



Filtered Belief Revision: Syntax and Semantics

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Abstract

In an earlier paper [Rational choice and AGM belief revision, *Artificial Intelligence*, 2009] a correspondence was established between the set-theoretic structures of revealed-preference theory (developed in economics) and the syntactic belief revision functions of the AGM theory (developed in philosophy and computer science). In this paper we extend the re-interpretation of those structures in terms of one-shot belief revision by relating them to the trichotomous attitude towards information studied in Garapa (Rev Symb Logic, 1–21, 2020) where information may be either (1) fully accepted or (2) rejected or (3) taken seriously but not fully accepted. We begin by introducing the syntactic notion of *filtered belief revision* and providing a characterization of it in terms of a mixture of both AGM revision and contraction. We then establish a correspondence between the proposed notion of filtered belief revision and the above-mentioned set-theoretic structures, interpreted as semantic *partial* belief revision structures. We also provide an interpretation of the trichotomous attitude towards information in terms of the degree of implausibility of the information.

Keywords Credible information · Allowable information · AGM belief revision · Choice function · Plausibility order

1 Introduction

The dominant paradigm in belief revision is the so-called AGM theory [Alchourrón et al. (1985) and Gärdenfors (1988)], which is a syntactic theory that represents the belief state of an agent as a pair $(K, *)$, where K is a consistent and deductively closed set of formulas in a propositional language (interpreted as the agent's initial beliefs) and

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$*$: $\Phi \rightarrow 2^\Phi$ (where Φ denotes the set of formulas and 2^Φ the set of subsets of Φ) is a function that associates with every formula $\phi \in \Phi$ (interpreted as new information) a set $K * \phi \subseteq \Phi$, representing the agent's revised beliefs in response to information ϕ . If the function $*$ satisfies a set of six properties, known as the basic AGM postulates, then it is called a *basic AGM belief revision function*, while if it satisfies two additional properties (the so-called supplementary postulates) then it is called a *supplemented AGM belief revision function*. One of the six basic postulates is the *success axiom*, according to which every item of information ϕ is incorporated into the revised beliefs: $\phi \in K * \phi$. In the past twenty years a large literature has emerged centered on relaxing the success axiom: the so-called *non-prioritized belief revision* approach [for surveys of this literature see Hansson (1999b), Fermé and Hansson (2011, Section 6.2, 2018, Chapter 8)]. This literature acknowledges the fact that there may be some items of information that an agent is not disposed to accept, for a number of reasons:

1. the agent may doubt the reliability of the source of the information,
2. the information may be in conflict with some highly entrenched beliefs of the agent,
3. the agent might judge the information to be too far-fetched or implausible.¹

In Case 1 it is not the intrinsic content of the information that leads the agent to reject it, but the lack of trust in its source; thus the same information may be accepted if it originates from source A but rejected if it originates from source B. For a recent investigation of the role of trust in belief revision see Booth and Hunter (2018).

In Case 2 it is the conflict of the information with some core beliefs of the agent that leads to the information being rejected: such core beliefs are given a privileged status by the agent and are essentially immune to revision. This possibility was first studied in Makinson (1997) and Fermé and Hansson (2001).²

In Case 3, the agent's belief state is described not only by the pair $(K, *)$, but also by a partition of the set of sentences into two sets: the set Φ_C of *credible* sentences and the set Φ_R of *rejected* sentences. The belief revision function $*$: $\Phi \rightarrow 2^\Phi$ is then such that $K * \phi = K$ if $\phi \in \Phi_R$ (if the information is rejected then beliefs remain unchanged), while the success postulate applies to credible information: $\phi \in K * \phi$ if $\phi \in \Phi_C$. This line of inquiry is pursued in Hansson et al. (2001).³ This approach is taken a step further in Garapa (2020) [and, independently, Bonanno (2019)] with the introduction of a third, and intermediate, possibility, namely formulas that are credible but with a lower level of credibility:

¹ For example, during the U.S. Presidential campaign in 2016, a "news" item appeared on several internet sites under the title "Pope Francis shocks world, endorses Donald Trump for president". While, perhaps, some people believed this claim, many discarded it as "fake news". In today's political climate, many items of "information" are routinely rejected as not credible.

² A related notion is that of "revision by comparison" studied in Fermé and Rott (2004). See also Rott (2012).

³ See also Fermé et al. (2003), Garapa et al. (2018), Booth et al. (2012), Booth et al. (2014), Boutilier et al. (1998) and Schlechta (1997). In Fermé and Hansson (1999) the possibility of *selective revision* is explored, according to which only part of the information received is accepted, while the rest is rejected; in a similar vein, Booth and Hunter (2018) propose the notion of "trust-sensitive revision", based on the idea that only part of the information is accepted, namely the part relative to which the source is qualified or competent.

When revising by a sentence that is considered to be at the second level of credibility, that sentence is not incorporated but all the beliefs that are inconsistent with it are removed. The intuition underlying this behavior is that, the belief is not credible enough to be incorporated in the agent’s belief set, but creates in the agent some doubt making him/her remove all the beliefs that are inconsistent with it. (Garapa (2020), p. 2.)

Garapa (2020) calls this trichotomous stance towards information “two level credibility-limited revision”. To illustrate these different attitudes towards a particular item of information, consider the case of three husbands, A, B and C, each of whom believes that his wife is, and has always been, faithful to him. Each husband is approached by a trusted friend who tells him “your wife is having an affair with another man”.

- Husband A accepts the information, comes to believe that his wife is unfaithful and files for divorce.
- Husband B is unshakable in his belief that his wife is faithful and discards the information as not credible.
- Husband C’s reaction is somewhere in the middle: he abandons his belief in his wife’s faithfulness, but does not go all the way to believing that she is unfaithful; in other words, he becomes open-minded about the possibility of her being unfaithful (and, perhaps, hires a private investigator to resolve the uncertainty).

Thus we partition the set of sentences Φ into *three* sets: the set Φ_C of *credible* sentences [these correspond to the “high credibility” sentences in Garapa (2020)], the set Φ_R of *rejected* sentences and the set Φ_A of sentences that we call “*allowable*” [these correspond to the “low credibility” sentences in Garapa (2020)].⁴ For the latter we have that $\{\phi, \neg\phi\} \cap (K * \phi) = \emptyset$, that is, when informed that ϕ the agent believes neither ϕ nor $\neg\phi$ (in other words, she is open-minded towards both ϕ and $\neg\phi$). While the approach in Garapa (2020) is entirely syntactic (see Sect. 2 for details), our focus is on the semantics (Sect. 3). However, in Sect. 2 we start with the AGM-style syntactic approach and put forward the notion of *filtered belief revision* and provide a characterization of it in terms of basic AGM belief revision, as follows (Proposition 1). Let $\circ : \Phi \rightarrow 2^\Phi$ be a filtered belief-revision function (Definition 2, Sect. 2); then for some basic AGM belief-revision function $* : \Phi \rightarrow 2^\Phi$, $\forall\phi \in \Phi$:

$$K \circ \phi = \begin{cases} K & \text{if } \phi \in \Phi_R \\ K * \phi & \text{if } \phi \in \Phi_C \\ K \cap (K * \phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{1}$$

Thus

1. if information ϕ is rejected then the original beliefs are maintained,
2. if ϕ is credible then revision is performed according to the basic AGM postulates and

⁴ Note that if $\Phi_A = \emptyset$ then we fall back to the binary case of “credibility limited revision” of Hansson et al. (2001).

3. if ϕ is allowable then revision is performed by contracting the original beliefs by the negation of ϕ (by the Harper identity [Harper (1976)] the contraction by $\neg\phi$ coincides with taking the intersection of the original beliefs with the revision by ϕ).⁵ Note that this implies that the underlying contraction function satisfies the *recovery* axiom: see Sect. 5 for a discussion of this point).

It is worth stressing that the focus of this paper is on one-shot belief revision (see Sect. 5, for a discussion of this point).

In Sect. 3 we turn to the main focus of this paper by proposing a semantics for filtered belief revision that extends the semantics for AGM belief revision put forward in Bonanno (2009), based on the structures of revealed-preference theory in economics.⁶ Revealed-preference theory considers *choice structures* $\langle \Omega, \mathcal{E}, f \rangle$ consisting of a non-empty set Ω (whose elements are interpreted as possible alternatives to choose from), a collection \mathcal{E} of subsets of Ω (interpreted as possible menus, or choice sets) and a function $f : \mathcal{E} \rightarrow 2^\Omega$ (2^Ω denotes the set of subsets of Ω), representing choices made by the agent, conditional on each menu. Given this interpretation, the following restriction on the function f is a natural requirement (the alternatives chosen from menu E should be elements of E): $\forall E \in \mathcal{E}$,

$$f(E) \subseteq E. \quad (2)$$

The objective of revealed-preference theory is to characterize choice structures that can be “rationalized” by a total pre-order \succsim on Ω , interpreted as a preference relation,⁷ in the sense that, for every $E \in \mathcal{E}$, $f(E)$ is the set of most preferred alternatives in E : $f(E) = \{\omega \in E : \omega \succsim \omega', \forall \omega' \in E\}$. In Bonanno (2009) a re-interpretation of choice structures in terms of *semantic partial belief revision functions* was put forward. A *model* of $\langle \Omega, \mathcal{E}, f \rangle$ (where Ω is now thought of as a set of *states*) is obtained by adding a valuation V that assigns to every atomic formula p the set of states at which p is true. Truth of an arbitrary formula at a state is then defined as usual. Given a model $\langle \Omega, \mathcal{E}, f, V \rangle$, the *initial beliefs* of the agent are taken to be the set of formulas ϕ such that $f(\Omega) \subseteq \|\phi\|$, where $\|\phi\|$ denotes the truth set of ϕ ; thus $f(\Omega)$ is interpreted as the set of states that are initially considered possible. The events (sets of states) in $\mathcal{E} \subseteq 2^\Omega$ are interpreted as possible items of information. If ϕ is a formula such that $\|\phi\| \in \mathcal{E}$, the *revised belief upon learning that ϕ* is defined as the set of formulas ψ such that $f(\|\phi\|) \subseteq \|\psi\|$. Thus the event $f(\|\phi\|)$ is interpreted as the set of states that are considered possible after learning that ϕ is the case. In light of this interpretation, condition (2) above corresponds to the *success postulate* of AGM theory. A model of a structure $\langle \Omega, \mathcal{E}, f \rangle$ thus gives rise to a partial syntactic belief revision function whose domain is typically a proper subset of the set of formulas. The objective of Bonanno (2009) was to find necessary and sufficient conditions on the structure that

⁵ Letting $K \div \neg\phi$ denote the contraction of belief set K by $\neg\phi$, the Harper identity states that $K \div \neg\phi = K \cap (K * \phi)$. Note that if $\phi \in K$ then $K \circ \phi = K$; in fact, it follows from the AGM axioms (see Sect. 2) that $K * \phi = K$ and thus also $K \cap (K * \phi) = K$.

⁶ See, for example, Rott (2001) and Suzumura (1983).

⁷ Thus the intended meaning of $\omega \succsim \omega'$ is “alternative ω is considered to be at least as good as alternative ω' ”.

guarantee the existence of an AGM belief-revision function that extends the partial belief revision function obtained from an arbitrary model of it.

In this paper we continue the above analysis by removing restriction (2). First of all, we allow for some events—in the set of potential items of information \mathcal{E} —to be treated as not credible, so that

$$f(E) = f(\Omega) \text{ if } E \in \mathcal{E} \text{ is rejected as not credible.} \tag{3}$$

Secondly, for information $E \in \mathcal{E}$ which is credible we postulate the success property (2):

$$f(E) \subseteq E \text{ if } E \in \mathcal{E} \text{ is credible.}$$

Finally, we also consider a third type of information, which is taken seriously but not given the same status as credible information, and call it *allowable*; we capture this possibility by means of the following condition, which says that allowable information is not ruled out by the revised beliefs:

$$f(E) \cap E \neq \emptyset \text{ if } E \in \mathcal{E} \text{ allowable.} \tag{4}$$

We model credibility, allowability and rejection by partitioning the set \mathcal{E} of possible items of information into three sets: the set \mathcal{E}_C of *credible* items, the set \mathcal{E}_A of *allowable* items and the set \mathcal{E}_R of *rejected* items. Thus we consider *partial belief revision structures* (PBRs for short) $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ such that: (1) $\Omega \neq \emptyset$, (2) $\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R$ are mutually disjoint subsets of 2^Ω with $\Omega \in \mathcal{E}_C$ and $\emptyset \notin \mathcal{E}_C \cup \mathcal{E}_A$ and (3) $f : \mathcal{E} \rightarrow 2^\Omega$ (where $\mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$) is such that $f(\Omega) \neq \emptyset$ and (a) if $E \in \mathcal{E}_R$ then $f(E) = f(\Omega)$, (b) if $E \in \mathcal{E}_C$ then $\emptyset \neq f(E) \subseteq E$ and (c) if $E \in \mathcal{E}_A$ then $f(E) \cap E \neq \emptyset$.

As explained above, we use valuations to link syntax and semantics and obtain, from every model of a PBRs, a syntactic partial belief revision function. We then define a PBRs to be *basic-AGM consistent* if, for every model of it, the associated partial belief revision function can be extended to a full-domain belief-revision function $\circ : \Phi \rightarrow 2^\Phi$ such that, for some basic AGM belief-revision function $* : \Phi \rightarrow 2^\Phi$, equation (1) is satisfied. Proposition 3 in Sect. 3 provides necessary and sufficient conditions for a PBRs to be basic-AGM consistent. In Sect. 4 we provide an interpretation of credible, allowable and rejected information in terms of the degree of implausibility of the information and then extend the analysis of the previous section to supplemented AGM consistency. Section 5 contains a discussion of several aspects of the proposed framework and of related literature.

