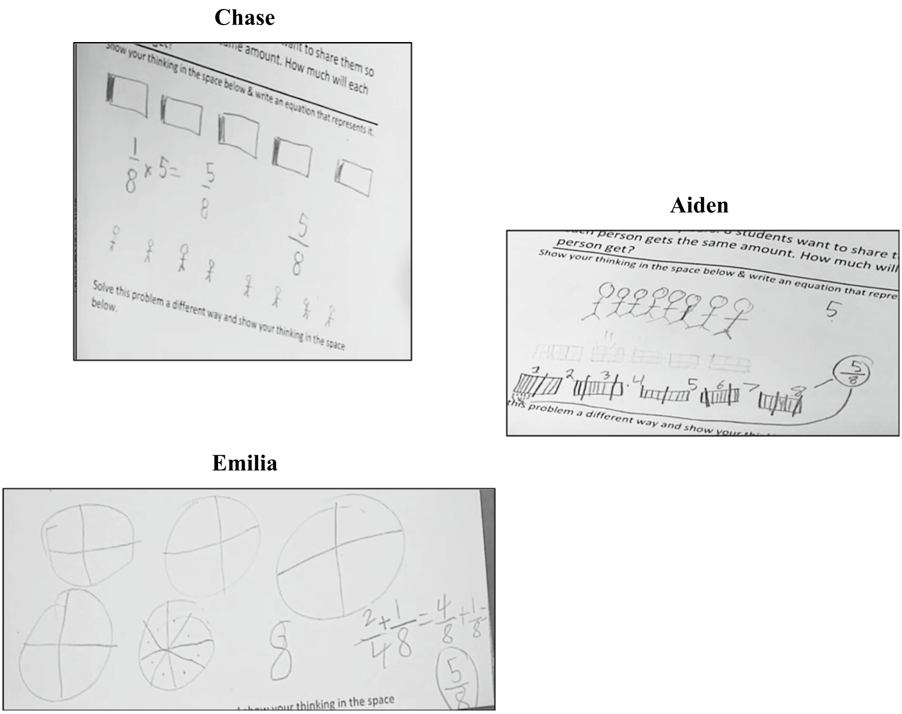

Fig. 2 Strategies Showcased in the Classroom Interactions Scenario

that were erased and a final picture with 5 candy bars partitioned into eighths. On these candy bars, he had created 8 groups of 5 consecutive one-eighths—one group for each of the 8 students. Specifically, he had grouped 5 one-eighths on the first candy bar, separated that group with a longer line, and written a "1" (for the first student) above that grouping. The second group included 3 one-eighths from the first candy bar and two one-eighths from the second candy bar, marked with a longer line and a "2" (for the second student) above that grouping. He continued similarly, making groups of 5 one-eighths until he had exhausted all the candy bars and finished with 8 groups of 5 one-eighths and an answer of 5/8 for each child. Throughout the interactions, Aiden's explanations were confusing because he described his groupings of 5 one-eighths in several ways, including "splitting them into fifths," "dividing it by 5," and "splitting all of them into groups of 5." Further, his rationale for using 5 was that he "just wanted to try that" because there were 5 candy bars. This focus on grouping by the number of items is less typical than the grouping in Chase's strategy in which he collected 1/8 from each candy bar (Empson & Levi, 2011). Aiden also shared that he had not yet attempted the equation part of the assignment.

Emilia's strategy was shared via her presentation of the strategy during the whole-group discussion. Her picture showed 5 candy bars (as circles), with the first 4 partitioned into fourths and the last one into eighths. She explained that she split all 5 candy bars into fourths but the last one she "split up more." She also shared that she had partitioned into fourths because she knew 4 × 2 is 8—she pointed to the first 2 candy bars for 8 (pieces) and the second 2 candy bars for another 8 (pieces). She did not explain how she partitioned the last candy bar into eighths, but she knew she needed to partition beyond fourths because there would not be enough pieces for each child to get another piece. Also visible on her paper was "2/4 + 1/8 = 4/8 + 1/8 = 5/8." To show where she got the 2/4, she pointed to the first 4 candy bars, and to show where she got the 1/8, she pointed to the last candy bar. She then read aloud her equation but did not explain how she knew 2/4 = 4/8 or how she added 4/8 + 1/8.

