Journal for General Philosophy of Science (2023) 54:125–148
https://doi.org/10.1007/s10838-021-09598-8

ARTICLE

Conceptual Structuralism

José Ferreirós[1]

Accepted: 29 November 2021 / Published online: 23 November 2022

Abstract
This paper defends a conceptualistic version of structuralism as the most convincing way of elaborating a philosophical understanding of structuralism in line with the classical tradition. The argument begins with a revision of the tradition of "conceptual mathematics", incarnated in key figures of the period 1850 to 1940 like Riemann, Dedekind, Hilbert or Noether, showing how it led to a structuralist methodology. Then the tension between the 'presuppositionless' approach of those authors, and the platonism of some recent versions of philosophical structuralism, is presented. In order to resolve this tension, we argue for the idea of 'logical objects' as a form of minimalist realism, again in the tradition of classical authors including Peirce and Cassirer, and we introduce the basic tenets of conceptual structuralism. The remainder of the paper is devoted to an open discussion of the assumptions behind conceptual structuralism, and—most importantly—an argument to show how the objectivity of mathematics can be explained from the adopted standpoint. This includes the idea that advanced mathematics builds on hypothetical assumptions (Riemann, Peirce, and others), which is presented and discussed in some detail. Finally, the ensuing notion of objectivity is interpreted as a form of particularly robust intersubjectivity, and it is distinguished from fictional or social ontology.

Keywords Philosophical structuralism · Conceptual mathematics · Methodological structuralism · Minimal realism · Objectivity · Mathematical practice · Peirce · Hilbert · Dedekind · Riemann

In Memoriam Sol Feferman

« Die Mathematik ist so im allgemeinsten Sinne die Wissenschaft der Verhältnisse» (Gauss in 1825).

Structuralism in the philosophy of mathematics explores the idea that what matters to a mathematical theory is not the inner nature of mathematical objects, be they numbers, points, functions, or spaces, but how those objects relate to each other. "In a sense, the thesis is that mathematical objects … simply have no intrinsic nature," as Shapiro said

José Ferreirós
josef@us.es

1 Dept. Lógica y Filosofía de la Ciencia, Universidad de Sevilla, Sevilla, Spain

in the Internet Encyclopedia of Philosophy (Shapiro, 2008). Hellman writes that, in some sense to be clarified, the objects "serve only as relata of key relations, and their "individual nature" is of no mathematical concern, if one can even speak of such a nature" (Hellman, 2005, 537).

In the practice of mathematics, structuralism is a methodology that has found more than one embodiment. Initially, around 1900, it was closely associated with axiomatics and set theory, a structure was a set of elements linked by a network of relations, which was specified in the axioms. Essentially, one can say that, in that sense, a structure is a *relational framework* (the language of set theory and the axiomatic method make it possible to describe it in detail). In the second half of the twentieth century, mathematicians took the next step, and attempted to characterize the structure of a mathematical "object" by its interrelations with other complex "objects"—this is called category-theoretic structuralism.[1] In the philosophy of mathematics, a number of different interpretations of the structural approach to mathematical systems have been elaborated, with different implications for ontology and epistemology.

In practice, structuralism is based on conceptual work: to apprehend structures is to elaborate concepts of structure, that is to say, conceptions of relational frameworks, by means of axiom systems describing them.[2] Indeed, it can be argued that mathematical work is conceptual work: *Mathematics is conceptual*, to work in mathematics means to study and clarify relations and relational systems—and, one step up, their interrelations. Yet in philosophical quarters the pursuit of structuralism has become entangled with platonistic assumptions, probably due to the philosophers' understandable concern for the objectivity and independence of mathematical knowledge.

I aim to defend the thesis that philosophical structuralism can and should be elaborated along lines that preserve its original *conceptualism* and resist the lure of metaphysics. To do so, we shall explore the tension between the relationalism of the early structuralists, and the postulation of "objects" (or meta-objects) such as structures, which happens e.g. in Shapiro's *ante rem* structuralism. If I am right, the form of conceptual structuralism that will be sketched here is closer to the classical forms of structuralism (as found in Dedekind, Hilbert or Noether) than either *ante rem* or *in re* structuralism.

