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Algebraic Completeness of Connexive and Bi-Intuitionistic Multilattice Logics

Yaroslav Petrukhin[1]

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Abstract

In this paper, we introduce the notions of connexive and bi-intuitionistic multilattices and develop on their base the algebraic semantics for Kamide, Shramko, and Wansing's connexive and bi-intuitionistic multilattice logics which were previously known in the form of sequent calculi and Kripke semantics. We prove that these logics are sound and complete with respect to the presented algebraic structures.

Keywords Connexive logic · Bi-intuitionistic logic · Multilattice logic · Algebraic logic · Sequent calculi

1 Introduction

Shramko introduced multilattice logic $\mathbf{ML}_n$ as a generalization of logics based on lattices. In particular, $\mathbf{ML}_n$ is a generalization of a class of many-valued logics based on four-valued Belnap-Dunn's logic of first degree entailment (Belnap, 1977a, b; Dunn, 1976) and its algebraic framework, De Morgan lattices. This class contains Arieli and Avron's four-valued bilattice logic (Arieli & Avron, 1996), Shramko and Wansing's sixteen-valued trilattice logic (Shramko & Wansing, 2005), and Zaitsev's eight-valued tetralattice logic (Zaitsev, 2009). Multilattice logic is based on the notion of n-lattice (multilattice), i.e., a lattice with n orders and some required relations between them. The algebraic completeness theorem for $\mathbf{ML}_n$ was only formulated in Shramko (2016), the proof was found later in Grigoriev and Petrukhin (2019b). The family of multilattice logics contains not only $\mathbf{ML}_n$, but its several modifications: bi-intuitionistic multilattice logic $\mathbf{BML}_n$ and its connexive version $\mathbf{CML}_n$ (Kamide et al., 2017), first-order multilattice logic $\mathbf{FML}_n$ (Kamide & Shramko, 2017b), modal multilattice logic $\mathbf{MML}_n$ (Kamide & Shramko, 2017a) and its modifications $\mathbf{MML}_n^{\mathbf{MNT4}}$, $\mathbf{MML}_n^{\mathbf{S4}}$, and $\mathbf{MML}_n^{\mathbf{S5}}$ (Grigoriev & Petrukhin, 2021, 2019a) as well as congruent and monotonic

✉ Yaroslav Petrukhin
iaroslav.petrukhin@adm.uni.lodz.pl; yaroslav.petrukhin@gmail.com

1 Department of Logic, University of Łodź, Lindleya 3/5, 90-131 Łódź, Poland

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modal multilattice logics (Grigoriev & Petrukhin, 2022), a fragment of $\mathbf{ML}_n$, called $\mathbf{MLL}_n$, determined by logical multilattices ($\mathbf{ML}_n$ itself is determined by ultralogical multilattices) (Grigoriev & Petrukhin, 2022), linear multilattice logics $\mathbf{EML}_n$ and $\mathbf{LML}_n$ (Kamide, 2019), and alternative multilattice logics $\mathbf{SM}_n$ and $\mathbf{IM}_n$ (Kamide, 2021). The algebraic completeness theorem has been established for $\mathbf{ML}_n$ and $\mathbf{MML}_n$ in Grigoriev and Petrukhin (2019b), for $\mathbf{MML}_n^{\mathbf{MNT4}}$, $\mathbf{MML}_n^{\mathbf{S4}}$, and $\mathbf{MML}_n^{\mathbf{S5}}$ in Grigoriev and Petrukhin (2021), for congruent and monotonic modal multilattice logics and $\mathbf{MLL}_n$ in Grigoriev and Petrukhin (2022), for $\mathbf{SM}_n$ and $\mathbf{IM}_n$ in Kamide (2021).

However, for a few members of the multilattice family of logics, the algebraic completeness theorem has not been established yet. Among such logics are connexive multilattice logic $\mathbf{CML}_n$ and bi-intuitionistic multilattice logic $\mathbf{BML}_n$. These two logics are considered a pair since $\mathbf{CML}_n$ is a connexive version of $\mathbf{BML}_n$. $\mathbf{BML}_n$ is a multilattice version of bi-intuitionistic logic **BiInt** introduced by Rauszer (1974, 1977, 1980); $\mathbf{CML}_n$ is a multilattice version of bi-intuitionistic connexive logic **BCL** introduced by Wansing (2008) as one of sixteen variants of bi-intuitionistic logic and separately studied later by Kamide and Wansing (2016).

Kamide et al. (2017) formulated $\mathbf{BML}_n$ and $\mathbf{CML}_n$ in the form of sequent calculi (based on sequent calculi for **BiInt** and **BCL** offered in Kamide and Wansing (2016)) and Kripke-style semantics (based on Rauszer's semantics for **BiInt** (Rauszer, 1977, 1980)), but algebraic semantics has not been developed. Moreover, the notions of bi-intuitionistic and connexive multilattices have not been introduced. This paper fills this gap: we introduce such notions and prove that sequent calculi for $\mathbf{BML}_n$ and $\mathbf{CML}_n$ are sound and complete with respect to bi-intuitionistic and connexive multilattices. Additionally, we show how the notions of bi-intuitionistic and connexive multilattices can be modified to get an alternative algebraic semantics for $\mathbf{ML}_n$ and $\mathbf{MLL}_n$.

The structure of the paper is as follows. The next section is devoted to the preliminaries regarding algebraic aspects of our topic. Section 3 describes Kamide, Shramko, and Wansing's sequent calculi for the logics in question. In Sect. 4, we introduce the notions of connexive and bi-intuitionistic multilattices. Section 5 contains a proof of the algebraic completeness theorem. Section 6 consists of concluding remarks.

2 Preliminaries

We begin with some preliminaries about lattices, following their presentation in Dunn and Restall (2002).

Definition 2.1 (Lattice) A *lattice* is a structure $\langle L, \cap, \cup \rangle$, where L is a non-empty set and $\cap$ and $\cup$ are binary operations on L, with the relation $a \leqslant b$ defined as $a \cap b = a$. Postulates characterising the operations are as follows, for each $a, b \in L$:

- Idempotence: $a \cap a = a$, $a \cup a = a$
- Commutativity: $a \cap b = b \cap a$, $a \cup b = b \cup a$
- Associativity: $a \cap (b \cap c) = (a \cap b) \cap c$, $a \cup (b \cup c) = (a \cup b) \cup c$
- Absorption: $a \cap (a \cup b) = a$, $a \cup (a \cap b) = a$.

Definition 2.2 (Distributive lattice) $\langle L, \cap, \cup \rangle$ is a *distributive* lattice iff it is a lattice satisfying the following postulate, for any $a, b, c \in L$: $a \cap (b \cup c) \leqslant (a \cap b) \cup c$.

Multilattices generalize the notion of lattice. We are ready now to present the notion of a multilattice and some other important related notions.

Definition 2.3 (Multilattice) (p. 204, Definition 4.1, Shramko 2016)

1. A *multilattice* (or *n-lattice*, or *n-dimensional multilattice*) is a structure $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$, where $n > 1$, $S \neq \emptyset$, $\leqslant_1, \ldots, \leqslant_n$ are partial orderings such that $\langle S, \leqslant_1 \rangle, \ldots, \langle S, \leqslant_n \rangle$ are lattices with the corresponding pairs of meet and join operations $\langle \cap_1, \cup_1 \rangle, \ldots, \langle \cap_n, \cup_n \rangle$.
2. A multilattice is called *complete* iff all meets and joins exist, with respect to all n orderings.
3. A multilattice is called *interlaced* iff each of the operations $\cap_1, \cup_1, \ldots, \cap_n, \cup_n$ is monotone with respect to all n orderings.
4. A multilattice is called *distributive* iff all $2(2n^2 - n)$ distributive laws are satisfied, i.e. $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, where $a, b, c \in S, \otimes, \oplus \in \{\cup_1, \cap_1, \ldots, \cup_n, \cap_n\}$, and $\otimes \neq \oplus$.

Remark 2.4 In what follows, we are going to deal with complete, interlaced, and distrubutive multilattices exclusively in our research.

Definition 2.5 (Multilattice with inversions) (p. 204, Definition 4.2, Shramko 2016) Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice. Then for any $j \leqslant n$ an unary operation $-_j$ on S is said to be a (pure) j-inversion iff for any $k \leqslant n$, $k \neq j$ the following conditions are satisfied, where $a, b \in S$:

$$a \leqslant_j b \text{ implies } -_j b \leqslant_j -_j a; \quad \text{(anti)}$$

$$a \leqslant_k b \text{ implies } -_j a \leqslant_k -_j b; \quad \text{(iso)}$$

$$-_j -_j a = a. \quad \text{(per2)}$$

Definition 2.6 (Multifilter) (p. 207, Definition 5.1, Shramko 2016) Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice, with pairs of meet and join operations $\langle \cap_1, \cup_1 \rangle, \ldots, \langle \cap_n, \cup_n \rangle$. $\mathcal{F}_n \subset S$ is a multifilter on $\mathcal{M}_n$ iff the following condition holds, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

$$a \cap_j b \in \mathcal{F}_n \text{ iff } a \in \mathcal{F}_n \text{ and } b \in \mathcal{F}_n; \quad \text{(filter)}$$

A multifilter $\mathcal{F}_n$ is a prime multifilter on $\mathcal{M}_n$ iff the following condition holds, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

$$a \cup_j b \in \mathcal{F}_n \text{ iff } a \in \mathcal{F}_n \text{ } or \text{ } b \in \mathcal{F}_n. \quad \text{(prime)}$$

Definition 2.7 (Logical multilattice) (p. 207, Definition 5.1, Shramko 2016) A pair $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ is a *logical multilattice* iff $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ is a multilattice and $\mathcal{F}_n$ is a prime multifilter.