Figure 3 presents the case teachers' responses to the prompts for each of the component skills for the Classroom Interactions scenario. Looking by column conveys a sense of each profile, across the component skills. Looking by row conveys a sense of each component skill, across the profiles. The following sections then provide an explanation of how each case teacher's responses illustrate the pattern of strengths and areas for growth for their profile. Note that each response also includes the score given to that response—robust, limited, or lack of evidence of engagement with children's mathematical thinking—thereby providing illustrations of our scoring rubrics.

Alicia: accomplished noticing profile

We begin with the *Accomplished Noticing* profile (column 1 in Fig. 3) to provide a sense of strong noticing expertise and, in particular, how attention to strategy details can permeate the other noticing component skills. Like the broader sample's performance in this profile, Alicia's responses consistently demonstrated strong expertise, with most scored as robust evidence of engaging with children's mathematical thinking. Further, her score for deciding on next problems showed the most room for growth.

For the component skill of attending to children's strategy details, Alicia's strategy descriptions (scored as robust evidence) highlighted the mathematically significant details and communicated them in ways that conveyed the strategies from start to finish. For Chase, she described how he partitioned the candy bars into eighths and combined 5 one-eighths to get the correct answer of 5/8, *without* needing to physically partition the candy bars. She also pointed to some of the teacher's questioning that supported him in constructing an equation to represent his thinking. For Aiden, she shared the trial-and-error nature of his strategy, his final picture which involved groupings of 5 consecutive one-eighths, his unclear explanation of these groupings, and his correct answer of 5/8. Note that many of the details identified in Alicia's strategy descriptions can be found in her responses to the assessment prompts linked to other noticing component skills.

For the component skill of interpreting children's understandings, Alicia's response (scored as robust evidence) differentiated the distinct understandings of Chase, Aiden, and Emilia. She also identified several understandings that remained unclear from the interactions, and her insights were consistently grounded in children's strategy details. Specifically, Alicia distinguished the children's understandings based on whether they needed to physically partition the candy bars or could enact the partitioning mentally—Chase "saw the overall picture without having to draw each individual piece," whereas Aiden and Emilia "needed to do the cutting up." For Chase, she further conveyed that his understanding of using equations was still developing as evidenced by the need for teacher prompting. For Aiden, she recognized that he "needed all the pieces equal in size to share." Thus, Aiden did not yet understand that items could be partitioned in multiple ways in the same strategy and that his earlier partitioning attempts, such as fourths, would have worked if he