I will be articulating a standpoint close to Feferman's "conceptual structuralism"—a viewpoint that this great logician proposed years ago, recently elaborated in two papers (Feferman, 2009; 2014). Other authors have elaborated related views, prominent among them Parsons (2004); indeed my proposal could be presented as an attempt to synthesize and combine some viewpoints of Feferman and Parsons with my own ideas. But here we shall be emphasizing above all some points of connection with classical figures in modern structural mathematics.

Section 1 introduces the history of "conceptual mathematics" and how it evolved into forms of structuralism. In Sect. 2 I discuss the relationalism of the classical authors, and its tension with the platonism of some philosophical elaborations. Section 3 presents conceptual structuralism as the best option for capturing the classical spirit, and identifies the key requirement that this position ought to satisfy. This is *objectivity*, discussed in more detail in Sect. 4, articulating the basis for a convincing and robust account. Here, as in previous

[1] We shall not deal with category theory in this article. See Marquis (2020) and also Awodey (2004), Krömer (2007), Marquis (2009).

[2] This can be substantiated in many ways, for instance through the nice presentation in Mac Lane (1996).

authors, the motto could be: “objectivity without objects” (Kreisel, Putnam) or better still, “objectivity before objects.” But a motto is just a way of gesturing towards the detailed argument that ought to occupy its place.

1 From “Conceptual Mathematics” to Relational Systems

Modern structuralism, the twentieth-century variety, emerged historically from the tradition of conceptual mathematics, a.k.a. the “conceptual methodology” in mathematics. In the early twentieth-century this was closely connected with the mathematical tradition of Göttingen. Aleksandrov talked about “*Begriffliche Mathematik*” (in German in the Russian original) in his obituary of Emmy Noether; in this case the approach is clearly tied to the structuralist method as elaborated in modern algebra: reliance on set-theoretic methods and axiomatics in the presentation of relational systems, the role of isomorphism and homomorphism results.[3] Noether herself used to emphasize the close similarities between her preferred methodology and the work of Dedekind (in particular his 1894 work on ideal theory), but the fact is that the denomination “conceptual mathematics” has both a previous history and later resonances.

The rubric has been associated with the names of some highly influential mathematicians (as it happens most of them Germans), notably Dirichlet, Riemann, Dedekind, Hilbert and E. Noether.[4] Their innovation consisted, initially, in reworking mathematical theories so as to base results more on far-reaching concepts than on extensive calculation,—as Dirichlet said, this was a tendency “to put thoughts in the place of calculations”—thus reconceiving previous theories and presenting the results in rather abstract terms. One of the outcomes was that the new “conceptual” theories admitted many instantiations of different kinds.

It may be argued that Riemann’s work in the 1850s was a turning point in this development. His definition e.g. of *analytic* functions (in the context of complex analysis) was clearly more “conceptual” in comparison to other contemporary alternatives, an aspect of his work that Riemann himself compared with the method used by Dirichlet in his study of the representation of functions by means of Fourier series. But perhaps we can use as a key example another one, Riemann’s approach to the analysis of space-forms by means of manifolds and differential geometry. Without getting into the technical details, the idea is that Riemann felt the need to illuminate the conception of “spatial magnitudes” by subsuming them under a more general “concept”, which turned out to be the idea of an *n-dimensional manifold*. The 3-dimensional space of the Euclidean tradition was to be conceived as a particular instance of an *n*-dimensional manifold, and to be analyzed by comparison with other possible instances (including the non-Euclidean space of Lobachevskii, but also other possibilities).

Roughly speaking, a *manifold* is just a point-set which forms a continuum of a certain dimensionality (those are *topological* properties), but Riemann focused especially on manifolds for which a *metric* structure had been defined with the means of differential geometry (these are called Riemannian manifolds). A typical (later) structuralist treatment of the notion would explain what a manifold is by presenting a group of axioms that determine its

[3] See Aleksandrov (1936, 101) and also Corry (1996), McLarty (2006).