Definition 2.8 (Ultramultifilter, ultralogical multilattice) (p. 207–208, Definition 5.2, Shramko 2016) Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice, with j-inversions defined with respect to every $\leqslant_j$ $(j \leqslant n)$. Then $\mathcal{F}_n$ is an n-ultrafilter (ultramultifilter) on $\mathcal{M}_n$ iff it is a prime multifilter, such that for every $j, k \leqslant n$, $j \neq k$, and $a \in S$:

$$a \in \mathcal{U}_n \text{ iff } -_k -_j a \notin \mathcal{U}_n. \tag{ultra}$$

A pair $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ is an *ultralogical multilattice* iff $\mathcal{M}_n$ is a multilattice and $\mathcal{U}_n$ is an ultramultifilter.

Definition 2.9 (Language) The formulas of $\mathbf{ML}_n$, $\mathbf{BML}_n$, and $\mathbf{CML}_n$ are built from the set $\mathcal{P} = \{p_n \mid n \in \mathbb{N}\}$ of propositional variables, negations $\neg_1, \ldots, \neg_n$, conjunctions $\wedge_1, \ldots, \wedge_n$, disjunctions $\vee_1, \ldots, \vee_n$, implications $\rightarrow_1, \ldots, \rightarrow_n$, and co-implications $\leftarrow_1, \ldots, \leftarrow_n$. The notion of a formula is defined in a standard inductive way.

Definition 2.10 (Valuation in $\mathbf{ML}_n$) Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice. A valuation v is defined as a mapping from $\mathcal{P}$ to S. It is extended into complex formulas as follows: $v(\neg_j A) = -_j v(A)$, $v(A \wedge_j B) = v(A) \cap_j v(B)$, $v(A \vee_j B) = v(A) \cup_j v(B)$, $v(A \rightarrow_j B) = -_k -_j v(A) \cup_j v(B)$, and $v(A \leftarrow_j B) = v(A) \cap_j -_k -_j v(B)$.

Remark 2.11 This definition of the valuation for implications and co-implications is applicable only to the case of ultralogical multilattices and $\mathbf{ML}_n$ (as well as its modal extensions studied in Grigoriev and Petrukhin (2019b, 2021, 2022)). We will need another definition for these connectives for the case of $\mathbf{BML}_n$ and $\mathbf{CML}_n$.

Definition 2.12 (Entailment in $\mathbf{ML}_n$) The entailment relation in $\mathbf{ML}_n$ is defined as follows:
$\Gamma \models_{\mathbf{ML}_n} \Delta$ iff for each ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{U}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{U}_n$, for some $D \in \Delta$.

Definition 2.13 (Entailment in $\mathbf{MLL}_n$) The entailment relation in $\mathbf{MLL}_n$ is defined as follows:
$\Gamma \models_{\mathbf{MLL}_n} \Delta$ iff for each logical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{U}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{U}_n$, for some $D \in \Delta$.

Remark 2.14 In $\mathbf{ML}_n$ if $j, k \leqslant n$ and $j \neq k$, then $\neg_k \neg_j A$ is equivalent to $\neg_j \neg_k A$; $\neg_k \neg_j$ behaves as Boolean negation. In $\mathbf{BML}_n$ and $\mathbf{CML}_n$, if $j, k \leqslant n$, $j \neq k$, and $j < k$, then $\neg_j \neg_k A$ behaves as intuitionistic negation and $\neg_k \neg_j A$ behaves as dual intuitionistic negation. In all the logics in question A is equivalent to $\neg_j \neg_j A$ and $\neg_j$ behaves as De Morgan negation.

3 Sequent Calculi

Let us describe Kamide, Shramko, and Wansing's sequent calculus for the logic $\mathbf{CML}_n$ Kamide et al. (2017). A sequent is understood as an ordered pair written as $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas. The axioms, for any propositional variable P:

$$(Ax) P \Rightarrow P \qquad (Ax_{\neg}) \neg_j P \Rightarrow \neg_j P$$

The structural rules:

$$(Cut)\frac{\Gamma \Rightarrow \Delta, A \quad A, \Theta \Rightarrow \Lambda}{\Gamma, \Delta \Rightarrow \Theta, \Lambda} \quad (W \Rightarrow)\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \quad (\Rightarrow W)\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

The non-negated logical rules:

$$(\wedge_j \Rightarrow)\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge_j B, \Gamma \Rightarrow \Delta} (\Rightarrow \wedge_j)\frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge_j B}$$

$$(\vee_j \Rightarrow)\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee_j B, \Gamma \Rightarrow \Delta} (\Rightarrow \vee_j)\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee_j B}$$

$$(\rightarrow_j \Rightarrow)\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Theta \Rightarrow \Lambda}{A \rightarrow_j B, \Gamma, \Theta \Rightarrow \Delta, \Lambda} (\Rightarrow \rightarrow_j)\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_j B}$$

$$(\leftarrow_j \Rightarrow)\frac{A \Rightarrow \Delta, B}{A \leftarrow_j B \Rightarrow \Delta} (\Rightarrow \leftarrow_j)\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, A \leftarrow_j B}$$

The jj-negated logical rules:

$$(\neg_j \wedge_j \Rightarrow)\frac{\neg_j A, \Gamma \Rightarrow \Delta \qquad \neg_j B, \Gamma \Rightarrow \Delta}{\neg_j (A \wedge_j B), \Gamma \Rightarrow \Delta} (\Rightarrow \neg_j \wedge_j)\frac{\Gamma \Rightarrow \Delta, \neg_j A, \neg_j B}{\Gamma \Rightarrow \Delta, \neg_j (A \wedge_j B)}$$

$$(\neg_j \vee_j \Rightarrow)\frac{\neg_j A, \neg_j B, \Gamma \Rightarrow \Delta}{\neg_j (A \vee_j B), \Gamma \Rightarrow \Delta} (\Rightarrow \neg_j \vee_j)\frac{\Gamma \Rightarrow \Delta, \neg_j A \qquad \Gamma \Rightarrow \Delta, \neg_j B}{\Gamma \Rightarrow \Delta, \neg_j (A \vee_j B)}$$

$$(\neg_j \rightarrow^c_j \Rightarrow)\frac{\Gamma \Rightarrow \Delta, A \qquad \neg_j B, \Theta \Rightarrow \Lambda}{\neg_j (A \rightarrow_j B), \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\Rightarrow \neg_j \rightarrow^c_j)\frac{A, \Gamma \Rightarrow \neg_j B}{\Gamma \Rightarrow \neg_j (A \rightarrow_j B)}$$

$$(\neg_j \leftarrow^c_j \Rightarrow)\frac{\neg_j A \Rightarrow \Delta, B}{\neg_j (A \leftarrow_j B) \Rightarrow \Delta} \qquad (\Rightarrow \neg_j \leftarrow^c_j)\frac{\Gamma \Rightarrow \Delta, \neg_j A \qquad B, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j (A \leftarrow_j B)}$$

$$(\neg_j \neg_j \Rightarrow)\frac{A, \Gamma \Rightarrow \Delta}{\neg_j \neg_j A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_j \neg_j)\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg_j \neg_j A}$$

The kj -negated logical rules (we presuppose that $j < k$ in the case of the rules for $\neg_j \neg_k$ and $\neg_k \neg_j$):

$$(\neg_k \wedge_j \Rightarrow)\frac{\neg_k A, \neg_k B, \Gamma \Rightarrow \Delta}{\neg_k (A \wedge_j B), \Gamma \Rightarrow \Delta} (\Rightarrow \neg_k \wedge_j)\frac{\Gamma \Rightarrow \Delta, \neg_k A \qquad \Gamma \Rightarrow \Delta, \neg_k B}{\Gamma \Rightarrow \Delta, \neg_k (A \wedge_j B)}$$

$$(\neg_k \vee_j \Rightarrow)\frac{\neg_k A, \Gamma \Rightarrow \Delta \qquad \neg_k B, \Gamma \Rightarrow \Delta}{\neg_k (A \vee_j B), \Gamma \Rightarrow \Delta} (\Rightarrow \neg_k \vee_j)\frac{\Gamma \Rightarrow \Delta, \neg_k A, \neg_k B}{\Gamma \Rightarrow \Delta, \neg_k (A \vee_j B)}$$

$$(\neg_k \rightarrow_j \Rightarrow)\frac{\Gamma \Rightarrow \Delta, \neg_k A \qquad \neg_k B, \Theta \Rightarrow \Lambda}{\neg_k (A \rightarrow_j B), \Gamma, \Theta \Rightarrow \Delta, \Lambda} \quad (\Rightarrow \neg_k \rightarrow_j)\frac{\neg_k A, \Gamma \Rightarrow \neg_k B}{\Gamma \Rightarrow \neg_k (A \rightarrow_j B)}$$

$$(\neg_k \leftarrow_j \Rightarrow)\frac{\neg_k A \Rightarrow \Delta, \neg_k B}{\neg_k (A \leftarrow_j B), \Rightarrow \Delta} \quad (\neg_k \leftarrow_j \Rightarrow)\frac{\Gamma \Rightarrow \Delta, \neg_k A \qquad \neg_k B, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_k (A \leftarrow_j B)}$$

$$(\neg_j \neg_k \Rightarrow)\frac{\Gamma \Rightarrow A}{\neg_j \neg_k A, \Gamma \Rightarrow} \qquad (\Rightarrow \neg_j \neg_k)\frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg_j \neg_k A}$$

$$(\neg_k \neg_j \Rightarrow)\frac{\Rightarrow \Delta, A}{\neg_k \neg_j A \Rightarrow \Delta} \qquad (\Rightarrow \neg_k \neg_j)\frac{A \Rightarrow \Delta}{\Rightarrow \Delta, \neg_k \neg_j A}$$

Kamide, Shramko, and Wansing's sequent calculus for $\mathbf{BML}_n$ Kamide et al. (2017) is obtained from the sequent calculus for $\mathbf{CML}_n$ by the replacement of the rules $(\neg_j \rightarrow_j^c \Rightarrow)$, $(\Rightarrow \neg_j \rightarrow_j^c)$, $(\neg_j \leftarrow_j^c \Rightarrow)$, and $(\Rightarrow \neg_j \leftarrow_j^c)$ with the following ones:

$$(\neg_j \rightarrow_j^b \Rightarrow)\frac{\neg_j B \Rightarrow \Delta, \neg_j A}{\neg_j (A \rightarrow_j B) \Rightarrow \Delta} (\Rightarrow \neg_j \rightarrow_j^b)\frac{\Gamma \Rightarrow \Delta, \neg_j B \qquad \neg_j A, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j (A \rightarrow_j B)}$$

$$(\neg_j \leftarrow_j^b \Rightarrow)\frac{\Gamma \Rightarrow \Delta, \neg_j B \qquad \neg_j A, \Theta \Rightarrow \Lambda}{\neg_j (A \leftarrow_j B), \Gamma, \Theta \Rightarrow \Delta, \Lambda} (\Rightarrow \neg_j \leftarrow_j^b)\frac{\neg_j B, \Gamma \Rightarrow \neg_j A}{\Gamma \Rightarrow \neg_j (A \leftarrow_j B)}$$

Let $\mathbf{L} \in \{\mathbf{CML}_n, \mathbf{BML}_n\}$. We write $\vdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$ iff there is a proof of the sequent $\Gamma \Rightarrow \Delta$ in the sequent calculus for the logic $\mathbf{L}$. The notion of the proof is defined in a standard manner for sequent calculi.

