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Investigating teachers' knowledge for teaching mathematics

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The teacher-knowledge perspective has become an important way to think about teachers and their work. While Shulman's (1986) notion of different types of knowledge permeates research on teacher education, other constructs of teachers' knowledge such as practical knowledge, personal knowledge, and craft knowledge have been used as conceptual frameworks for empirical studies. In mathematics education, two of Shulman's categories of knowledge have played central roles in shaping the direction of research on mathematics teachers: content knowledge and pedagogical content knowledge. This makes sense in terms of making the content (mathematics) the focus of the work. Through the work of Ball and colleagues (e.g. Ball et al. 2008; Hill et al. 2008), these two types of knowledge have been *mathematized* under an umbrella label of *mathematics knowledge for teaching*, which has become one of the central constructs in research on the development of understandings for teaching mathematics.

For mathematics knowledge for teaching, Ball et al. (2008) have defined three types of content knowledge (i.e. common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon) and three types of pedagogical content knowledge (i.e. knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum). Thus, as a research construct, this perspective of mathematics knowledge for teaching offers six types of knowledge to investigate in studying the mathematics teacher and knowledge for teaching mathematics. For us to have a comprehensive understanding of mathematics teachers' knowledge, all six should be investigated not only individually but in terms of relationships among them in supporting teacher learning and practice. Some types also should be further investigated in terms of addressing potential limitations in defining them based on Ball et al.'s theoretical framework that is grounded in knowledge of practice. For example, Ball et al. point out that there are potential limitations due to the variety in curricula and associated classroom implementations. It is also not clear how cultural variability across and among teachers and students is accounted for in their model. However, while these different types of knowledge offer

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useful constructs to investigate teachers' knowledge for teaching mathematics, in general, focusing solely on such a set of knowledge is likely to limit the scope of our understanding of what happens in mathematics classrooms, why it happens, what ought to happen, and how to help teachers to achieve it. The teacher-personal perspective in terms of other constructs such as intention, motivation, value, and identity should be connected to the teacher-knowledge perspective. More attention should be given to this connection in investigating mathematics teachers' knowledge in order to extend it beyond being a set of knowledge analogous to Shulman's (1986) model.

The ongoing trend in research on mathematics teachers' knowledge for teaching mathematics is to investigate the nature of the knowledge teachers possess and approaches to support further development of the knowledge they hold or lack. Both of these are important to inform our understanding and efforts to reform mathematics education. However, the body of work resulting from such investigations needs to go beyond what teachers are able or not able to do to encompass a broader scope of ways of making sense of that knowledge, for example, what knowledge they hold; how and why they hold it; how did it develop; how, when, and why do they use it; how does it impact students' learning and achievement; how does it impact their classroom actions; how can it be supported, changed, or enhanced; and how can it be measured.

Regarding the development of teachers' knowledge, recent investigations of it are shifting from the traditional knowledge-based tendency to think of teachers' knowledge as knowledge generated by others that is then given to teachers to more learner-focused, teacher-perspective approaches that are more consistent with the growing teacher research movement or collaborative relations among teachers to improve their practice. For example, investigations of the ways that teachers generate knowledge, in particular ways to strengthen their mathematical understandings and mathematics knowledge for teaching, have included experiences involving collaborating with other teachers to plan or discuss mathematics lessons in terms of instruction or conducting lesson studies; studying students' mathematical thinking based on students' written work or video cases; reflecting on implementation/use of new curricular materials or new mathematical tasks based on interaction with students in the classroom and colleagues doing the same; and engaging in other formal researcher-facilitated professional conversations and practice-based activities. However, the investigations generally focus on formal learning contexts usually facilitated by the researcher or expert leaders. They do not account for the fact that knowledge about teaching mathematics grows through informal exchanges among mathematics teachers, that is, teachers learning from other teachers, for example, through the exchanges among colleagues that occur during teacher professional conferences, in hallways, or "over a cup of coffee". Through interactions such as these, teachers form bonds that result in an exchange of knowledge and in the generation of new knowledge. Thus, if we want to improve the quality of knowledge development in professional communities, investigation needs to go beyond the formal level of knowledge development to better understand the ways in which mathematics teachers use their own experiences and those of their colleagues to become better teachers and the nature of the knowledge and practice resulting from it.