[4] See among others Ferreirós (2007, ch. 1), Laugwitz (1996), Goldstein (1989). A very valuable recent addition to the literature on early structuralism is Reck & Schiemer (2020).

topological structure, and another group of axioms that characterize its metric properties. This is in essence the methodology that Hilbert made famous with his 1899 work *Foundations of geometry*.

Riemann was explicit about methods and goals: he remarked that previous studies (e.g. of complex functions) were based on an *expression* for the function, which allowed to compute every value; but his approach was to be *independent* of any definition by means of operations or analytical expressions. One starts from a general concept [*Begriff*] of a complex (analytic) function, and adds to it only the characteristics [*Merkmale*] that are *necessary* for determining the function—the analytical expressions will be obtained only *as a result* of the development of the theory (see Ferreirós, 2007, 30, which includes a quote).

The novel ideas and style of work of Riemann were very influential in the last third of the nineteenth century, leaving their mark on the work of mathematicians like Klein or Poincaré, and even on philosophers such as Frege and Husserl ("manifold" was a key term in Husserl's reflections on mathematics). Riemann's work was proposed as a methodological model by Dedekind, Klein and Hilbert, all of them names linked to Göttingen.

Dedekind was another pivotal figure, particularly relevant for Noether; he was directly and heavily influenced by Dirichlet and Riemann. In 1895 he speaks about the "Riemannian definition of functions by means of characteristic inner properties, from which the outer forms of representation arise with necessity" and says that his efforts in advanced number theory were oriented in just the same way—to base the investigation, "not on accidental forms of representation, but on simple basic concepts" (Ferreirós, 2007, 29). Hilbert in the Preface to his famous *Zahlbericht*, while considering some results of Kummer as the "highest peak" ever reached in number theory, goes on to say that he has tried to *avoid* the great calculational apparatus of Kummer, "so that also here the basic principle of Riemann can be realized, that one should produce the proofs not by calculation, but exclusively by means of thoughts" (Hilbert, 1897, vi).

We may offer a rather simple example of "basic concept" (or structure) from Dedekind's work: the concept of a *number-field*. At some point around 1860, he started thinking about what is common to different systems of numbers, examples being the rationals **Q**, the reals **R**, the complex numbers **C** (but also systems of numbers of the form $a+b\sqrt{-5}$ with $a, b \in \mathbf{Q}$, and so on indefinitely). Thus he became interested in a certain abstract "form" that was crucial for Galois theory, for algebraic number theory, indeed (he thought) for algebra in general. What Dedekind did was to introduce the name, *Körper* (*corps*, field) and to characterize the relevant kind of number system, as being so "closed and complete" that one can perform the 'four species' (sum, product, rest, division) unlimitedly.[5]

There are many different concrete instances of number-fields, in fact infinitely many. The smallest is **Q**, the largest is **C**. Some *Körper* are totally ordered (an example is **Q**), some are not (the complex numbers **C**); some have a *dense* ordering (say, the algebraic reals **A**) while some furthermore are *continuous or complete* (the reals **R**). These points were carefully discussed in Dedekind's famous essay on the concept of continuity and the irrational numbers. Some *Körper* are substructures of others, in fact Dedekind realized that there is a whole lattice of fields in between **Q** and **C**. In 1871, he also presented the idea of *isomorphism* (but not under this name) when he discussed how a number-field A has a "conjugate field" $\mathrm{B}=\Phi(\mathrm{A})$ obtained through a "substitution" Φ (what he later called an

5 That is, the system has closure under the four basic operations (except division by 0); together with their usual laws including distributivity. This is equivalent to laying down axioms for some well-known algebraic relations between the elements of the number-field, as Hilbert later did.