These calculi are multilattice versions of a sequent calculus **BL** for bi-intuitionistic logic and a sequent calculus **BCL** for bi-intuitionistic connexive logic, respectively, developed by Kamide and Wansing (2016). As mentioned in Kamide et al. (2017), since the cut-elimination theorem does not hold for **BCL** and **BL** (Kamide & Wansing, 2016), the cut-elimination theorem also does not hold for $\mathbf{CML}_n$ and $\mathbf{BML}_n$.

To obtain Kamide and Shramko (2017b) sequent calculus for $\mathbf{ML}_n$ from the sequent calculus for $\mathbf{BML}_n$ one needs to change the rules of which have only one formula in ancedent or consequent of a sequent: this restriction should be rejected; the rules for $\neg_j \neg_k$ and $\neg_k \neg_j$ will coincide, the condition that in their formulation $j < k$, should be omitted. The sequent calculus for $\mathbf{ML}_n$ is cut-free (Kamide & Shramko, 2017b).

To obtain the sequent calculus for the logic $\mathbf{MLL}_n$ from (Grigoriev & Petrukhin, 2022) from the sequent calculus for $\mathbf{ML}_n$ one needs to delete the rules for $\neg_j \neg_k$ $(\neg_k \neg_j)$ as well as all the rules for implications, coimplications, and their negations. The sequent calculus for $\mathbf{MLL}_n$ is cut-free (Grigoriev & Petrukhin, 2022).

4 Connexive and Bi-Intuitionistic Multilattices

Definition 4.1 (De Morgan multifilter) Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice (cf. Definition 2.3) and $\mathcal{F}_n$ be a prime multifilter on $\mathcal{M}_n$ (cf. Definition 2.6). Then for any $j \leqslant n$ an unary operation $-_j$ on S is said to be a *j-pseudo*-inversion and $\mathcal{F}_n$ is called *De Morgan* multifilter iff for any $k \leqslant n$, $k \neq j$ the following conditions are satisfied, where $a, b \in S$:

$$-_j (a \cap_j b) \in \mathcal{F} \quad \text{iff} \quad -_j a \cup_j -_j b \in \mathcal{F}; \qquad \text{(DM1)}$$

$$-_j\,(a \cup_j b) \in \mathcal{F} \quad \text{iff } -_j\, a \cap_j -_j b \in \mathcal{F}; \tag{DM2}$$

$$-_k\,(a \cap_j b) \in \mathcal{F} \quad \text{iff } -_k\, a \cap_j -_k b \in \mathcal{F}; \tag{DM3}$$

$$-_k\,(a \cup_j b) \in \mathcal{F} \quad \text{iff } -_k\, a \cup_j -_k b \in \mathcal{F}; \tag{DM4}$$

$$-_j -_j a \in \mathcal{F} \quad \text{iff } a \in \mathcal{F}. \tag{per2}$$

A De Morgan multifilter $\mathcal{F}_n$ is called *De Morgan ultra*multifilter iff it satisifes the condition (ultra). A pair $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ is called *De Morgan logical* (resp. *ultralogical*) multilattice iff $\mathcal{M}_n$ is a multilattice and $\mathcal{F}_n$ is a De Morgan multifilter (resp. ultramultifilter).

Definition 4.2 (Connexive multilattice and connexive multifilter) Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice with the j-pseudo-inversion operations $-_1, \ldots, -_n$ and $\mathcal{F}_n$ be a De Morgan multifilter. Then for any $j, k \leqslant n$ such that $j \neq k$ the corresponding pairs $\langle \supset_1, \subset_1 \rangle, \ldots, \langle \supset_n, \subset_n \rangle$ of binary operations called *relative pseudo-complement* and *relative pseudo-difference* operations are defined as follows, where $a, b, c \in S$ and $j < k$ in the conditions $(-_j -_k)$ and $(-_k -_j)$:

$$c \in \mathcal{F}_n \text{ implies } a \supset_j b \in \mathcal{F}_n \text{ iff } a \cap_j c \in \mathcal{F}_n \text{ implies } b \in \mathcal{F}_n; \tag{$\supset_j$}$$

$$a \subset_j b \in \mathcal{F}_n \text{ implies } c \in \mathcal{F}_n \text{ iff } a \in \mathcal{F}_n \text{ implies } b \cup_j c \in \mathcal{F}_n; \tag{$\subset_j$}$$

$$c \in \mathcal{F}_n \text{ implies } -_j\,(a \supset_j b) \in \mathcal{F}_n \text{ iff } a \cap_j c \in \mathcal{F}_n \text{ implies } -_j\, b \in \mathcal{F}_n; \tag{$-_j\supset_j^c$}$$

$$-_j(a \subset_j b) \in \mathcal{F}_n \text{ implies } c \in \mathcal{F}_n \text{ iff } -_j\, a \in \mathcal{F}_n \text{ implies } b \cup_j c \in \mathcal{F}_n; \tag{$-_j\subset_j^c$}$$

$$c \in \mathcal{F}_n \text{ implies } -_k\,(a \supset_j b) \in \mathcal{F}_n \text{ iff } -_k\, a \cap_j c \in \mathcal{F}_n \text{ implies } -_k\, b \in \mathcal{F}_n; \tag{$-_k\supset_j$}$$

$$-_k(a \subset_j b) \in \mathcal{F}_n \text{ implies } c \in \mathcal{F}_n \text{ iff } -_k\, a \in \mathcal{F}_n \text{ implies } -_k\, b \cup_j c \in \mathcal{F}_n; \tag{$-_k\subset_j$}$$

$$-_j -_k a \in \mathcal{F}_n \text{ iff } a \supset_k (a \subset_k a) \in \mathcal{F}_n; \tag{$-_j-_k$}$$

$$-_k -_j a \in \mathcal{F}_n \quad \text{iff } (a \supset_k a) \subset_k a \in \mathcal{F}_n. \tag{$-_k-_j$}$$

A multilattice $\mathcal{M}_n$ is called *connexive* iff $\mathcal{M}_n$ is a multilattice with the j-pseudo-inversion operations $-_1, \ldots, -_n$ and the pairs $\langle \supset_1, \subset_1 \rangle, \ldots, \langle \supset_n, \subset_n \rangle$ of relative pseudo-complement and relative pseudo-difference operations (where $j, k \leqslant n$ and $j \neq k$). A De Morgan multifilter $\mathcal{F}_n$ is called a *connexive* multifilter iff it satisfies the above presented conditions. A pair $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ is called *connexive* logical multilattice iff $\mathcal{M}_n$ is a connexive multilattice and $\mathcal{F}_n$ is a connexive multifilter.

Definition 4.3 (Bi-intuitionistic multilattice and bi-intuitionistic multifilter) *Bi-intuitionistic* multilattice, multiliter, logical multilattice satisfy all those conditions which hold for their connexive counterparts, except $(-_j\supset_j^c)$ and $(-_j\subset_j^c)$, as well as satisfy the following ones, for each $j, k \leqslant n$, $j \neq k$, and $a, b, c \in S$:

$$-_j(a \supset_j b) \in \mathcal{F}_n \text{ implies } c \in \mathcal{F}_n \text{ iff } -_j b \in \mathcal{F}_n \text{ implies } -_j a \cup_j c \in \mathcal{F}_n; \quad (-_j{\supset}_j^b)$$

$$c \in \mathcal{F}_n \text{ implies } -_j (a \subset_j b) \in \mathcal{F}_n \text{ iff } -_j b \cap_j c \in \mathcal{F}_n \text{ implies } -_j a \in \mathcal{F}_n. \quad (-_j{\subset}_j^b)$$

Definition 4.4 (Valuation in $\mathbf{CML}_n$ and $\mathbf{BML}_n$) Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a connexive (resp. bi-intuitionistic) multilattice. A valuation v is defined as a mapping from $\mathcal{P}$ to S. It is extended into complex formulas as follows: $v(\neg_j A) = -_j v(A)$, $v(A \wedge_j B) = v(A) \cap_j v(B)$, $v(A \vee_j B) = v(A) \cup_j v(B)$, $v(A \rightarrow_j B) = v(A) \supset_j v(B)$, and $v(A \leftarrow_j B) = v(A) \subset_j v(B)$.

Definition 4.5 (Entailment in $\mathbf{CML}_n$ and $\mathbf{BML}_n$) The entailment relation in $\mathbf{CML}_n$ and $\mathbf{BML}_n$ is be defined as follows, for any finite sets of formulas Γ and Δ:

- $\Gamma \models_{\mathbf{CML}_n} \Delta$ iff for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta$.
- $\Gamma \models_{\mathbf{BML}_n} \Delta$ iff for each logical bi-intuitionistic multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta$.