2 The syntactic approach

Let Φ be the set of formulas of a propositional language based on a countable set At of atomic formulas.⁸ We write $\vdash \phi$ to denote that the formula ϕ is a tautology, that

⁸ Thus Φ is defined recursively as follows: if $p \in \text{At}$ then $p \in \Phi$ and if $\phi, \psi \in \Phi$ then $\neg\phi \in \Phi$ and $(\phi \vee \psi) \in \Phi$. The connectives \wedge, \rightarrow and \leftrightarrow are then defined as usual.

is, a theorem of Propositional Logic. Given a subset $K \subseteq \Phi$, its PL-deductive closure $[K]^{PL}$ (where ‘PL’ stands for Propositional Logic) is defined as follows: $\psi \in [K]^{PL}$ if and only if there exist $\phi_1, \dots, \phi_n \in K$ (with $n \geq 0$) such that $\vdash (\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$.⁹ A set $K \subseteq \Phi$ is *consistent* if $[K]^{PL} \neq \Phi$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to $[K]^{PL}$). A set $K \subseteq \Phi$ is *deductively closed* if $K = [K]^{PL}$.

Let $K \subseteq \Phi$ be a consistent and deductively closed set of formulas representing the agent’s initial beliefs. A (syntactic) *belief revision function* is a function $* : \Phi \rightarrow 2^\Phi$ that associates with every formula $\phi \in \Phi$ (thought of as new information) a set $K * \phi \subseteq \Phi$ (thought of as the revised beliefs upon learning that ϕ). A belief revision function $* : \Phi \rightarrow 2^\Phi$ is called a *basic AGM function* if it satisfies the first six of the following properties and it is called a *supplemented AGM function* if it satisfies all of them. The following properties are known as the *AGM postulates*: $\forall \phi, \psi \in \Phi$,

(AGM1) (<i>closure</i>)	$K * \phi = [K * \phi]^{PL}$.
(AGM2) (<i>success</i>)	$\phi \in K * \phi$.
(AGM3) (<i>inclusion</i>)	$K * \phi \subseteq [K \cup \{\phi\}]^{PL}$.
(AGM4) (<i>vacuity</i>)	if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq K * \phi$.
(AGM5) (<i>consistency</i>)	$K * \phi = \Phi$ if and only if ϕ is a contradiction.
(AGM6) (<i>extensionality</i>)	if $\vdash \phi \leftrightarrow \psi$ then $K * \phi = K * \psi$.
(AGM7) (<i>superexpansion</i>)	$K * (\phi \wedge \psi) \subseteq [(K * \phi) \cup \{\psi\}]^{PL}$.
(AGM8) (<i>subexpansion</i>)	if $\neg\psi \notin K * \phi$, then $[(K * \phi) \cup \{\psi\}]^{PL} \subseteq K * (\phi \wedge \psi)$.

AGM1 requires the revised belief set to be deductively closed.

AGM2 postulates that the information be believed.

AGM3 says that beliefs should be revised minimally, in the sense that no new formula should be added unless it can be deduced from the information received and the initial beliefs.¹⁰

AGM4 says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs.

AGM5 requires the revised beliefs to be consistent, unless the information ϕ is a contradiction (that is, unless $\vdash \neg\phi$).

AGM6 requires that if ϕ is propositionally equivalent to ψ then the result of revising by ϕ be identical to the result of revising by ψ .

AGM1-AGM6 are called the *basic AGM postulates*, while AGM7 and AGM8 are called the *supplementary AGM postulates*.

AGM7 and AGM8 are a generalization of AGM3 and AGM4 that concern composite belief revisions of the form $K * (\phi \wedge \psi)$:

The idea is that, if K is to be changed minimally so as to include two sentences ϕ and ψ , such a change should be possible by first revising K with respect to ϕ

⁹ Note that, if F is a set of formulas, $\psi \in [F \cup \{\phi\}]^{PL}$ if and only if $(\phi \rightarrow \psi) \in [F]^{PL}$.

¹⁰ Note that (see Footnote 9) $\psi \in [K \cup \{\phi\}]^{PL}$ if and only if $(\phi \rightarrow \psi) \in K$ (since, by hypothesis, $K = [K]^{PL}$).

and then expanding $K * \phi$ by ψ —provided that ψ does not contradict the beliefs in $K * \phi$ (Gärdenfors & Rott, 1995, p. 54).

For an extensive discussion of the rationale behind the AGM postulates see Gärdenfors (1988), Gärdenfors and Rott (1995).

We now extend the notion of belief revision by allowing the agent to discriminate among different items of information.

Definition 1 Let Φ be the set of formulas of a propositional language. A *credibility partition* is a partition $\{\Phi_C, \Phi_A, \Phi_R\}$ of Φ such that

1. Φ_C is the set of *credible* formulas and is such that
 - (a) if $\vdash \phi$ then $\phi \in \Phi_C$,
 - (b) if $\phi \in \Phi_C$ then ϕ is consistent,
 - (c) if $\phi \in \Phi_C$ and $\vdash \phi \leftrightarrow \psi$ then $\psi \in \Phi_C$, that is, Φ_C is closed under logical equivalence.
2. Φ_A is the (possibly empty) set of *allowable* formulas. We assume that if $\phi \in \Phi_A$ then ϕ is consistent and that Φ_A is closed under logical equivalence.
3. Φ_R is the set of *rejected* formulas, which contains (at least) all the contradictions.

The properties that we have postulated for Φ_C and Φ_A are called “element consistency” and “closure under logical equivalence” in Hansson et al. (2001). There are other, natural, properties to consider, but for the time being we restrict attention to a *minimal set* of properties that are sufficient for the representation result of Proposition 1. We will address the issue of what are additional “natural” properties for the sets Φ_C , Φ_A and Φ_R in Sect. 5.

Following Hansson et al. (2001), given a belief set K , we use the symbol \circ to denote a general belief revision function based on K (that is, a function $\circ : \Phi \rightarrow 2^\Phi$ that associates with every $\phi \in \Phi$ a new belief set $K \circ \phi$) and the symbol $*$ to denote a (basic or supplemented) AGM belief-revision function based on K .

Definition 2 Let K be a consistent and deductively closed set of formulas (representing the initial beliefs) and $\{\Phi_C, \Phi_A, \Phi_R\}$ a credibility partition of Φ (Definition 1). A belief-revision function $\circ : \Phi \rightarrow 2^\Phi$ is called a *filtered belief revision function* (based on K and $\{\Phi_C, \Phi_A, \Phi_R\}$) if it satisfies the following properties: $\forall \phi, \psi \in \Phi$,

- (F1) if $\phi \in \Phi_R$ then $K \circ \phi = K$,
otherwise
- (F2) if $\neg\phi \notin K$ then
 - (a) if $\phi \in \Phi_C$ then $K \circ \phi = [K \cup \{\phi\}]^{PL}$,
 - (b) if $\phi \in \Phi_A$ then $K \circ \phi = K$,
- (F3) if $\neg\phi \in K$ then $K \circ \phi$ is consistent and deductively closed and
 - (a) if $\phi \in \Phi_C$ then $\phi \in K \circ \phi$,
 - (b) if $\phi \in \Phi_A$ then $K \circ \phi \subseteq K \setminus \{\neg\phi\}$
and $[(K \circ \phi) \cup \{\neg\phi\}]^{PL} = K$,
- (F4) if $\vdash \phi \leftrightarrow \psi$ then $K \circ \phi = K \circ \psi$.

By (F1), if information ϕ is rejected ($\phi \in \Phi_R$), then the original beliefs are preserved. The next two properties deal with the case where $\phi \notin \Phi_R$.

(F2) says that if, initially, the agent did not believe $\neg\phi$, then (a) if ϕ is credible then the new beliefs are given by the expansion of K by ϕ , while (b) if ϕ is allowable then the agent does not change her beliefs.

(F3) says that if, initially, the agent believed $\neg\phi$, then (a) if ϕ is credible, then the agent switches from believing $\neg\phi$ to believing ϕ , (b) if ϕ is allowable, then the agent *suspends her belief in $\neg\phi$ in a conservative way*, that is, she contracts her belief set by $\neg\phi$ in such a way as to retain as much as possible of her original beliefs and if she were to re-introduce $\neg\phi$ into her revised beliefs and close under logical consequence then she would go back to her initial beliefs. As noted in the introduction, this means that contraction satisfies the so-called *recovery* postulate. See Sect. 5 for a discussion of this assumption.

By (F4) belief revision satisfies extensionality: if ϕ is logically equivalent to ψ then revision by ϕ coincides with revision by ψ .

The following proposition provides a characterization of filtered belief revision in terms of basic AGM belief revision. The proof is given in Appendix A.

Proposition 1 *Let K be a consistent and deductively closed set of formulas, $\{\Phi_C, \Phi_A, \Phi_R\}$ a credibility partition of Φ and $\circ : \Phi \rightarrow 2^\Phi$ a belief revision function based on K and $\{\Phi_C, \Phi_A, \Phi_R\}$. Then the following are equivalent:*

- (A) \circ is a filtered belief revision function,
- (B) there exists a basic AGM belief revision function $* : \Phi \rightarrow 2^\Phi$ such that, $\forall \phi \in \Phi$, equation (1) holds.

We take the belief state of the agent to be represented by the triple $(K, \{\Phi_C, \Phi_A, \Phi_R\}, \circ)$: initial beliefs, credibility partition and revision function.¹¹

Garapa (2020) shows that the credibility partition can be derived from the pair (K, \circ) by defining $\Phi_C = \{\phi : \phi \in K \circ \phi\}$, $\Phi_A = \{\phi : \neg\phi \notin K \circ \phi\} \setminus \Phi_C$ and $\Phi_R = \Phi \setminus (\Phi_C \cup \Phi_A)$. Indeed Garapa (2020) proves the following equivalence.

Proposition 2 Garapa (2020, Theorem 3.14, p.10) *Let K be a consistent and deductively closed set of formulas and $\circ : \Phi \rightarrow 2^\Phi$ a belief revision function based on K . Then the following are equivalent:*

(A) \circ satisfies the following properties:

1. *weak relative success: either $\phi \in K \circ \phi$ or $K \circ \phi \subseteq K$*
2. *closure: $K \circ \phi = [K \circ \phi]^{PL}$*
3. *inclusion: $K \circ \phi \subseteq [K \cup \{\phi\}]^{PL}$*
4. *consistency preservation: if K is consistent then $K \circ \phi$ is consistent*
5. *weak vacuity: if $\neg\phi \notin K$ then $K \subseteq K \circ \phi$*
6. *extensionality: if $\vdash \phi \leftrightarrow \psi$ then $K \circ \phi = K \circ \psi$*
7. *N-relative success: if $\neg\phi \in K \circ \phi$ then $K \circ \phi = K$*

¹¹ Thus, returning to the example of the three husbands given in the introduction, letting p be the sentence “my wife is faithful”, we have that (1) for all three husbands, $p \in K$, (2) for Husband A, $\neg p \in \Phi_C$, while, for Husband B, $\neg p \in \Phi_R$ and, for Husband C, $\neg p \in \Phi_A$ so that (3) for Husband A, $\neg p \in K \circ \neg p$, for Husband B, $p \in K \circ \neg p = K$ and, for Husband C, $p \notin K \circ \neg p$ but also $\neg p \notin K \circ \neg p$.

8. *containment*: if K is consistent then $K \cap ((K \circ \phi) \cup \{\phi\})^{PL} \subseteq K \circ \phi$.

(B) there exists a basic AGM belief revision function $* : \Phi \rightarrow 2^\Phi$ and a credibility partition $\{\Phi_C, \Phi_A, \Phi_R\}$ of Φ (Definition 1) such that Eq. (1) is satisfied.

While in the AGM approach the domain of a belief revision function is the entire set of formulas (that is, every formula is viewed as a potential item of information), in the following sections we will consider the possibility that the domain of a belief revision function is a *subset* of the set of formulas Φ . Let K be a consistent and deductively closed set of formulas, representing the agent’s initial beliefs, and let $\Psi \subseteq \Phi$ be a set of formulas representing possible items of information. Let $\circ : \Psi \rightarrow 2^\Phi$ be a function that associates with every formula $\psi \in \Psi$ a set $K \circ \psi \subseteq \Phi$. If $\Psi \neq \Phi$ then \circ is called a *partial* belief revision function, while if $\Psi = \Phi$ then it is called a *full-domain* belief revision function. If \circ' is a partial belief revision function with domain Ψ and \circ is a full-domain belief revision function, we say that \circ is an *extension* of \circ' if, for all $\psi \in \Psi$, $K \circ \psi = K \circ' \psi$. The rationale for considering partial belief revision functions is discussed in Sect. 5.

We now turn to the main focus of this paper, namely the semantic structures outlined in the introduction.

3 Semantics: belief revision structures

Definition 3 A *partial belief revision structure* (PBRS) is a tuple $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ such that:

1. $\Omega \neq \emptyset$,¹²
2. $\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R$ are mutually disjoint subsets of 2^Ω with $\Omega \in \mathcal{E}_C$ and $\emptyset \notin \mathcal{E}_C \cup \mathcal{E}_A$,¹³
3. $f : \mathcal{E} \rightarrow 2^\Omega$ (where $\mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$) is such that
 - (a) $f(\Omega) \neq \emptyset$,
 - (b) if $E \in \mathcal{E}_R$ then $f(E) = f(\Omega)$,
 - (c) if $E \in \mathcal{E}_C$ then $\emptyset \neq f(E) \subseteq E$,
 - (d) if $E \in \mathcal{E}_A$ then $f(E) \cap E \neq \emptyset$.

Next we turn to the notion of a model, or interpretation, of a PBRS.

Fix a propositional language based on a countable set At of atomic formulas and let Φ be the set of formulas. A *valuation* is a function $V : \text{At} \rightarrow 2^\Omega$ that associates with every atomic formula $p \in \text{At}$ the set of states at which p is true. Truth of an arbitrary formula at a state is defined recursively as follows ($\omega \models \phi$ means that formula ϕ is true at state ω):

- (1) for $p \in \text{At}$, $\omega \models p$ if and only if $\omega \in V(p)$,

¹² We do not assume that Ω is finite.