	Alicia *Accomplished Noticing* Profile	Monica *Mixed Noticing* Profile	Erica *Emerging Noticing* Profile
Attending to Children's Strategy Details **Prompt:** *Please describe in detail what you think each child did in response to the problem.*[a] **Problem:** *There are 5 candy bars. 8 students want to share them so that each person gets the same amount. How much will each person get?*	**Robust Evidence (*Chase's Strategy*)**  Looked like he drew wholes and never divided them—when questioned, he colored in what he said was 1/8 of a candy bar. He said the answer was 5/8. When questioned, he counted 1/8, 1/8, 1/8, 1/8, 1/8 [Teacher] "What did you do with these?" [Chase] "Added" [Teacher] "Equation" [Chase] "1/8 × 5 = 5/8" **Robust Evidence (*Aiden's Strategy*)** Started with 5 candy bars—broke into 1/2s—"wouldn't work." Then split again—never said 1/4s but pretty sure that was what he meant. Then split again. He said he kept splitting until he got 1/8s.  Said that he split them in 1/5s—Think he meant that 5 pieces for each of 8 kids.	**Limited Evidence (*Chase's Strategy*)**  He took the 5 candy bars and took 1/8 of each. Then put the 1/8s together to get 5/8. He struggled to write a number sentence showing his work. **Limited Evidence (*Aiden's Strategy*)**  Aiden used trial and error approach. He started by trying to divide into fifths. Then it looked like halves until he finally was able to get to 1/8s. He used a model of the students as well as the candy bars to help him see the amount each would get.	**Limited Evidence (*Chase's Strategy*)** Chase wrote 5 boxes with 8 stick people. He was looking at a visual and then came up with 5/8. He then wrote 1/8 but wasn't sure what to do next. He was looking at the candy bars but still thinking about the amount for each student. **Lack of Evidence (*Aiden's Strategy*)** Aiden shared 5 boxes and divided the boxes into 6 parts. He then counted his amount based on the equal parts of his candy bars. He used trial and error because he first did thirds then kept dividing the candy bars.
	Alicia ***Accomplished Noticing* Profile**	**Monica** ***Mixed Noticing* Profile**	**Erica** ***Emerging Noticing* Profile**
Interpreting Children's Understandings **Prompt:** *Explain what you learned about these children's understandings.*	**Robust Evidence** They all got to 5/8 with different ways. Chase saw the overall big picture without having to draw each individual piece, but he had trouble writing an equation on his own. Once prompted, he was good with it. Aiden needed to do the cutting up and wanted to start with bigger pieces. Interesting that when he got to 1/4s, he didn't see the 2/4s for everyone. He needed all pieces equal in size to share. His expression of cutting the 1/8s into 1/5s needs to be explored more. Emilia needed to cut them up but could start with bigger chunks. She also saw the relationship between 2, 4, and 8. Would have liked for the question, "How do you know that 2/4 = 4/8?" to have been asked.	**Limited Evidence** These 3 students had a solid understanding of fractional amounts and what they represented. They also understood that something can be divided different ways (1/2, 1/4, 1/8, etc.). Chase struggled with putting together a number sentence. Emilia seemed to have a strong understanding of taking apart and putting fractions back together. Chase did not understand that you can break the items into larger pieces and smaller parts and put those together. I would want to work on fractional equivalency with him.	**Lack of Evidence** I think all these students have a great understanding of how to divide food into equal shares. They also can reduce and add fractions well and understand how to get to the right answer. They stayed with their strategies and were able to articulate their thinking to others which probably helped some students in the class who may not be at that point yet.

Fig. 3 Case Teachers' Responses (and Scores) for the Classroom Interactions Scenario (continued on the next page)

had partitioned the final candy bar in a different way (as Emilia did). Alicia also recognized that Aiden's confusing comments about "cutting the 1/8s into 1/5s" needed further clarification for her to fully grasp his understanding of partitioning. For Emilia, Alicia identified her understanding of "the relationship between 2, 4, and 8," which was communicated in her explanation for partitioning the first four candy bars into fourths ($4 \times 2 = 8$) and her equation work. However, Alicia also wanted more information about how Emilia knew "2/4 = 4/8" to clarify for herself Emilia's understanding of this fraction equivalence.

For the component skill of deciding on follow-up questions, Alicia chose to focus on Aiden. In her response (scored as robust evidence), she used his strategy details as her starting point for further clarifying his understandings. Specifically, she wanted to explore his sequence of partitioning attempts to confirm her suspicion that he was partitioning

Deciding How to Respond	**Alicia** *Accomplished Noticing* **Profile**	**Monica** *Mixed Noticing* **Profile**	**Erica** *Emerging Noticing* **Profile**
Deciding on Follow-Up Questions **Abbreviated Prompt:** *Choose one child for a one-on-one conversation. Describe some ways you might respond to their work on this problem and explain why.*	**Robust Evidence** *Follow-Up Questions for Aiden* I would ask him to show me what he means when he says keep splitting until I get 1/8s to see if he was really doing halving. Question the "fifths" to see if he meant what I thought he did—that he did 8 groups of 5/8s—also to see if he could write an equation to show that.	**Lack of Evidence** *Follow-Up Questions for Chase* I may ask him to show me the remaining students and their parts of the candy bar from his model. I would also want him to show me a number sentence based on how he was describing his work to me. He might be a student I would choose to double the numbers in the problem to 10 candy bars and 16 students to see if he could see the pattern or relationship and then I would want him to write another number sentence. I would hope that may lead to fractional equivalency.	**Limited Evidence** *Follow-Up Questions for Chase* 1) Chase I would like to talk to about his work and have him talk through why he wrote 5/8. He couldn't really articulate that reason of why he did his problem that way so some one on one may be good for him to talk through more. 2) Can he do this problem another way? Can he show me another way to solve the problem based on the same numbers? This would give me a better clue of his understanding of fractions and the problem.
Deciding on Next Problems **Abbreviated Prompt:** *What problem or problems might you pose next? What is your rationale?*	**Limited Evidence** *There are 2 pizzas being delivered to a birthday party. There will be 6 kids at the party. How much pizza will each kid get if they share the pizza equally? Show your thinking and write an equation to match your work.* In watching the kids, it seemed like they were pretty familiar with splitting up the objects equally—especially with halves and fourths. It looks like there needs to be more discussion in writing equations to match the problem. Maybe fewer items to cut up would make the equation easier to see.	**Lack of Evidence** 1) *I may try 10 candy bars and 16 students, then 15 and 24.* (I would want to see if a number of the students could see a pattern, where they could take their initial work and build on it.) 2) *I may flip the numbers to 5 students and 8 candy bars.* (I would want them to see that there can be a fraction greater than 1, with a whole number and a fraction.) 3) *For some students, I may use 4 candy bars and 8 students.* (To see if students can break into halves or other fractional amounts.)	**Lack of Evidence** *I would pose a problem that has an odd number of students such as 7 or 9 sharing 5 candy bars.* It seems the students who shared have a good understanding of even numbers and dividing them up so this may be the next step to take them. I would also look at the thinking of lower students who may have struggled and do another set of numbers to help them.