For any finite sets of formulas Γ and Δ, we write $\bigwedge_j \Gamma$ for the j-conjunction of all formulas from Γ and $\bigvee_j \Delta$ for the j-disjunction of all formulas from Δ. If $\bigwedge_j \Gamma = \emptyset$, then $\bigwedge_j \Gamma = p \subset_j p$. If $\bigvee_j \Delta = \emptyset$, then $\bigvee_j \Delta = p \supset_j p$.

Definition 4.6 (Validity of sequents. The case of $\mathbf{CML}_n$ and $\mathbf{BML}_n$) A sequent $\Gamma \Rightarrow \Delta$ is valid in the logic $\mathbf{L} \in \{\mathbf{CML}_n, \mathbf{BML}_n\}$ (symbolically, $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$) iff $\Gamma \models_{\mathbf{L}} \Delta$.

In a similar fashion, modifying the above presented algebraic semantics for $\mathbf{CML}_n$ and $\mathbf{BML}_n$, we can propose an alternative algebraic semantics for the sequent calculi for $\mathbf{ML}_n$ and $\mathbf{MLL}_n$.

Definition 4.7 (Classical multilattice and classical multifilter) Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice with the j-pseudo-inversion operations $-_1, \ldots, -_n$ and $\mathcal{F}_n$ be a De Morgan ultramultifilter. Then for any $j, k \leqslant n$ such that $j \neq k$ the pairs $\langle \supset_1, \subset_1 \rangle, \ldots, \langle \supset_n, \subset_n \rangle$ of binary operations called *pseudo-complement* and *pseudo-difference* operations are defined as follows, where $a, b, c \in S$:

$$a \supset_j b \in \mathcal{F}_n \text{ iff } -_k -_j a \cup_j b \in \mathcal{F}_n; \quad (\supset_j')$$

$$a \subset_j b \in \mathcal{F}_n \text{ iff } a \cap_j -_k -_j b \in \mathcal{F}_n; \quad (\subset_j')$$

$$-_j(a \supset_j b) \in \mathcal{F}_n \text{ iff } -_j b \subset_j -_j a \in \mathcal{F}_n; \quad (-_j{\supset}_j')$$

$$-_j(a \subset_j b) \in \mathcal{F}_n \text{ iff } -_j b \supset_j -_j a \in \mathcal{F}_n; \quad (-_j{\subset}_j')$$

$$-_k(a \supset_j b) \in \mathcal{F}_n \text{ iff } -_k a \supset_j -_k b \in \mathcal{F}_n; \quad (-_k{\supset}_j{}')$$

$$-_k(a \subset_j b) \in \mathcal{F}_n \text{ iff } -_k a \subset_j -_k b \in \mathcal{F}_n. \qquad (-_k\subset_j{}')$$

A multilattice $\mathcal{M}_n$ is called *classical* iff $\mathcal{M}_n$ is a multilattice with the j-pseudo-inversion operations $-_1, \ldots, -_n$ and the pairs $\langle \supset_1, \subset_1 \rangle, \ldots, \langle \supset_n, \subset_n \rangle$ of pseudo-complement and pseudo-difference operations (where $j, k \leqslant n$ and $j \neq k$). A De Morgan ultramultifilter $\mathcal{F}_n$ is called a *classical* ultramultifilter iff it satisfies the above presented conditions. A pair $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ is called *classical* ultralogical multilattice iff $\mathcal{M}_n$ is a classical multilattice and $\mathcal{F}_n$ is a classical ultramultifilter.

Definition 4.8 (Entailment in $\mathbf{ML}_n$ and $\mathbf{MLL}_n$) The entailment relation in $\mathbf{ML}_n$ and $\mathbf{MLL}_n$ can be defined as follows, for any finite sets of formulas Γ and Δ[1]:

- $\Gamma \models^A_{\mathbf{ML}_n} \Delta$ iff for each ultralogical classical multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta$.
- $\Gamma \models^A_{\mathbf{MLL}_n} \Delta$ iff for each logical De Morgan multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta$.

Definition 4.9 (Validity of sequents. The case of $\mathbf{ML}_n$ and $\mathbf{MLL}_n$) A sequent $\Gamma \Rightarrow \Delta$ is valid in the logic $\mathbf{L} \in \{\mathbf{ML}_n, \mathbf{MLL}_n\}$ (symbolically, $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$) iff $\Gamma \models^A_{\mathbf{L}} \Delta$.

5 Soundness and Completeness Proofs

Lemma 5.1 *All the rules of the sequent calculus for* $\mathbf{CML}_n$ *are sound with respect to logical connexive multilattices.*

Proof Consider the rule $(\Rightarrow \neg_j \wedge_j)$. Suppose $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, \neg_j A, \neg_j B$. Thus, for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{\neg_j A, \neg_j B\}$. Assume that $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$. Thus, $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{\neg_j A, \neg_j B\}$. If $v(D) \in \mathcal{F}_n$, for some $D \in \Delta$, then $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, \neg_j(A \wedge_j B)$. If $v(D) \in \mathcal{F}_n$, for some $D \in \{\neg_j A, \neg_j B\}$, that is $v(\neg_j A) \in \mathcal{F}_n$ or $v(\neg_j B) \in \mathcal{F}_n$, then, since $\mathcal{F}_n$ is prime, $v(\neg_j A) \cup_j v(\neg_j B) \in \mathcal{F}_n$. By Definition 4.4, $-_j v(A) \cup_j -_j v(B) \in \mathcal{F}_n$. By (DM1), $-_j(v(A) \cap_j v(B)) \in \mathcal{F}_n$. By Definition 4.4, $v(\neg_j(A \wedge_j B)) \in \mathcal{F}_n$. Therefore, $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, \neg_j(A \wedge_j B)$.

Consider the rule $(\Rightarrow \neg_k \wedge_j)$. Suppose that $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, \neg_k A$ and $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, \neg_k B$. Then for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{\neg_k A\}$ as well as for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{\neg_k B\}$. Assume that $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$. Thus, $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{\neg_k A\}$ as well as $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{\neg_k B\}$. If $v(D) \in \mathcal{F}_n$, for some $D \in \Delta$, then $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, \neg_k(A \wedge_j B)$. If $v(\neg_k A) \in \mathcal{F}_n$ and $v(\neg_k B) \in \mathcal{F}_n$,

[1] (A in $\models^A$ emphasize the fact that we deal with an alternative algebraic semantics in order not to confuse it with the original one, cf. Definitions 2.12 and 2.13)

then, since $\mathcal{F}_n$ is a filter, $v(\neg_k A) \cap_j v(\neg_k B) \in \mathcal{F}_n$. By Definition 4.4 and (DM3), $\neg_k v(A \wedge_j B) \in \mathcal{F}_n$. Thus, $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, \neg_k(A \wedge_j B)$.

The cases of the other rules for $\vee_j$, $\wedge_j$, $\neg_j\vee_j$, $\neg_j\wedge_j$, $\neg_k\vee_j$, and $\neg_k\wedge_j$ are considered similarly with the help of the fact that $\cup_j$ and $\cap_j$ are lattice operations and with the use of (DM1)–(DM4). The rules for $\neg_j\neg_j$ are easily checked due to (per2).

Consider the rule $(\Rightarrow\rightarrow_j)$. Suppose that $\models_{\mathbf{CML}_n} A, \Gamma \Rightarrow B$. Then for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma \cup \{A\}$, then $v(B) \in \mathcal{F}_n$. Since $\mathcal{F}_n$ is a multifilter, we conclude that for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(A \wedge_j \bigwedge_j \Gamma) \in \mathcal{F}_n$, then $v(B) \in \mathcal{F}_n$. Using Definition 4.4 and $(\supset_j)$, we infer that for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(\bigwedge_j \Gamma) \in \mathcal{F}_n$, then $v(A \rightarrow_j B) \in \mathcal{F}_n$. Hence, $\models_{\mathbf{CML}_n} \Gamma \Rightarrow A \rightarrow_j B$.

Consider the rule $(\rightarrow_j\Rightarrow)$. Suppose that $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, A$ and $\models_{\mathbf{CML}_n} B, \Theta \Rightarrow \Lambda$. Then, for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{A\}$; as well as for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Theta \cup \{B\}$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Lambda$. Since $\mathcal{F}_n$ is a prime multifilter, we have that for any valuation v, if $v(\bigwedge_j \Gamma) \in \mathcal{F}_n$, then $v(\bigvee_j \Delta) \in \mathcal{F}_n$ or $v(A) \in \mathcal{F}_n$; if $v(B) \in \mathcal{F}_n$ and $v(\bigwedge_j \Theta) \in \mathcal{F}_n$, then $v(\bigvee_j \Lambda) \in \mathcal{F}_n$. Assume that $v(A \rightarrow_j B), v(\bigwedge_j \Gamma), v(\bigwedge_j \Theta) \in \mathcal{F}_n$. Therefore, $v(\bigvee_j \Delta) \in \mathcal{F}_n$ or $v(A) \in \mathcal{F}_n$. If $v(\bigvee_j \Delta) \in \mathcal{F}_n$, then $\models_{\mathbf{CML}_n} A \rightarrow_j B, \Gamma, \Theta \Rightarrow \Delta, \Lambda$. If $v(A) \in \mathcal{F}_n$, then $v(B) \in \mathcal{F}_n$, since $v(A \rightarrow_j B) \in \mathcal{F}_n$ (this is a consequence of $(\supset_j)$). Since $v(B) \in \mathcal{F}_n$ and $v(\bigwedge_j \Theta) \in \mathcal{F}_n$, then $v(\bigvee_j \Lambda) \in \mathcal{F}_n$. Thus, $\models_{\mathbf{CML}_n} A \rightarrow_j B, \Gamma, \Theta \Rightarrow \Delta, \Lambda$.