Another consideration that requires attention in investigating development of teachers' knowledge, even if from the learner-focused, teacher-perspective, is the role of the researchers' knowledge when researchers and teachers foster knowledge production in mathematics professional development settings. This topic was the basis of a Discussion Group on "Researchers' and teachers' knowledge in mathematics professional development" co-led by Sztajn et al. (2012) at the 36th conference of the International Group for the Psychology of Mathematics Education. The goal of this discussion group was for participants to examine the role of researchers' and practicing teachers' knowledge in both the design and the research of mathematics professional development, with particular attention to different ways in which researchers and practicing teachers interact, and to how their knowledge contributions are interrelated and exchanged. There was general consensus that professional development initiatives can be organized around different views of the role of the researcher's knowledge in relation to teachers' knowledge. Two examples are as follows:

1. Researchers as "Generators" of mathematical professional development: Role of researcher is to share researcher-generated new knowledge with teachers in ways that respect and build on teachers' current knowledge.
2. Researchers as "Reactors" in mathematical professional development work: Role of researcher is "expert" to provide support by responding to the teachers' needs in the professional development rather then imposing direction.

Both of these, and other possibilities, need to be investigated in terms of the role of the researcher's knowledge in mathematics professional development, the role of both the researchers' and teachers' knowledge (how they interact) in mathematics professional development settings, the role of mathematical professional development in fostering knowledge exchange between researchers and teachers, and the design for mathematics professional development to support knowledge sharing and creation. In general, more attention is needed in investigating how mathematics teachers appropriate and transform such knowledge in practice.

The articles in this issue of this journal make contributions to the field of teachers' knowledge for teaching mathematics in terms of investigations of assessment of teachers' knowledge, teachers' knowledge of tasks, and teachers' knowledge of constructivism to frame teaching. Michael Steele investigated a set of assessment tasks designed to measure teachers' mathematical knowledge for teaching geometry and measurement. Esther Levenson investigated teachers' choices of tasks when their aim was to promote mathematical creativity in the classroom and what lies behind these choices. John O'Shea and Aisling Leavy investigated the extent to which teachers' detailed understanding of emergent constructivism and its implications for classroom practice informed their teaching practices within the context of teaching problem solving. Collectively, they contribute to our understanding of: (1) teachers' knowledge of an underexplored area of school mathematics (geometry) and a possible way to assess this knowledge that also helps to fill the need for assessment instruments that effectively evaluate and allow teachers to demonstrate this knowledge; (2) teachers' knowledge of creativity and tasks to support it, in particular what teachers may take into consideration as they choose a task with the aim of promoting mathematical creativity in the classroom; and (3) the challenges for teachers to implement their theoretical knowledge of constructivism for teaching problem solving. Thus, they deal with important areas of mathematics education with implications for teacher education and professional development.

Michael Steele's article, "Exploring the mathematical knowledge for teaching geometry and measurement through the design and use of rich assessment tasks", reports on a study of the development of a set of tasks designed to investigate and measure secondary teachers' mathematical knowledge for teaching geometry and measurement. He drew on current research on teachers' mathematical knowledge and the construct of mathematics knowledge for teaching to design these tasks. In particular, he used the common content knowledge and specialized content knowledge constructs to describe mathematics knowledge for teaching for the relationships between length, perimeter, and area. The tasks adhered to three key design principles: tasks are grounded in the context of teaching; as a set, measure aspects of and relationships between common and specialized content knowledge related to geometry and measurement; and capture nuances of teacher's knowledge beyond correct and incorrect answers. The tasks were created as part of a 6-week Masters-level content-focused methods course designed as an opportunity for teachers to enhance their mathematics knowledge for teaching geometry and measurement in the middle grades. Twenty-five secondary mathematics teachers participated in the course. The instructional intervention included working on solving mathematical tasks, analysing students' work, and planning lessons around geometry and measurement content. Data from an administration of these tasks at the start of the course were used to illustrate the ways in which the tasks differentiated teachers' performance and revealed important aspects of teacher's knowledge.

