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Double Negation as Minimal Negation

Satoru Niki[1]

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Abstract
N. Kamide introduced a pair of classical and constructive logics, each with a peculiar type of negation: its double negation behaves as classical and intuitionistic negation, respectively. A consequence of this is that the systems prove contradictions but are non-trivial. The present paper aims at giving insights into this phenomenon by investigating subsystems of Kamide's logics, with a focus on a system in which the double negation behaves as the negation of minimal logic. We establish the negation inconsistency of the system and embeddability of contradictions from other systems. In addition, we attempt at an informational interpretation of the negation using the dimathematical framework of H. Wansing.

Keywords Contradiction · Double negation · Dimathematism · Minimal logic · Negation inconsistency · Strong negation

1 Introduction

In Kamide (2017) introduced the systems **IP** and **CP**, which are variants of the intuitionistic system of constructible falsity **N4** (Almukdad & Nelson, 1984) and its classical extension $\mathbf{B}_4^{\rightarrow}$ (Odintsov, 2005). The difference between these systems is that a doubly negated formula $\sim\sim A$ in **IP** and **CP** corresponds to the intuitionistic/classically negated formula $\neg A$, instead of A. In this sense, the negation $\sim$ is closely related to a type of connective called *demi-negation*, introduced by Humberstone (1995). The negation is also studied algebraically by Paoli (2019), who introduced another system with $\sim$ in a different language. Later, it was pointed out by Omori and Wansing (2018, 2022) that these systems prove a formula A as well as its negation $\sim A$. That is to say, **IP** and **CP** contain *provable contradictions*. Consequently, they are negation inconsistent but non-trivial systems. Their paper also offers a defence of

✉ Satoru Niki
Satoru.Niki@rub.de

1 Department of Philosophy I, Ruhr University Bochum, Universitätsstraße 150, Bochum 44801, North Rhine-Westphalia, Germany

reading $\sim$ as negation, by referring to A. Avron's view (Avron, 2005) that a negation represents the falsehood in the sense that a negated formula is true iff its negand is false. In addition, Omori and Wansing suggest that the double negation can be understood as representing the phenomenon of *negative concord* in natural languages.

Given this kind of view, Kamide's systems appear to be of interest from both formal and philosophical perspectives. When it comes to the former aspect, one essential task is to identify the source of negation inconsistency of the systems. One natural methodology for this would be to look into subsystems of **IP** in which the negation has more restricted properties.

In this paper, we shall first observe that the negation inconsistency still holds when the double negation is made to correspond to the negation of minimal logic (Johansson, 1937), with a system we shall call **MP**. In addition, we shall see that provable contradictions in **CP** can be embedded to provable contradictions in the weak system. We then investigate further the method of obtaining provable contradictions via translation, by turning our attention into P. Ruet's *quarter turn* operation (Ruet, 1996).

This is followed by the observation concerning some subsystems of **MP**. We will observe the effects of restricting axiom schemata into rules on negation inconsistency.

Finally, we shall attempt at giving an interpretation of the negation in **MP** which complements the interpretation in Omori and Wansing (2018) for **IP** and **CP**. One characteristic of minimal negation is that it behaves like an implication to a propositional variable that does not have to exhibit a 'negative' property such as never being forced in a world of a Kripke model. This lack of a 'negative' flavour gives a more philosophical motivation to consider a negation whose double negation behaves as a minimal negation. It allows an interpretation of the double negation that is more 'positive', and so closer to usual kinds of negations which are better understood. We shall in particular attempt to understand **MP** from a more constructive and informational point of view, by employing the *dimathematic* perspective of Wansing (Wansing, 2022). For this purpose, we shall give a 'positive' interpretation of the double negation, according to which the support of falsity of $\sim A$ is equated with the regularity that the support of truth of A must be a *strong* one. We shall also discuss a modification to the semantic clause motivated by this interpretation, and how it fares with the negation inconsistency.

2 Minimal Variant of IP

Let $\mathsf{PROP} = \{p_i : i \in \mathbb{N}\}$ be a set of *propositional variables*, and $()'$ be a mapping which assigns for each p_i another propositional variable $(p_i)'$ in such a way that $\mathsf{PROP}' := \{(p_i)' : i \in \mathbb{N}\}$ is a set distinct from PROP. We shall use $p, q, r, \ldots$ and $p', q', r', \ldots$ as the metavariables of the elements of PROP and PROP'. In what follows, we shall use the next three propositional languages.

$$(\mathcal{L}_{\sim})\ A ::= p \mid A \wedge A \mid A \vee A \mid A \to A \mid {\sim}A.$$
$$(\mathcal{L}_{\neg})\ A ::= p \mid p' \mid A \wedge A \mid A \vee A \mid A \to A \mid \neg A.$$
$$(\mathcal{L}_{\circlearrowleft})\ A ::= p \mid A \wedge A \mid A \vee A \mid A \to A \mid \circlearrowleft A.$$

We shall use $\equiv$ for the literal identity of formulas. The first language $\mathcal{L}_{\sim}$ is the main language we shall consider. $\mathcal{L}_{\neg}$ is a language with intuitionistic/minimal negation as well as duplicate propositional variables, which will be used for some arguments via translation. $\mathcal{L}_{\circlearrowleft}$ will be used for systems with the quarter turn operator. In each of the languages, we will use the abbreviations $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$, $\circ^{i+1}A$ for $\circ(\circ^{i} A)$ and $\circ^{0}A$ for A where $\circ \in \{\sim, \neg, \circlearrowleft\}$. The *complexity* $|A|$ of formulas is inductively defined as follows:

$$|p| = |p'| = 0. \qquad |\circlearrowleft(A \circ B)| = |A| + |B| + 2.$$
$$|{\sim}p| = |\circlearrowleft p| = 0. \qquad |{\sim}{\sim}A| = |{\sim}A| + 1.$$
$$|A \circ B| = |A| + |B| + 1. \qquad |\circlearrowleft\circlearrowleft A| = |\circlearrowleft A| + 1.$$
$$|{\sim}(A \circ B)| = |A| + |B| + 2. \qquad |\neg A| = |A| + 1.$$

where $\circ \in \{\wedge, \vee, \to\}$.

2.1 Sequent Calculi

The systems **CP** and **IP** are introduced in Kamide (2017) as sequent calculi. We introduce our system **MP** following the paradigm of these systems. One point to notice in these calculi is that $\Gamma, \Delta, \ldots$ will denote finite *sets* of formulas, rather than finite multisets as is often the case with sequent calculi.

Definition 1 (**MP**) The system **MP** in $\mathcal{L}_{\sim}$ is defined by the following rules.

$$p \Rightarrow p \ (\mathrm{Ax}) \qquad {\sim}p \Rightarrow {\sim}p \ (\mathrm{Ax}{\sim})$$

$$\frac{\Gamma \Rightarrow A \qquad A, \Sigma \Rightarrow C}{\Gamma, \Sigma \Rightarrow C}\ (\mathrm{Cut}) \qquad \frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ (\mathrm{LW})$$

$$\frac{A_i, \Gamma \Rightarrow C}{A_1 \wedge A_2, \Gamma \Rightarrow C}\ (\mathrm{L}\wedge) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\mathrm{R}\wedge)$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ (\mathrm{L}\vee) \qquad \frac{\Gamma \Rightarrow C_i}{\Gamma \Rightarrow C_1 \vee C_2}\ (\mathrm{R}\vee)$$

$$\frac{\Gamma \Rightarrow A \qquad B, \Sigma \Rightarrow C}{A \to B, \Gamma, \Sigma \Rightarrow C}\ (\mathrm{L}{\to}) \qquad \frac{A, \Gamma \Rightarrow C}{\Gamma \Rightarrow A \to C}\ (\mathrm{R}{\to})$$

$$\frac{{\sim}A, \Gamma \Rightarrow C \qquad {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \wedge B), \Gamma \Rightarrow C}\ (\mathrm{L}{\sim}\wedge) \qquad \frac{\Gamma \Rightarrow {\sim}C_i}{\Gamma \Rightarrow {\sim}(C_1 \wedge C_2)}\ (\mathrm{R}{\sim}\wedge)$$

$$\frac{{\sim}A_i, \Gamma \Rightarrow C}{{\sim}(A_1 \vee A_2), \Gamma \Rightarrow C}\ (\mathrm{L}{\sim}\vee) \qquad \frac{\Gamma \Rightarrow {\sim}A \qquad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \vee B)}\ (\mathrm{R}{\sim}\vee)$$

$$\frac{A, \Gamma \Rightarrow C}{{\sim}(A \to B), \Gamma \Rightarrow C}\ (\mathrm{L}{\sim}{\to}1) \qquad \frac{{\sim}B, \Gamma \Rightarrow C}{{\sim}(A \to B), \Gamma \Rightarrow C}\ (\mathrm{L}{\sim}{\to}2)$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \rightarrow B)}\ (\mathrm{R}{\sim}{\rightarrow})$$

$$\frac{A, \Gamma \Rightarrow B}{{\sim}{\sim}B, \Gamma \Rightarrow {\sim}{\sim}A}\ ({\sim}{\sim}1) \qquad \frac{A, \Gamma \Rightarrow {\sim}{\sim}A}{\Gamma \Rightarrow {\sim}{\sim}A}\ ({\sim}{\sim}2)$$

where $i \in \{1, 2\}$.

We shall write **MP** $\vdash \Gamma \Rightarrow C$ when a sequent $\Gamma \Rightarrow C$ is derivable following the rules of **MP**. In particular, we shall write **MP** $\vdash \Rightarrow C$ when $\Gamma = \emptyset$. Similar conventions apply for other systems in the paper.

For a system L and a rule (R), L-(R) will denote the system obtained by eliminating (R) from the rules of L. We say a rule is *admissible* in L, if the derivability of the premises in L implies that of the conclusion. In particular, a rule is *derivable* if a derivation of the conclusion is obtainable by continuing from any derivations of the premises.

We can readily check that **MP**-(Cut) $\vdash A, \Gamma \Rightarrow A$. Also, the system **IP** is defined in the following way.

Definition 2 (**IP**) The system **IP** is definable from **MP** by (i) replacing (${\sim}{\sim}$1) and (${\sim}{\sim}$2) with (L${\sim}{\sim}$), (R${\sim}{\sim}$) below; (ii) adding the next structural rule (RW).

$$\frac{\Gamma \Rightarrow C}{{\sim}{\sim}C, \Gamma \Rightarrow}\ (\mathrm{L}{\sim}{\sim}) \qquad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow {\sim}{\sim}A}\ (\mathrm{R}{\sim}{\sim})$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (\mathrm{RW})$$

Proposition 1 *If* **MP** $\vdash \Gamma \Rightarrow C$ *then* **IP** $\vdash \Gamma \Rightarrow C$.