[a]Only the teachers' descriptions (and pictorial re-creations) for Chase's and Aiden's strategies are presented. The teachers' responses for Emilia's strategy were not included because they did not help distinguish the profiles—all were scored as Limited Evidence

Fig. 3 (continued)

by repeated halving—creating halves, then taking half of each half to create fourths, and finally taking half of each fourth to create eighths—which is typical of children's early partitioning strategies (Empson & Levi, 2011). To confirm her understanding of his final solution, Alicia also wanted to probe Aiden's confusing explanation, which involved the term "fifths." She believed Aiden had created 8 groups of 5/8 which meant that "fifths" referred not to the size of the partitions but to the "5" one-eighths in a group. Her final question gave Aiden an opportunity to represent his thinking with an equation, because he had not yet reached that part of the assignment. Note that all these follow-up questions consistently centered Aiden's thinking.

For the component skill of deciding on next problems, Alicia's response (scored as limited evidence) included a problem linked to children's strategies and understandings, but somewhat generally. For instance, she built on the children's familiar partitioning of halves and fourths (as seen in the strategies of Aiden and Emilia) by strategically proposing a problem that would likely involve other partitioning. By choosing a similar problem structure with 6 kids as the number of sharers, children would be likely to partition into thirds or sixths to generate enough pieces for each sharer (Empson & Levi, 2011). Alicia also built on the children's difficulty writing equations (as seen with Chase and Aiden) by strategically selecting a smaller number of items—2 items instead of the 5 items used in the candy bar problem—because she hypothesized that partitioning a smaller number of items might be easier to see as an equation. Overall, Alicia's response built on the children's thinking and left space for the children to solve the proposed problem in ways that made sense to them, but her response was scored as limited evidence of engagement with children's thinking because of the limited connection to the strategy details or understandings of individual children. When teachers' responses were scored as robust evidence, they typically considered strategy details in more depth or differentiated for the three children who had demonstrated different understandings—either by differentiating their goals for each child on a single problem or by proposing different number choices or completely different problems for each child. Alicia's relatively lower score on this component skill was typical for this profile, which consistently showed the most room for growth with deciding on next problems.

Monica: mixed noticing profile

We use Monica's responses to illustrate the *Mixed Noticing* profile (column 2 in Fig. 3). Like the broader sample's performance in this profile, Monica's responses overall showed less expertise than responses in the *Accomplished Noticing* profile. Her performance also reflected the pattern of strengths and areas for growth for this profile—relative strengths in attending to children's strategy details and interpreting children's understandings, but room for growth in using this information to decide on follow-up questions and next problems.