The article describes the tasks, focusing on the design features, and offers what teacher work on the tasks showed with respect to their common and specialized content knowledge. The examples of teacher performance on the tasks illustrate the ways in which the tasks can differentiate teacher performance and provide insights into the connections between common and specialized content knowledge in terms of the ways in which the former can influence how teachers make use of the latter in teaching. For example, for one of the tasks, teachers with stronger abilities to describe the relationships between length, perimeter, and area were more likely to use multiple representations in their response. Another task also revealed a relationship between common and specialized content knowledge and teachers' ability to write goals for a mathematical lesson. Teachers with stronger mathematical performances on the task were better able to write more specific goals for another task with students. While these secondary teachers were expected to be able to calculate perimeter and area and make basic connections between length, perimeter, and area, items that measured those aspects of common content knowledge from multiple angles (e.g. using both rectangles and parallelograms, reasoning from diagrammatic and contextual starting situations) and that provided opportunities to mobilize both common content knowledge and specialized content knowledge in the service of the same task also revealed important nuances and connections.

As Steele points out, investigation of teachers' knowledge of geometry and measurement has been nearly non-existent in the research literature and in particular secondary mathematics teachers' mathematical knowledge for teaching geometry and measurement almost entirely unexplored. Thus, this study, in addition to contributing to an underrepresented area of school mathematics, illustrates important connections between common content knowledge and specialized content knowledge, a relationship that previous work on mathematical knowledge for teaching has not fully explored. However, ongoing efforts are needed to conceptualize, develop, and test measures of teachers' knowledge, not only for common and specialized content knowledge, as in this study, but for all of the different types of knowledge that comprise mathematical knowledge for teaching (Ball et al. 2008). The three design features exemplified by the tasks in this study can be useful to researchers interested in such an investigation.

Esther Levenson's article, "Tasks that may occasion mathematical creativity: teachers' choices", investigated the tasks teachers choose to promote mathematical creativity in the classroom. Specifically, three questions framed the work: On what basis do teachers choose mathematical tasks when their aim is to promote mathematical creativity? What task characteristics do they look for? Do they consider affective issues, and if so, what affective constructs may be related to their choices? In framing the study, Levenson shows that there are several ways of characterizing both what is and what is not mathematical creativity and makes the case that, as manifested in the mathematics classroom, it is multifaceted. For example, the product of mathematical creativity in the classroom may be original ideas that are personally meaningful to the students and appropriate for the mathematical activity being considered. Thus, the teacher who aims to promote mathematical creativity may take into consideration several factors, one being the task around which much of student learning is centred. The participants were 43 graduate students in Israel working towards a Master's degree in *Mathematics, Science, and Technology Education*. Some of them had no experience in teaching and were concurrently studying towards their teaching degree, while others were experienced teachers. None of them had previously taken a formal course related to creativity. The research tool was an assignment that included choosing a task or activity from a mathematics textbook or workbook that in the participant's opinion promotes mathematical creativity and writing one paragraph to explain why, in his or her opinion, this task has the potential to promote mathematical creativity. Five cases are described in the article to exemplify the wide range of tasks chosen by participants as well as to exemplify the variety of reasons participants listed for choosing the tasks.

Key findings for the five cases include the following: Sylvie chose tasks that associate creativity with being different from the norm, taking the students outside of the classroom, and having them do a mathematical activity that does not feel like mathematics. Randy noted features of the task as well as cognitive demands which are to some extent also inherent in the task and added affective issues which she believes may add to the promotion of mathematical creativity. For Erwin, creativity was mostly about being different—presenting a different or unusual question that will promote a different and new way of thinking. Miriam associates creativity with being able to make connections. For Ava, flexibility was the key to creativity. Findings of common trend across the cases consisted of task features, cognitive demands, and affective issues (emotions and values) that may be involved in the promotion of mathematical creativity that was mentioned by participants. For example, some participants argued that their tasks would promote curiosity and fun, which in turn would promote creativity. Approximately half of the tasks had only one correct answer, but almost all had several ways in which a final answer could be reached, which was not necessarily noted by the participants. Participants sought out tasks that might encourage students to think differently, in non-routine non-imitative paths, in ways that connect different mathematical domains and connect mathematics to other areas, implying that creativity pertains to being different and unusual.