Proof It suffices to check that (${\sim}{\sim}$1) and (${\sim}{\sim}$2) are derivable in **IP**. The former case is immediate, and for the latter case, if **IP** $\vdash A, \Gamma \Rightarrow {\sim}{\sim}A$ then by applying (L${\sim}{\sim}$), (R${\sim}{\sim}$) to A, (L${\sim}{\sim}$) and finally (R${\sim}{\sim}$) to ${\sim}^4A$, we obtain **IP** $\vdash \Gamma \Rightarrow {\sim}^6A$. Also, **IP** $\vdash {\sim}^6A \Rightarrow {\sim}{\sim}A$ by a similar argument. Thus by (Cut) **IP** $\vdash \Gamma \Rightarrow {\sim}{\sim}A$. □

In order to establish some results later, we need to look into minimal logic as well. For the sequent calculus formalisation of minimal logic, we shall use the following system, based on Bílková and Colacito (2020) and Colacito et al. (2017, Proposition 2) but again using finite sets rather than finite multisets.

Definition 3 (**LM**) The system **LM** in $\mathcal{L}_{\neg}$ is defined by (Ax),[1] (Cut)–(R→) and the following rules.

$$\frac{A, \Gamma \Rightarrow B}{\neg B, \Gamma \Rightarrow \neg A}\ (\neg 1) \qquad \frac{A, \Gamma \Rightarrow \neg A}{\Gamma \Rightarrow \neg A}\ (\neg 2)$$

[1] We let (Ax) to include the cases of the additional propositional variables p', q' etc.

Theorem 2 *(Cut) is admissible in* **IP**-*(Cut) and* **LM**-*(Cut).*

Proof Respectively see Kamide (2017, Theorem 11) and Colacito (2020, Theorem 4.1). In the latter case, it is not difficult to check that the difference in the presentation, namely that the antecedent of sequents is a finite set rather than a finite multiset, does not affect the structure of the argument. □

Then the admissibility of (Cut) for **MP**-(Cut) can be obtained similarly to that of **IP**-(Cut), using an argument via translation.

Definition 4 We define a translation f of formulas in $\mathcal{L}_{\sim}$ into those of $\mathcal{L}_{\neg}$ by the following clauses:

$$f(p) = p. \qquad f(\sim(A \wedge B)) = f(\sim A) \vee f(\sim B).$$
$$f(A \circ B) = f(A) \circ f(B). \qquad f(\sim(A \vee B)) = f(\sim A) \wedge f(\sim B).$$
$$f(\sim p) = p'. \qquad f(\sim(A \to B)) = f(A) \wedge f(\sim B).$$
$$f(\sim\sim A) = \neg f(A).$$

where $\circ \in \{\wedge, \vee, \to\}$.

This translation justifies the view that a double negation in **MP** represents minimal negation. In what follows, given a finite set Γ we shall use the notation $f(\Gamma)$ for the set $\{f(A) : A \in \Gamma\}$: similar conventions apply for later translations as well.

Theorem 3 *The following statements hold.*

1. **MP** $\vdash \Gamma \Rightarrow A$ *if and only if* **LM** $\vdash f(\Gamma) \Rightarrow f(A)$.
2. **MP**-*(Cut)*$\vdash \Gamma \Rightarrow A$ *if and only if* **LM**-*(Cut)* $\vdash f(\Gamma) \Rightarrow f(A)$.
3. *(Cut) is admissible in* **MP**-*(Cut).*

Proof Analogous to Kamide (2017, Theorem 1–3). □

One corollary of this theorem is the *disjunction property*, which indicates the constructivity of the system.

Corollary 4 *If* **MP** $\vdash \Rightarrow A \vee B$ *then either* **MP** $\vdash \Rightarrow A$ *or* **MP** $\vdash \Rightarrow B$.

Proof Consider a proof of $\Rightarrow A \vee B$ in **MP**-(Cut). Then the last rule applied must be (R$\vee$), whose premise has either the form $\Rightarrow A$ or $\Rightarrow B$. □

2.2 Hilbert-Style System and Semantics

We next introduce a Hilbert-style system for **MP**, which is obtained simply by combining the ones for **IP** (Omori & Wansing, 2018) and minimal logic.

Definition 5 (**H-MP**) The following axiomatisation defines the calculus **H-MP**.

$$(A\to(B\to C))\to((A\to B)\to(A\to C))$$
$$A\to(B\to A)$$
$$A\to(B\to(A\wedge B))$$
$$A_1\wedge A_2\to A_i$$
$$A_i\to A_1\vee A_2$$
$$(A\to C)\to((B\to C)\to(A\vee B\to C))$$
$$\sim(A\wedge B)\leftrightarrow(\sim A\vee\sim B)$$
$$\sim(A\vee B)\leftrightarrow(\sim A\wedge\sim B)$$
$$\sim(A\to B)\leftrightarrow(A\wedge\sim B)$$
$$(A\to B)\to(\sim\sim B\to\sim\sim A)$$
$$(A\to\sim\sim A)\to\sim\sim A$$
$$\frac{A \qquad A\to B}{B}\ \text{(MP)}$$

where $i\in\{1,2\}$. A derivation of A from a set of formulas Γ is a finite sequence $B_1,\ldots,B_n\equiv A$, where each B_i is either an element of Γ, an instance of one of the axiom schemata, or obtained from the preceding entries by a rule (in this case, (MP)). The derivability of A from Γ in **H-MP** is denoted by $\Gamma\vdash_h A$. Then by straightforwardly modifying (Omori & Wansing, 2018, Proposition 3.11, 3.12), we can show the following.

Theorem 5 *Let Γ be a finite set. Then $\Gamma\vdash_h A$ if and only if* **MP** $\vdash\Gamma\Rightarrow A$.

For semantics, a Kripke semantics for **MP** is obtained from those of **IP** (Kamide, 2017; Omori & Wansing, 2018) and minimal logic by Segerberg (1968).

Definition 6 (Kripke semantics for **MP**) We define a *frame* to be a triple $(W,\leq,Q)$ where W is a non-empty set, $\leq$ is a partial ordering on W, and $Q\subseteq W$ is an *upward closed* set, i.e. $w\in Q$ and $w'\geq w$ implies $w'\in Q$.

A *model* then is a pair $(\mathcal{F},\mathcal{V})$ where $\mathcal{F}$ is a frame and $\mathcal{V}$ assigns a pair of upward closed sets $\mathcal{V}^+(p)$ and $\mathcal{V}^-(p)$ to each propositional variable p. $\mathcal{V}$ is extended to forcings $(\Vdash^+,\Vdash^-)$ of all formulas by the next clauses.

$w\Vdash^+ p$ iff $w\in\mathcal{V}^+(p)$. $\quad w\Vdash^- p$ iff $w\in\mathcal{V}^-(p)$.

$w\Vdash^+ A\wedge B$ iff $w\Vdash^+ A$ and $w\Vdash^+ B$. $\quad w\Vdash^- A\wedge B$ iff $w\Vdash^- A$ or $w\Vdash^- B$.

$w\Vdash^+ A\vee B$ iff $w\Vdash^+ A$ or $w\Vdash^+ B$. $\quad w\Vdash^- A\vee B$ iff $w\Vdash^- A$ and $w\Vdash^- B$.

$w\Vdash^+ A\to B$ iff $\forall w'\geq w(w'\Vdash^+ A\Rightarrow w'\Vdash^+ B)$. $\quad w\Vdash^- A\to B$ iff $w\Vdash^+ A$ and $w\Vdash^- B$.

$w\Vdash^+\sim A$ iff $w\Vdash^- A$. $\quad w\Vdash^-\sim A$ iff $\forall w'\geq w(w'\Vdash^+ A\Rightarrow w'\in Q)$.

We write $\Gamma\vDash A$ if for any model and $w\in W$, $w\Vdash^+ B$ for all $B\in\Gamma$ implies $w\Vdash^+ A$.

The last equivalence might appear mysterious; in fact, the right hand side just mirrors the condition for negation in minimal logic. Later in section 7, we shall discuss how the use of Q in **MP** enables an informal interpretation of the negation that is not available in **IP**.

Lemma 6 *(upward closure) For $*\in\{+,-\}$, if $w\Vdash^* A$ and $w'\geq w$ then $w'\Vdash^* A$.*

Proof By induction on the complexity of A. In particular, if A is $\sim B$ and $w'\geq w$, then $w\Vdash^-\sim B$ implies $x\Vdash^+ B\Rightarrow x\in Q$ for all $x\geq w$ and *a fortiori* for all $x\geq w'$. Thus $w'\Vdash^-\sim B$. □

Then the relationship between the Hilbert-style system and the semantics can be established in a standard manner.

Theorem 7 *(completeness)* $\Gamma \vdash_h A$ *if and only if* $\Gamma \models A$.

Proof The left-to-right direction is shown by induction on the depth of derivations. For the converse direction, the outline is as in the case for **IP** (Omori & Wansing, 2018, Theorem 3.9).[2] We can likewise construct a canonical model $((W, \leq, Q), \mathcal{V})$, where:

- $(W, \leq)$ is a set of collections of formulas ordered by $\leq := \{(\Sigma, \Delta) \in W \times W : \Sigma \subseteq \Delta\}$, and each $\Sigma \in W$ satisfies:
  - there is A such that $A \notin \Sigma$
  - if $\Sigma \vdash_h A$ then $A \in \Sigma$.
  - if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$.
- $Q = \{\Sigma : {\sim}^2 A, {\sim}^4 A \in \Sigma$ for some $A.\}$.
- $\Sigma \in \mathcal{V}^+(p)$ iff $p \in \Sigma$.
- $\Sigma \in \mathcal{V}^-(p)$ iff ${\sim}p \in \Sigma$.

It is immediate that Q is upward closed. We also have to check that:

$$\Sigma \Vdash^+ A \;\text{ if and only if }\; A \in \Sigma.$$
$$\Sigma \Vdash^- A \;\text{ if and only if }\; {\sim}A \in \Sigma.$$

Here we consider the latter equivalence for the case A is ${\sim}B$. By I.H. $\Sigma \Vdash^- {\sim}B$ if and only if $\forall \Delta \geq \Sigma (B \in \Delta \Rightarrow \exists C({\sim}^2 C, {\sim}^4 C \in \Delta))$. We need to check this is equivalent to ${\sim}{\sim}B \in \Sigma$. For the forward direction, we show the contrapositive. If ${\sim}{\sim}B \notin \Sigma$, then $B \to {\sim}{\sim}B \notin \Sigma$. Now we can argue along (Omori & Wansing, 2018, Lemma 3.8) to conclude that there is $\Delta \in W$ s.t. $\Delta \supseteq \Sigma$, $B \in \Delta$ and ${\sim}{\sim}B \notin \Delta$. But then, using the fact that $\vdash_h (D \wedge {\sim}^2 D) \to {\sim}^2 E$ follows from $\vdash_h D \to (E \to D)$ and $\vdash_h (E \to D) \to ({\sim}^2 D \to {\sim}^2 E)$, we see ${\sim}^2 C, {\sim}^4 C \in \Delta$ implies ${\sim}^2 B \in \Delta$, a contradiction. Hence $\neg \forall \Delta \geq \Sigma (B \in \Delta \Rightarrow \exists C({\sim}^2 C, {\sim}^4 C \in \Delta))$. For the backward direction, if ${\sim}{\sim}B \in \Sigma$ and $B \in \Delta \supseteq \Sigma$, then ${\sim}^2 B, {\sim}^4 B \in \Delta$, as required.