¹³ These sets may be “small”, that is, we do not assume that the union $\mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$ covers the entire set 2^Ω . Note that if $\mathcal{E}_A = \mathcal{E}_R = \emptyset$ then the above definition of PBRS coincides with the definition of choice structure in Bonanno (2009), which we will now call a *simple* PBRS.

- (2) $\omega \models \neg\phi$ if and only if $\omega \not\models \phi$,
- (3) $\omega \models (\phi \vee \psi)$ if and only if either $\omega \models \phi$ or $\omega \models \psi$ (or both).

The truth set of formula ϕ is denoted by $\|\phi\|$, that is, $\|\phi\| = \{\omega \in \Omega : \omega \models \phi\}$.

Given a valuation V , define:¹⁴

$$K = \{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|\} \tag{5}$$

$$\Psi = \{\phi \in \Phi : \|\phi\| \in \mathcal{E}\} \tag{6}$$

$$\circ_\Psi : \Psi \rightarrow 2^\Phi \text{ given by: } K \circ_\Psi \phi = \{\chi \in \Phi : f(\|\phi\|) \subseteq \|\chi\|\}. \tag{7}$$

Since $f(\Omega)$ is interpreted as the set of states that the individual initially considers possible, (5) is the initial belief set. It is straightforward to show that K is consistent (since, by 3(a) of Definition 3, $f(\Omega) \neq \emptyset$) and deductively closed.

(6) is the set of formulas that are potential items of information.

(7) is the partial belief revision function encoding the agent’s disposition to revise her beliefs in response to items of information in Ψ (for $E \in \mathcal{E}$, $f(E)$ is interpreted as the set of states that the individual considers possible after receiving information represented by event E).

Definition 4 Given a PBRS $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$, a *model* or *interpretation* of it is obtained by adding to it a pair $(\{\Phi_C, \Phi_A, \Phi_R\}, V)$ (where $\{\Phi_C, \Phi_A, \Phi_R\}$ is a credibility partition of Φ and V is a valuation) such that, $\forall \phi \in \Phi$,

1. if $\|\phi\| \in \mathcal{E}_C$ then $\phi \in \Phi_C$,
2. if $\|\phi\| \in \mathcal{E}_A$ then $\phi \in \Phi_A$,
3. if $\|\phi\| \in \mathcal{E}_R$ then $\phi \in \Phi_R$.

Definition 5 A partial belief revision structure $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ is *basic-AGM consistent* if, for every model of it, letting \circ_Ψ be the corresponding partial belief revision function (defined by (7)), there exist

1. a full-domain belief revision function $\circ : \Phi \rightarrow 2^\Phi$ that extends \circ_Ψ (that is, for every $\psi \in \Psi$, $K \circ \psi = K \circ_\Psi \psi$) and
2. a basic AGM belief revision function $* : \Phi \rightarrow 2^\Phi$ such that, $\forall \phi \in \Phi$, equation (1) is satisfied.

Equivalently, by Proposition 1, a PBRS is basic-AGM consistent if, for every model of it, there exists a filtered belief revision function (Definition 2) that extends the partial belief revision function generated by the model.

The following proposition gives necessary and sufficient conditions for a PBRS to be basic-AGM consistent. The proof is given in Appendix B.

Proposition 3 *Let $\mathcal{C} = \langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ be a partial belief revision structure. Then the following are equivalent:*

- (A) \mathcal{C} is basic-AGM consistent (Definition 5).

¹⁴ All these objects, including the truth sets of formulas, are dependent on the valuation V and thus ought to be indexed by it; however, in order to keep the notation simple, we omit the subscript V .

(B) \mathcal{C} satisfies the following properties: for every $E \in \mathcal{E}_C \cup \mathcal{E}_A$,

1. if $E \cap f(\Omega) \neq \emptyset$ then
 - (a) if $E \in \mathcal{E}_C$ then $f(E) = E \cap f(\Omega)$,
 - (b) if $E \in \mathcal{E}_A$ then $f(E) = f(\Omega)$,
2. if $E \cap f(\Omega) = \emptyset$ and $E \in \mathcal{E}_A$ then $f(E) = f(\Omega) \cup E'$ for some $\emptyset \neq E' \subseteq E$.

4 Plausibility and supplemented AGM consistency

In this section we investigate what additional properties a PBRS needs to satisfy in order to obtain a correspondence result analogous to Proposition 3 but involving *supplemented*, rather than basic, AGM belief revision (that is, belief revision functions that satisfy the six basic AGM postulates as well as the two supplementary ones).¹⁵ To obtain a characterization in terms of supplemented AGM belief revision we need to add more structure. In particular, we are interested in determining when a PBRS can be “rationalized” by a plausibility order.¹⁶

In parallel to the partition of the set of sentences Φ into three sets (Φ_C , the set of credible sentences, Φ_A , the set of allowable sentences, and Φ_R , the set of rejected sentences), we postulate a “plausibility-based” partition of the set of states Ω into three sets: the set Ω_C of *credible states*, which—in a precise sense explained below – are the states with low **im**plausibility, the set Ω_A of *allowable states*, which are the states with intermediate **im**plausibility, and the set Ω_R of *rejected states*, which are the states with high **im**plausibility. Note that, throughout this section, we shall *assume that the set of states Ω is finite*.¹⁷

Definition 6 A *plausibility order* is a total pre-order on Ω , that is, a binary relation $\succsim \subseteq \Omega \times \Omega$ which is complete or total ($\forall \omega, \omega' \in \Omega$ either $\omega \succsim \omega'$ or $\omega' \succsim \omega$ or both) and transitive ($\forall \omega, \omega', \omega'' \in \Omega$ if $\omega \succsim \omega'$ and $\omega' \succsim \omega''$ then $\omega \succsim \omega''$). The interpretation of $\omega \succsim \omega'$ is that state ω is at least as plausible as state ω' . We denote by \succ the strict component of \succsim : $\omega \succ \omega'$ if $\omega \succsim \omega'$ and $\omega' \not\succeq \omega$ (thus $\omega \succ \omega'$ means that ω is *more* plausible than ω') and by \sim the equivalence component of \succsim : $\omega \sim \omega'$ if $\omega \succsim \omega'$ and $\omega' \succsim \omega$ (thus $\omega \sim \omega'$ means that ω is *just as* plausible as ω').

For every $F \subseteq \Omega$ we denote by $best_{\succsim} F$ the set of most plausible elements of F , that is,

$$best_{\succsim} F = \{\omega \in F : \omega \succsim \omega', \forall \omega' \in F\}. \tag{8}$$

Given a plausibility order \succsim we can partition Ω into ranked equivalence classes as follows.¹⁸

¹⁵ In Theorem 3.16 (p. 11) Garapa (2020) provides a syntactic characterization that is analogous to Proposition 2 but in terms of supplemented, rather than basic, AGM belief revision.

¹⁶ The notion of a pre-order on the set of possible worlds as a semantics for the AGM theory of belief revision first appeared in Katsuno and Mendelzon (1991).

¹⁷ Alternatively, we could allow Ω to be infinite and restrict attention to *well-founded* plausibility orders.

¹⁸ Plausibility orders are equivalent to a special case of the ordinal ranking functions introduced by Spohn (1988), namely ranking functions $r : \Omega \rightarrow \mathbb{N}$ (where \mathbb{N} denotes the set of non-negative integers) with no gaps, that is, if $r(\omega) = k$ for some $k > 0$ then there exists an ω' such that $r(\omega') = k - 1$. Starting from

– Let Ω_0 be the set of most plausible states in Ω :

$$\Omega_0 = \text{best}_{\succsim} \Omega = \{\omega \in \Omega : \omega \succsim \omega', \forall \omega' \in \Omega\}.$$

– Let Ω_1 be the set of most plausible states in $\Omega \setminus \Omega_0$:

$$\Omega_1 = \text{best}_{\succsim} (\Omega \setminus \Omega_0) = \{\omega \in \Omega \setminus \Omega_0 : \omega \succsim \omega', \forall \omega' \in \Omega \setminus \Omega_0\}.$$

– In general, for $k \geq 1$, let Ω_k be the set of most plausible states in $\Omega \setminus \bigcup_{j=0}^{k-1} \Omega_j$:

$$\Omega_k = \text{best}_{\succsim} \left(\Omega \setminus \bigcup_{j=0}^{k-1} \Omega_j \right) = \left\{ \omega \in \Omega \setminus \bigcup_{j=0}^{k-1} \Omega_j : \omega \succsim \omega', \forall \omega' \in \Omega \setminus \bigcup_{j=0}^{k-1} \Omega_j \right\}.$$

If $\omega \in \Omega_k$ we say that k is the degree of **implausibility** of state ω (thus the most plausible states are those with degree of implausibility 0, the next most plausible states are those with degree of implausibility 1, etc.). Clearly, for every $k \geq 0$, if $\omega, \omega' \in \Omega_k$ then $\omega \sim \omega'$. Let \hat{n} be the largest degree of implausibility, that is, \hat{n} is such that $\Omega_{\hat{n}} \neq \emptyset$ and $\bigcup_{j=0}^{\hat{n}} \Omega_j = \Omega$.

Given two integers m and n such that $0 \leq m \leq n \leq \hat{n}$, let Ω_C be the set of states with degree of implausibility at most m , Ω_A be the set of states with degree of implausibility greater than m but at most n and Ω_R the set of the remaining states:¹⁹

$$\Omega_C = \bigcup_{j=0}^m \Omega_j \quad \Omega_A = \bigcup_{j=m+1}^n \Omega_j \quad \Omega_R = \Omega \setminus (\Omega_C \cup \Omega_A) \quad (9)$$

Definition 7 When a partition $\{\Omega_C, \Omega_A, \Omega_R\}$ of Ω is obtained from a plausibility order \succsim of Ω as explained above, we call it a *plausibility-based partition*; clearly, the following holds:

$$\begin{aligned} &\text{if } \omega \in \Omega_C \text{ and } \omega' \in \Omega_A \cup \Omega_R \text{ then } \omega \succ \omega', \text{ and} \\ &\text{if } \omega \in \Omega_A \text{ and } \omega' \in \Omega_R \text{ then } \omega \succ \omega'. \end{aligned} \quad (10)$$

That is, credible states (those in Ω_C) are more plausible than allowable or rejected states (those in $\Omega_A \cup \Omega_R$) and allowable states (those in Ω_A) are more plausible than rejected states (those in Ω_R).

From now on we will focus on basic-AGM-consistent PBRs, which—in virtue of Definitions 3 and 5 and Proposition 3—can be redefined as follows.

such a ranking function one would define $\Omega_j = \{\omega \in \Omega : r(\omega) = j\}$. Furthermore, when Ω is taken to be the set of maximally consistent sets of sentences, plausibility orders are also equivalent to the systems of spheres introduced by Grove (1988).

¹⁹ If $n = m$ then $\Omega_A = \emptyset$ and if $n = \hat{n}$ then $\Omega_R = \emptyset$.

Definition 8 A basic-AGM-consistent partial belief revision structure (BPBRS) is a tuple $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ such that:

1. $\Omega \neq \emptyset$,
2. $\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R$ are mutually disjoint subsets of 2^Ω with $\Omega \in \mathcal{E}_C$ and $\emptyset \notin \mathcal{E}_C \cup \mathcal{E}_A$,
3. $f : \mathcal{E} \rightarrow 2^\Omega$ (where $\mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$) is such that
 - (a) $f(\Omega) \neq \emptyset$,
 - (b) if $E \in \mathcal{E}_R$ then $f(E) = f(\Omega)$,
 - (c) if $E \in \mathcal{E}_C$ then $\emptyset \neq f(E) \subseteq E$ and if $E \cap f(\Omega) \neq \emptyset$ then $f(E) = E \cap f(\Omega)$,
 - (d) if $E \in \mathcal{E}_A$ then
 - i. if $E \cap f(\Omega) \neq \emptyset$ then $f(E) = f(\Omega)$,
 - ii. if $E \cap f(\Omega) = \emptyset$ then $f(E) = f(\Omega) \cup E'$ for some $\emptyset \neq E' \subseteq E$.

Definition 9 A BPBRS is called *plausibility based* if there is a plausibility-based partition $\{\Omega_C, \Omega_A, \Omega_R\}$ of Ω such that

1. If $E \in \mathcal{E}_C$ then
 - (a) $E \cap \Omega_C \neq \emptyset$,
 - (b) $E \cap \Omega_C \in \mathcal{E}_C$,²⁰
 - (c) $f(E) = f(E \cap \Omega_C) \subseteq \Omega_C$.
2. If $\Omega_A \neq \emptyset$ then $\Omega_A \in \mathcal{E}_A$. Furthermore, if $E \in \mathcal{E}_A$ then
 - (a) $E \cap \Omega_C = \emptyset$,
 - (b) $E \cap \Omega_A \neq \emptyset$,
 - (c) $E \cap \Omega_A \in \mathcal{E}_A$,
 - (d) $f(E) = f(E \cap \Omega_A)$.
3. If $E \in \mathcal{E}_R$ then $E \subseteq \Omega_R$.

Thus,

- by Point 1, if information E has a credible content ($E \cap \Omega_C \neq \emptyset$), then the agent revises her beliefs based exclusively on the credible content of the information ($f(E) = f(E \cap \Omega_C)$) and incorporates it into her revised beliefs ($f(E) \subseteq E \cap \Omega_C$),²¹
- by Point 2, if information E does not have a credible content ($E \cap \Omega_C = \emptyset$) but does not consist entirely of rejected states either ($E \cap \Omega_A \neq \emptyset$), then the agent revises her beliefs based exclusively on the “allowable” content of the information ($f(E) = f(E \cap \Omega_A)$),
- by Point 3, if information E is rejected then it consists entirely of rejected states ($E \subseteq \Omega_R$).