Abbildung, a mapping or function). He underscored the fact that the relation of conjugation is an equivalence relation: "two fields conjugate to a third are also conjugate of each other, and every field is a conjugate of itself" (quoted in Ferreirós, 2007, 92).[6]

Notice that we have started with the conceptual determination of a relational system, the kind of network-of-relations called a "number-field", but then we have moved to interrelations among those systems (such as isomorphism). Naturally, concrete fields can be regarded as "objects" of a complex kind, and we go on to analyzing relations between them, and so on. Mathematical thought is always iterative, from the basic level of the natural numbers, all the way up.

Being thus equipped, Dedekind could also realize that the system of algebraic functions has the *Körper* (field) structure, at which point the new methodology was becoming the source of significant mathematical advances.[7] The analogue of an ideal theory here was the basis for a totally new way of grounding results on algebraic functions, culminating in a new algebraic proof of the Riemann–Roch theorem (in joint work of Dedekind & Weber, 1882). This is a beautiful, and mathematically highly productive, example of the feature that we discussed at the beginning, namely that the new theories of "conceptual mathematics" admitted many different instantiations.

Let us take stock. To apprehend structures is to elaborate *concepts of structure,* general notions *of relational frameworks*, by means of axiom systems describing or characterizing them (which also requires the selection of primitive concepts and the corresponding symbolism). Such was the notion of a differentiable manifold which emerged from Riemann's work, or the different notions of space (Archimedian, non-Archimedian, Euclidean, non-Euclidean) that Hilbert presented in his famous work on the foundations of geometry.

Hilbert, by the way, often expressed himself saying that the axioms, which in our parlance characterize a structure, make precise a mathematical *concept*.[8] This again underscores the importance of the tradition of "conceptual methodology" in mathematics.

What about mathematical objects in this tradition? As Shapiro said, what matters to mathematics from this standpoint is not the inner nature of mathematical objects, but how those objects relate to each other. As Hellman underscored, in some sense the objects serve only as relata of key relations, and their "individual nature" is of no mathematical concern, if one can even speak of such a nature. The example of Dedekind's treatment of the natural numbers is well known: numbers are not singular objects as in Frege,[9] but just "the abstract elements" of a *simply infinite system*; ordinal numbers, the *ordinal* relations among numbers (determined by the successor function) are the key; even in our intuitive arithmetical development, "the concept five is only reached via the concept four" (letter to Weber, January 1888; Ewald, 1996, II, 835).

Dedekind insisted that mathematical objects are "free creations" of the human mind, but understood this to mean that they are thought-objects (*Dinge*, elements of the *Gedankenwelt*) whose existence is legitimized by the general laws of logic. The creation is free but

[6] Following along those lines, Dedekind introduced more advanced ideas such as the set-theoretic notion of an *ideal* (a certain kind of subset of the ring of integers in a given number-field), which became the basis for his solution to the general problem of the number theory of algebraic integers.

[7] This is not a number-field, but a more general kind of instance with the same "form".

[8] See Ferreirós, 2009, 56–57.

[9] Frege was interested in characterizing each number as a uniquely specified object (the Caesar problem). See Reck, 2003 and Ferreirós, 2017 for some more subtle issues about Dedekind's structuralist approach that I skip here. See also Reck's chapter in Reck & Schiemer (2020) for Cassirer's relational and structuralist views and his reaction to Dedekind.

strictly bounded by the laws of logic.[10] This is how the irrational numbers are introduced as new objects, but it also applies to space and its continuity, as Dedekind explains in an interesting passage:

> If space has a real existence at all it is *not* necessary for it to be continuous; many of its properties would remain the same even if it were discontinuous.[11] And if we knew for certain that space were discontinuous there would be nothing to prevent us, in case we so desired, from filling up its gaps in thought and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be carried out in accordance with the above principle. (1872, 772, Sect. 3)