Now if $\Gamma \nvdash_h A$, then there is (arguing like (Omori & Wansing, 2018, Lemma 3.7)) $\Gamma' \supseteq \Gamma$ in the canonical model such that $A \notin \Gamma'$. Therefore $\Gamma' \not\models A$. □

3 Provable Contradictions in MP

As is mentioned in the introduction, (Omori & Wansing, 2018) observed that the systems **CP** and **IP** prove contradictions, and thus are negation inconsistent. We shall observe in this section that **MP** satisfies the same property. To be more precise, we shall say a formula A is a *provable contradiction* in a sequent calculus if the sequents

[2] As a notational difference, we are using use $w \Vdash^+ A$ and $w \Vdash^- A$ in place of $1 \in I(w, A)$ and $0 \in I(w, A)$, respectively.

$\Rightarrow A$ and $\Rightarrow {\sim}A$ are derivable. When it is a Hilbert-style system, then A being a *provable contradiction* will mean that both A and ${\sim}A$ are derivable.[3]

We will often appeal to the derivability of the sequents below, which correspond to well-known equivalences in minimal logic.

Proposition 8 *The following sequents are derivable in* **MP**.

- $\Rightarrow {\sim}^4(A \wedge B) \leftrightarrow ({\sim}^4 A \wedge {\sim}^4 B)$
- $\Rightarrow {\sim}^4(A \vee B) \leftrightarrow {\sim}^4({\sim}^4 A \vee {\sim}^4 B)$
- $\Rightarrow {\sim}^4(A \to {\sim}^4 B) \leftrightarrow ({\sim}^4 A \to {\sim}^4 B)$

In Omori and Wansing (2018), ${\sim}^5(A \wedge {\sim}{\sim}A)$ is given as an example of provable contradiction for **IP**. The same formula can be used to show the negation inconsistency of **MP**.

Proposition 9 ${\sim}^5(A \wedge {\sim}^2 A)$ *is a provable contradiction in* **MP**.

Proof We need to show that $\mathbf{MP} \vdash \Rightarrow {\sim}^5(A \wedge {\sim}{\sim}A)$ and $\mathbf{MP} \vdash \Rightarrow {\sim}^6(A \wedge {\sim}{\sim}A)$. For the former:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{\sim}A \Rightarrow {\sim}A}{{\sim}A \Rightarrow {\sim}(A \wedge {\sim}^2 A)}\,(\mathrm{R}{\sim}\wedge)}{{\sim}^3(A \wedge {\sim}^2 A) \Rightarrow {\sim}^3 A}\,({\sim}{\sim}1)}{{\sim}^3(A \wedge {\sim}^2 A) \Rightarrow {\sim}(A \wedge {\sim}^2 A)}\,(\mathrm{R}{\sim}\wedge)}{{\sim}^3(A \wedge {\sim}^2 A) \Rightarrow {\sim}^5(A \wedge {\sim}^2 A)}\,({\sim}{\sim}1)}{\Rightarrow {\sim}^5(A \wedge {\sim}^2 A)}\,({\sim}{\sim}2)$$

For the latter (where a double line indicates a repeated application of a rule):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \wedge {\sim}^2 A \Rightarrow A}\,(\mathrm{L}\wedge)}{{\sim}^2 A \Rightarrow {\sim}^2(A \wedge {\sim}^2 A)}\,({\sim}{\sim}1)}{A \wedge {\sim}^2 A \Rightarrow {\sim}^2(A \wedge {\sim}^2 A)}\,(\mathrm{L}\wedge)}{{\sim}^4(A \wedge {\sim}^2 A) \Rightarrow {\sim}^6(A \wedge {\sim}^2 A)}\,({\sim}{\sim}1)}{\Rightarrow {\sim}^6(A \wedge {\sim}^2 A)}\,({\sim}{\sim}2)$$

□

Hence the formula ${\sim}^5(A \wedge {\sim}^2 A)$ is in a sense not a provable contradiction that is characteristic of **IP**. A natural question then is whether there is a contradiction that is provable in **IP** but not in **MP**.

Proposition 10 ${\sim}^3({\sim}^2(A \to A) \to B)$ *is a provable contradiction in* **IP**.

Proof It suffices to show that $\mathbf{IP} \vdash \Rightarrow {\sim}^3({\sim}^2(A \to A) \to B)$ and $\mathbf{IP} \vdash \Rightarrow {\sim}^4({\sim}^2(A \to A) \to B)$. For the former:

[3] Another option is to define it as the derivability of the conjunction of a contradictory pair. The two formulations are equivalent in the current setting, and we shall use the alternative formulation as well when it is more convenient for presentation.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A \to A}\ (\text{R}{\to})}{\sim^2(A \to A) \Rightarrow}\ (\text{L}{\sim\sim})}{\sim(\sim^2(A \to A) \to B) \Rightarrow}\ (\text{L}{\sim}{\to})}{\Rightarrow \sim^3(\sim^2(A \to A) \to B)}\ (\text{R}{\sim\sim})$$

For the latter:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A \to A}\ (\text{R}{\to})}{\sim^2(A \to A) \Rightarrow}\ (\text{L}{\sim\sim})}{\sim^2(A \to A) \Rightarrow B}\ (\text{RW})}{\Rightarrow \sim^2(A \to A) \to B}\ (\text{R}{\to})}{\sim^2(\sim^2(A \to A) \to B) \Rightarrow}\ (\text{L}{\sim\sim})}{\Rightarrow \sim^4(\sim^2(A \to A) \to B)}\ (\text{R}{\sim\sim})$$

□

Proposition 11 **MP** $\nvdash \Rightarrow \sim^4(\sim^2(A \to A) \to B)$.

Proof By Theorem 3, if the sequent is provable then so is $\Rightarrow \neg\neg(\neg(p \to p) \to q)$ in **LM**, which is known not to be the case.[4] □

The observations above confirm that **MP** is strong enough to be a non-trivial negation inconsistent system. Where, however, does the contra-classicality[5] come from? The relative weakness of minimal negation enables us to give the following explanation to this question. We may observe (in the manner of (Colacito, 2016, Proposition 1.2.5)) that we do not lose the strength of the system by replacing (∼∼1) with the rule:

$$\dfrac{A, \Gamma \Rightarrow B \qquad B, \Gamma \Rightarrow A}{\sim\sim A, \Gamma \Rightarrow \sim\sim B}\ (\sim\sim 3)$$

It is straightforward to check that (∼∼1) is derivable from (∼∼2) and (∼∼3). One difference between (∼∼1) and (∼∼3) is that the latter rule is admissible in a classical sequent calculus when ∼ is identified with the classical negation. Thus in this alternative formulation, the cause of the contra-classicality is isolated to (∼∼2) alone.

4 Embedding Provable Contradictions

So far we have looked at provable contradictions in **IP** and **MP** on a more or less individual basis. Another, more general way to obtain provable contradictions in **IP**

[4] The addition of $\neg\neg(\bot \to q)$ to minimal logic (note $\bot$ is definable as $\neg(p \to p)$) is considered by Segerberg (1968).

[5] We here follow Omori and Wansing (2018, pp.816-817) in using the term in the 'superficial' sense of Humberstone (2000, p.438), namely that a contra-classical logic is a logic that is not a sublogic of classical logic. This formulation however takes some aspects for granted, such as that the logic shares the same language as classical logic. As one referee rightly pointed out, it may therefore leave some imprecision for a more robust understanding of the concept.

and **MP** is to appeal to negative translations extended to treat $\sim$. Using this method, we shall observe that **MP** proves, in a sense, as many contradictions as **CP**. We start with recalling **CP** and the Glivenko-like theorem in Kamide (2017), which embeds the theorems of **CP** into those of **IP**.

Definition 7 (**CP**) The system **CP** in $\mathcal{L}_{\sim}$ is defined by (Ax), (Ax$\sim$) and the following rules.

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{mCut})$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\text{mLW}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, C}\ (\text{mRW})$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\text{mL}\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\text{mR}\wedge)$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\text{mL}\vee) \qquad \frac{\Gamma \Rightarrow \Delta, C, D}{\Gamma \Rightarrow \Delta, C \vee D}\ (\text{mR}\vee)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Sigma \Rightarrow \Pi}{A \to B, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{mL}\to) \qquad \frac{A, \Gamma \Rightarrow \Delta, C}{\Gamma \Rightarrow \Delta, A \to C}\ (\text{mR}\to)$$

$$\frac{\sim A, \Gamma \Rightarrow \Delta \quad \sim B, \Gamma \Rightarrow \Delta}{\sim(A \wedge B), \Gamma \Rightarrow \Delta}\ (\text{mL}\sim\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, \sim C, \sim D}{\Gamma \Rightarrow \Delta, \sim(C \wedge D)}\ (\text{mR}\sim\wedge)$$

$$\frac{\sim A, \sim B, \Gamma \Rightarrow \Delta}{\sim(A \vee B), \Gamma \Rightarrow \Delta}\ (\text{mL}\sim\vee) \qquad \frac{\Gamma \Rightarrow \Delta, \sim A \quad \Gamma \Rightarrow \Delta, \sim B}{\Gamma \Rightarrow \Delta, \sim(A \vee B)}\ (\text{mR}\sim\vee)$$

$$\frac{A, \sim B, \Gamma \Rightarrow \Delta}{\sim(A \to B), \Gamma \Rightarrow \Delta}\ (\text{mL}\sim\to) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \sim B}{\Gamma \Rightarrow \Delta, \sim(A \to B)}\ (\text{mR}\sim\to)$$

$$\frac{\Gamma \Rightarrow \Delta, C}{\sim\sim C, \Gamma \Rightarrow \Delta}\ (\text{mL}\sim\sim) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\sim A}\ (\text{mR}\sim\sim)$$

Glivenko's theorem (Glivenko, 1998) states that if A is a classical theorem, then $\neg\neg A$ is an intuitionistic theorem. The *Glivenko-like theorem* (Kamide, 2017, Theorem 19) similarly states the following.

Theorem 12 **CP** $\vdash \Rightarrow A$ *if and only if* **IP** $\vdash \Rightarrow \sim^4 A$.

This immediately implies that provable contradictions on **CP** can be embedded into **IP**.

Corollary 13 *If A is a provable contradiction in* **CP**, *then $\sim^4 A$ is a provable contradiction in* **IP**.

The question is whether we can have a similar result with respect to **MP**. While the Glivenko-like theorem cannot be extended to **MP**, it is possible to extend the Gödel-Gentzen-like theorem, also considered by Kamide.