For the component skill of attending to children's strategy details, Monica's strategy descriptions (scored as limited evidence) contained some of the mathematically significant details, but some significant details were vague or absent. For example, Monica described how Chase used eighths and combined 5 one-eighths to get the correct answer of 5/8, but missing was any explicit mention of how his initial strategy did not require any physical partitioning of the candy bars—a defining feature of his strategy. She also described Chase's struggle to write an equation, but any specificity about that equation was absent. Similarly, Monica mentioned Aiden's overall approach using trial and error, but confused the order of his sequence—sharing that he started with fifths when Aiden had explained that he started with halves. Further, Monica's description of his final solution highlighted use of 1/8s but was vague in terms of how those 1/8s were grouped and counted to get an answer ("He used a model of the students as well as the candy bars to help him see the amount each would get."). Thus, absent from her description (and drawing) was Aiden's final solution which involved 8 groupings of 5 consecutive one-eighths.

For the component skill of interpreting children's understandings, Monica's response (scored as limited evidence) demonstrated some important considerations of the understandings reflected in strategy details and some differentiation among the children's understandings. However, compared to responses scored as robust evidence of engagement with children's mathematical thinking, her response was less complete and sometimes misleading. Monica began by sharing a general understanding for all three children ("solid understanding of fractional amounts and what they represented"). She then addressed important understandings for Aiden and Emilia: "something can be divided different ways." However, she attributed this understanding to all three children, which was an overgeneralization given that Chase only partitioned into eighths. Monica also seemed to privilege Emilia's strategy and understandings. She highlighted Emilia's understanding of equivalency, which was visible when Emilia combined pieces of different sizes ("a strong understanding of taking apart and putting fractions back together"). In contrast, Monica never described individual understandings for the other children. Aiden was not individually mentioned, and comments about Chase focused on what he did *not* understand (put together a number sentence) or had *not* done (use fraction equivalence to put together pieces of different sizes).

For the component skill of deciding on follow-up questions, Monica chose to focus on Chase, and she demonstrated less engagement with children's mathematical thinking than with the previous two component skills, as was typical for this profile. Her follow-up questions (scored as lack of evidence) funneled Chase's thinking toward her preferred strategy rather than exploring Chase's thinking. Specifically, Monica wanted Chase to complete the physical partitioning of his candy bars, thereby not appreciating the advanced understanding Chase had shown in his original strategy in which he had solved the problem by partitioning mentally. She also wanted a new equation, but what was not clear was how this equation was related to the equation he had already produced. An additional suggestion was to ask Chase to solve a related problem, with the hope that issues of equivalency would become relevant—perhaps again reflective of Monica's privileging of Emilia's strategy. It was unclear whether Monica had considered whether equivalency would make sense with how Chase was thinking (versus how she was thinking). Overall, Monica's follow-up questions involved important mathematics, but they did not build on Chase's thinking and did not convey to Chase that his current strategy was valued. What was missing was exploration and appreciation of Chase's reasoning about what he had already done.

For the component skill of deciding on next problems, Monica's response (scored as lack of evidence) had an emphasis on the teacher's goals over engagement with the children's thinking, which was similar to her emphasis when deciding on follow-up questions. Specifically, she wanted the children to see a relationship between problems in which numbers were multiples of each other, solve problems in which the number of sharers was greater than the number of items, and encourage partitioning into halves by strategically selecting numbers (8 students sharing 4 candy bars). These proposed problems addressed important mathematical content and demonstrated purposeful number selection. However, they were not customized, or even linked, to the children's strategies and understandings. In fact, the problems could have been created without even seeing Chase's, Aiden's, and Emilia's work on the candy bar problem. Overall, while mathematically interesting, Monica's response showed less engagement with children's mathematical thinking than her responses to the prompts for attending to children's strategy details and interpreting children's understandings, as was typical for this profile.

Erica: emerging noticing profile

We use Erica's responses to illustrate the *Emerging Noticing* profile (column 3 in Fig. 3). Like the broader sample's performance in this profile, Erica's responses overall showed less expertise than the other two profiles. Her performance also reflected the pattern of strengths and areas for growth for this profile—relative strengths in attending to children's strategy details and deciding on follow-up questions, but substantial room for growth in interpreting children's understandings and deciding on next problems.