Consider the rule $(\neg_j\leftarrow^c_j \Rightarrow)$. Suppose that $\models_{\mathbf{CML}_n} \neg_j A \Rightarrow \Delta, B$. Then, for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(\neg_j A) \in \mathcal{F}_n$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{B\}$. Since $\mathcal{F}_n$ is a prime multifilter, $v(\neg_j A) \in \mathcal{F}_n$ implies $v(\bigvee_j \Delta) \cup_j v(B) \in \mathcal{F}_n$. By Definition 4.4 and $(-_j\subset^c_j)$, $v(\neg_j(A \leftarrow_j B)) \in \mathcal{F}_n$ implies $v(\bigvee_j \Delta) \in \mathcal{F}_n$. Hence, $\neg_j(A \leftarrow_j B) \models_{\mathbf{CML}_n} \Delta$.

Consider the rule $(\Rightarrow \neg_j\leftarrow^c_j)$. Suppose that $\models_{\mathbf{CML}_n} \Gamma \Rightarrow \Delta, \neg_j A$ and $\models_{\mathbf{CML}_n} B, \Theta \Rightarrow \Lambda$. Then, for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Delta \cup \{\neg_j A\}$; as well as for each logical connexive multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{F}_n$, for each $C \in \Theta \cup \{B\}$, then $v(D) \in \mathcal{F}_n$, for some $D \in \Lambda$. Suppose that $v(\bigwedge_j \Gamma), v(\bigwedge_j \Theta) \in \mathcal{F}_n$. Then $v(\bigvee_j \Delta) \in \mathcal{F}_n$ or $v(\neg_j A) \in \mathcal{F}_n$. If $v(\bigvee_j \Delta) \in \mathcal{F}_n$, then $\models_{\mathbf{CML}_n} \Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j(A \leftarrow_j B)$. If $v(\neg_j A) \in \mathcal{F}_n$, then $v(\neg_j(A \leftarrow_j B)) \in \mathcal{F}_n$ or $v(B) \in \mathcal{F}_n$ (this is a consequence of $(-_j\supset^c_j)$). If $v(\neg_j(A \leftarrow_j B)) \in \mathcal{F}_n$, then $\models_{\mathbf{CML}_n} \Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j(A \leftarrow_j B)$. If $v(B) \in \mathcal{F}_n$, then $v(\bigvee_j \Lambda) \in \mathcal{F}_n$, since $v(\bigwedge_j \Theta) \in \mathcal{F}_n$; $v(\bigvee_j \Lambda) \in \mathcal{F}_n$ implies $\models_{\mathbf{CML}_n} \Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j(A \leftarrow_j B)$.

The rules for $\leftarrow_j$, $\neg_j\rightarrow_j$, $\neg_k\rightarrow_k$, $\neg_k\leftarrow_k$, $\neg_j\neg_k$, and $\neg_k\neg_j$ are considered similarly, use the conditions $(\subset_j)$, $(-_j\supset^c_j)$, $(-_k\supset_j)$, $(-_k\subset_j)$, $(-_j-_k)$, and $(-_k-_j)$, respectively.

□

Lemma 5.2 *All the rules of the sequent calculus for* $\mathbf{BML}_n$ *are sound with respect to logical bi-intuitionistic multilattices.*

Proof For most of the rules the proof is the same as in Lemma 5.1, for the rules for $\neg_j\!\rightarrow_j$ and $\neg_j\!\leftarrow_j$ use the conditions ($-_j\supset^b_j$) and ($-_j\subset^b_j$). □

Theorem 5.3 (Soundness) *Let* $\mathbf{L} \in \{\mathbf{BML}_n, \mathbf{CML}_n\}$. *For every pair of finite sets of formulas* Γ *and* Δ, *it holds that if* $\mathbf{L} \vdash \Gamma \Rightarrow \Delta$, *then* $\Gamma \models_{\mathbf{L}} \Delta$.

Proof By the fact that both axioms of $\mathbf{L}$ are valid, and the induction on the length of derivation with the help of Lemmas 5.1 and 5.2. □

Definition 5.4 (Class of equivalence) The class of equivalence $[A]$ of a formula A is the set of formulas $\{B \mid \mathbf{L} \vdash A \Rightarrow B$ and $\mathbf{L} \vdash B \Rightarrow A$ and $\mathbf{L} \vdash \neg_j A \Rightarrow \neg_j B$ and $\mathbf{L} \vdash \neg_j B \Rightarrow \neg_j A\}$, for any $j \leqslant n$, where $\mathbf{L} \in \{\mathbf{BML}_n, \mathbf{CML}_n\}$. The class of equivalence $[\Gamma]$ of a set of formulas Γ is the set $\{[C] \mid C \in \Gamma\}$.

Definition 5.5 (Lindenbaum-Tarski algebra) A Lindenbaum-Tarski algebra (LT-algebra) is a structure $\mathcal{M}^{\mathbf{L}}_n = \langle [\mathscr{F}], \leqslant_1, \ldots, \leqslant_n \rangle$, where $\mathbf{L} \in \{\mathbf{BML}_n, \mathbf{CML}_n\}$ and $\mathscr{F}$ is the set of all formulas, which satisfies the following conditions, for any formulas $A, B \in \mathscr{F}$:

$$[A] \leqslant_j [B] \text{ iff } [A] = [A \wedge_j B];$$
$$-_j[A] = [\neg_j A];$$
$$[A] \cap_j [B] = [A \wedge_j B];$$
$$[A] \cup_j [B] = [A \vee_j B];$$
$$[A] \supset_j [B] = [A \rightarrow_j B];$$
$$[A] \subset_j [B] = [A \leftarrow_j B].$$

Fact 5.6 Let $\mathbf{L} \in \{\mathbf{BML}_n, \mathbf{CML}_n\}$. For any formulas A and B, any $j \leqslant n$, it holds that

- $\mathbf{L} \vdash A \Rightarrow B$, $\mathbf{L} \vdash B \Rightarrow A$, $\mathbf{L} \vdash \neg_j A \Rightarrow \neg_j B$, and $\mathbf{L} \vdash \neg_j B \Rightarrow \neg_j A$ iff $[A] = [B]$;
- $\mathbf{L} \vdash A \Rightarrow B$ and $\mathbf{L} \vdash \neg_j B \Rightarrow \neg_j A$ iff $[A] \leqslant_j [B]$.

Lemma 5.7 *The following sequents are provable in* $\mathbf{L} \in \{\mathbf{CML}_n, \mathbf{BML}_n\}$, *where* $j, k \leqslant n$ *and in* (10) *and* (11) *we suppose that* $j < k$:

(1) $A \wedge_j A \Rightarrow A$; $A \Rightarrow A \wedge_j A$; $\neg_k(A \wedge_j A) \Rightarrow \neg_k A$; $\neg_k A \Rightarrow \neg_k(A \wedge_j A)$;
(2) $A \vee_j A \Rightarrow A$; $A \Rightarrow A \vee_j A$; $\neg_k(A \vee_j A) \Rightarrow \neg_k A$; $\neg_k A \Rightarrow \neg_k(A \vee_j A)$;
(3) $A \wedge_j B \Rightarrow B \wedge_j A$; $B \wedge_j A \Rightarrow A \wedge_j B$; $\neg_k(A \wedge_j B) \Rightarrow \neg_k(B \wedge_j A)$; $\neg_k(B \wedge_j A) \Rightarrow \neg_k(A \wedge_j B)$;
(4) $A \vee_j B \Rightarrow B \vee_j A$; $B \vee_j A \Rightarrow A \vee_j B$; $\neg_k(A \vee_j B) \Rightarrow \neg_k(B \vee_j A)$; $\neg_k(B \vee_j A) \Rightarrow \neg_k(A \vee_j B)$;
(5) $A \wedge_j (B \wedge_j C) \Rightarrow (A \wedge_j B) \wedge_j C$; $(A \wedge_j B) \wedge_j C \Rightarrow A \wedge_j (B \wedge_j C)$; $\neg_k(A \wedge_j (B \wedge_j C)) \Rightarrow \neg_k((A \wedge_j B) \wedge_j C)$; $\neg_k((A \wedge_j B) \wedge_j C) \Rightarrow \neg_k(A \wedge_j (B \wedge_j C))$;
(6) $A \vee_j (B \vee_j C) \Rightarrow (A \vee_j B) \vee_j C$; $(A \vee_j B) \vee_j C \Rightarrow A \vee_j (B \vee_j C)$; $\neg_k(A \vee_j (B \vee_j C)) \Rightarrow \neg_k((A \vee_j B) \vee_j C)$; $\neg_k((A \vee_j B) \vee_j C) \Rightarrow \neg_k(A \vee_j (B \vee_j C))$;
(7) $A \wedge_j (A \vee_j B) \Rightarrow A$; $A \Rightarrow A \wedge_j (A \vee_j B)$; $\neg_k(A \wedge_j (A \vee_j B)) \Rightarrow \neg_k A$; $\neg_k A \Rightarrow \neg_k(A \wedge_j (A \vee_j B))$;

(8) $A \vee_j (A \wedge_j B) \Rightarrow A$; $A \Rightarrow A \vee_j (A \wedge_j B)$; $\neg_k(A \vee_j (A \wedge_j B)) \Rightarrow \neg_k A$; $\neg_k A \Rightarrow \neg_k(A \vee_j (A \wedge_j B))$;

(9) $A \otimes (B \oplus C) \Rightarrow (A \otimes B) \oplus (A \otimes C)$; $(A \otimes B) \oplus (A \otimes C) \Rightarrow A \otimes (B \oplus C)$; $\neg_k(A \otimes (B \oplus C)) \Rightarrow \neg_k((A \otimes B) \oplus (A \otimes C))$; $\neg_k((A \otimes B) \oplus (A \otimes C)) \Rightarrow \neg_k(A \otimes (B \oplus C))$, *where* $\otimes, \oplus \in \{\wedge_1, \vee_1, \ldots, \wedge_n, \vee_n\}$ *and* $\otimes \neq \oplus$;

(10) $\neg_j \neg_k A \Rightarrow A \rightarrow_k (A \leftarrow_k A)$; $A \rightarrow_k (A \leftarrow_k A) \Rightarrow \neg_j \neg_k A$;

(11) $\neg_k \neg_j A \Rightarrow (A \rightarrow_k A) \leftarrow_k A$; $(A \rightarrow_k A) \leftarrow_k A \Rightarrow \neg_k \neg_j A$.