Definition 8 (Gödel-Gentzen-like translation) We define a translation h of formulas in $\mathcal{L}_\sim$ into itself by the following clauses.

$$\begin{aligned} h(p) &= \sim^4 p. & h(\sim(A\wedge B)) &= \sim^2(\sim^2 h(\sim A)\wedge\sim^2 h(\sim B)). \\ h(A\circ B) &= h(A)\circ h(B). & h(\sim(A\vee B)) &= h(\sim A)\wedge h(\sim B). \\ h(A\vee B) &= \sim^2(\sim^2 h(A)\wedge\sim^2 h(B)). & h(\sim(A\to B)) &= h(A)\wedge h(\sim B). \\ h(\sim p) &= \sim^5 p. & h(\sim\sim A) &= \sim\sim h(A). \end{aligned}$$

where $\circ \in \{\wedge, \to\}$.

Lemma 14 $\mathbf{MP} \vdash \sim^4 h(A) \Rightarrow h(A)$.

Proof By induction on the complexity of A. For cases where $h(A)$ has the form $\sim\sim B$, the statement follows from $\mathbf{MP} \vdash \sim^6 B \Rightarrow \sim^2 B$. When $A \equiv B \to C$, use the derivability of $\sim^4(h(B) \to h(C)) \Rightarrow h(B) \to \sim^4 h(C)$ and (by I.H.) $\sim^4 h(C) \Rightarrow h(C)$. For other cases, use the equivalence for conjunction in Proposition 8 via (Cut). □

Given a finite set Γ of formulas, we define $\sim\Gamma := \{\sim A : A \in \Gamma\}$. Then we have the following lemma.

Lemma 15 *If* $\mathbf{CP} \vdash \Gamma \Rightarrow \Delta$ *then* $\mathbf{MP} \vdash h(\Gamma), \sim^2 h(\Delta) \Rightarrow \sim^2(p \to p)$.

Proof By induction on the depth of derivation in **CP**. □

Theorem 16 $\mathbf{CP} \vdash \Rightarrow A$ *if and only if* $\mathbf{MP} \vdash \Rightarrow h(A)$.

Proof For the left-to-right direction, by Lemma 15 if $\mathbf{CP} \vdash \Rightarrow A$ then $\mathbf{MP} \vdash \sim\sim h(A) \Rightarrow \sim\sim(p \to p)$. Thus by $(\sim\sim 1)$, $\mathbf{MP} \vdash \sim^4(p \to p) \Rightarrow \sim^4 h(A)$. Apply (Cut) and Lemma 14 to conclude $\mathbf{MP} \vdash\Rightarrow h(A)$. The right-to-left direction is obtained by showing $\mathbf{CP} \vdash h(A) \Rightarrow A$; see also (Kamide, 2017, Lemma 10, Theorem 20). □

The translation h is not sufficient to embed provable contradictions in **CP** into provable contradictions in **MP**. For instance, while $\sim(p \wedge \sim^2 p)$ is known Omori and Wansing (2018) to be a provable contradiction in **CP**, the translations of the formulas do not preserve the form A and $\sim A$.

$$\begin{aligned} h(\sim(p \wedge \sim^2 p)) &= \sim^2(\sim^7 p \wedge \sim^9 p). \\ h(\sim^2(p \wedge \sim^2 p)) &= \sim^2(\sim^4 p \wedge \sim^6 p). \end{aligned}$$

What would be desirable, in order to preserve the form of contradiction, is to have $h(\sim A) = \sim h(A)$ instead. However, this modification does not work for the Gödel-Gentzen-like translation, because $h(\sim(p\wedge\sim^2 p))$ would then become $\sim(\sim^4 p\wedge\sim^6 p)$, which is not provable in **MP**.

What can be done instead is to use a different translation. We shall use a translation based on the *minimal Kuroda*-translation by Ferreira and Oliva (2011), which is a generalisation of Kuroda's translation by Kuroda (1951) to minimal logic.

Definition 9 (minimal-Kuroda-like translation) We define a translation A^k of formulas in $\mathcal{L}_\sim$ into itself by the following clauses.

$$\begin{aligned} p^k &= p. \\ (A \circ B)^k &= A^k \circ B^k. \\ (A \to B)^k &= A^k \to {\sim}^4 B^k. \\ ({\sim}A)^k &= {\sim}A^k. \end{aligned}$$

where $\circ \in \{\wedge, \vee\}$. Then we define $k(A) = {\sim}^4 A^k$.

Lemma 17 $\mathbf{MP} \vdash \Rightarrow h(A) \leftrightarrow k(A)$.

Proof By induction on the complexity of A. The crucial case is when A has the form ${\sim}(B \to C)$. In this case,

$$\begin{aligned} h({\sim}(B \to C)) &= h(B) \wedge h({\sim}C), \\ k({\sim}(B \to C)) &= {\sim}^5(B^k \to {\sim}^4 C^k). \end{aligned}$$

By I.H., $\mathbf{MP} \vdash \Rightarrow h(B) \leftrightarrow {\sim}^4 B^k$ (since $k(B) \equiv {\sim}^4 B^k$) and $\mathbf{MP} \vdash \Rightarrow h({\sim}C) \leftrightarrow {\sim}^5 C^k$ (since $k({\sim}C) \equiv {\sim}^4({\sim}C)^k$). Then we can show the equivalence

$$\begin{aligned} h(B) \wedge h({\sim}C) &\leftrightarrow ({\sim}^4 B^k \wedge {\sim}^9 C^k) \\ &\leftrightarrow {\sim}^4(B^k \wedge {\sim}^5 C^k) \\ &\leftrightarrow {\sim}^5(B^k \to {\sim}^4 C^k). \end{aligned}$$

□

Theorem 18 $\mathbf{CP} \vdash \Rightarrow A$ *if and only if* $\mathbf{MP} \vdash \Rightarrow k(A)$.

Proof Immediate from Theorem 16 and Lemma 17. □

Corollary 19 *If A is a provable contradiction in* **CP**, *then $k(A)$ is a provable contradiction in* **MP**.

Proof If $\mathbf{CP} \vdash \Rightarrow A$ and $\mathbf{CP} \vdash \Rightarrow {\sim}A$, then by Theorem 18 $\mathbf{MP} \vdash \Rightarrow {\sim}^4 A^k$ and $\mathbf{MP} \vdash \Rightarrow {\sim}^4({\sim}A)^k$, i.e. ${\sim}^4 A^k (\equiv k(A))$ is a provable contradiction in **MP**. □

Consequently, **MP** has, in a sense, no less advantage than **CP** in producing provable contradictions. This suggests that the properties classical/intuitionistic negation adds to minimal negation, such as the law of excluded middle or explosion, have only a limited effect for deriving provable contradictions.

5 Comparison with the Quarter Turn Operator

5.1 Classical Case

In **CP**, a triple negation ${\sim}^3$ gives the following equivalences:

Fig. 1 Rotation of the values

A	$\sim A$
t	**n**
b	**t**
n	**f**
f	**b**

A	$\circlearrowleft A$
t	**b**
b	**f**
n	**t**
f	**n**

- $\sim^3(A \wedge B) \leftrightarrow (\sim^3 A \wedge \sim^3 B)$
- $\sim^3(A \vee B) \leftrightarrow (\sim^3 A \vee \sim^3 B)$
- $\sim^3(A \to B) \leftrightarrow (A \to \sim^3 B)$

It thus resembles the *conflation* operator of Fitting (1991) except that we do not have $\sim^3\sim^3 A \leftrightarrow A$. In the semantics for **CP** given in Omori and Wansing (2018), the truth tables of Fig. 1 are given for $\sim$ and the *quarter turn* operator $\circlearrowleft$ Ruet (1996).[6] It is observed in Omori and Wansing (2018) that $\sim$ rotates the values of **FDE**-style four-valued semantics in the opposite direction to $\circlearrowleft$. This immediately implies that $\circlearrowleft$ coincides with $\sim^3$ and $\sim$ coincides with $\circlearrowleft^3$. The correspondence can be to our advantage, because the equivalences above show that $\circlearrowleft$ makes a system more contra-classical than $\sim$ (when it is seen as a negation). One methodology to produce provable contradictions for $\sim$ then is to obtain it for $\circlearrowleft$ and then make a translation.

The proof theory of Ruet's operator has been investigated by Belikov et al. (2022) under the name of *connegation*. The history of this type of operator can be traced back to the *cyclical* negation of Post; see Karpenko (2017), Post (1921) for the details.

In what follows, we first define a classical system with $\circlearrowleft$ taken as primitive (i.e. in $\mathcal{L}_{\circlearrowleft}$). As we shall see, this system is only a slight variant to the system **dCP** in Belikov et al. (2022), and as such we adopt the same name. It will be established that we can embed the provable contradictions of each system (**dCP**, **CP**) into the other system. We then extend the idea to **IP** by introducing an intuitionistic system **dIP** and show an analogous result, this time with an additional help of the negative translations in the previous section.

Definition 10 (**dCP**) The system **dCP** in $\mathcal{L}_{\circlearrowleft}$ is defined by (Ax), (mCut)–(mR→) and the following rules.

$$\circlearrowleft p \Rightarrow \circlearrowleft p \ (\text{Ax}\circlearrowleft)$$

$$\frac{\circlearrowleft A, \circlearrowleft B, \Gamma \Rightarrow \Delta}{\circlearrowleft(A \wedge B), \Gamma \Rightarrow \Delta}\ (\text{mL}\circlearrowleft\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, \circlearrowleft A \qquad \Gamma \Rightarrow \Delta, \circlearrowleft B}{\Gamma \Rightarrow \Delta, \circlearrowleft(A \wedge B)}\ (\text{mR}\circlearrowleft\wedge)$$

$$\frac{\circlearrowleft A, \Gamma \Rightarrow \Delta \qquad \circlearrowleft B, \Gamma \Rightarrow \Delta}{\circlearrowleft(A \vee B), \Gamma \Rightarrow \Delta}\ (\text{mL}\circlearrowleft\vee) \qquad \frac{\Gamma \Rightarrow \Delta, \circlearrowleft C, \circlearrowleft D}{\Gamma \Rightarrow \Delta, \circlearrowleft(C \vee D)}\ (\text{mR}\circlearrowleft\vee)$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad \circlearrowleft B, \Sigma \Rightarrow \Pi}{\circlearrowleft(A \to B), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{mL}\circlearrowleft\to) \qquad \frac{A, \Gamma \Rightarrow \Delta, \circlearrowleft C}{\Gamma \Rightarrow \Delta, \circlearrowleft(A \to C)}\ (\text{mR}\circlearrowleft\to)$$

[6] Sometimes the symbol $\circlearrowright$ is used instead.

$$\frac{\Gamma \Rightarrow, \Delta, C}{\circlearrowleft\circlearrowleft C, \Gamma \Rightarrow \Delta}\ (\mathrm{mL}\circlearrowleft\circlearrowleft) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \circlearrowleft\circlearrowleft A}\ (\mathrm{mR}\circlearrowleft\circlearrowleft)$$

It is straightforward to check that **dCP** $\vdash A, \Gamma \Rightarrow \Delta, A$.