We now turn to the issue of what properties of a plausibility-based BPBRS are necessary and sufficient for *supplemented* AGM consistency. The following definition mirrors Definition 5.

²⁰ Thus, since $\Omega \in \mathcal{E}_C$, $\Omega_C \cap \Omega = \Omega_C$ and $\Omega_C \neq \emptyset$, it follows that $\Omega_C \in \mathcal{E}_C$.

²¹ Since, by 3(c) of Definition 8, $f(E) \subseteq E$, it follows that $f(E) \subseteq E \cap \Omega_C$. In particular, $f(\Omega) = f(\Omega_C) \subseteq \Omega_C$.

Definition 10 A plausibility-based BPBRS $\langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ is *supplemented-AGM consistent* if, for every model $\langle \{\Phi_C, \Phi_A, \Phi_R\}, V \rangle$ of it, letting \circ_Ψ be the corresponding partial belief revision function, there exist

1. a full-domain belief revision function $\circ : \Phi \rightarrow 2^\Phi$ that extends \circ_Ψ (that is, for every $\phi \in \Psi$, $K \circ \phi = K \circ_\Psi \phi$) and
2. two supplemented AGM belief revision functions $*_C : \Phi \rightarrow 2^\Phi$ and $*_A : \Phi \rightarrow 2^\Phi$ such that, for every $\phi \in \Phi$,

$$K \circ \phi = \begin{cases} K & \text{if } \phi \in \Phi_R \\ K *_C \phi & \text{if } \phi \in \Phi_C \\ K \cap (K *_A \phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{11}$$

Definition 11 A plausibility-based BPBRS $\langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ is *rationalizable* if the plausibility order \succsim on Ω on which the partition $\{\Omega_C, \Omega_A, \Omega_R\}$ is based is such that, $\forall E \in \mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$,

$$f(E) = \begin{cases} best_{\succsim} E & \text{if } E \cap \Omega_C \neq \emptyset \\ best_{\succsim} \Omega \cup best_{\succsim} E & \text{if } E \cap \Omega_C = \emptyset \text{ and } E \cap \Omega_A \neq \emptyset. \\ best_{\succsim} \Omega & \text{if } E \subseteq \Omega_R. \end{cases} \tag{12}$$

If (12) is satisfied, we say that the plausibility order \succsim *rationalizes* the BPBRS.

Note that, by 2(c) of Definition 6, $best_{\succsim} \Omega \subseteq \Omega_C$ and thus $best_{\succsim} \Omega = best_{\succsim} \Omega_C$; furthermore, Properties 1 and 2 of Definition 9 are consistent with (12): for example, if $E \cap \Omega_C \neq \emptyset$ then $best_{\succsim} E = best_{\succsim} (E \cap \Omega_C)$, so that $f(E) = f(E \cap \Omega_C)$.

Proposition 5 in Appendix C provides necessary and sufficient conditions for a plausibility-based BPBRS to be rationalizable. Since that result is rather technical and independent of the characterization of supplemented AGM consistency (Proposition 4 below), we have relegated it to an appendix.

The following proposition, which mirrors Proposition 3 of the previous section, extends Propositions 7 and 8 in Bonanno (2009) to the current framework. The proof is given in Appendix D.

Proposition 4 Let $\mathcal{C} = \langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ be a plausibility-based BPBRS where Ω is finite. Then the following are equivalent:

- (A) \mathcal{C} is supplemented AGM consistent,
- (B) \mathcal{C} is rationalizable (Definition 11).

As explained in Bonanno (2009), the assumption that Ω is finite is needed to ensure that $best_{\succsim} E \neq \emptyset$, for every $\emptyset \neq E \subseteq \Omega$. If one strengthens the definition of plausibility order by requiring that, $\forall E \subseteq \Omega$, if $E \neq \emptyset$ then $best_{\succsim} E \neq \emptyset$, that is, if one requires the plausibility order to be well founded, then the assumption of finiteness of Ω can be dropped.

5 Discussion

We investigated a notion of belief revision that, besides acceptance of information, allows for two additional possibilities: (1) that an item of information be discarded as not credible and thus not allowed to affect one's beliefs and (2) that an item of information be treated as a serious possibility without assigning full credibility to it. We started with a syntactic version of this notion, which we called "filtered belief revision", and characterized it in terms of basic AGM belief revision. We then introduced the notion of partial belief revision structure, which provides a simple set-theoretic semantics for belief revision, and provided a characterization of filtered belief revision in terms of properties of these structures. Finally, we considered the notion of rationalizability of a choice structure in terms of a plausibility order and established a correspondence between rationalizability and AGM consistency in terms of the full set of AGM postulates (that is, the six basic postulates together with the supplementary ones). We also provided an interpretation of credibility, allowability and rejection of information in terms of the degree of implausibility of the information.

In this Section we discuss related literature and several aspects of the proposed approach.

1. **Alternative theories of non-prioritized belief revision.** Several theories of "non-prioritized" belief revision have been proposed in which the AGM success postulate is relaxed.²² Fermé and Hansson (1999) put forward a syntactic model of "selective revision" in which it is possible for the agent to accept only part of the information. In a similar vein, Booth and Hunter (2018) develop a semantic model where only part of the information originating from a given source is accepted, namely that part relative to which the source is qualified or competent. Fermé and Rott (2004) [see also Rott (2012)] introduce the notion of "revision by comparison" based on the idea that the degree of entrenchment of the information ϕ ought to be compared with the degree of entrenchment of a reference sentence ψ which is currently believed; the revision is performed according to the principle "see to it that the entrenchment of ϕ is at least as firm as the entrenchment of ψ " (Fermé and Rott (2004), p. 8).²³

Our contribution is along the lines of Hansson et al. (2001) where the set of sentences is partitioned into two sets, the set of credible sentences and the set of rejected sentences: if information is credible then it is incorporated in the new beliefs while if it is rejected then the initial beliefs are unaffected. Garapa (2020) [and, independently, Bonanno (2019)] goes a step further by separating credible sentences into two categories: the high-credibility sentences, for which the success property holds, and the low-credibility sentences which induce a contraction of the original belief set by the negation of the information (instead of 'high credibility' and 'low credibility' we used the terms 'credibility' and 'allowability', respectively). While the analysis in Garapa (2020) is exclusively focused on the

²² For an overview of the literature on non-prioritized belief revision see Chapter 8 of Fermé and Hansson (2018).

²³ If, however, the reference sentence ψ is too weakly entrenched relative to the negation of the input sentence ϕ , then the attempted revision will fail and end up with a contraction of the belief set, more precisely a severe withdrawal of the reference sentence in the sense of Rott and Pagnucco (1999).

AGM-style syntactic approach, our focus has been on set-theoretic structures that can be viewed as the semantic counterpart of *partial* belief revision functions.

2. **Why a semantics based on *partial* belief revision functions?** The semantic structures we considered in Sections 3 and 4 are such that not every conceivable proposition (or event) constitutes a possible item of information, so that – typically – any given interpretation (or valuation) gives rise to a *partial* syntactic belief revision function, that is, one whose domain is not the entire set of sentences Φ . What is the rationale for the proposed semantics? It can be viewed as complementing the traditional syntactic approach by “inverting” it, in the following sense. The syntactic approach starts with an initial belief set and determines, for *every* input sentence ϕ , a new belief set representing the revised beliefs in response to ϕ . On the other hand, the partial belief revision structures considered in Sections 3 and 4 can be thought of as providing a (possibly small) collection of “revision scenarios”, each of which consists of a piece of new information and its corresponding revision result (the set of pairs $(E, f(E))$, for every E in some set $\mathcal{E} \subseteq 2^\Omega$) and the question is “does there exist an AGM belief-revision function that rationalizes those scenarios?” In other words, is the given collection of revision scenarios consistent with a rational belief-revision policy? For example, an agent might introspectively consider how she would react to a number of hypothetical items of information and ask herself “would it be rational for me to revise my beliefs in this way?” Alternatively, one could probe an agent with some hypothetical informational inputs to see how she would react to those inputs and ask the question “are the agent’s answers consistent with a rational disposition to revise her beliefs?” The characterization results of Propositions 3 and 4 provide an answer to these questions.²⁴
3. **Properties of the credibility partition.** In Sect. 2 (Definition 1) we gave a list of properties for a credibility partition $\{\Phi_C, \Phi_A, \Phi_R\}$. The intention there was to identify a *minimal* set of properties that would be sufficient for our results. It should be noted that the two main representation results provided in Garapa (2020) (Theorems 3.14, p. 10 and 3.16, p. 11) also make use of the same two properties that we considered for Φ_C and Φ_A . However, there are other, natural, properties that have been discussed in the literature concerning the set of credible/allowable sentences [Hansson et al. (2001); Fermé et al. (2003); Garapa et al. (2018); Garapa (2020)] and we now turn to a discussion of this issue. Let us start with the set of credible sentences Φ_C . In Definition 1 we required this set to contain only consistent sentences and to be closed under logical equivalence [Hansson et al. (2001) call the first requirement “element consistency” and the second requirement “closure under logical equivalence”]. Another natural property to consider is closure under logical consequence, which Hansson et al. (2001) call “single sentence closure”: if $\phi \in \Phi_C$ and $\vdash \phi \rightarrow \psi$ then $\psi \in \Phi_C$; that is, if ϕ is credible then any sentence that is logically implied by ϕ is also credible. Hansson et al. (2001), Fermé et al. (2003) and Garapa et al. (2018) consider three more

²⁴ There is an alternative setting in which partial belief revision functions make sense: when the language used to define revision inputs is not rich enough to distinguish every state, so that the set \mathcal{E} of possible inputs consists of those subsets that are definable in the language. This is one of the main technical hurdles inspiring the recent work on belief change in Horn Logic: see Delgrande et al. (2018).

properties: “disjunctive completeness”: if $\phi \vee \psi \in \Phi_C$ then either $\phi \in \Phi_C$ or $\psi \in \Phi_C$, “negation completeness”: either $\phi \in \Phi_C$ or $\neg\phi \in \Phi_C$, and “expansive credibility”: if $\phi \notin K$ then $\neg\phi \in \Phi_C$. In order to gain some insight as to whether these four additional properties (closure under logical consequence, disjunctive completeness, negation completeness and expansive credibility) represent natural requirements and—at the same time—confirm the appropriateness of the two assumed properties (element consistency and closure under logical equivalence), let us take a semantic point of view. Consider an arbitrary finite set of states Ω , an arbitrary plausibility partition $\{\Omega_C, \Omega_A, \Omega_R\}$ of Ω (Definition 7) and an arbitrary valuation $V : \Phi \rightarrow 2^\Omega$.

Label a sentence ϕ

- credible if $\|\phi\| \cap \Omega_C \neq \emptyset$,
- allowable if $\|\phi\| \cap \Omega_C = \emptyset$ and $\|\phi\| \cap \Omega_A \neq \emptyset$,
- rejected if $\|\phi\| \subseteq \Omega_R$.

With this interpretation and focusing first on the set Φ_C of credible sentences, it is clear that the properties assumed in Definition 1 are satisfied: (1) element consistency (if ϕ is a contradiction then $\|\phi\| = \emptyset$ and thus $\|\phi\| \cap \Omega_C = \emptyset$, so that ϕ is not credible), (2) closure under logical equivalence (if $\vdash \phi \leftrightarrow \psi$ then $\|\phi\| = \|\psi\|$ and thus $\|\phi\| \cap \Omega_C \neq \emptyset$ if and only if $\|\psi\| \cap \Omega_C \neq \emptyset$). Now let us go through the additional properties: (3) closure under logical consequence is satisfied (if $\vdash \phi \rightarrow \psi$ then $\|\phi\| \subseteq \|\psi\|$ and thus $\|\phi\| \cap \Omega_C \neq \emptyset$ implies $\|\psi\| \cap \Omega_C \neq \emptyset$), (4) disjunctive completeness is satisfied (since $\|\phi \vee \psi\| = \|\phi\| \cup \|\psi\|$, if $\|\phi \vee \psi\| \cap \Omega_C \neq \emptyset$ then either $\|\phi\| \cap \Omega_C \neq \emptyset$ or $\|\psi\| \cap \Omega_C \neq \emptyset$); (5) negation completeness is also satisfied (if $\|\phi\| \cap \Omega_C = \emptyset$ then $\Omega_C \subseteq \|\neg\phi\|$ and thus, since $\Omega_C \neq \emptyset$, $\Omega_C \cap \|\neg\phi\| = \Omega_C \neq \emptyset$); on the other hand, (6) expansive credibility does not hold: it is possible that, for a credible sentence ϕ , $\phi \notin K$ and yet $\|\neg\phi\| \cap \Omega_C = \emptyset$, so that $\neg\phi$ is not credible. Thus natural properties to impose on Φ_C —besides the two given in Definition 1—are: closure under logical consequence (or single sentence closure: if $\phi \in \Phi_C$ then $[\phi]^{PL} \subseteq \Phi_C$), disjunctive completeness (if $\phi \vee \psi \in \Phi_C$ then either $\phi \in \Phi_C$ or $\psi \in \Phi_C$) and negation completeness (either $\phi \in \Phi_C$ or $\neg\phi \in \Phi_C$).