For the component skill of attending to children's strategy details, Erica's descriptions identified a few strategy details. Still, many were absent, and the details were often conveyed in ways that were unclear or incomplete, leaving us wondering which pieces of the strategies Erica had understood. For example, in her description of Chase's strategy (scored as limited evidence), she mentioned that he provided the correct answer of 5/8, knew that 1/8s were involved, and hinted that he had solved the problem mentally ("looking at a visual" described as "5 boxes with 8 stick people"). However, the connecting steps for these pieces were absent. Her strategy description for Aiden (scored as lack of evidence) was even less clear and included some inaccuracies. She mentioned his trial-and-error approach but highlighted thirds and sixths ("6 parts"), neither of which were mentioned in Aiden's strategy. Further, Aiden's final solution—including partitioning, groupings, and answer—were absent. Overall, this component skill was a relative strength for teachers in this profile, and Erica's responses illustrate how the quality of the teachers' descriptions was often uneven and depended on the strategy—some strategy descriptions reflected initial attention to mathematically significant details (e.g., Chase's strategy description) and other strategies were not yet accessible (e.g., Aiden's strategy description). From an assessment point of view, we appreciated that all teachers successfully described at least some strategies, which meant that all teachers understood the task of describing children's strategies. Thus, the profile differences reflected distinctions in expertise in attending to children's strategy details rather than a misunderstanding of our task.

For the component skill of interpreting children's understandings, Erica's response (scored as lack of evidence) was characterized by generality, lack of differentiation among the children, and misinterpretation. For instance, she described the children's understandings of "how to divide food into equal shares," "how to get to the right answer," and their ability to "articulate their thinking." These assertions were general and could likely be applied to many strategies and problems—consideration of the children's thinking specifically related to the candy bar problem was absent. Erica also mentioned that the children could "reduce and add fractions well," but none of the children "reduced" fractions in their strategies. Further, although all three strategies involved combining fractions, there was no mention of the different understandings needed to combine like fractions versus unlike fractions (done only by Emilia). This distinction was important given Aiden's trial-and-error approach to eventually allow him to partition all candy bars into pieces of only one size.

For the component skill of deciding on follow-up questions, Erica chose to focus on Chase. Her follow-up questions (scored as limited evidence) broadly addressed Chase's thinking on the candy-bar problem, illustrating how this component skill is an area of relative strength for teachers in this profile. Specifically, Erica wanted Chase to "talk through why he wrote 5/8" thereby giving him a chance to articulate his strategy more fully. She also wanted him to solve the problem another way to learn more about his understandings. Neither of these questions took advantage of the details in Chase's strategy. However, both provided opportunities for Chase to further explore the mathematics and for Erica to learn more about his understandings, because Chase's thinking was centered rather than funneled in a direction consistent with Erica's thinking.

For the component skill of deciding on next problems, Erica's response (scored as lack of evidence) included suggestions and rationales that were confusing and lacked specificity, leaving us wondering how clear Erica was about her own goals. She wanted to pose a problem with an odd number of sharers to build on the children's "good understanding of even numbers and dividing them up." However, the relevance of even and odd numbers in relation to children's strategies and understandings was unclear. Further, Erica's assessment of this "good understanding" was left unexplained as was her selection of 7 and 9 sharers—why did she choose these odd numbers versus other odd numbers? An additional suggestion was targeted at a specific group of students ("lower students who may have struggled") but left unexplained was in what ways these students were "lower" or "struggled." Further, her suggestion for them to "do another set of numbers" was vague. Teachers in this profile often proposed problems without specific numbers or without explanations for the numbers chosen, suggesting that they did not yet appreciate the need for purposeful selection of numbers (Empson et al., 2021; Land, 2017; Land et al., 2015).

Discussion

We began this study with the assumption that all participating teachers had strengths as teachers. They chose to engage in our PD to enhance their teaching by learning about children's thinking and its pivotal role in instruction—learning to notice children's mathematical thinking was a piece of that learning. By assessing teachers' noticing expertise and identifying profiles of expertise, we hoped to better understand the variety of ways teachers were taking up the PD focused on professional noticing of children's mathematical thinking, so that ultimately teachers' development could be better supported. In the following sections, we consider the contributions of these profiles, which include the multi-dimensional characterization of noticing expertise and the elaboration of the component skill of deciding how to respond. We conclude with study limitations coupled with suggestions for future research.

Profiles as a multi-dimensional characterization of noticing expertise

We empirically identified three distinct profiles of teachers' expertise in professional noticing of children's mathematical thinking among teachers engaged in multi-year PD.