Remark 1 The difference between the current system and that of Belikov et al. (2022) is in the rules for $\circlearrowleft \rightarrow$, for which they have:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, \circlearrowleft^3 B}{\Gamma \Rightarrow \Delta, \circlearrowleft^3(A \rightarrow B)} \qquad \frac{A, \circlearrowleft^3 B, \Gamma \Rightarrow \Delta}{\circlearrowleft^3(A \rightarrow B), \Gamma \Rightarrow \Delta}$$

It is easy to see that the systems become equivalent by (mCut) and the derivability of $\circlearrowleft^5 A \Rightarrow \circlearrowleft A$ and $\circlearrowleft A \Rightarrow \circlearrowleft^5 A$ in both of the systems, if (Ax) and (Ax$\circlearrowleft$) in the formulation of Belikov et al. (2022) are replaced [7] with $A \Rightarrow A$ and $\circlearrowleft A \Rightarrow \circlearrowleft A$.

Recall that a rule is said to be *invertible* (w.r.t. a premise) if the derivability of the conclusion of the rule implies that of the premise, i.e. its converse rule (w.r.t. the premise) is admissible. We need the next inversion lemma.

Lemma 20 *(mL$\circ$),(mR$\circ$) for $\circ \in \{\wedge, \vee\}$ and (mR$\rightarrow$) are invertible in both* **CP**-*(mCut) and* **dCP**-*(mCut); (mL$\sim\circ$) and (mR$\sim\circ$) for $\circ \in \{\wedge, \vee, \rightarrow, \sim\}$ are invertible in* **CP**-*(mCut); and (mL$\circlearrowleft\circ$), (mR$\circlearrowleft\circ$) for $\circ \in \{\wedge, \vee, \circlearrowleft\}$ and (mR$\circlearrowleft\rightarrow$) are invertible in* **dCP**-*(mCut). Moreover, the following rules are admissible in* **CP**-*(mCut) and* **dCP**-*(mCut).*

1. For both **CP**-(mCut) and **dCP**-(mCut)

- *If* $\vdash A \rightarrow B, \Gamma \Rightarrow \Delta$, *then* $\vdash \Gamma \Rightarrow \Delta, A$ *and* $\vdash B, \Gamma \Rightarrow \Delta$.

2. For **dCP**-(mCut):

- *If* $\vdash \circlearrowleft(A \rightarrow B), \Gamma \Rightarrow \Delta$, *then* $\vdash \Gamma \Rightarrow \Delta, A$ *and* $\vdash \circlearrowleft B, \Gamma \Rightarrow \Delta$.

Proof By induction on the depth of derivation. □

We now define a pair of translations between $\mathcal{L}_\sim$ and $\mathcal{L}_\circlearrowleft$.

Definition 11 We define translations $()^t : \mathcal{L}_\sim \longrightarrow \mathcal{L}_\circlearrowleft$ and $()^s : \mathcal{L}_\circlearrowleft \longrightarrow \mathcal{L}_\sim$ by the next clauses.

$$p^t = p. \qquad p^s = p.$$
$$(A \circ B)^t = A^t \circ B^t. \qquad (A \circ B)^s = A^s \circ B^s.$$
$$(\sim A)^t = \circlearrowleft^3 A^t. \qquad (\circlearrowleft A)^s = \sim^3 A^s.$$

where $\circ \in \{\wedge, \vee, \rightarrow\}$.

[7] This modification is necessary, as the general versions are not derivable in the formulation of Belikov et al. (2022). We can show by induction on the depth of derivation that the derivability of $\Gamma \Rightarrow \Delta, \circlearrowleft(p \rightarrow p)$ (or $\circlearrowleft(p \rightarrow p), \Gamma \Rightarrow \Delta$) implies that of $\Gamma \Rightarrow \Delta$; so $\circlearrowleft(p \rightarrow p) \Rightarrow \circlearrowleft(p \rightarrow p)$ is underivable if their system is non-trivial. This indeed is the case since it is weaker than the modification suggested in Remark 1 and thus than our formulation of **dCP** whose non-triviality is assured by Corollary 23.

Lemma 21 *The following statements hold.*

1. *If* **CP**-*(mCut)* $\vdash \Gamma \Rightarrow \Delta$, *then* **dCP**-*(mCut)* $\vdash \Gamma^t \Rightarrow \Delta^t$.
2. *If* **dCP**-*(mCut)* $\vdash \Gamma \Rightarrow \Delta$, *then* **CP**-*(mCut)* $\vdash \Gamma^s \Rightarrow \Delta^s$.

Proof By induction on the depth of derivation. □

Lemma 22 *The following statements hold.*

1. *If* **CP**-*(mCut)* $\vdash (A^t)^s, \Gamma \Rightarrow \Delta$, *then* **CP**-*(mCut)* $\vdash A, \Gamma \Rightarrow \Delta$.
2. *If* **CP**-*(mCut)* $\vdash \Gamma \Rightarrow \Delta, (A^t)^s$, *then* **CP**-*(mCut)* $\vdash \Gamma \Rightarrow \Delta, A$.
3. *If* **dCP**-*(mCut)* $\vdash (A^s)^t, \Gamma \Rightarrow \Delta$, *then* **dCP**-*(mCut)* $\vdash A, \Gamma \Rightarrow \Delta$.
4. *If* **dCP**-*(mCut)* $\vdash \Gamma \Rightarrow \Delta, (A^s)^t$, *then* **dCP**-*(mCut)* $\vdash \Gamma \Rightarrow \Delta, A$.

Proof By induction on the complexity of A. For 1.–2., we must simultaneously show:

1'. If **CP**-(mCut) $\vdash {\sim}(A^t)^s, \Gamma \Rightarrow \Delta$, then **CP**-(mCut) $\vdash {\sim}A, \Gamma \Rightarrow \Delta$.
2'. If **CP**-(mCut) $\vdash \Gamma \Rightarrow \Delta, {\sim}(A^t)^s$, then **CP**-(mCut) $\vdash \Gamma \Rightarrow \Delta, {\sim}A$.

Here we consider the case for 1'. when A is $B \to C$. Since $((B \to C)^t)^s \equiv (B^t)^s \to (C^t)^s$, assume **CP**-(mCut) $\vdash {\sim}((B^t)^s \to (C^t)^s), \Gamma \Rightarrow \Delta$. Then by Lemma 20, we have **CP**-(mCut) $\vdash (B^t)^s, {\sim}(C^t)^s, \Gamma \Rightarrow \Delta$. Now by I.H. for 1. and 1'., **CP**-(mCut) $\vdash B, {\sim}C, \Gamma \Rightarrow \Delta$. Thus by (mL$\sim\to$) we conclude **CP**-(mCut) $\vdash {\sim}(B \to C), \Gamma \Rightarrow \Delta$. Other cases are similarly argued. For 3.–4., the induction must simultaneously show:

3'. If **dCP**-(mCut) $\vdash \circlearrowleft(A^s)^t, \Gamma \Rightarrow \Delta$, then **dCP**-(mCut) $\vdash \circlearrowleft A, \Gamma \Rightarrow \Delta$.
4'. If **dCP**-(mCut) $\vdash \Gamma \Rightarrow \Delta, \circlearrowleft(A^s)^t$, then **dCP**-(mCut) $\vdash \Gamma \Rightarrow \Delta, \circlearrowleft A$.

Otherwise the argument is analogous. □

One consequence of the above lemmas is that (mCut) is eliminable in our formulation of **dCP**.

Corollary 23 *(mCut) is admissible in* **dCP**-*(mCut).*

Proof Suppose **dCP** $\vdash \Gamma \Rightarrow \Delta$. Then it is easy to check that Lemma 21 2. holds even with the presence of (mCut), because it is admissible in **CP**-(mCut) (Kamide, 2017, Theorem 2). Then Lemma 21 1. implies **dCP**-(mCut) $\vdash (\Gamma^s)^t \Rightarrow (\Delta^s)^t$. Use Lemma 22 to conclude **dCP**-(mCut) $\vdash \Gamma \Rightarrow \Delta$. □

We also obtain the desired correspondence between the two systems.

Theorem 24 *The following statements hold.*

1. **CP** $\vdash \Gamma \Rightarrow \Delta$ *if and only if* **dCP** $\vdash \Gamma^t \Rightarrow \Delta^t$.
2. **dCP** $\vdash \Gamma \Rightarrow \Delta$ *if and only if* **CP** $\vdash \Gamma^s \Rightarrow \Delta^s$.

Proof For 1., by Lemma 21, it suffices to show that **dCP** $\vdash \Gamma^t \Rightarrow \Delta^t$ implies **CP** $\vdash \Gamma \Rightarrow \Delta$. This holds by Lemma 22. The argument for 2. is analogous. □

We can now use the correspondence to obtain a characterisation of provable contradictions in the systems.

Corollary 25 *The following statements hold.*

1. $\mathbf{CP} \vdash \Rightarrow A \wedge {\sim}A$ *if and only if* $\mathbf{dCP} \vdash \Rightarrow \circlearrowleft^3 A^t \wedge \circlearrowleft^4 A^t$.
2. $\mathbf{dCP} \vdash \Rightarrow A \wedge \circlearrowleft A$ *if and only if* $\mathbf{CP} \vdash \Rightarrow {\sim}^3 A^s \wedge {\sim}^4 A^s$.

Proof 1. is an immediate consequence of Theorem 24 as well as $\mathbf{dCP} \vdash C \Rightarrow \circlearrowleft^4 C$ and $\mathbf{dCP} \vdash \circlearrowleft^4 C \Rightarrow C$. The case for 2. is analogous. □

Corollary 25 allows us to obtain some provable contradictions in **CP** relatively simply. For instance, it is easy to note that $\mathbf{dCP} \vdash \Rightarrow p \vee \circlearrowleft^2 p$ and $\mathbf{dCP} \vdash \Rightarrow \circlearrowleft(p \vee \circlearrowleft^2 p)$. Then Corollary 25 tells that ${\sim}^3(p \vee {\sim}^6 p)$ is a provable contradiction in **CP**. In addition, Corollary 25 clarifies that $\circlearrowleft$, which makes the system seem quite contra-classical if understood as a negation, does not produce more provable contradictions than $\sim$.

5.2 Intuitionistic Case

Let us turn our attention back to **IP**. The main questions here is to what extent it is possible to have a corresponding system like for **CP**. A natural starting point is to restrict **dCP** and move on to a single-conclusion system.