Proof We prove the case (10).

$$\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A}{A \leftarrow_k A \Rightarrow}\;(\leftarrow_k\Rightarrow)}{A,\, A \rightarrow_k (A \leftarrow_k A) \Rightarrow}\;(\rightarrow_k\Rightarrow)}{A \rightarrow_k (A \leftarrow_k A) \Rightarrow \neg_j \neg_k A}\;(\Rightarrow\neg_j\neg_k) \qquad \dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A}{\neg_j \neg_k A,\, A \Rightarrow}\;(\neg_j\neg_k\Rightarrow)}{\neg_j \neg_k A,\, A \Rightarrow A \leftarrow_k A}\;(\Rightarrow\leftarrow_k)}{\neg_j \neg_k A \Rightarrow A \rightarrow_k (A \leftarrow_k A)}\;(\Rightarrow\rightarrow_k)$$

We prove the case (11).

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad A \Rightarrow A}{A,\, A \rightarrow_k A \Rightarrow A}\;(\rightarrow_k\Rightarrow)}{A,\, (A \rightarrow_k A) \leftarrow_k A \Rightarrow}\;(\leftarrow_k\Rightarrow)}{(A \rightarrow_k A) \leftarrow_k A \Rightarrow \neg_k \neg_j A}\;(\Rightarrow\neg_k\neg_j) \qquad (\neg_k\neg_j \Rightarrow)\;\dfrac{(\Rightarrow\leftarrow_k)\;\dfrac{(\Rightarrow\rightarrow_k)\;\dfrac{A \Rightarrow A}{\Rightarrow A \rightarrow_k A} \qquad A \Rightarrow A}{\Rightarrow (A \rightarrow_k A) \leftarrow_k A,\, A}}{\neg_k \neg_j A \Rightarrow (A \rightarrow_k A) \leftarrow_k A}$$

The other cases are proved similarly. □

Lemma 5.8 *Let* $\widetilde{v}$ *be a valuation introduced in Definition 4.4 such that* $\widetilde{v}(P) = [P]$, *for all* $P \in \mathcal{P}$ *(such a valuation is said to be a canonic one). Then* $\widetilde{v}(A) = [A]$, *for any formula* A.

Proof By a structural induction on a formula A. Use Definition 5.5. □

Lemma 5.9 (Lindenbaum lemma for $\mathbf{CML}_n$) *Let* $\mathbf{L}$ *be* $\mathbf{CML}_n$. *For every pair of finite sets of formulas* Γ *and* Δ, *it holds that* $\nvdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$ *implies that there is a connexive multifilter* $\mathcal{F}_n^{\mathbf{L}}$ *on the Lindenbaum-Tarski algebra* $\mathcal{M}_n^{\mathbf{L}}$ *and* $[C] \in \mathcal{F}_n^{\mathbf{L}}$, *for each* $C \in \Gamma$, *while* $[D] \notin \mathcal{F}_n^{\mathbf{L}}$, *for each* $D \in \Delta$.

Proof We follow the standard strategy of the proof of the Lindenbaum lemma which was adopted for the case of multilattice logic $\mathbf{ML}_n$ in (Lemma 4.12, Grigoriev and Petrukhin 2019b).

Suppose that $\nvdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$. Let $F_1, \ldots, F_m, \ldots$ be an enumeration of the set of all formulas. We postulate the following identities:

$$\Omega_1 = \Gamma$$

$$\Omega_{i+1} = \begin{cases} \Omega_i \cup \{F_{i+1}\}, & \text{if} \nvdash_{\mathbf{L}} \Omega_i, F_{i+1} \Rightarrow; \\ \Omega_i & \text{otherwise;} \end{cases}$$

$$\Sigma = \bigcup_{i=1}^{\infty} \Omega_i.$$

By Definition 5.4, we have $[\Sigma] = \{[B] \mid B \in \Sigma\}$. We need to show that $[\Sigma]$ is the required connexive multifilter on $\mathcal{M}_n^{\mathbf{L}}$.

By the induction on i, one may easily prove that ($\circledast$) for each i, it holds that $\nvdash_{\mathbf{L}} \Omega_i \Rightarrow \Sigma$. Moreover, $\nvdash_{\mathbf{L}} \Omega_i \Rightarrow$, otherwise, by ($\Rightarrow$W), $\vdash_{\mathbf{L}} \Omega_i \Rightarrow \Sigma$. It is easy to justify that $[\Gamma] \subseteq [\Sigma]$, i.e. $[C] \in [\Sigma]$ (for each $C \in \Gamma$), and $[D] \notin [\Sigma]$ (for each $D \in \Delta$).

Let us show that $[\Sigma]$ *satisfies condition* (filter). Suppose that $[A], [B] \in [\Sigma]$. Then $A, B \in \Sigma$ and there are l and m such that $\vdash_{\mathbf{L}} \Omega_l \Rightarrow A$ and $\vdash_{\mathbf{L}} \Omega_m \Rightarrow B$. Assume that $[A] \cap_j [B] \notin [\Sigma]$. Then, by Definition 5.5, $[A \wedge_j B] \notin [\Sigma]$ which yields $A \wedge_j B \notin \Sigma$. Then there is i such that $A \wedge_j B = F_{i+1}$ and $\vdash_{\mathbf{L}} \Omega_i, F_{i+1} \Rightarrow$. We have (double lines indicate multiple applications of a rule):

$$\cfrac{\cfrac{\cfrac{\Omega_l \Rightarrow A}{\Omega_l, \Omega_m \Rightarrow A}\,(\mathrm{W}\Rightarrow) \qquad \cfrac{\Omega_m \Rightarrow B}{\Omega_l, \Omega_m \Rightarrow B}\,(\mathrm{W}\Rightarrow)}{\Omega_l, \Omega_m \Rightarrow A \wedge_j B}\,(\Rightarrow \wedge_j) \qquad \Omega_i, A \wedge_j B \Rightarrow}{\Omega_l, \Omega_m, \Omega_i \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⁎). Thus, $[A] \cap_j [B] \in [\Sigma]$.

Suppose that $[A] \cap_j [B] \in [\Sigma]$, but $[A] \notin [\Sigma]$ or $[B] \notin [\Sigma]$. Then $\vdash_{\mathbf{L}} \Omega_l \Rightarrow A \wedge_j B$, for some l. Assume that $[A] \notin [\Sigma]$. Then $A = F_{i+1}$ and $\Omega_i, F_{i+1} \Rightarrow$, for some i. We have:

$$\cfrac{\Omega_l \Rightarrow A \wedge_j B \qquad \cfrac{\cfrac{\Omega_i, A \Rightarrow}{\Omega_i, A, B \Rightarrow}\,(\mathrm{W}\Rightarrow)}{\Omega_i, A \wedge_j B \Rightarrow}\,(\Rightarrow \wedge_j)}{\Omega_l, \Omega_i \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⁎). Hence, $[A] \in [\Sigma]$. The case $[B] \notin [\Sigma]$ is treated similarly. Therefore, $[\Sigma]$ is a multifilter.

Let us show that $[\Sigma]$ *satisfies condition* (prime). Assume that $[A] \cup_j [B] \in [\Sigma]$ while $[A], [B] \notin [\Sigma]$. We have $\vdash_{\mathbf{L}} \Omega_i \Rightarrow A \vee_j B$ as well as $\vdash_{\mathbf{L}} \Omega_l, A \Rightarrow$ and $\vdash_{\mathbf{L}} \Omega_m, B \Rightarrow$, for some i, l, and m. Thus,

$$\cfrac{\Omega_i \Rightarrow A \vee_j B \qquad \cfrac{\cfrac{\Omega_l, A \Rightarrow}{\Omega_l, \Omega_m, A \Rightarrow}\,(\mathrm{W}\Rightarrow) \qquad \cfrac{\Omega_m, B \Rightarrow}{\Omega_l, \Omega_m, B \Rightarrow}\,(\mathrm{W}\Rightarrow)}{\Omega_l, \Omega_m, A \vee_j B \Rightarrow}\,(\Rightarrow \vee_j)}{\Omega_i, \Omega_l, \Omega_m, \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⁎). Hence, $[A] \in [\Sigma]$ or $[B] \in [\Sigma]$.

Suppose that $[A] \in [\Sigma]$ while $[A] \cup_j [B] \notin [\Sigma]$. Then we have:

$$\cfrac{\cfrac{\cfrac{\Omega_i \Rightarrow A}{\Omega_i \Rightarrow A, B}\,(\Rightarrow \mathrm{W})}{\Omega_i \Rightarrow A \vee_j B}\,(\Rightarrow \vee_j) \qquad \Omega_l, A \vee_j B \Rightarrow}{\Omega_i, \Omega_l \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⁎). The case when $[B] \in [\Sigma]$ is treated similarly. Therefore, $[\Sigma]$ is a prime multifilter.