- Teachers with an *Accomplished Noticing* profile demonstrated strong expertise across the component skills, but still with room to grow, especially in deciding on next problems.
- Teachers with a *Mixed Noticing* profile demonstrated some expertise, with more expertise in attending to children's strategy details and interpreting children's understandings than with the two forms of deciding how to respond.
- Teachers with an *Emerging Noticing* profile had substantial room to grow in all component skills, but they showed relative strengths in attending to children's strategy details and deciding on follow-up questions.

Our profiles contribute to the work on teacher noticing by providing a multi-dimensional characterization of noticing expertise that integrates teacher expertise in each of the component skills. Prior research on professional noticing of children's mathematical thinking has typically assessed teacher expertise in each of the component skills separately, without integrating them (see, e.g., Coskun et al., 2021; Haltiwanger & Simpson, 2014; Jacobs et al., 2010; Krupa et al., 2017; LaRochelle et al., 2019; Lee, 2020; Schack et al., 2013). Thus, the profiles extend this work by providing a concise way of characterizing teacher noticing expertise across the component skills. In doing so, they reveal common patterns of strengths and areas for growth, and they showcase the multi-dimensional nature of expertise in professional noticing of children's mathematical thinking. Additionally, the integration of the component skills into profiles supports researchers in exploring connections between teachers' noticing expertise and their expertise in other instructional practices[3] or student achievement (see also, Blömeke et al., 2020).

[3] For example, in our larger study, we also conducted classroom observations for 49 teachers, and we used a 4-point scale to evaluate the teachers' capacity for questioning to build on children's thinking during instruction. We found a significant, moderate correlation between teachers' noticing profiles and their questioning scores (r (47) = .56, $p < .05$). For more information, see Empson & Jacobs (2021).

Teacher educators can also benefit from the profiles. The profiles' patterns of strengths and areas for growth not only help teacher educators understand the variety of teacher expertise but also provide a starting point for differentiating learning opportunities for teachers, such as customization of activities and follow-up questions (see also Beattie et al., 2017; Munson, 2020). In short, we appreciate the profiles as tools to inform teacher educators' efforts, but we also caution that variability exists within each profile, so teacher educators always need to be mindful of the needs of the individual teachers they support.

Elaboration of the component skill of deciding how to respond

We chose to systematically explore two forms of deciding how to respond, and in all three noticing profiles, teachers demonstrated more expertise when deciding on follow-up questions than when deciding on next problems. This distinction may reflect the differential complexity inherent in each form. Deciding on follow-up questions requires teachers to focus on only one child's strategy and understandings, whereas deciding on next problems requires multiple children's strategies and understandings to be coordinated. The distinction may also relate to the relationship teachers typically have with each form of deciding how to respond. Teachers regularly decide on follow-up questions in the midst of instruction in response to children's work, but when deciding on next problems, they are often required to use problems from mandated textbook materials and pacing guides. Thus, teachers may have limited experience deciding on next problems on the basis of children's understandings, and they may even feel that they lack the agency to do so (Amador & Lamberg, 2013).

Our comparison of the two forms of deciding how to respond contributes to the work on teacher noticing by elaborating this component skill and by suggesting the need for a line of research focused on different forms of deciding how to respond. Prior research on professional noticing of children's mathematical thinking has typically explored only one form of deciding how to respond per study and thus direct comparisons have been limited. Future research needs to continue to map this terrain by further exploring our two forms of deciding how to respond as well as additional forms. Given the importance of facilitating mathematical discourse among children, we specifically suggest exploration of teachers' decision making when the goal is to strategically connect children to talk about their strategies, (Franke et al., 2007; Mercer & Littleton, 2007; National Council of Teachers of Mathematics, 2014; Webb et al., 2014). We also suggest exploration of teachers' deciding how to respond with children's *invalid* strategies. Our work focused exclusively on valid strategies, and we expect that teachers' inclinations for taking over or funneling children's thinking may be stronger when faced with invalid strategies, and these inclinations would likely have implications for their decisions about how to respond.