Definition 12 (**dIP**) The system **dIP** in $\mathcal{L}_{\circlearrowleft}$ is defined by (Ax), (Ax$\circlearrowleft$), (Cut)–(R$\to$) and the following rules.

$$\frac{\circlearrowleft A, \circlearrowleft B, \Gamma \Rightarrow C}{\circlearrowleft(A \wedge B), \Gamma \Rightarrow C}\ (\mathrm{L}\circlearrowleft\wedge) \qquad \frac{\Gamma \Rightarrow \circlearrowleft A \qquad \Gamma \Rightarrow \circlearrowleft B}{\Gamma \Rightarrow \circlearrowleft(A \wedge B)}\ (\mathrm{R}\circlearrowleft\wedge)$$

$$\frac{\circlearrowleft A, \Gamma \Rightarrow C \qquad \circlearrowleft B, \Gamma \Rightarrow C}{\circlearrowleft(A \vee B), \Gamma \Rightarrow C}\ (\mathrm{L}\circlearrowleft\vee) \qquad \frac{\Gamma \Rightarrow \circlearrowleft C_i}{\Gamma \Rightarrow \circlearrowleft(C_1 \vee C_2)}\ (\mathrm{R}\circlearrowleft\vee)$$

$$\frac{\Gamma \Rightarrow A \qquad \circlearrowleft B, \Sigma \Rightarrow C}{\circlearrowleft(A \to B), \Gamma, \Sigma \Rightarrow C}\ (\mathrm{L}\circlearrowleft\to) \qquad \frac{A, \Gamma \Rightarrow \circlearrowleft C}{\Gamma \Rightarrow \circlearrowleft(A \to C)}\ (\mathrm{R}\circlearrowleft\to)$$

$$\frac{\Gamma \Rightarrow C}{\circlearrowleft\circlearrowleft C, \Gamma \Rightarrow}\ (\mathrm{L}\circlearrowleft\circlearrowleft) \qquad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \circlearrowleft\circlearrowleft A}\ (\mathrm{R}\circlearrowleft\circlearrowleft)$$

where $i \in \{1, 2\}$.

We shall establish some basic properties of **dIP**.

Definition 13 We define a translation g of formulas in $\mathcal{L}_{\circlearrowleft}$ into those of $\mathcal{L}_{\neg}$ by the following clauses:

$$g(p) = p. \qquad g(\circlearrowleft(A \wedge B)) = g(\circlearrowleft A) \wedge g(\circlearrowleft B).$$
$$g(A \circ B) = g(A) \circ g(B). \qquad g(\circlearrowleft(A \vee B)) = g(\circlearrowleft A) \vee g(\circlearrowleft B).$$
$$g(\circlearrowleft p) = p'. \qquad g(\circlearrowleft(A \to B)) = g(A) \to g(\circlearrowleft B).$$
$$g(\circlearrowleft\circlearrowleft A) = \neg g(A).$$

where $\circ \in \{\wedge, \vee, \to\}$.

Theorem 26 *(Cut) is admissible in* **dIP**.

Proof The argument is analogous to Kamide (2017, Theorem 11,12) except that we need to use g instead of f in Definition 4. □

Theorem 27 **dCP** $\vdash \Rightarrow A$ *if and only if* **dIP** $\vdash \Rightarrow \circlearrowleft^4 A$.

Proof Analogous to (Kamide, 2017, Theorem 19). □

The translations t and s do not work for **IP** and **dIP**, because, for instance, (a) **IP** $\vdash \sim(p \wedge q) \Rightarrow \sim p \vee \sim q$ but **dIP** $\nvdash \circlearrowleft^3(p \wedge q) \Rightarrow \circlearrowleft^3 p \vee \circlearrowleft^3 q$; and (b) **dIP** $\vdash \circlearrowleft(p \vee q) \Rightarrow \circlearrowleft p \vee \circlearrowleft q$ but **IP** $\nvdash \sim^3(p \vee q) \Rightarrow \sim^3 p \vee \sim^3 q$, as can be checked easily by searching possible derivations. Nonetheless, it is still possible to give a restricted version of Corollary 25.

Corollary 28 *The following statements hold.*

1. **IP** $\vdash \Rightarrow \sim^2 A \wedge \sim^3 A$ *if and only if* **dIP** $\vdash \Rightarrow \circlearrowleft^5 A^t \wedge \circlearrowleft^6 A^t$.
2. **dIP** $\vdash \Rightarrow \circlearrowleft^2 A \wedge \circlearrowleft^3 A$ *if and only if* **IP** $\vdash \Rightarrow \sim^5 A^s \wedge \sim^6 A^s$.

Proof (i) If **IP** $\vdash \Rightarrow \sim^2 A \wedge \sim^3 A$ then so is it derivable in **CP**. Hence by Theorem 24, **dCP** $\vdash \Rightarrow \circlearrowleft^6 A^t \wedge \circlearrowleft^9 A^t$. This is equivalent to **dCP** $\vdash \Rightarrow \circlearrowleft A^t \wedge \circlearrowleft^2 A^t$. Then by Theorem 27 and the distributivity of $\circlearrowleft^4$ over conjunction, we obtain **dIP** $\vdash \Rightarrow \circlearrowleft^5 A^t \wedge \circlearrowleft^6 A^t$. The converse direction is argued along the same path, this time using the equivalence between $\sim^6$ and $\sim^2$. The argument for (ii) is analogous. □

On the other hand, it is not possible to obtain Corollary 25 fully for **IP** and **dIP**. In order to see this, note that **dIP** $\vdash \Rightarrow \circlearrowleft^6(p \wedge \circlearrowleft^6 p) \wedge \circlearrowleft^7(p \wedge \circlearrowleft^6 p)$ and so we can show **dIP** $\vdash \Rightarrow \circlearrowleft^3(\sim(p \wedge \sim^2 p))^t \wedge \circlearrowleft^4(\sim(p \wedge \sim^2 p))^t$. However **IP** $\nvdash \Rightarrow \sim(p \wedge \sim^2 p)$. Similarly, since it holds that **IP** $\vdash \Rightarrow \sim^3(p \vee \sim^6 p) \wedge \sim^4(p \vee \sim^6 p)$, we have **IP** $\vdash \Rightarrow \sim^3(p \vee \circlearrowleft^2 p)^s \wedge \sim^4(p \vee \circlearrowleft^2 p)^s$. However **dIP** $\nvdash \Rightarrow p \vee \circlearrowleft^2 p$.

Remark 2 If we write both $\sim\sim$ and $\circlearrowleft\circlearrowleft$ as $\neg$ (representing intuitionistic negation), then the above proof establishes that **IP** $\vdash \Rightarrow \neg A$ iff **dIP** $\vdash \Rightarrow \neg A^t$ as well as **dIP** $\vdash \Rightarrow \neg A$ iff **IP** $\vdash \Rightarrow \neg A^s$. Hence the intuition that $\sim$ and $\circlearrowleft$ represent rotations of values from opposite sides can be seen to be partially alive: for instance, in **CP**, applying $\sim$ to ($\sim$-free) A corresponds to applying $\circlearrowleft$ three times to it. In **IP**, applying $\sim$ to $\neg A$ (i.e. $\sim^3 A \equiv \neg\sim A$) corresponds to applying $\circlearrowleft$ three times to it (i.e. $\circlearrowleft^5 A \equiv \neg\circlearrowleft^3 A \equiv \neg(\sim A)^t$).

Another possible restriction is to disallow applications of certain rules.

Proposition 29 *The following statements hold.*

1. *If* **IP**-*(Cut)* $\vdash \Gamma \Rightarrow A$ *with no applications of* $(L{\sim}\wedge)$ *or* $(L{\sim}{\to}1)$*, then* **dIP** $\vdash \Gamma^t \Rightarrow A^t$.
2. *If* **dIP**-*(Cut)* $\vdash \Gamma \Rightarrow A$ *with no applications of* $(L{\circlearrowleft}\vee)$*, then* **IP** $\vdash \Gamma^s \Rightarrow A^s$.

Proof For 1., by induction on the depth of derivation. For instance, when the last rule applied is an instance of $(L{\sim}{\to}2)$,

$$\frac{\sim B, \Gamma \Rightarrow C}{\sim(A \to B), \Gamma \Rightarrow C}$$

then by I.H. we have a derivation on **IP** of $\circlearrowleft^3 B^t, \Gamma^t \Rightarrow C^t$. Then since **dIP**-(Cut) $\vdash \circlearrowleft^3(A^t \to B^t) \Rightarrow \circlearrowleft^3 B^t$, by the admissibility of (Cut) we infer $\circlearrowleft^3(A^t \to B^t), \Gamma^t \Rightarrow C^t$. 2. is argued likewise. □

Corollary 30 *If* **dIP** $\vdash \Rightarrow A \wedge \circlearrowleft A$ *and there is no occurrence of disjunction in A, then* **IP** $\vdash \Rightarrow \sim^3 A^s \wedge \sim^4 A^s$.

Proof From Proposition 29 and the fact that if a disjunction occurs in a derivation of **dIP**-(Cut), it must occur in the endsequent. □

For example, we can show **dIP** $\vdash ((p \wedge \circlearrowleft p) \to p) \wedge \circlearrowleft((p \wedge \circlearrowleft p) \to p)$, and the corollary then tells that $\sim^3((p \wedge \sim^3 p) \to p)$ is a provable contradiction in **IP**.

6 Some Subsystems of MP

In this section, we shall look at some examples of subsystems of **MP**, mainly ones which do not prove a contradiction, to point out certain limits for weakening the property of the double negation. We first give a general characterisation for a set of formulas to contain a contradictory pair, assuming that it satisfies **N4**-like conditions.

Theorem 31 *Let L be a set of formulas in $\mathcal{L}_\sim$ such that:*

1. $A \notin L$ *for some* A.
2. $A_1 \wedge A_2 \in L \Rightarrow A_i \in L$ *for* $i \in \{1, 2\}$.
3. $A, A \to B \in L \Rightarrow B \in L$.
4. $A \in L \Rightarrow A[p/B] \in L$.
5. $A \vee B \in L \Rightarrow A \in L$ *or* $B \in L$.
6. $\sim(A \wedge B) \in L \Rightarrow \sim A \vee \sim B \in L$.
7. $\sim(A \vee B) \in L \Rightarrow \sim A \wedge \sim B \in L$.
8. $\sim(A \to B) \in L \Rightarrow A \wedge \sim B \in L$.

Then $A, \sim A \in L$ for some A if and only if there is B such that

- $\sim B, \sim\sim B \in L$.
- *for any C such that $|C| < |\sim B|$, either $C \notin L$ or $\sim C \notin L$.*

Proof The ‘if’ direction is immediate. For the ‘only if’ direction, we show the contrapositive. So suppose for all B, if $\sim B, \sim\sim B \in L$ then there is C such that $|C| < |\sim B|$ and $C, \sim C \in L$. We shall argue by induction on the complexity of A that $A, \sim A \in L$ leads to contradiction.

If A is p or $\sim p$, then $A, \sim A \in L$ implies $\sim p \in L$. By the condition 4. for L, we infer $\sim(D \to D) \in L$ and so $D \in L$ for all D by the conditions 2. and 8.. This contradicts the condition 1..