Turning now to the set Φ_A of allowable sentences, we postulated (Definition 1) that, like Φ_C , also Φ_A contains only consistent sentences and is closed under logical equivalence. These two properties pass the semantic test suggested above. On the other hand, closure under logical consequence does *not* apply to Φ_A . To see this, let $\phi \in \Phi_A$ and note that $\vdash \phi \rightarrow (p \vee \neg p)$; yet, since $(p \vee \neg p)$ is a tautology, $(p \vee \neg p)$ is credible, rather than allowable. A property that *does* satisfy the semantic test is the following: if $\phi \in \Phi_A$ then $\neg\phi \in \Phi_C$ (indeed, if ϕ is allowable then $\|\phi\| \cap \Omega_C = \emptyset$ and thus, as shown above, $\Omega_C \cap \|\neg\phi\| = \Omega_C \neq \emptyset$). Another property that passes the semantic test is: if $\phi \in \Phi_A$ and $\psi \in \Phi_A$ then $\phi \vee \psi \in \Phi_A$ (indeed, if ϕ and ψ are allowable then $\|\phi\| \cap \Omega_C = \|\psi\| \cap \Omega_C = \emptyset$, $\|\phi\| \cap \Omega_A \neq \emptyset$ and $\|\psi\| \cap \Omega_A \neq \emptyset$, so that $(\|\phi\| \cup \|\psi\|) \cap \Omega_C = \emptyset$ and $(\|\phi\| \cup \|\psi\|) \cap \Omega_A \neq \emptyset$).

The properties discussed so far pertain to the sets Φ_C and Φ_A . One can also relate properties of the belief operator to properties of the sets Φ_C and Φ_A : Garapa (2020, Proposition 3.12, p. 9) provides a thorough analysis of 14 such possible links.

Finally, we turn to the set Φ_R of rejected sentences. For this set natural properties are: (1) closure under logical equivalence (if $\vdash \phi \leftrightarrow \psi$ then $\|\phi\| = \|\psi\|$ and thus $\|\phi\| \subseteq \Phi_R$ if and only if $\|\psi\| \subseteq \Phi_R$), (2) if $\phi \in \Phi_R$ then $\neg\phi \in \Phi_C$ (indeed, if ϕ is rejected then $\|\phi\| \subseteq \Omega_R$ and thus, since $\Omega_R \cap \Omega_C = \emptyset$, $\|\phi\| \cap \Omega_C = \emptyset$ and therefore, as shown above, $\|\neg\phi\| \cap \Omega_C \neq \emptyset$), (3) if $\phi \in \Phi_R$ and $\psi \in \Phi_R$ then $\phi \vee \psi \in \Phi_R$.

4. **Categorical matching.** We identified the belief state of an agent with a triple $\langle K, \{\Phi_C, \Phi_A, \Phi_R\}, \circ \rangle$ where
- $K \subset \Phi$ is a consistent and deductively closed set of sentences, representing the initial beliefs,
 - $\{\Phi_C, \Phi_A, \Phi_R\}$ is a partition of the set of sentences into credible, allowable and rejected sentences, representing the disposition of the agent whether to accept any given sentence as information,
 - $\circ : \Phi \rightarrow 2^\Phi$ is a belief revision function transforming the initial belief set K into a new belief set $K \circ \phi$ in response to a sentence ϕ (thought of as information).

However, a belief-revision theory should satisfy what Gärdenfors and Rott (1995, p. 37) called the *principle of categorical matching*, that is, the principle that “the representation of a belief state after a belief change has taken place should be of the same format as the representation of the belief state before the change” [see also Rott (1999)]. This is particularly important in the context of iterated belief revision [see Darwiche and Pearl (1997)]. In our context, if \mathcal{S} is the set of belief states of the form $\langle K, \{\Phi_C, \Phi_A, \Phi_R\}, \circ \rangle$ then a belief-revision function ought to be defined as a function $f : \Phi \times \mathcal{S} \rightarrow \mathcal{S}$. The function $\circ : \Phi \rightarrow 2^\Phi$ gives only one of the three components of $f(\phi, \langle K, \{\Phi_C, \Phi_A, \Phi_R\}, \circ \rangle)$, namely the new belief set $K \circ \phi$, but—in general—also the function \circ and the credibility partition $\{\Phi_C, \Phi_A, \Phi_R\}$ could change as a result of receiving informational input ϕ . Since this paper is not concerned with iterated belief revision, that is, its focus is on “one-shot” revision, our analysis (which follows the traditional approach) is without loss of generality. We leave the extension of the analysis to the context of iterated belief revision as a topic for future research.

5. **The recovery postulate.** We proposed a notion of belief change, called “filtered belief revision”, which was shown to be equivalent to a combination of classical AGM belief revision and contraction, the latter defined via the Harper Identity. Contraction by $\neg\phi$ occurs when the agent is informed of a sentence ϕ which she deems not credible but allowable. The agent’s reaction is then to suspend judgment about ϕ , in particular to abandon her belief in $\neg\phi$ if the latter was in her initial belief set. For such a belief change we postulated the recovery property, according to which the agent would return to her initial belief set if she were to then accept $\neg\phi$. The recovery postulate appears to be a natural way of capturing a “minimal” way of suspending belief in $\neg\phi$, but has been subject to scrutiny (see Makinson, 1987; Fuhrmann, 1991; Hansson, 1991, 1996, 1999a; Levi 1991; Lindström

& Rabinowicz, 1991; Niederée, 1991). In Makinson’s terminology Makinson (1987), contraction operations that do not satisfy the recovery postulates are called *withdrawals*. Alternative types of withdrawal operators have been studied in the literature: contraction without recovery (Fermé (1998)), semi-contraction (Fermé and Rodriguez (1998)), severe withdrawal (Rott and Pagnucco (1999)), systematic withdrawal (Meyer et al. (2002)), mild contraction (Levi (2004)). A potential topic for future research is whether one can relax the recovery postulate and still obtain a simple characterization, in terms of AGM belief revision, of the appropriately modified notion of filtered belief revision.

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A Proof of Proposition 1

(A) **implies** (B). Given a belief set K , a credibility partition $\{\Phi_C, \Phi_A, \Phi_R\}$ and a filtered belief revision function $\circ : \Phi \rightarrow 2^\Phi$, define the function $* : \Phi \rightarrow 2^\Phi$ as follows:

$$K * \phi = \begin{cases} \Phi & \text{if } \phi \text{ is a contradiction} \\ [(K \circ \phi) \cup \{\phi\}]^{PL} & \text{if } \phi \text{ is consistent.} \end{cases} \tag{13}$$

First we show that the function $*$ so defined is a basic AGM belief revision function. Fix an arbitrary $\phi \in \Phi$.

1. Suppose first that ϕ is a contradiction, so that, by (13), $K * \phi = \Phi$. Then
 - (AGM1) is satisfied since $\Phi = [\Phi]^{PL}$.
 - (AGM2) is satisfied since $\phi \in \Phi$.
 - (AGM3) is satisfied since $[K \cup \{\phi\}]^{PL} = \Phi$ (because, by hypothesis, ϕ is a contradiction).
 - (AGM4) is satisfied trivially, since, by hypothesis, $\vdash \neg\phi$ and thus $\neg\phi \in K$ because K is deductively closed.
 - The ‘if’ part of (AGM5) is satisfied by construction.
 - (AGM6) is satisfied because if $\vdash (\phi \leftrightarrow \psi)$ then ψ is also a contradiction and thus $K * \psi = K * \phi = \Phi$.
2. Suppose now that ϕ is consistent, so that, by (13), $K * \phi = [(K \circ \phi) \cup \{\phi\}]^{PL}$. Then
 - (AGM1) is satisfied because $[(K \circ \phi) \cup \{\phi\}]^{PL} = [[(K \circ \phi) \cup \{\phi\}]^{PL}]^{PL}$.
 - (AGM2) is satisfied because $\phi \in [(K \circ \phi) \cup \{\phi\}]^{PL}$.
 - (AGM3) is satisfied because,

- (1) if $\neg\phi \in K$ then $[K \cup \{\phi\}]^{PL} = \Phi$ and,
- (2) if $\neg\phi \notin K$ then, by Definition 2,
 - if $\phi \in \Phi_C$ then $K \circ \phi = [K \cup \{\phi\}]^{PL}$ and thus $K * \phi = [[K \cup \{\phi\}]^{PL} \cup \{\phi\}]^{PL} = [K \cup \{\phi\}]^{PL}$,
 - if $\phi \in \Phi_A \cup \Phi_R$ then $K \circ \phi = K$ and thus $K * \phi = [(K \circ \phi) \cup \{\phi\}]^{PL} = [K \cup \{\phi\}]^{PL}$.
- (AGM4) is satisfied, because - as shown above - if $\neg\phi \notin K$ then $[(K \circ \phi) \cup \{\phi\}]^{PL} = [K \cup \{\phi\}]^{PL}$.
- The ‘only if’ part of (AGM5) is satisfied because
 - if $\neg\phi \in K$ then, by Definition 2, $K \circ \phi$ is consistent and thus not equal to Φ ,
 - if $\neg\phi \notin K$ then $K \circ \phi = \begin{cases} [K \cup \{\phi\}]^{PL} & \text{if } \phi \in \Phi_C \\ K & \text{if } \phi \in \Phi_A \cup \Phi_R \end{cases}$ and thus, since K is consistent and does not imply $\neg\phi$, $K \circ \phi \neq \Phi$.
- (AGM6) is satisfied because if $\vdash \phi \leftrightarrow \psi$ then, by (F4) of Definition 2, $K \circ \phi = K \circ \psi$ and thus $[(K \circ \phi) \cup \{\phi\}]^{PL} = [(K \circ \psi) \cup \{\psi\}]^{PL}$.

Since, by (F1) of Definition 2, $K \circ \phi = K$ when $\phi \in \Phi_R$, we only need to show that

- (a) if $\phi \in \Phi_C$ then $K \circ \phi = K * \phi$, and
 - (b) if $\phi \in \Phi_A$ then $K \circ \phi = K \cap (K * \phi)$.
- (a) Fix an arbitrary $\phi \in \Phi_C$.
- If $\neg\phi \notin K$ then, by (F2a) of Definition 2, $K \circ \phi = [K \cup \{\phi\}]^{PL}$; furthermore, $[K \cup \{\phi\}]^{PL} = [[K \cup \{\phi\}]^{PL} \cup \{\phi\}]^{PL}$. Hence $K \circ \phi = [(K \circ \phi) \cup \{\phi\}]^{PL}$ and, by (13), $[(K \circ \phi) \cup \{\phi\}]^{PL} = K * \phi$.
 - If $\neg\phi \in K$ then, by (F3a) of Definition 2, $\phi \in K \circ \phi$ and thus $[(K \circ \phi) \cup \{\phi\}]^{PL} = [K \circ \phi]^{PL} = K \circ \phi$ (the last equality holds because, by (F3) of Definition 2, $K \circ \phi$ is deductively closed); thus, by (13), $K * \phi = K \circ \phi$.
- (b) Fix an arbitrary $\phi \in \Phi_A$. We need to show that $K \circ \phi = K \cap (K * \phi)$. First of all, note that, by (F2) and (F3) of Definition 2, $K \circ \phi$ is deductively closed, that is, $K \circ \phi = [K \circ \phi]^{PL}$.
- If $\neg\phi \notin K$ then, by (F2b) of Definition 2, $K \circ \phi = K$; furthermore, $K \circ \phi \subseteq [(K \circ \phi) \cup \{\phi\}]^{PL} = K * \phi$; hence $K \circ \phi = K \cap (K * \phi)$.
 - If $\neg\phi \in K$ then, by (F3b) of Definition 2,

$$K \circ \phi \subseteq K \setminus \{\neg\phi\} \text{ and} \tag{14}$$

$$[(K \circ \phi) \cup \{\neg\phi\}]^{PL} = K. \tag{15}$$

Since $K \circ \phi \subseteq [(K \circ \phi) \cup \{\phi\}]^{PL} = K * \phi$ it follows from this and (14) that $K \circ \phi \subseteq K \cap (K * \phi)$.

It remains to prove that $K \cap (K * \phi) \subseteq K \circ \phi$. By (15), $\forall \psi \in \Phi$,

$$\psi \in K \text{ if and only if } (\neg\phi \rightarrow \psi) \in K \circ \phi. \tag{16}$$

Fix an arbitrary $\psi \in K \cap (K * \phi)$. Since $\psi \in K$, by (16), $(\neg\phi \rightarrow \psi) \in K \circ \phi$. Since $\psi \in K * \phi = [(K \circ \phi) \cup \{\phi\}]^{PL}$, $(\phi \rightarrow \psi) \in K \circ \phi$. Thus, since $K \circ \phi$ is deductively closed $(\neg\phi \rightarrow \psi) \wedge (\phi \rightarrow \psi) \in K \circ \phi$; hence, since $\vdash ((\neg\phi \rightarrow \psi) \wedge (\phi \rightarrow \psi) \rightarrow \psi)$ and $K \circ \phi$ is deductively closed, $\psi \in K \circ \phi$.

(B) implies (A). Let $*$: $\Phi \rightarrow 2^\Phi$ be a belief revision function based on K that satisfies the six basic AGM postulates and let \circ : $\Phi \rightarrow 2^\Phi$ be such that, $\forall \phi \in \Phi$,

$$K \circ \phi = \begin{cases} K & \text{if } \phi \in \Phi_R \\ K * \phi & \text{if } \phi \in \Phi_C \\ K \cap (K * \phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{17}$$

We need to show that \circ is a filtered belief revision function, that is, that, $\forall \phi, \psi \in \Phi$,

- (F1) if $\phi \in \Phi_R$ then $K \circ \phi = K$,
- (F2) if $\neg\phi \notin K$ then
 - (a) if $\phi \in \Phi_C$ then $K \circ \phi = [K \cup \{\phi\}]^{PL}$,
 - (b) if $\phi \in \Phi_A$ then $K \circ \phi = K$,
- (F3) if $\neg\phi \in K$ then $K \circ \phi$ is consistent and deductively closed and
 - (a) if $\phi \in \Phi_C$ then $\phi \in K \circ \phi$,
 - (b) if $\phi \in \Phi_A$ then $K \circ \phi \subseteq K \setminus \{\neg\phi\}$ and $[(K \circ \phi) \cup \{\neg\phi\}]^{PL} = K$,
- (F4) if $\vdash \phi \leftrightarrow \psi$ then $K \circ \phi = K \circ \psi$.