Teachers' differential expertise with the two forms of deciding how to respond also has implications for teacher educators. Not only does deciding on follow-up questions appear to be a more accessible practice, but it can also lead to opportunities for teacher learning. When teachers pose follow-up questions to children during instruction, they have opportunities to learn more about specific strategies, specific children's thinking, and—over time—children's mathematical thinking in general (Franke et al., 1998, 2001). This learning potential was particularly visible in the *Emerging Noticing* profile. Deciding on follow-up questions was these teachers' strongest skill, suggesting that even when teachers have not yet made complete sense of children's strategies or understandings, they can still propose follow-up questions that honor and leave space for children's thinking, which in turn provides opportunities for teachers to learn about children's thinking. Thus, while teachers need experiences with both forms of deciding how to respond, initially emphasizing deciding on follow-up questions may be a productive starting point for helping teachers learn how to use children's strategies and understandings to inform their next instructional steps.

Study limitations and suggestions for future research

In this section, we highlight limitations related to our participant sample and our inability to collect longitudinal noticing data in the context of our broader study. We describe each limitation in terms of future research that could begin to overcome the limitations and move the field forward.

Explorations with a broader set of participants

Our study involved 72 teachers who had completed 1–3 years of PD focused on children's fraction thinking. Our findings provide a solid starting point for conversations about profiles of noticing expertise for teachers engaged in multi-year PD, especially given that our sample size was relatively large compared to much of the research on teacher noticing (König et al., 2022). However, because all teachers in our study had already completed at least one year of PD, we were unable to consider baseline information—the noticing expertise of teachers who were interested in, but had not yet engaged in, intensive study of children's mathematical thinking. Future studies of noticing profiles would benefit from including these teachers as well as teachers involved in multi-year PD focused on a content area other than fractions.

Explorations of the longitudinal development of teacher noticing expertise

Our profile means had a consistent ordering—both overall and within each component skill—suggesting that teachers may progress from an *Emerging Noticing* profile to a *Mixed Noticing* profile to an *Accomplished Noticing* profile. We recognize that this developmental interpretation must be tentative because the profiles were created using data that captured teachers' expertise at only one point in time (versus longitudinal data). However, data connecting the profiles and teachers' years of PD provide some initial evidence for a developmental interpretation of our profiles.[4]

The distribution of teachers into profiles varied as one might expect with a developmental interpretation—the *Emerging Noticing* profile consisted of more teachers having completed 1 year of PD and the *Accomplished Noticing* profile consisted of more teachers having completed 3 years of PD. In fact, the membership percentages in the two profiles were essentially mirror images of each other. The *Emerging Noticing* profile had 56%, 36%, and 8% of teachers who had completed 1, 2, or 3 years of PD, respectively, whereas the *Accomplished Noticing* profile had 7%, 36%, and 57% of teachers who had completed 1, 2, or 3 years of PD, respectively. The *Mixed Noticing* profile was in-between, with a more even distribution.

A developmental interpretation of the profiles is reminiscent of the widely cited van Es (2011) framework for learning to notice student mathematical thinking. Using longitudinal data from a

[4] We also wondered if differences in years of teaching experience could help explain our profiles, but the mean number of years of teaching was 12 years for all three profiles, suggesting that teaching experience was not a source of our profile differences.

year-long video club involving 7 upper elementary school teachers, van Es described the development of the *group's* noticing expertise with 4 ordered levels—baseline, mixed, focused, and extended—in terms of *what and how teachers notice*. Our ordered profiles potentially provide a complementary alternative for characterizing the development of noticing expertise. Specifically, we focused on the development of an *individual's* noticing expertise, showcasing patterns of strengths and areas for growth in terms of the *noticing component skills*. Our development also extended over a longer period of time given that the 72 teachers had up to 3 years of PD. However, longitudinal work is needed to not only confirm the developmental interpretation of our profiles but also to explore what supports teachers in moving from one profile to the next.

Final thoughts

This study contributes to the growing body of research that highlights the usefulness of professional noticing of children's mathematical thinking for understanding the complexity and variability in teaching expertise. By foregrounding teachers' noticing of children's thinking with fraction problem solving in grades 3–5, we identified profiles of teachers' noticing expertise and elaborated the component skill of deciding how to respond. Overall, this study enhanced our appreciation for the richness of teacher noticing—an invisible, but critical, instructional practice that is foundational for a vision of instruction in which teaching is responsive to children's mathematical thinking.

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