Let us show that $[\Sigma]$ *satisfies condition* (DM1). Assume that $-_j([A] \cap_j [B]) \in [\Sigma]$, while $-_j[A] \cup_j -_j[B] \notin [\Sigma]$. Since we already know that $[\Sigma]$ is a prime multifilter, $-_j[A] \notin [\Sigma]$ and $-_j[B] \notin [\Sigma]$. We have $\vdash_{\mathbf{L}} \Omega_i \Rightarrow \neg_j(A \wedge_j B)$ as well as $\vdash_{\mathbf{L}} \Omega_l, \neg_j A \Rightarrow$ and $\vdash_{\mathbf{L}} \Omega_m, \neg_j B \Rightarrow$, for some i, l, and m. Thus,

$$\dfrac{\Omega_i \Rightarrow \neg_j(A \wedge_j B) \qquad \dfrac{\dfrac{\Omega_l, \neg_j A \Rightarrow}{\Omega_l, \Omega_m, \neg_j A \Rightarrow}\,(\mathrm{W}\Rightarrow) \qquad \dfrac{\Omega_m, \neg_j B \Rightarrow}{\Omega_l, \Omega_m, \neg_j B \Rightarrow}\,(\mathrm{W}\Rightarrow)}{\Omega_l, \Omega_m, \neg_j(A \wedge_j B) \Rightarrow}\,(\neg_j\wedge_j \Rightarrow)}{\Omega_i, \Omega_l, \Omega_m, \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⋇). Hence, $-_j[A] \in [\Sigma]$ or $-_j[B] \in [\Sigma]$.

Assume that $-_j[A] \cup_j -_j[B] \in [\Sigma]$, while $-_j([A] \cap_j [B]) \notin [\Sigma]$. Since $[\Sigma]$ is a prime multifilter, $-_j[A] \in [\Sigma]$ or $-_j[B] \in [\Sigma]$. Hence, $\vdash_{\mathbf{L}} \neg_j(A \wedge_j B), \Omega_i \Rightarrow$ as well as $\vdash_{\mathbf{L}} \Omega_l \Rightarrow \neg_j A$ or $\vdash_{\mathbf{L}} \Omega_m \Rightarrow \neg_j B$, for some i, l, and m. Suppose that $\vdash_{\mathbf{L}} \Omega_l \Rightarrow \neg_j A$. Thus,

$$\dfrac{\dfrac{\dfrac{\Omega_l \Rightarrow \neg_j A}{\Omega_l \Rightarrow \neg_j A, \neg_j B}\,(\Rightarrow \mathrm{W})}{\Omega_l \Rightarrow \neg_j(A \wedge_j B)}\,(\Rightarrow \neg_j\wedge_j) \qquad \Omega_i, \neg_j(A \wedge_j B) \Rightarrow}{\Omega_i, \Omega_l \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⋇). The case when $-_j[B] \in [\Sigma]$ is treated similarly. Therefore, $[\Sigma]$ satisfies condition (DM1).

By a similar reasoning, one can show that $[\Sigma]$ satisfies conditions (DM2)–(DM4) as well.

Let us show that $[\Sigma]$ *satisfies condition* (per2). Assume that $-_j -_j [A] \in [\Sigma]$, while $[A] \notin [\Sigma]$. We have $\vdash_{\mathbf{L}} \Omega_i \Rightarrow \neg_j\neg_j A$ as well as $\vdash_{\mathbf{L}} \Omega_l, A \Rightarrow$, for some i and l. Thus,

$$\dfrac{\Omega_i \Rightarrow \neg_j\neg_j A \qquad \dfrac{\Omega_l, A \Rightarrow}{\Omega_l, \neg_j\neg_j A \Rightarrow}\,(\neg_j\neg_j \Rightarrow)}{\Omega_i, \Omega_l, \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⋇). Hence, $[A] \in [\Sigma]$.

Assume that $[A] \in [\Sigma]$, while $-_j -_j [A] \notin [\Sigma]$. Hence, $\vdash_{\mathbf{L}} \neg_j\neg_j A, \Omega_i \Rightarrow$ as well as $\vdash_{\mathbf{L}} \Omega_l \Rightarrow A$, for some i and l. Thus,

$$\dfrac{\dfrac{\Omega_l \Rightarrow A}{\Omega_l \Rightarrow \neg_j\neg_j A}\,(\Rightarrow \neg_j\neg_j) \qquad \Omega_i, \neg_j\neg_j A \Rightarrow}{\Omega_i, \Omega_l \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⋇). Hence, $-_j-_j[A] \in [\Sigma]$. Thus, $[\Sigma]$ satisfies condition (per2).

Let us show that $[\Sigma]$ *satisfies condition* ($-_j-_k$). Assume that $-_j -_k [A] \in [\Sigma]$, while $[A] \supset_k ([A] \subset_k [A]) \notin [\Sigma]$. We have $\vdash_{\mathbf{L}} \Omega_i \Rightarrow \neg_j\neg_k A$ as well as $\vdash_{\mathbf{L}} \Omega_l, A \rightarrow_k (A \leftarrow_k A) \Rightarrow$, for some i and l. Recall that, by Lemma 5.7, $\vdash_{\mathbf{L}} \neg_j\neg_k A \Rightarrow A \rightarrow_k (A \leftarrow_k A)$. Thus,

$$\dfrac{\dfrac{\Omega_i \Rightarrow \neg_j\neg_k A \qquad \neg_j\neg_k A \Rightarrow A \rightarrow_k (A \leftarrow_k A)}{\Omega_i \Rightarrow A \rightarrow_k (A \leftarrow_k A)}\,(\mathrm{Cut}) \qquad \Omega_l, A \rightarrow_k (A \leftarrow_k A) \Rightarrow}{\Omega_i, \Omega_l, \Rightarrow}\,(\mathrm{Cut})$$

It contradicts the fact (⋇). Hence, $[A] \supset_k ([A] \subset_k [A]) \in [\Sigma]$.

Assume that $[A] \supset_k ([A] \subset_k [A]) \in [\Sigma]$, while $-_j -_k [A] \notin [\Sigma]$. Hence, $\vdash_{\mathbf{L}} \neg_j\neg_k A, \Omega_i \Rightarrow$ as well as $\vdash_{\mathbf{L}} \Omega_l \Rightarrow A \to_k (A \leftarrow_k A)$, for some i and l. Recall that, by Lemma 5.7, $\vdash_{\mathbf{L}} A \to_k (A \leftarrow_k A) \Rightarrow \neg_j\neg_k A$. Thus,

$$\dfrac{\dfrac{\Omega_l \Rightarrow A \to_k (A \leftarrow_k A) \qquad A \to_k (A \leftarrow_k A) \Rightarrow \neg_j\neg_k A}{\Omega_l \Rightarrow \neg_j\neg_k A}\,(\text{Cut}) \qquad \Omega_i, \neg_j\neg_k A \Rightarrow}{\Omega_i, \Omega_l \Rightarrow}\,(\text{Cut})$$

It contradicts the fact (⋇). Hence, $-_j-_k[A] \in [\Sigma]$. Thus, $[\Sigma]$ satisfies condition $(-_j-_k)$.

Similarly, one can show that $[\Sigma]$ satisfies condition $(-_k-_j)$.

Let us show that $[\Sigma]$ *satisfies condition* $(\supset_j)$. Assume that $C \notin [\Sigma]$ or $[A] \supset_j [B] \in [\Sigma]$. Suppose that $[A] \cap_j [C] \in [\Sigma]$, while $[B] \notin [\Sigma]$. Since we already know that $[\Sigma]$ is a multifilter, $[A] \in [\Sigma]$ and $[C] \in [\Sigma]$. We have $\vdash_{\mathbf{L}} \Omega_i, B \Rightarrow$ as well as $\vdash_{\mathbf{L}} \Omega_l \Rightarrow A$ and $\vdash_{\mathbf{L}} \Omega_m \Rightarrow C$, for some i, l, and m. Suppose that $C \notin [\Sigma]$. Then $\vdash_{\mathbf{L}} \Omega_o, C \Rightarrow$, for some o. Thus,

$$\dfrac{\Omega_m \Rightarrow C \qquad \Omega_o, C \Rightarrow}{\Omega_m, \Omega_o \Rightarrow}\,(\text{Cut})$$

It contradicts the fact (⋇). Assume that $[A] \supset_j [B] \in [\Sigma]$. Then $\vdash_{\mathbf{L}} \Omega_t \Rightarrow A \to_j B$, for some t. Thus,

$$\dfrac{\Omega_t \Rightarrow A \to_j B \qquad \dfrac{\Omega_l, \Rightarrow A \qquad \Omega_i, B \Rightarrow}{\Omega_l, \Omega_i, A \to_j B \Rightarrow}\,(\Rightarrow\to_j)}{\Omega_t, \Omega_l, \Omega_i, \Rightarrow}\,(\text{Cut})$$

It contradicts the fact (⋇). Hence, $[B] \in [\Sigma]$. Consequently, $[A] \cap_j [C] \in [\Sigma]$ implies $[B] \in [\Sigma]$. Therefore, if $C \notin [\Sigma]$ or $[A] \supset_j [B] \in [\Sigma]$, then $[A] \cap_j [C] \in [\Sigma]$ implies $[B] \in [\Sigma]$.

Assume that $[A] \cap_j [C] \notin [\Sigma]$ or $[B] \in [\Sigma]$. Since $[\Sigma]$ is a multifilter, $[A] \notin [\Sigma]$ or $[C] \notin [\Sigma]$. Suppose that $[C] \in [\Sigma]$, while $[A] \supset_j [B] \notin [\Sigma]$. We have $\vdash_{\mathbf{L}} \Omega_l \Rightarrow C$ and $\vdash_{\mathbf{L}} \Omega_m, A \to_j B \Rightarrow$, for some l, and m. Suppose that $[A] \notin [\Sigma]$. Then $\vdash_{\mathbf{L}} \Omega_i, A \Rightarrow$, for some i. Thus,

$$\dfrac{\dfrac{\dfrac{\Omega_i, A \Rightarrow}{\Omega_i, A \Rightarrow B}\,(\Rightarrow \text{W})}{\Omega_i \Rightarrow A \to_j B}\,(\Rightarrow\to_j) \qquad \Omega_m, A \to_j B \Rightarrow}{\Omega_i, \Omega_m \Rightarrow}\,(\text{Cut})$$

It contradicts the fact (⋇). Suppose that $[C] \notin [\Sigma]$. Then $\vdash_{\mathbf{L}} \Omega_t, C \Rightarrow$, for some t. Thus,

$$\dfrac{\Omega_l \Rightarrow C \qquad \Omega_t, C \Rightarrow}{\Omega_l, \Omega_t \Rightarrow}\,(\text{Cut})$$

Consequently, $[A] \supset_j [B] \in [\Sigma]$. Hence, if $[A] \cap_j [C] \notin [\Sigma]$ or $[B] \in [\Sigma]$, then $[C] \in [\Sigma]$ implies $[A] \supset_j [B] \in [\Sigma]$. Therefore, $[\Sigma]$ satisfies condition ($\supset_j$).