(F1) is the first line in (17). Fix an arbitrary $\phi \in \Phi_C \cup \Phi_A$; then, by Definition 1, ϕ is consistent.

Suppose first that $\neg\phi \notin K$. Then, by AGM3 and AGM4, $K * \phi = [K \cup \{\phi\}]^{PL}$ so that, if $\phi \in \Phi_C$, Part (a) of (F2) follows from the second line of (17) and, if $\phi \in \Phi_A$, then also Part (b) of (F2) is satisfied because $K \subseteq [K \cup \{\phi\}]^{PL} = K * \phi$ so that $K \cap (K * \phi) = K$.

Suppose now that $\neg\phi \in K$. Since ϕ is consistent, by AGM1 and AGM5, $K * \phi$ is deductively closed and consistent; since, by hypothesis, K is deductively closed and consistent it follows that $K \cap (K * \phi)$ is also deductively closed and consistent, so that $K \circ \phi$ is deductively closed and consistent. If $\phi \in \Phi_C$ then Part (a) of (F3) is satisfied because, by AGM2, $\phi \in K * \phi$. Suppose that $\phi \in \Phi_A$. Since $K * \phi$ is consistent and $\phi \in K * \phi$ it follows that $\neg\phi \notin K * \phi$ and thus $\neg\phi \notin K \cap (K * \phi) = K \circ \phi$, so that $K \circ \phi \subseteq K \setminus \{\neg\phi\}$. Next we show that $[(K \circ \phi) \cup \{\neg\phi\}]^{PL} = K$. Since, by hypothesis, $\neg\phi \in K$, and, by construction, $K \circ \phi \subseteq K$, it follows that $((K \circ \phi) \cup \{\neg\phi\}) \subseteq K$ and thus $[(K \circ \phi) \cup \{\neg\phi\}]^{PL} \subseteq [K]^{PL} = K$. It remains to prove that $K \subseteq [(K \circ \phi) \cup \{\neg\phi\}]^{PL}$. Fix an arbitrary $\psi \in K$. Since, by hypothesis, $K = [K]^{PL}$ and $\vdash \psi \rightarrow (\neg\phi \rightarrow \psi)$, $(\neg\phi \rightarrow \psi) \in K$. Since $\phi \in K * \phi$ and $K * \phi$ is deductively closed, $(\phi \vee \psi) \in K * \phi$ and since $(\phi \vee \psi)$ is logically equivalent to $(\neg\phi \rightarrow \psi)$, it follows that $(\neg\phi \rightarrow \psi) \in K * \phi$. Thus $(\neg\phi \rightarrow \psi) \in K \cap (K * \phi)$ and thus $\psi \in [(K \cap (K * \phi)) \cup \{\neg\phi\}]^{PL} = [(K \circ \phi) \cup \{\neg\phi\}]^{PL}$.

Finally, if ψ is logically equivalent to ϕ then $\psi \in \Phi_C \cup \Phi_A$ because both sets are closed under logical equivalence and, by hypothesis, $\phi \in \Phi_C \cup \Phi_A$. Since, by AGM4, $K * \phi = K * \psi$ it follows that (F4) is satisfied.

B Proof of Proposition 3

(A) implies (B). Fix a basic-AGM-consistent PBRs $\langle \Omega, \{ \mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R \}, f \rangle$ and an arbitrary $E \in \mathcal{E}_C \cup \mathcal{E}_A$. Let p, q and r be atomic propositions and consider a model $(\{ \Phi_C, \Phi_A, \Phi_R \}, V)$ (Definition 4) where $\|p\| = E, \|q\| = f(E)$ and $\|r\| = f(\Omega)$. Let $K = \{ \phi \in \Phi : f(\Omega) \subseteq \|\phi\| \}, \Psi = \{ \phi \in \Phi : \|\phi\| \in \mathcal{E} \}$ and define $\circ_\Psi : \Psi \rightarrow 2^\Phi$ by $K \circ_\Psi \phi = \{ \chi \in \Phi : f(\|\phi\|) \subseteq \|\chi\| \}$. Thus $r \in K, p \in \Psi$ and $q \in K \circ_\Psi p$. Let $\circ : \Phi \rightarrow 2^\Phi$ be a full-domain extension of $\circ_\Psi : \Psi \rightarrow 2^\Phi$ and $*$: $\Phi \rightarrow 2^\Phi$ a basic AGM revision function such that, for every $\phi \in \Phi$,

$$K \circ \phi = \begin{cases} K & \text{if } \phi \in \Phi_R \\ K * \phi & \text{if } \phi \in \Phi_C \\ K \cap (K * \phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{19}$$

- Suppose first that

$$E \cap f(\Omega) \neq \emptyset. \tag{20}$$

We need to show that

$$\text{if } E \in \mathcal{E}_C \text{ then } f(E) = E \cap f(\Omega). \tag{21}$$

and

$$\text{if } E \in \mathcal{E}_A \text{ then } f(E) = f(\Omega). \tag{22}$$

By (20), $f(\Omega) \not\subseteq \Omega \setminus E = \Omega \setminus \|p\| = \|\neg p\|$, that is,

$$\neg p \notin K \tag{23}$$

so that, by AGM3 and AGM4,

$$K * p = [K \cup \{p\}]^{PL}. \tag{24}$$

- Consider first the case where $E \in \mathcal{E}_C$, so that $p \in \Phi_C$. Since $K \circ_\Psi p = K * p$ and $q \in K \circ_\Psi p, q \in K * p$, so that, by (24), $q \in [K \cup \{p\}]^{PL}$; hence $(p \rightarrow q) \in K$ (recall that K is deductively closed), that is, $f(\Omega) \subseteq \|\neg p \vee q\| = (\Omega \setminus E) \cup f(E)$; thus, intersecting both sides with $E, E \cap f(\Omega) \subseteq f(E) \cap E = f(E)$ (recall that, by Definition 3, since $E \in \mathcal{E}_C, f(E) \subseteq E$). Next we show that $f(E) \subseteq E \cap f(\Omega)$. Since $f(\Omega) = \|r\|, r \in K$ and thus, since K is deductively closed, $(p \rightarrow r) \in K$, from which it follows that $r \in [K \cup \{p\}]^{PL} = K * p$ (by (24)); thus, since $K * p = K \circ_\Psi p, r \in K \circ_\Psi p$, that is, $f(E) \subseteq \|r\| = f(\Omega)$. Hence, since $f(E) \subseteq E, f(E) \subseteq E \cap f(\Omega)$. This completes the proof of (21).

- Consider next the case where $E \in \mathcal{E}_A$, so that $p \in \Phi_A$. By (19), since $q \in K \circ_\Psi p$, $q \in K \circ p = K \cap (K * p)$. From $q \in K$ it follows that $f(\Omega) \subseteq \|q\| = f(E)$. It remains to prove that the converse is also true, namely that $f(E) \subseteq f(\Omega)$. Since $f(\Omega) = \|r\|$, $r \in K$. Thus, since K is deductively closed, $(p \rightarrow r) \in K$, from which it follows that $r \in [K \cup \{p\}]^{PL} = K * p$ (by (24)). Thus $r \in K \cap (K * p)$, so that, since (by (19)) $K \cap (K * p) = K \circ p = K \circ_\Psi p$, $r \in K \circ_\Psi p$, that is, $f(E) \subseteq \|r\| = f(\Omega)$. This completes the proof of (22).
- Suppose now that $E \in \mathcal{E}_A$ (thus, by Point 2 of Definition 3, $E \neq \emptyset$) and

$$E \cap f(\Omega) = \emptyset. \tag{25}$$

We need to show that $f(E) = f(\Omega) \cup E'$ for some $\emptyset \neq E' \subseteq E$. Since $E \in \mathcal{E}_A$ and $\|p\| = E$, $p \in \Phi_A$. Thus, by (19),

$$K \circ_\Psi p = K \circ p = K \cap (K * p). \tag{26}$$

Since $E = \|p\|$ and $f(E) = \|q\|$, $q \in K \circ_\Psi p$ and thus, by (26), $q \in K$, that is, $f(\Omega) \subseteq \|q\| = f(E)$. It follows from this and the fact that $f(E) \cap E \subseteq f(E)$, that

$$f(\Omega) \cup (f(E) \cap E) \subseteq f(E). \tag{27}$$

Next we show that $f(E) \subseteq f(\Omega) \cup (f(E) \cap E)$. Since $f(\Omega) = \|r\|$, $r \in K$ and thus, since K is deductively closed, $(r \vee p) \in K$. Since $p \in K * p$ and $K * p$ is deductively closed, $(r \vee p) \in K * p$. Thus, by (26), $(r \vee p) \in K \circ_\Psi p$, that is, $f(E) \subseteq \|r \vee p\| = \|r\| \cup \|p\| = f(\Omega) \cup E$; hence (intersecting both sides with $\Omega \setminus E$),

$$\begin{aligned} f(E) \cap (\Omega \setminus E) &\subseteq (f(\Omega) \cup E) \cap (\Omega \setminus E) \\ &= (f(\Omega) \cap (\Omega \setminus E)) \cup (E \cap (\Omega \setminus E)) \\ &= f(\Omega) \cap (\Omega \setminus E) \stackrel{\text{(by (25))}}{=} f(\Omega). \end{aligned} \tag{28}$$

Thus,

$$f(E) = (f(E) \cap (\Omega \setminus E)) \cup (f(E) \cap E) \stackrel{\text{(by (28))}}{\subseteq} f(\Omega) \cup (f(E) \cap E). \tag{29}$$

It follows from (27) and (29) that $f(E) = f(\Omega) \cup E'$ with $E' = f(E) \cap E$. Finally, by (d) of definition of PBRS (Definition 3), $f(E) \cap E \neq \emptyset$.

(B) implies (A). Fix a PBRS that satisfies the properties of part (B) of Proposition 2 and an arbitrary model $(\{\Phi_C, \Phi_A, \Phi_R\}, V)$ of it (Definition 4). As usual, let

$$K = \{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|\}, \tag{30}$$

$$\Psi = \{\phi \in \Phi : \|\phi\| \in \mathcal{E}\}, \tag{31}$$

$$\circ_\Psi : \Psi \rightarrow 2^\Phi \text{ given by: } K \circ_\Psi \phi = \{\chi \in \Phi : f(\|\phi\|) \subseteq \|\chi\|\}. \tag{32}$$

Let $*$: $\Phi \rightarrow 2^\Phi$ be the following full domain belief revision function: $\forall \phi \in \Phi$,

$$K * \phi = \begin{cases} 1. \ \Phi & \text{if and only if } \phi \text{ is a contradiction} \\ 2. \ [\phi]^{PL} & \text{if } \|\phi\| \notin \mathcal{E}_C \cup \mathcal{E}_A \\ 3. \ [K \cup \{\phi\}]^{PL} & \text{if } \|\phi\| \in \mathcal{E}_C \cup \mathcal{E}_A \text{ and } \neg\phi \notin K \\ 4. \ [\{\psi \in \Phi : f(\|\phi\|) \subseteq \|\psi\|\}]^{PL} & \text{if } \|\phi\| \in \mathcal{E}_C \text{ and } \neg\phi \in K \\ 5. \ [\{\psi \in \Phi : (f(\|\phi\|) \cap \|\phi\|) \subseteq \|\psi\|\}]^{PL} & \text{if } \|\phi\| \in \mathcal{E}_A \text{ and } \neg\phi \in K \end{cases} \quad (33)$$

First we show that $*$ is a basic AGM belief revision function.

- AGM1 is satisfied by construction.
- AGM2 is clearly satisfied in cases 1–3 and 5 of (33). As for case 4, since $\|\phi\| \in \mathcal{E}_C$, by definition of PBRS $f(\|\phi\|) \subseteq \|\phi\|$.
- AGM3 is clearly satisfied in cases 1–3 of (33). In cases 4 and 5, since $\neg\phi \in K$, $[K \cup \{\phi\}]^{PL} = \Phi$ and the property holds trivially.
- AGM4 is clearly satisfied in cases 1–3 of (33) since $[(K * \phi) \cup \{\phi\}]^{PL} = K * \phi$. In cases 4 and 5 the property holds trivially since $\neg\phi \in K$.
- AGM5 is satisfied by construction.
- AGM6 is satisfied because if $\vdash \phi \leftrightarrow \psi$ then

1. if ϕ is a contradiction then so is ψ and thus, by construction, $K * \phi = K * \psi = \Phi$.
2. $[\phi]^{PL} = [\psi]^{PL}$.
3. $[K \cup \{\phi\}]^{PL} = [K \cup \{\psi\}]^{PL}$.
4. and 5. $\|\phi\| = \|\psi\|$.

Next define the following full-domain belief revision function \circ : $\Phi \rightarrow 2^\Phi$:

$$K \circ \phi = \begin{cases} K & \text{if } \phi \in \Phi_R \\ K * \phi & \text{if } \phi \in \Phi_C \\ K \cap (K * \phi) & \text{if } \phi \in \Phi_A \end{cases} \quad (34)$$

where $K * \phi$ is given by (33). Then, by definition of basic-AGM consistent PBRS (Definition 5), it only remains to prove that \circ is an extension of \circ_Ψ (given by (32)), that is, that, for every $\phi \in \Psi$, $\chi \in K \circ_\Psi \phi$ if and only if $\chi \in K \circ \phi$. Fix an arbitrary $\phi \in \Psi$, that is, a formula ϕ such that $\|\phi\| \in \mathcal{E}$.