If A is $D \wedge E$, then $D \wedge E, \sim(D \wedge E) \in L$ and so $D, E \in L$ by the condition 2.. Also $\sim D \vee \sim E \in L$ and so either $\sim D \in L$ or $\sim E \in L$ by the conditions 5. and 6.. Consequently either $D, \sim D \in L$ or $E, \sim E \in L$, But by I.H. both lead to a contradiction. The case when A is $D \vee E$ is analogous.

If A is $D \to E$, then $D \to E, \sim(D \to E) \in L$ and so $D, \sim E \in L$ by the condition 8.. Also $D, D \to E \in L$ means $E \in L$ by the condition 3, so $E, \sim E \in L$ and by I.H. we obtain a contradiction.

If A is $\sim D$, then the case $|\sim D| = 0$ is already treated. Otherwise, by our initial supposition there has to be a formula C of lower complexity such that $C, \sim C \in L$, to which we can apply I.H.. □

6.1 Systems with Double Negation Rules

We will formulate our examples in this section axiomatically. For the first example, we keep the axiom schema $(A \to {\sim\sim}A) \to {\sim\sim}A$ in **MP**, but the other double negation axiom schema is weakened to a rule corresponding to $(A \leftrightarrow B) \to ({\sim\sim}A \leftrightarrow {\sim\sim}B)$[8].

Definition 14 (**H-WP**) The system **H-WP** is defined from **H-MP** by replacing the axiom schema $(A \leftrightarrow B) \to ({\sim\sim}B \leftrightarrow {\sim\sim}A)$ by:

$$\frac{A \leftrightarrow B}{{\sim\sim}A \leftrightarrow {\sim\sim}B}$$

We check that **H-WP** satisfies the disjunction property, using the technique of the *Aczel slash* Aczel (1968).

Definition 15 (Aczel slash) We define the notion of $|A$ for formulas in $\mathcal{L}_{\sim}$ by the following clauses.

$$\begin{aligned} |p &\Leftrightarrow \mathbf{H\text{-}WP} \vdash p. & |{\sim}p &\Leftrightarrow \mathbf{H\text{-}WP} \vdash {\sim}p.\\ |A \wedge B &\Leftrightarrow |A \text{ and } |B. & |{\sim}(A \wedge B) &\Leftrightarrow |{\sim}A \text{ or } |{\sim}B.\\ |A \vee B &\Leftrightarrow |A \text{ or } |B. & |{\sim}(A \vee B) &\Leftrightarrow |{\sim}A \text{ and } |{\sim}B.\\ |A \to B &\Leftrightarrow (|A \text{ implies } |B) & |{\sim}(A \to B) &\Leftrightarrow |A \text{ and } |{\sim}B.\\ &\quad \text{and } \mathbf{H\text{-}WP} \vdash A \to B. & |{\sim\sim}A &\Leftrightarrow \mathbf{H\text{-}WP} \vdash {\sim\sim}A. \end{aligned}$$

Lemma 32 *$|A$ if and only if **H-WP** $\vdash A$.*

Proof The left-to-right direction is shown by induction on the complexity of formulas. The right-to-left direction is shown by induction on the depth of derivation. In particular, for the axiom schema $(A \to {\sim\sim}A) \to {\sim\sim}A$, it suffices to show that $|A \to {\sim\sim}A$ implies $|{\sim\sim}A$. The former implies **H-WP** $\vdash A \to {\sim\sim}A$, so **H-WP** $\vdash {\sim\sim}A$ and thus $|{\sim\sim}A$, as required. For the rule $\frac{A \leftrightarrow B}{{\sim\sim}A \leftrightarrow {\sim\sim}B}$, if $|{\sim\sim}A$ then **H-WP** $\vdash {\sim\sim}A$ and so **H-WP** $\vdash {\sim\sim}B$. Thus $|{\sim\sim}B$ and consequently $|{\sim\sim}A \to {\sim\sim}B$. Similarly, $|{\sim\sim}B \to {\sim\sim}A$ and so $|{\sim\sim}A \leftrightarrow {\sim\sim}B$. □

Proposition 33 *If **H-WP** $\vdash A \vee B$ then either **H-WP** $\vdash A$ or **H-WP** $\vdash B$.*

Proof If **H-WP** $\vdash A \vee B$, then by Lemma 32 $|A \vee B$ and so $|A$ or $|B$. By the same lemma, this implies either **H-WP** $\vdash A$ or **H-WP** $\vdash B$. □

We now show that no formula of the form ${\sim\sim}A$ is derivable in the system, using a Gödel-Dummett style infinite-valued matrix (cf. e.g. Ono (2019)).

[8] In the following, we shall assume that **H-MP** is axiomatised with these double negation axiom schemata, rather than with $(A \to B) \to ({\sim\sim}B \to {\sim\sim}A)$. The equivalence between the two axiomatisations is again checkable straightforwardly.

Definition 16 Let $T = \{2^{-i} : i \in \mathbb{N}\} \cup \{0\}$. Let v be an assignment of values $v^+(p), v^-(p) \in T$ to each propositional variable p. For compound formulas, we set:

$$v^+(A \wedge B) = min(v^+(A), v^+(B)) \qquad v^-(A \wedge B) = max(v^-(A), v^-(B)).$$
$$v^+(A \vee B) = max(v^+(A), v^+(B)) \qquad v^-(A \vee B) = min(v^-(A), v^-(B)).$$
$$v^+(A \to B) = \begin{cases} 1 & \text{if } v^+(A) \leq v^+(B). \\ v^+(B) & \text{otherwise.} \end{cases} \qquad v^-(A \to B) = min(v^+(A), v^-(B)).$$
$$v^+({\sim}A) = v^-(A) \qquad v^-({\sim}A) = \begin{cases} 1 & \text{if } v^+(A)=0. \\ 2^{-(i+1)} & \text{if } v^+(A)=2^{-i}. \end{cases}$$

A formula A is then said to be *valid* if $v^+(A) = 1$ for any assignment v.

Proposition 34 ***H-WP*** $\nvdash {\sim}{\sim}A$ *for any* A.

Proof It is straightforward to check that **H-WP** is sound with respect to the above matrix. For instance, if $v^+(A) = 0$ then $v^+({\sim}{\sim}A) = v^-({\sim}A) = 1$. Otherwise, $v^+(A) = 2^{-i} > 2^{-(i+1)} = v^+({\sim}{\sim}A)$ and so $v^+(A \to {\sim}{\sim}A) = v^+({\sim}{\sim}A)$. So $v^+((A \to {\sim}{\sim}A) \to {\sim}{\sim}A) = 1$ under any assignment v. Also, if $A \leftrightarrow B$ is valid, then for any v we have $v^+(A) = v^+(B)$, from which it also follows that $v^+({\sim}{\sim}A \leftrightarrow {\sim}{\sim}B) = 1$, and so the formula is valid.

Now, it is easy to see that if v is an assignment such that $v^*(p) > 0$ for all p and $* \in \{+, -\}$, then $v^*(A) > 0$ for all A. Hence under this assignment, for no A we have $v^+({\sim}{\sim}A) = 1$. The statement then follows by soundness. □

Corollary 35 **H-WP** *does not have a provable contradiction.*

Proof It readily from Proposition 33 that the set of theorems of **H-WP** satisfies the conditions of Theorem 31. Then the statement follows using Proposition 34. □

This example in particular shows that $(A \to {\sim}{\sim}A) \to {\sim}{\sim}A$ is insufficient by itself to cause negation inconsistency.

At this point, one might wonder what happens for the other way of weakening **MP**, namely to retain $(A \leftrightarrow B) \to ({\sim}{\sim}A \leftrightarrow {\sim}{\sim}B)$ while $(A \to {\sim}{\sim}A) \to {\sim}{\sim}A$ is made into a rule $\dfrac{A \to {\sim}{\sim}A}{{\sim}{\sim}A}$. Let us call this system **H-XP**. We can confirm that this system is negation inconsistent.

Proposition 36 ${\sim}({\sim}(A \wedge {\sim}^2 A) \wedge {\sim}^3(A \wedge {\sim}^2 A))$ *is a provable contradiction in* **H-XP**.

Proof We first observe that **H-XP** $\vdash {\sim}{\sim}(A \wedge {\sim}{\sim}A)$. This follows from $(A \wedge {\sim}{\sim}A) \to (A \leftrightarrow (A \wedge {\sim}{\sim}A))$ by first applying the double negation axiom schema of **H-XP**, giving $(A \wedge {\sim}{\sim}A) \to ({\sim}{\sim}A \to {\sim}{\sim}(A \wedge {\sim}{\sim}A))$. Hence **H-XP** $\vdash (A \wedge {\sim}{\sim}A) \to {\sim}{\sim}(A \wedge {\sim}{\sim}A)$, to which we can apply the double negation rule. This implies then **H-XP** $\vdash {\sim}({\sim}(A \wedge {\sim}^2 A) \wedge {\sim}^3(A \wedge {\sim}^2 A))$. On the other hand, the negation of the formula is obtained by taking ${\sim}(A \wedge {\sim}{\sim}A)$ in place of A in ${\sim}{\sim}(A \wedge {\sim}{\sim}A)$. □

Neg. Inconsistent?	$(A\leftrightarrow B)\to(\sim\sim A\leftrightarrow\sim\sim B)$	$\dfrac{A\leftrightarrow B}{\sim\sim A\leftrightarrow\sim\sim B}$
$(A\to\sim\sim A)\to\sim\sim A$	Yes (**H-MP**)	No (**H-WP**)
$\dfrac{A\to\sim\sim A}{\sim\sim A}$	Yes (**H-XP**)	No

Fig. 2 The effect of changing axioms to rules

Figure 2 summarises what happens to negation inconsistency when we weaken the axioms schemata of **H-MP** corresponding to ($\sim\sim$2) and ($\sim\sim$3) into the corresponding rules. Alternatively, we can also consider the weakening of the axiom schema corresponding to ($\sim\sim$1) as well. In that case, the negation inconsistency is kept even when both of the axiom schemata are weakened to rules: we have $\sim^5(A\wedge\sim^2 A)$ as a provable contradiction, as can be checked by mimicking the derivations in Proposition 9.

7 Informational Interpretation of MP

The introduction by Kamide of the kind of negation we are considering seems to have been motivated more from a technical perspective. Hence it did not necessarily come with a (non-logical) philosophical project to which the formalisation of **CP** and **IP** is dedicated. Nonetheless, it will be of a considerable interest if these logics with unusual features get tied with a robust philosophical interpretation. At the same time, it may be unlikely to immediately reach a definite interpretation, so we shall aim at a more modest goal of enriching our understanding of the logics by suggesting an interpretation from a point of view alternative to the pre-existing one.