The cases regarding conditions ($\subset_j$), ($-_k\supset_j$), ($-_k\subset_j$), ($-_j\supset^c_j$), and ($-_j\subset^c_j$) are considered similarly. Therefore, $[\Sigma]$ is a connexive multifilter. □

Lemma 5.10 (Lindenbaum lemma for $\mathbf{BML}_n$) *Let* $\mathbf{L}$ *be* $\mathbf{BML}_n$. *For every pair of finite sets of formulas* Γ *and* Δ, *it holds that* $\nvdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$ *implies that there is a bi-intuitionistic multifilter* $\mathcal{F}^{\mathbf{L}}_n$ *on the Lindenbaum-Tarski algebra* $\mathcal{M}^{\mathbf{L}}_n$ *and* $[C] \in \mathcal{F}^{\mathbf{L}}_n$, *for each* $C \in \Gamma$, *while* $[D] \notin \mathcal{F}^{\mathbf{L}}_n$, *for each* $D \in \Delta$.

Proof Similarly to Lemma 5.9. □

Lemma 5.11 $\langle \mathcal{M}^{\mathbf{CML}}_n, \mathcal{F}^{\mathbf{CML}}_n \rangle$, *where* $\mathcal{F}^{\mathbf{CML}}_n$ *is a connexive multifilter constructed in Lemma* 5.9, *is a connexive logical multilattice.*

Proof Due to Lemmas 5.7 and 5.9 operations $-_j, \cap_j, \cup_j, \supset_j$, and $\subset_j$ on $[\mathscr{F}]$ satisfy the conditions listed in Definition 4.2. To be more exact, the correspondence between the properties required by the definition and the provable sequents from the lemmas is as follows: the condition that $\langle \cap_1, \cup_1 \rangle, \ldots, \langle \cap_n, \cup_n \rangle$ are pairs of lattice meet and join operations satisfying distributivity is justified by the provability of (1)–(9) (Lemma 5.7). The conditions regarding both the behaviour of the connective and the properties of a multifilter (that is (filter), (prime), (DM1)–(DM4), (per2), ($\supset_j$), ($\subset_j$), ($-_k\supset_j$), ($-_k\subset_j$), ($-_j-_k$), ($-_k-_j$), ($-_j\supset^c_j$), and ($-_j\subset^c_j$)) are justified by Lemma 5.9 and in the case of ($-_j-_k$) and ($-_k-_j$) by Lemma 5.7 as well. □

Lemma 5.12 $\langle \mathcal{M}^{\mathbf{BML}}_n, \mathcal{F}^{\mathbf{BML}}_n \rangle$, *where* $\mathcal{F}^{\mathbf{BML}}_n$ *is a bi-intuitionistic multifilter constructed in Lemma* 5.10, *is a bi-intuitionistic logical multilattice.*

Proof Follows from Lemmas 5.7 and 5.10. □

Theorem 5.13 (Soundness and completeness) *Let* $\mathbf{L} \in \{\mathbf{BML}_n, \mathbf{CML}_n\}$. *For every pair of finite sets of formulas* Γ *and* Δ, *it holds that* $\Gamma \models_{\mathbf{L}} \Delta$ *iff* $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$.

Proof The equivalence $\Gamma \models_{\mathbf{L}} \Delta$ iff $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$ holds due to Definition 4.6. As for the equivalence $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$, its soundness part is justified by Theorem 5.3. As for the completeness part, assume that $\mathbf{L} \nvdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$. By Lemma 5.9, there is a connexive (resp. bi-intuitionistic) multifilter $\mathcal{F}_n$ on $\mathcal{M}^{\mathbf{L}}_n$ such that $[C] \in \mathcal{F}_n$, for all $C \in \Gamma$, and $[D] \notin \mathcal{F}_n$, for all $D \in \Delta$. By Lemma 5.11, if $\mathbf{L} = \mathbf{CML}_n$, then $\langle \mathcal{M}^{\mathbf{L}}_n, \mathcal{F}^{\mathbf{L}}_n \rangle$ is a connexive logical multilattice. By Lemma 5.12, if $\mathbf{L} = \mathbf{BML}_n$, then $\langle \mathcal{M}^{\mathbf{L}}_n, \mathcal{F}^{\mathbf{L}}_n \rangle$ is a bi-intuitionistic logical multilattice. By Lemma 5.8, there is a canonic valuation $\widetilde{v}$ such that $\widetilde{v}(C) \in \mathcal{F}_n$, for all $C \in \Gamma$, and $\widetilde{v}(D) \notin \mathcal{F}_n$, for all $D \in \Delta$, i.e. $\mathbf{L} \not\models \Gamma \Rightarrow \Delta$. □

Theorem 5.14 (Soundness and completeness) *Let* $\mathbf{L} \in \{\mathbf{ML}_n, \mathbf{MLL}_n\}$. *For every pair of finite sets of formulas* Γ *and* Δ, *it holds that* $\Gamma \models^A_{\mathbf{L}} \Delta$ *iff* $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$.

Proof Similarly to Theorem 5.13. □

6 Conclusion

We offered the algebraic semantics for connexive and bi-intuitionistic multilattice logics previously being formulated only with the help of sequent calculi and Kripke semantics. As for topics for future research, we leave an investigation of modal extensions of $\mathbf{CML}_n$ and $\mathbf{BML}_n$ by Tarski, Kuratowski, and Halmos closure and interior operators (see (Grigoriev & Petrukhin, 2021) for a systematic study of the extensions of $\mathbf{ML}_n$ by these operators). Yet another topic is the study of congruent and monotonic modal multilattice logics on the basis of $\mathbf{CML}_n$ and $\mathbf{BML}_n$ (such modal logics on the basis of $\mathbf{ML}_n$ and $\mathbf{MLL}_n$ were explored in Grigoriev and Petrukhin (2022)).

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Declarations

Conflict of interest No potential Conflict of interest was reported by the author.

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References

Arieli, O., & Avron, A. (1996). Reasoning with logical bilattices. *Journal of Logic, Language and Information, 5*, 25–63.

Belnap, N. D. (1977). A useful four-valued logic. In: Dunn, J. M., Epstein, G. (Eds.), *Modern uses of multiple-valued logic* (pp. 7–37). Reidel Publishing Company.

Belnap, N. D. (1977). How a computer should think. In: Rule, G. (Ed.), *Contemporary aspects of philosophy* (pp. 30–56). Oriel Press.

Dunn, J. M. (1976). Intuitive semantics for first-degree entailment and coupled trees. *Philosophical Studies, 29*, 149–168.

Dunn, J. M., & Restall, G. (2002). *Relevance logic. Handbook of philosophical logic*. New York: Springer.

Grigoriev, O., & Petrukhin, Y. (2022). Basic modal congruent and monotonic multilattice logics. *Journal of Logic and Computation* (online first article).

Grigoriev, O., & Petrukhin, Y. (2019). On a multilattice analogue of a hypersequent S5 calculus. *Logic and Logical Philosophy, 28*(4), 683–730.

Grigoriev, O., & Petrukhin, Y. (2019). Two proofs of the algebraic completeness theorem for multilattice logic. *Journal of Applied Non-Classical Logics, 29*(4), 358–381.

Grigoriev, O., & Petrukhin, Y. (2021). Modal multilattice logics with Tarski, Kuratowski, and Halmos operators. *Logic and Logical Philosophy, 30*(3), 385–415.

Kamide, N. (2019). A substructural view of multilattice logic. *Journal of Multiple-Valued Logic and Soft Computing, 33*(3), 239–270.

Kamide, N. (2021). Alternative multilattice logics: An approach based on monosequent and indexed monosequent calculi. *Studia Logica, 109*(6), 1241–1271.
Kamide, N., & Shramko, Y. (2017). Modal multilattice logic. *Logica Universalis, 11*(3), 317–343.
Kamide, N., & Shramko, Y. (2017). Embedding from multilattice logic into classical logic and vice versa. *Journal of Logic and Computation, 27*(5), 1549–1575.
Kamide, N., Shramko, Y., & Wansing, H. (2017). Kripke completeness of biintuitionistic multilattice logic and its connexive variant. *Studia Logica, 105*(5), 1193–1219.
Kamide, N., & Wansing, H. (2016). Completeness of connexive Heyting-Brouwer logic. *IFCoLog Journal of Logic and their Applications, 3*, 441–466.
Rauszer, C. (1974). A formalization of the propositional calculus of H-B logic. *Studia Logica, 33*(1), 23–34.
Rauszer, C. (1977). Applications of kripke models to heyting-brouwer logic. *Studia Logica, 36*(1–2), 61–71.
Rauszer, C. (1980). An algebraic and Kripke-style approach to a certain extension of intuitionistic logic. *Dissertationes Mathematicae, 167*, 1–62.
Shramko, Y. (2016). Truth, falsehood, information and beyond: the American plan generalized. In: Bimbo, K. (Ed.), *J. Michael Dunn on information based logics, outstanding contributions to logic* (pp. 191–212). Dordrecht: Springer.
Shramko, Y., & Wansing, H. (2005). Some useful sixteen-valued logics: How a computer network should think. *Journal of Philosophical Logic, 34*, 121–153.
Wansing, H. (2008). Constructive negation, implication, and co-implication. *Journal of Applied Non-Classical Logics, 18*(2–3), 341–364.
Zaitsev, D. (2009). A few more useful 8-valued logics for reasoning with tetralattice EIGHT4. *Studia Logica, 92*(2), 265–280.

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