- If $\|\phi\| \in \mathcal{E}_R$ (so that $\phi \in \Phi_R$) then (by definition of PBRS: Definition 3) $f(\|\phi\|) = f(\Omega)$ and thus, $\forall \chi \in \Phi$, $\chi \in K \circ_\Psi \phi$ if and only if $f(\Omega) \subseteq \|\chi\|$ if and only if $\chi \in K$ and, by (34), $K \circ \phi = K$.
- If $\|\phi\| \in \mathcal{E}_C$ (so that $\phi \in \Phi_C$) then (recall that $\forall \chi \in \Phi$, $\chi \in K \circ_\Psi \phi$ if and only if $f(\|\phi\|) \subseteq \|\chi\|$)
 - if $\neg\phi \in K$ then, by 4 of (33), $f(\|\phi\|) \subseteq \|\chi\|$ if and only if $\chi \in K * \phi = K \circ \phi$,
 - if $\neg\phi \notin K$ then $f(\Omega) \cap \|\phi\| \neq \emptyset$ and thus, by hypothesis (1(a) of Part (B) of Proposition 3), $f(\|\phi\|) = f(\Omega) \cap \|\phi\|$ so that $\chi \in K \circ_\Psi \phi$ if and only if $f(\Omega) \cap \|\phi\| \subseteq \|\chi\|$ if and only if $f(\Omega) \subseteq (\Omega \setminus \|\phi\|) \cup \|\chi\| = \|\phi \rightarrow \chi\|$ if and only if $(\phi \rightarrow \chi) \in K$, if and only if $\chi \in [K \cup \{\phi\}]^{PL} = K * \phi = K \circ \phi$.

- If $\|\phi\| \in \mathcal{E}_A$ (so that $\phi \in \Phi_A$) then,
 - if $\neg\phi \in K$, that is, $f(\Omega) \cap \|\phi\| = \emptyset$ then, by hypothesis, $f(\|\phi\|) = f(\Omega) \cup E'$ for some $\emptyset \neq E' \subseteq \|\phi\|$ so that $E' = f(\|\phi\|) \cap \|\phi\|$; hence $\chi \in K \circ_{\Psi} \phi$ if and only if $f(\|\phi\|) \subseteq \|\chi\|$ if and only if $f(\Omega) \subseteq \|\chi\|$ and $f(\|\phi\|) \cap \|\phi\| \subseteq \|\chi\|$, if and only if $\chi \in K$ and, by 5 of (33), $\chi \in K * \phi$, that is, $\chi \in K \cap (K * \phi) = K \circ \phi$,
 - if $\neg\phi \notin K$ then $f(\Omega) \cap \|\phi\| \neq \emptyset$ and thus, by hypothesis, $f(\|\phi\|) = f(\Omega)$ so that $\chi \in K \circ_{\Psi} \phi$ if and only if $f(\Omega) \subseteq \|\chi\|$ if and only if $\chi \in K = K \cap [K \cup \{\phi\}]^{PL} = K \cap (K * \phi) = K \circ \phi$. □

C Necessary and sufficient conditions for rationalizability of a BPBRs

In this appendix we provide necessary and sufficient conditions for a plausibility-based BPBRs to be rationalizable.²⁵

Proposition 5 *A plausibility-based BPBRs $\langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ is rationalizable if and only if, for every sequence $\langle E_1, \dots, E_n, E_{n+1} \rangle$ in \mathcal{E} with $E_{n+1} = E_1$, conditions (A) and (B) below are satisfied:*

- (A) if $(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) \neq \emptyset, \forall k = 1, \dots, n$, then $(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) = f(E_k \cap \Omega_C) \cap (E_{k+1} \cap \Omega_C), \forall k = 1, \dots, n$.
- (B) if $E_k \cap \Omega_C = \emptyset, \forall k = 1, \dots, n$, and $(E_k \cap \Omega_A) \cap f(E_{k+1} \cap \Omega_A) \neq \emptyset, \forall k = 1, \dots, n$, then $(E_k \cap \Omega_A) \cap f(E_{k+1} \cap \Omega_A) = f(E_k \cap \Omega_A) \cap (E_{k+1} \cap \Omega_A), \forall k = 1, \dots, n$.

The proof of Proposition 5 makes repeated use of Proposition 6 below due to Hansson (1968, Theorem 7, p. 455). We begin with a definition.

Definition 12 *A simple choice structure is a triple $\langle W, \mathcal{F}, h \rangle$ where W is a non-empty set, $\mathcal{F} \subseteq 2^W$, with $\emptyset \notin \mathcal{F}$ and $W \in \mathcal{F}$, and $h : \mathcal{F} \rightarrow 2^W$ satisfies $\emptyset \neq h(F) \subseteq F$, for all $F \in \mathcal{F}$.*

Proposition 6 (Hansson, 1968) *Let $\langle W, \mathcal{F}, h \rangle$ be a simple choice structure. Then the following conditions are equivalent:*

1. *there exists a total pre-order $\succsim \subseteq W \times W$ such that, for all $F \in \mathcal{F}$,*

$$h(F) = \text{best}_{\succsim} F \stackrel{\text{def}}{=} \{w \in F : w \succsim w', \forall w' \in F\},$$

²⁵ Conditions A and B in Proposition 5 below are a generalization of what is known in the revealed preference literature as the Strong Axiom of Revealed Preference (SARP), which is a necessary, but not sufficient, condition for rationalizability by a total pre-order (see Hansson, 1968). Let $\langle \Omega, \mathcal{E}, f \rangle$ be a simple choice structure (Definition 12 below) and let $\langle E_1, \dots, E_n, E_{n+1} \rangle$ be a sequence in \mathcal{E} with $E_{n+1} = E_1$. Then SARP is the following condition: if $E_k \cap f(E_{k+1}) \neq \emptyset, \forall k \in \{1, \dots, n\}$, then there exists a $j \in \{1, \dots, n\}$ such that $E_j \cap f(E_{j+1}) = f(E_j) \cap E_{j+1}$. For the case where $n = 2$, SARP says the following (interpreting $f(E)$ as the set of alternatives chosen from E): if from a set of alternatives E_1 , x is chosen when y and z are available, and if from some other set of alternatives E_2 , y is chosen while z is available, then there can be no set of alternatives containing x and z where z is chosen and x is not (SARP extends this condition to chains of any finite length).

2. for every sequence $\langle F_1, \dots, F_n, F_{n+1} \rangle$ in \mathcal{F} with $F_{n+1} = F_1$, if $F_k \cap h(F_{k+1}) \neq \emptyset$, $\forall k = 1, \dots, n$, then $F_k \cap h(F_{k+1}) = h(F_k) \cap F_{k+1}$, $\forall k = 1, \dots, n$.

Proof of Proposition 5 First we show that if \succsim rationalizes the plausibility-based BPBRS $\langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ then (A) of Proposition 5 is satisfied. Construct the following simple choice structure $\langle W, \mathcal{F}, h \rangle$:

$$\begin{cases} W = \Omega_C \\ \mathcal{F} = \{E \cap \Omega_C : E \in \mathcal{E}_C\} \\ h : \mathcal{F} \rightarrow 2^W \text{ is the restriction of } f \text{ to } \mathcal{F}. \end{cases} \tag{35}$$

By Definition 9, $\Omega_C \neq \emptyset$. By 2 of Definition 9, if $E \in \mathcal{E}_C$ then $E \cap \Omega_C \in \mathcal{E}_C$ and by 2 of Definition 8, $\emptyset \notin \mathcal{F}$ and $\Omega_C \in \mathcal{F}$. By 3(c) of Definition 8, $h(F) \neq \emptyset, \forall F \in \mathcal{F}$. By hypothesis, since the BPBRS is rationalized by the total pre-order $\succsim \subseteq \Omega \times \Omega$, if $E \in \mathcal{E}_C$ then $f(E) = f(E \cap \Omega_C) = \text{best}_{\succsim}(E \cap \Omega_C) \subseteq E \cap \Omega_C$ and thus, letting $F = E \cap \Omega_C, h(F) \subseteq F$. Hence we have indeed defined a simple choice structure. Let \succsim_C be the restriction of \succsim to Ω_C (that is, $\succsim_C = \succsim \cap (\Omega_C \times \Omega_C)$). By construction, since \succsim rationalizes the given PBRS, we have that

$$\forall F \in \mathcal{F}, h(F) = \text{best}_{\succsim_C} F \stackrel{\text{def}}{=} \{\omega \in F : \omega \succsim_C \omega', \forall \omega' \in F\}. \tag{36}$$

Now fix an arbitrary sequence $\langle E_1, \dots, E_n, E_{n+1} \rangle$ in \mathcal{E}_C with $E_{n+1} = E_1$ such that, $\forall k = 1, \dots, n, (E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) \neq \emptyset$. Let $\langle F_1, \dots, F_n, F_{n+1} \rangle$ be the corresponding sequence in \mathcal{F} , that is, for every $k = 1, \dots, n, F_k = E_k \cap \Omega_C$ (thus $F_{n+1} = F_1$). Then, for every $k = 1, \dots, n, F_k \cap h(F_{k+1}) \neq \emptyset$. Thus, by (36) and Proposition 6, $F_k \cap h(F_{k+1}) = h(F_k) \cap F_{k+1}, \forall k = 1, \dots, n$, that is, $(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) = f(E_k \cap \Omega_C) \cap (E_{k+1} \cap \Omega_C), \forall k = 1, \dots, n$; that is, (A) of Proposition 5 holds.

Next we show that if \succsim rationalizes the plausibility-based BPBRS then Part (B) of Proposition 5 is satisfied. If $\Omega_A = \emptyset$ there is nothing to prove. Assume, therefore, that $\Omega_A \neq \emptyset$ (so that, by 3 of Definition 9, $\Omega_A \in \mathcal{E}$). Construct the following simple choice structure $\langle W, \mathcal{G}, g \rangle$:

$$\begin{cases} W = \Omega_A \\ \mathcal{G} = \{E \cap \Omega_A : E \in \mathcal{E}_A\} \\ g : \mathcal{G} \rightarrow 2^W \text{ is the restriction of } f \text{ to } \mathcal{G}. \end{cases} \tag{37}$$

By 3(d) of Definition 8, for every $G \in \mathcal{G}, g(G) \neq \emptyset$; furthermore, by hypothesis, since the BPBRS is rationalized by the total pre-order $\succsim \subseteq \Omega \times \Omega$, if $E \in \mathcal{E}_A$ then $f(E) = f(E \cap \Omega_A) = \text{best}_{\succsim}(E \cap \Omega_A) \subseteq E \cap \Omega_A$ and thus, letting $G = E \cap \Omega_A, g(G) \subseteq G$. Hence we have indeed defined a simple choice structure. Let \succsim_A be the restriction of \succsim to Ω_A (that is, $\succsim_A = \succsim \cap (\Omega_A \times \Omega_A)$). By construction, since \succsim rationalizes the given PBRS, we have that

$$\forall G \in \mathcal{G}, g(G) = \text{best}_{\succsim_A} G \stackrel{\text{def}}{=} \{\omega \in G : \omega \succsim_A \omega', \forall \omega' \in G\}. \tag{38}$$

Now fix an arbitrary sequence $\langle E_1, \dots, E_n, E_{n+1} \rangle$ in \mathcal{E}_A with $E_{n+1} = E_1$ (thus, by 3(a) of Definition 9 $E_k \cap \Omega_C = \emptyset, \forall k = 1, \dots, n$) such that $(E_k \cap \Omega_A) \cap f(E_{k+1} \cap \Omega_A) \neq \emptyset, \forall k = 1, \dots, n$. Let $\langle G_1, \dots, G_n, G_{n+1} \rangle$ be the corresponding sequence in \mathcal{G} , that is, for every $k = 1, \dots, n, G_k = E_k \cap \Omega_A$ (thus $G_{n+1} = G_1$). Then, for every $k = 1, \dots, n, G_k \cap g(G_{k+1}) \neq \emptyset$. Thus, by (38) and Proposition 6, $G_k \cap g(G_{k+1}) = g(G_k) \cap G_{k+1}, \forall k = 1, \dots, n$, that is, $(E_k \cap \Omega_A) \cap f(E_{k+1} \cap \Omega_A) = f(E_k \cap \Omega_A) \cap (E_{k+1} \cap \Omega_A), \forall k = 1, \dots, n$, that is, (B) of Proposition 5 holds.

Next we show that if the BPBRS $\langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ satisfies Properties (A) and (B) of Proposition 5 then it can be rationalized by a plausibility order $\succsim \subseteq \Omega \times \Omega$.

Let $\langle W, \mathcal{F}, h \rangle$ be the simple choice structure defined in (35). Fix an arbitrary sequence $\langle E_1, \dots, E_n, E_{n+1} \rangle$ in \mathcal{E}_C with $E_{n+1} = E_1$ such that $(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) \neq \emptyset, \forall k = 1, \dots, n$, and let $\langle F_1, \dots, F_n, F_{n+1} \rangle$ be the corresponding sequence in \mathcal{F} (that is, $F_k = E_k \cap \Omega_C$, for all $k = 1, \dots, n$). By the Property (A) of Proposition 5, $(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) = f(E_k \cap \Omega_C) \cap (E_{k+1} \cap \Omega_C), \forall k = 1, \dots, n$, that is, $F_k \cap h(F_{k+1}) = h(F_k) \cap F_{k+1}, \forall k = 1, \dots, n$. Hence, since the sequence was chosen arbitrarily, it follows from Proposition 6 that there exists a total pre-order \succsim_C on $W \times W = \Omega_C \times \Omega_C$ such that

$$\forall F \in \mathcal{F}, h(F) = best_{\succsim_C} F \stackrel{def}{=} \{\omega \in F : \omega \succsim_C \omega', \forall \omega' \in F\}. \tag{39}$$

Two cases are possible.