As mentioned in the introduction, the double negation in **CP** and **IP** are already explained in Omori and Wansing (2018) in terms of negative concord. This interpretation is however linguistic in character, and a more constructive interpretation is perhaps also beneficial for systems like **IP** and **MP**. A canonical example of such an interpretation is the BHK-interpretation for intuitionistic logic, which explains the meaning of a connective by the proof condition of a formula with the connective as the main connective (see e.g. Troelstra and van Dalen (1988)). For systems related to **N4**, an extended interpretation is often used, in which the explanation is given by the parallel notion of refutation/disproof condition as well (Nelson, 1949; López-Escobar, 1972).

If we allow ourselves to understand by a proof/disproof a construction that provides an evidence in favour of a statement's truth/falsity, BHK-style interpretation can be connected to an interpretation based on the notion of information. The recent proposal of *dimathematism* by Wansing (2022) appears to be particularly suitable for **IP** and **MP**. As observed in that paper, **IP** (and also **MP**) satisfies the qualification of *strong dimathematism*:

> *Strong dimathematism* is an informational view: some languages L and some contradictory L-formulas are such that their truth is supported by every state from every L-model. That is, where $\sim$ is negation, there are L-sentences A, such that for every L-model $\mathfrak{M}$ and every state w from $\mathfrak{M}$, in $\mathfrak{M}$ state w supports the truth

of both A and $\sim A$. Given that a state supports the falsity of A iff it supports the truth of $\sim A$, this is to say that there are some L-formulas A, such that every L-model $\mathfrak{M}$ and state w from $\mathfrak{M}$ are such that in $\mathfrak{M}$ state w supports both the truth and the falsity of A. [Wansing (2022)]

The view thus liberates us from the perhaps constructively untenable notions of truth and falsity by replacing them with the more tangible notions of supports of truth and falsity. An interpretation of **MP** (or **IP**) from this viewpoint then has to answer how the falsity of $\sim A$ is supported.

7.1 Double Negation and Strong Support

Limiting our attention to **MP**, we have to explain the condition:

$$w \Vdash^{-} \sim A \text{ iff } \forall w' \geq w (w' \Vdash^{+} A \Rightarrow w' \in Q).$$

In minimal logic, Q is used to capture the absurdity constant $\bot$, and as a consequence Q is best understood as a set of worlds which disfavour the truth of A. Nonetheless, it is also possible to consider Q as a set of worlds which favour A. Although we in no way expect it to be a definitive interpretation, we tentatively suggest to deem Q as a set of worlds in which the supported formulas obtain a higher informational status or a *strong support*.

Once we understand Q in such a way, then the equivalence above seems less perplexing. One can perhaps defend it on the following ground: it is something unusual if any later state that supports the truth of A is one that is capable of giving a strong support. This regularity challenges the claim that A is false; so the falsity of $\sim A$ is supported,[9]. Then conversely the falsity of $\sim A$ can be taken to be supported only if the above situation holds.

Remark 3 To understand the naïve intuition behind the interpretation, it may help to see each world metaphorically as an agent (ordered by what they support), and Q as the set of ones who are *trustworthy*. Then consider as an example the case of an agent assessing the falsity of the statement 'global warming is not a major issue.'. In order to evaluate the statement, the agent looks at other agents with whom he shares what to support (i.e. later worlds), and discovers that every agent who supports the truth of the statement 'global warming is a major issue' is trustworthy.[10] Based on this evidence, the agent supports the falsity of the original statement. This method of evaluation seems rather natural, as we often refer to people making relevant claims and check their credibility, when we judge a claim.

[9] Needless to say, the acceptability of this reasoning depends much on the relationship between the supports of truth and falsity. This is left unspecified in **MP** which is perhaps unsatisfactory. On the other hand, **N4** may be argued to have a similar problem as well.

[10] Grigory Olkhovikov pointing out that it can happen that such an agent is in fact in the minority among the agents in Q, in which case concluding the support of falsity may seem odd. One possible defence to the objection would be to argue that the rule of majority should not play a role when it comes to evaluating the opinions of trustworthy agents.

It is hard to apply a similar kind of interpretation to **IP**, because in the intuitionistic setting, Q has to be empty. We can not, as a result, have a 'positive' reading of the support of falsity condition for negation, unlike in **MP**. This may be hinting that **MP**, which in a way liberates from the view that the double negation must resemble a demi-negation, is more preferable than **IP** from the informational point of view.

7.2 Extending the Interpretation

One possible objection to the support of falsity condition for negation in **MP**, when understood in the above manner, is that $w \Vdash^- {\sim}A$ holds even when there is no $w' \geq w$ such that $w' \Vdash^+ A$. It may be suggested that the falsity of ${\sim}A$ should not be supported when the truth of A will not ever be supported. Accepting this criticism motivates one to make the following modification to the forcing conditions in the semantics for **MP**:

$$w \Vdash^- {\sim}A \text{ iff } \forall w' \geq w((w' \Vdash^+ A \Rightarrow w' \in Q) \text{ and } \exists x \geq w'(x \Vdash^+ A)).$$

This condition ensures that that the aforementioned situation does not occur, and also preserves the upward closure of the forcing relation. Let us denote the consequence for the modified semantics by $\vDash_m$.

We can show using a standard method (see e.g. van Dalen (2014)) that the modification does not affect the constructive character of the semantics.

Proposition 37 *If* $\vDash_m A \vee B$ *then* $\vDash_m A$ *or* $\vDash_m B$.

Proof We show the contrapositive. So suppose $\nvDash_m A$ and $\nvDash_m B$. Then there are countermodels $\mathcal{M}_i = ((W_i, \leq_i, Q_i), \mathcal{V}_i)$ for $i \in \{1, 2\}$ such that there are $w_i \in W_i$ with $\mathcal{M}_1, w_1 \nVdash^+ A$ and $\mathcal{M}_2, w_2 \nVdash^+ B$. We may assume $W_1 \cap W_2 = \emptyset$. Define a new model $\mathcal{M} = ((W, \leq, Q), \mathcal{V})$ where:

- $W = W_1 \cup W_2 \cup \{g\}$, where g is a new element.
- $w \leq w'$ if $w, w' \in W_i$ and $w \leq_i w'$, or $w = g$.
- $Q = Q_1 \cup Q_2$.
- $\mathcal{V}^*(p) = \mathcal{V}_1^*(p) \cup \mathcal{V}_2^*(p)$ for $* \in \{+, -\}$.

Then for $i \in \{1, 2\}$ and $* \in \{+, -\}$, it holds that for all $w \in W_i$, we have $\mathcal{M}, w \Vdash^* A$ iff $\mathcal{M}_i, w \Vdash^* A$. Consider as an example the case when A is ${\sim}B$, $w \in W_1$ and $* = -$. Then $\mathcal{M}, w \Vdash^- {\sim}B$ iff $\forall w' \geq w((\mathcal{M}, w' \Vdash^+ B \Rightarrow w' \in Q)$ and $\exists x \geq w'(\mathcal{M}, x \Vdash^+ B))$. By I.H., this is equivalent to $\forall w' \geq w((\mathcal{M}_1, w' \Vdash^+ B \Rightarrow w' \in Q_1)$ and $\exists x \geq w'(\mathcal{M}_1, x \Vdash^+ B))$ and thus to $\mathcal{M}_1, w \Vdash^- {\sim}B$.

Now we have $\mathcal{M}, w_1 \nVdash^+ A$, $\mathcal{M}, w_2 \nVdash^+ B$ and so $\mathcal{M}, g \nVdash^+ A \vee B$; thus $\nvDash_m A \vee B$. □

On the other hand, if one is interested in a negation-inconsistent system, then the semantics is not adequate.

Proposition 38 $\nvDash_m {\sim}{\sim}A$ *for any* A.

Proof Consider a model with $Q = \emptyset$. In the model, $\mathcal{M}, w \Vdash^+ {\sim}{\sim}A$ iff $\forall w' \geq w(w' \nVdash^+ A$ and $\exists x \geq w'(x \Vdash^+ A))$. Hence ${\sim}{\sim}A$ is never forced in such a model. □

Corollary 39 *There is no formula A such that $\vDash_m A$ and $\vDash_m {\sim}A$.*

Proof Let $L = \{A : \vDash_m A\}$. Then the statement follows from Theorem 31 as well as Proposition 37 and 38. □

Remark 4 It may be suggested that Proposition 38 can be avoided by putting an extra condition that $Q \neq \emptyset$. In that scenario, when $W = \{w\}$ we obtain the equivalence $w \Vdash^- {\sim}A$ if and only if $w \Vdash^+ A$. In a model based on such a frame[11], if $w \Vdash^+ A$ and $w \Vdash^+ {\sim}A$, where $|A| \geq 1$, we can always find a strict subformula B of A such that $w \Vdash^+ B$ and $w \Vdash^+ {\sim}B$. It follows then that we do not have such A in a model where $\mathcal{V}^+(p) = \mathcal{V}^-(p) = \emptyset$. This gives a countermodel for $A \wedge {\sim}A$ in the suggested semantics, and so we again end up with a negation-consistent system.

The above observations suggest that it is essential for the negation inconsistency of **MP** that the falsity of negation can be 'vacuously' supported. Is it possible to make sense of such a support?

One reply to this question might be to note that a vacuous support does not mean it contains no information. Observing that the truth of A is never going to be supported provides enough information to rule out a counterexample to the support of falsity for ${\sim}A$, in the sense of a later world which supports the truth of A but not in Q. Hence a vacuous support is something more than a non-support. Nonetheless, one might argue that we cannot regard the observation as a support, because it gives less information than the non-vacuous cases. However, this should not be too much of a problem, as our interpretation presupposes that supports come in different degrees.

8 Concluding Remarks

This paper explored some subsystems of Kamide's logics **CP** and **IP**, in order to analyse the cause of the non-trivial negation inconsistency of the systems. We formulated the system **MP**, whose double negation can be seen as the minimal negation. **MP**, as we found out, is not only negation inconsistent, but also is a system into which any provable contradiction in **CP** can be embedded, preserving the status of provable contradiction. This result was also used to show how provable contradictions in **CP** and **IP** correspond to those of the systems with quarter turn operator, **dCP** and **dIP**.

As the Hilbert-style system **H-MP** clarifies, $(A \to {\sim}{\sim}A) \to {\sim}{\sim}A$ is isolated as the sole axiom schema inducing the contra-classicality. Then to understand the role of the schema further, we looked at some subsystems of **MP** in which double negation axiom schemata are turned into rules. We observed the status of negation inconsistency depends on which axiom schema is weakened.

Lastly, we attempted to give an interpretation of **MP** from an informational, and more specifically dimathematic point of view. We suggested that the support of falsity for ${\sim}A$ can perhaps be understood as the regularity that the support of truth for A being always strong. We then observed that the negation consistency is lost if some stronger conditions are imposed.

[11] A model of this kind is equivalent to the one for the system **CLoNs** in Batens et al. (1999).

MP may perhaps not rival **N4** in terms of naturalness, but given the interpretation, the difference might be not as wide as it appears at first sight. The crucial question then is under which contexts it can work as a complementary system. This needs to be explored further in order to justify the importance of Kamide's negation and the contradictions it creates.

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