Levenson wonders whether the affective issues arose due to the association of the task with mathematical creativity or whether they are implicitly connected to task choices in general. However, she concludes that in our endeavour to increase mathematical creativity in the classroom, we need to take into consideration that some teachers also relate to the affective side of implementing tasks that may occasion mathematical creativity; thus, it is important to discuss with them both the affective aspect of tasks and the affective aspect of mathematical creativity. She also notes that the study provides a framework for analysing tasks that may be used with teachers in professional development to discuss how a task may afford or constrain mathematical creativity.

John O'Shea and Aisling Leavy's article, "Teaching mathematical problem solving from an emergent constructivist perspective: the experiences of Irish primary teachers", explores the extent to which teachers' understanding of an *emergent constructivist perspective* and its implication for the classroom inform their teaching of mathematical problem solving consistent with the perspective of their official mathematics curriculum.

The focus is on primary teachers' understanding of organizing learning following their engagement with constructivism and mathematical problem solving in a professional development initiative. The emergent constructivist perspective used to frame the study considers mathematical knowledge as both an individual and a social construction and that individual and social dimensions of learning complement each other. The five participating teachers taught upper levels of primary school (10–12 years of age). They did not have previous experience with emergent constructivist theory and for the most part embraced traditional classroom approaches in their teaching, that is, regimented, controlled environments. The professional development activities consisted of the researcher and teachers reading and reflecting on seminal literature in constructivist and problem-solving fields and engaging in video analysis of problem-solving lessons in primary and middle school classrooms. The researchers also provided support to the teachers in incorporating the emergent constructivist perspective on problem-solving in their day-to-day classroom practices.

All five participants found discussing and examining the nature of constructivism and its implications for their practice exciting and were keen to engage with it in practice. In engaging with this project, they indicated a keen interest in mathematical problem-solving and constructivist practices and acknowledged the significance of mathematics and, in particular, problem solving. However, their enthusiasm for constructivism was challenged as they endeavoured to organize learning from the constructivist perspective and struggled to make links between it and its manifestation in classroom practice. The creation of a constructivist classroom was a significant task for them. Their traditional-oriented instructional way of thinking and teaching was still more dominant. For example, although they used a problem-solving heuristic model that could allow students to think for themselves, they engaged students in it in a traditional manner by helping them to arrive at answers in a specified way that limited their understanding of the answers and the process used to get the answers. The teachers faced many dilemmas throughout the project that included finding a balance between individual and group learning and the definition of appropriate constructivist learning experiences. The most profound challenges that emerged for the teachers are the following: to make personal sense of constructivism, to reorientate the culture of the classroom to accommodate constructivist philosophy, and to deal with conservatism that works against teaching for understanding.

The study reveals the challenges of teachers attempting to organize learning from an emergent constructivist perspective after participating in a professional development. At the end of their engagement with the project, it was clear that the traditional methods of instruction and traditional focus on content remained deeply rooted in their teaching practices. Although they were inspired by learning from a constructivist perspective, it was evident that methodologies that reflect constructivist principles could not usurp the traditional methods used in their classrooms. O'Shea and Leavy conclude that teachers need to understand their beliefs and those of their students and to work on changing both if necessary. They point out that the study affirms that teachers' knowledge, beliefs, identify, school contexts, and school curriculum are all important factors to consider. The study has implications for mathematics teacher education in that it provides further evidence to support the position that professional development experiences focused on developing teachers' understanding of constructivism is not sufficient to help them to overcome the challenges of adopting it in their teaching of problem solving.

To conclude, while the articles by Steele and by Levenson provide further insights about specific aspects of teachers' knowledge for teaching mathematics, O'Shea and Leavy's article reminds us of how difficult it could be to change aspects of teachers' knowledge of teaching mathematics in a way to transform their practice to a more desirable perspective. Collectively, they cover three important aspects of teachers' knowledge that have implications for teacher education and further investigation of the knowledge teachers hold and how to change that knowledge when necessary to better to support students' development of deep mathematical understanding. While they contribute to our understanding of different types of knowledge for teaching mathematics, there is also awareness of the need to extend beyond this as future investigations are considered in these areas of mathematics education.

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