Case 1: $\Omega_A = \emptyset$. In this case, define $\succsim \subseteq \Omega \times \Omega$ as follows:

$$\succsim = \succsim_C \cup \{(\omega, \omega') : \omega \in \Omega_C \text{ and } \omega' \in \Omega_R\} \cup \{(\omega, \omega') : \omega, \omega' \in \Omega_R\}. \tag{40}$$

Then \succsim is a plausibility order (Definition 6) and satisfies the properties given in (10). Fix an arbitrary $E \in \mathcal{E}$. If $E \cap \Omega_C \neq \emptyset$ then, by 2(c) of Definition 9, $f(E) = f(E \cap \Omega_C) \stackrel{def}{=} h(E \cap \Omega_C)$ and by (39) $h(E \cap \Omega_C) = best_{\succsim_C}(E \cap \Omega_C)$. By (40), if $\omega \in E \cap \Omega_C$ and $\omega' \in E \setminus \Omega_C$ then $\omega \succ \omega'$ so that $best_{\succ} E = best_{\succ}(E \cap \Omega_C) = best_{\succ_C}(E \cap \Omega_C)$; thus $f(E) = best_{\succ} E$. If $E \cap \Omega_C = \emptyset$ then $E \subseteq \Omega_R$ and, by 3(b) Definition 3,

$f(E) = f(\Omega)$. Since $\Omega \cap \Omega_C = \Omega_C \neq \emptyset, f(\Omega) = f(\Omega \cap \Omega_C) \stackrel{def}{=} h(\Omega_C)$ and by (39) $h(\Omega_C) = best_{\succ_C}(\Omega_C)$; by (40), $best_{\succ} \Omega = best_{\succ_C} \Omega_C$, so that $f(\Omega) = best_{\succ} \Omega$.

Case 2: $\Omega_A \neq \emptyset$. In this case let $\langle \tilde{W}, \mathcal{G}, g \rangle$ be the choice structure defined in (37). Fix an arbitrary sequence $\langle E_1, \dots, E_n, E_{n+1} \rangle$ in \mathcal{E} with $E_{n+1} = E_1$ such that $E_k \cap \Omega_C = \emptyset, \forall k = 1, \dots, n$, and $(E_k \cap \Omega_A) \cap g(E_{k+1} \cap \Omega_A) \neq \emptyset, \forall k = 1, \dots, n$ and let $\langle G_1, \dots, G_n, G_{n+1} \rangle$ be the corresponding sequence in \mathcal{G} (that is, $G_k = E_k \cap \Omega_A$, for all $k = 1, \dots, n$). By Property (B) of Proposition 5, $(E_k \cap \Omega_A) \cap g(E_{k+1} \cap \Omega_A) = g(E_k \cap \Omega_A) \cap (E_{k+1} \cap \Omega_A), \forall k = 1, \dots, n$, that is, $G_k \cap g(G_{k+1}) = g(G_k) \cap G_{k+1}, \forall k = 1, \dots, n$. Hence, since the sequence was chosen arbitrarily, it follows from Proposition 6 that there exists a total pre-order \succsim_A on $W \times W = \Omega_A \times \Omega_A$ such that

$$\forall G \in \mathcal{G}, g(G) = best_{\succsim_A} G \stackrel{def}{=} \{\omega \in G : \omega \succsim_A \omega', \forall \omega' \in G\}. \tag{41}$$

Define $\succsim \subseteq \Omega \times \Omega$ as follows (where \succsim_C is the total pre-order on $\Omega_C \times \Omega_C$ that satisfies (39)):

$$\begin{aligned} \succsim &= \succsim_C \cup \succsim_A \\ &\cup \{(\omega, \omega') : \omega \in \Omega_C \text{ and } \omega' \in \Omega_A \cup \Omega_R\} \\ &\cup \{(\omega, \omega') : \omega \in \Omega_A \text{ and } \omega' \in \Omega_R\} \\ &\cup \{(\omega, \omega') : \omega, \omega' \in \Omega_R\}. \end{aligned} \tag{42}$$

Then \succsim is a plausibility order (Definition 6) and satisfies the properties given in (10). Fix an arbitrary $E \in \mathcal{E}$. If $E \cap \Omega_C \neq \emptyset$ or $E \subseteq \Omega_R$, then $f(E) = best_{\succsim} E$ by the argument developed in Case 1. If $E \cap \Omega_C = \emptyset$ and $E \cap \Omega_A \neq \emptyset$, then, by 3(d) of Definition 9, $f(E) = f(E \cap \Omega_A) \stackrel{def}{=} g(E \cap \Omega_A)$ and by (41) $g(E \cap \Omega_A) = best_{\succsim_A}(E \cap \Omega_A)$. By (42), if $\omega \in E \cap \Omega_A$ and $\omega' \in \Omega_R$ then $\omega \succ \omega'$ so that $best_{\succsim} E = best_{\succsim}(E \cap \Omega_A) = best_{\succsim_A}(E \cap \Omega_A)$; thus $f(E) = best_{\succsim} E$. \square

D Proof of Proposition 4

(A) implies (B). Let \mathcal{C} (with a finite set of states Ω) be a plausibility-based BPBRS which is supplemented AGM consistent, that is, there exist supplemented AGM belief revision functions $*_C : \Phi \rightarrow 2^\Phi$ and $*_A : \Phi \rightarrow 2^\Phi$ such that the function $\circ : \Phi \rightarrow 2^\Phi$ defined by

$$K \circ \phi = \begin{cases} K & \text{if } \phi \in \Phi_R \\ K *_C \phi & \text{if } \phi \in \Phi_C \\ K \cap (K *_A \phi) & \text{if } \phi \in \Phi_A \end{cases} \tag{43}$$

is an extension of $\circ_\Psi : \Psi \rightarrow 2^\Phi$, where, as usual, $K = \{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|\}$, $\Psi = \{\phi \in \Phi : \|\phi\| \in \mathcal{E}\}$ and $K \circ_\Psi \phi = \{\chi \in \Phi : f(\|\phi\|) \subseteq \|\chi\|\}$. We need to show that \mathcal{C} is rationalizable by a plausibility order \succsim on Ω , in the sense that, for every $E \in \mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$,

$$f(E) = \begin{cases} best_{\succsim} E & \text{if } E \cap \Omega_C \neq \emptyset \\ best_{\succsim} \Omega \cup best_{\succsim} E & \text{if } E \cap \Omega_C = \emptyset \text{ and } E \cap \Omega_A \neq \emptyset \\ best_{\succsim} \Omega & \text{if } E \subseteq \Omega_R \end{cases} \tag{44}$$

(as usual, for every $F \subseteq \Omega$, $best_{\succsim} F \stackrel{def}{=} \{\omega \in F : \omega \succsim \omega', \forall \omega' \in F\}$). Extract from \mathcal{C} the simple choice structure $\langle \Omega_C, \mathcal{F}, h \rangle$ where $\mathcal{F} = \{E \cap \Omega_C : E \in \mathcal{E}_C\}$ and h is the restriction of f to \mathcal{F} . By 2(c) of Definition 9, (1) $f(\Omega) = f(\Omega \cap \Omega_C) = f(\Omega_C) = h(\Omega_C)$, so that $\{\phi \in \Phi : h(\Omega_C) \subseteq \|\phi\|\} = K$ and (2) $\Psi_{\mathcal{F}} \stackrel{def}{=} \{\phi \in \Phi : \|\phi\| \in \mathcal{F}\} \subseteq \Psi$. For every $\phi \in \Phi$, let $K \circ_{\Psi_{\mathcal{F}}} \phi = \{\chi \in \Phi : h(\|\phi\|) \subseteq \|\chi\|\}$. Then, by (43), the supplemented AGM function $*_C$ is an extension of $\circ_{\Psi_{\mathcal{F}}}$ and thus the simple structure $\langle \Omega_C, \mathcal{F}, h \rangle$ is AGM consistent in the sense of Definition 3 in Bonanno (2009) so that, by Proposition 8 in Bonanno (2009), there exists a total preorder \succsim_C on Ω_C such that,

for every $F \in \mathcal{F}$,

$$h(F) = \text{best}_{\succsim_C} F \stackrel{\text{def}}{=} \{\omega \in F : \omega \succsim_C \omega', \forall \omega' \in F\}. \tag{45}$$

Two cases are possible.

Case 1: $\Omega_A = \emptyset$. In this case, define $\succsim \subseteq \Omega \times \Omega$ as in (39) and the argument to show that, $\forall E \in \mathcal{E}$, $f(E) = \text{best}_{\succsim} E$ is a repetition of the argument used in Case 1 of the proof of Proposition 5.

Case 2: $\Omega_A \neq \emptyset$. In this case extract from \mathcal{C} the simple choice structure $\langle \Omega, \mathcal{G}, g \rangle$ where $\mathcal{G} = \{E \cap \Omega_A : E \in \mathcal{E}_A\} \cup \{\Omega\}$ and g is the restriction of f to \mathcal{G} . By 3(d) of Definition 9, $\Psi_{\mathcal{G}} \stackrel{\text{def}}{=} \{\phi \in \Phi : \|\phi\| \in \mathcal{G}\} \subseteq \Psi$. By construction, since $g(\Omega) = f(\Omega)$, $\{\phi \in \Phi : g(\Omega) \subseteq \|\phi\|\} = K$. Then, by (43), the supplemented AGM function $*_A$ is an extension of $\circ_{\Psi_{\mathcal{G}}}$ and thus the simple structure $\langle \Omega, \mathcal{G}, g \rangle$ is AGM consistent in the sense of Definition 3 in Bonanno (2009) so that, by Proposition 8 in Bonanno (2009), there exists a total preorder \succsim'_A on Ω such that, for every $G \in \mathcal{G}$,

$$g(G) = \text{best}_{\succsim'_A} G \stackrel{\text{def}}{=} \{\omega \in G : \omega \succsim'_A \bar{\omega}, \forall \bar{\omega} \in G\}. \tag{46}$$

Let $\succsim_A = \succsim'_A \cap (\Omega_A \times \Omega_A)$ and define $\succsim \subseteq \Omega \times \Omega$ as in (42). Then the argument to show that, $\forall E \in \mathcal{E}$, $f(E) = \text{best}_{\succsim} E$ is a repetition of the argument used in Case 2 of the proof of Proposition 5.

(B) implies (A). Let \mathcal{C} (with a finite set of states Ω) be a plausibility-based BPBRs which is rationalized by a plausibility order \succsim on Ω (Definition 6), in the sense that, for every $E \in \mathcal{E}$,

$$f(E) = \begin{cases} \text{best}_{\succsim} E & \text{if } E \cap \Omega_C \neq \emptyset \\ \text{best}_{\succsim} \Omega \cup \text{best}_{\succsim} E & \text{if } E \cap \Omega_C = \emptyset \text{ and } E \cap \Omega_A \neq \emptyset \\ \text{best}_{\succsim} \Omega & \text{if } E \subseteq \Omega_R. \end{cases} \tag{47}$$

We want to show that \mathcal{C} is supplemented AGM consistent (Definition 10). Let $\langle \Omega_C, \mathcal{F}, h \rangle$ be the simple choice structure described above ($\mathcal{F} = \{E \cap \Omega_C : E \in \mathcal{E}_C\}$ and h is the restriction of f to \mathcal{F}). Then, by (47), $\langle \Omega_C, \mathcal{F}, h \rangle$ is rationalized by the total pre-order $\succsim_C \stackrel{\text{def}}{=} \succsim \cap (\Omega_C \times \Omega_C)$, so that, by Proposition 7 in Bonanno (2009), there exists a supplemented AGM function $*_C$ that extends the function $\circ_{\Psi_{\mathcal{F}}}$ defined above ($K \circ_{\Psi_{\mathcal{F}}} \phi = \{\chi \in \Phi : h(\|\phi\|) \subseteq \|\chi\|\}$). Similarly, let $\langle \Omega, \mathcal{G}, g \rangle$ be the other simple choice structure described above ($\mathcal{G} = \{E \cap \Omega_A : E \in \mathcal{E}_A\} \cup \{\Omega\}$ and g is the restriction of f to \mathcal{G}). Then, by (47), $\langle \Omega, \mathcal{G}, g \rangle$ is rationalized by the total pre-order $\succsim_A \stackrel{\text{def}}{=} \succsim \cap (\Omega_A \times \Omega_A)$, so that, by Proposition 7 in Bonanno (2009), there exists a supplemented AGM function $*_A$ that extends the function $\circ_{\Psi_{\mathcal{G}}}$ defined above (that is, $K \circ_{\Psi_{\mathcal{G}}} \phi = \{\chi \in \Phi : g(\|\phi\|) \subseteq \|\chi\|\}$). Now define $\circ : \Phi \rightarrow 2^\Phi$ by

$$K \circ \phi = \begin{cases} K & \text{if } \phi \in \Phi_R \\ K *_C \phi & \text{if } \phi \in \Phi_C \\ K \cap (K *_A \phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{48}$$

We need to show that \circ is an extension of \circ_{Ψ} . This is a consequence of the following facts:

1. by Definitions 3 and 4, if $\|\phi\| \in \mathcal{E}_R$, then $\phi \in \Phi_R$ and $K \circ_{\Psi} \phi = \{\chi \in \Phi : f(\Omega) \subseteq \|\chi\|\} = K$,
2. $*_C$ is an extension of \circ_{Ψ_C} (recall that, by Definition 4, if $\|\phi\| \in \mathcal{E}_C$ then $\phi \in \Phi_C$),
3. $*_A$ is an extension of \circ_{Ψ_A} (recall that, by Definition 4, if $\|\phi\| \in \mathcal{E}_A$ then $\phi \in \Phi_A$),
4. by Definition 9, if $E \in \mathcal{E}_A$ then $\emptyset \neq E \cap \Omega_A \in \mathcal{E}_A$ and $E \cap \Omega_C = \emptyset$, so that by (47) $f(E) = \text{best}_{\succsim} \Omega \cup \text{best}_{\succsim} E = f(\Omega) \cup \text{best}_{\succsim} E$; thus if $\|\phi\| \in \mathcal{E}_A$ then $f(\|\phi\|) \subseteq \|\chi\|$ if and only if $f(\Omega) \subseteq \|\chi\|$ (that is, $\chi \in K$) and $\text{best}_{\succsim} E \subseteq \|\chi\|$ (so that $\chi \in K *_A \phi$) and thus $K \circ_{\Psi} \phi \subseteq K \cap (K *_A \phi)$. \square

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