Interestingly, this is exactly parallel to the way Hilbert handles the problem of the infinite in his well-known paper of 1925. First Hilbert discusses the results of physics at the time, arguing that there is no evidence of the physical existence of the infinite, either in the extremely large (cosmology) or the extremely small (quantum physics). But then, he claims that the infinite may have a well-justified place "*in our thinking*" and the role of "an indispensable concept" (Hilbert, 1926, 372), the reality of mathematics being quite unlike 'existence' in the naïve sense. In the paper he goes on to introduce the ideas of metamathematics by highlighting the central role of *ideal elements*, as distinct from contentual elements and relations, and ultimately he lays out the plan for justifying the infinite as *an idea* (almost in the Kantian sense, 1926, 392), a basic ideal element, justified by metamathematics and proof theory. In Hilbert's approach, the cornerstone is a consistency proof, which plays a role parallel to Dedekind's "logical proof of existence".

Those ideas were perceptively understood by some philosophers, most notably perhaps Cassirer in *Substance and Function* (1910).[12] In this work he offers an interesting philosophical exegesis of some early structuralist contributions in math, for instance of Dedekind's views. About his analysis of natural numbers Cassirer writes that everything depends on the structure of a progression, i.e. what Dedekind called a simply infinite system. And he goes on:

> What is here expressed is just this: that there is a system of ideal objects whose whole content is exhausted in their mutual relations. The 'essence' of the numbers is completely expressed in their positions. (Cassirer, 1910, 39)

At several places he explains that the "things", the "ideal objects" that are spoken of, are not assumed as independent existences anterior to any relation, but gain their whole being in and with the relations which are predicated of them (Cassirer, 1910, 36). The whole 'certitude' or 'solidity' (*Bestand*) of numbers "rests upon the relations, the interrelations

[10] Without a "logical proof of existence", it would always remain dubious whether the assumption of such objects may not involve contradictions (letter to Keferstein, February 1890). For, as he had said already long time before (letter to Lipschitz, July 1876), "nothing is more dangerous in mathematics than to *assume* existence without sufficient proof".

[11] In a letter (to Lipschitz, July 1876), he explained that "the concept of space is totally independent, completely separable from the representation of continuity, and property (C) serves only to select, starting from the *general* concept of space, the *special* one of continuous space." Property (C) is continuity as defined by Dedekind's cut principle (1872, sec. 3).

[12] See the corresponding chapter in Reck & Schiemer, 2020. Cassirer does not employ the term 'structure', nor talk of structuralism, but it is quite natural to elucidate his views using this word.

between themselves, and not upon any relation to an outer objective reality" (Cassirer, 1910, 38). Cassirer went so far as to say that the reality of those ideal objects does not depend on physical reality (the outer world) nor on mental reality (the inner world).

To some extent, that is reminiscent of Hilbert. It is worthwhile to remind the reader that in 1927 Hilbert would state that "mathematics is a presuppositionless science":

> To found it I do not need God, as does Kronecker, or the assumption of a special faculty of our understanding attuned to the principle of mathematical induction, as does Poincaré, or the primal intuition of Brouwer, or, finally, as do Russell and Whitehead, axioms of infinity, reducibility, or completeness... (Hilbert, 1927, 479)

As one can see, the reality of mathematical objects is independent from metaphysical considerations. Math is presuppositionless, its requirements are minimal—pure logic according to Dedekind, the intuition of symbols or finitary objects, plus logic, in the case of Hilbert. The classical variants of structuralism thus emphasized how this "conceptual methodology" discharges any kind of external consideration of 'real existence' in the naïve sense of these words.[13]

2 Two Interpretations: Platonism and Relationalism

So much for history. Let us now turn to philosophical structuralism. It is well known that the structuralist methodology can be interpreted philosophically in many different ways. Here I would like to emphasize two significant and very different interpretations: one of them is platonistic, the other builds on a form of relationalism. The two seem to pull in opposite directions. But it is the last interpretation that seems to be in line with the spirit of the structuralist viewpoint, at least in its early decades.

Let us call the first interpretation *p*-structuralism, for platonist structuralism. Shapiro has written (2008, Sect. 2):

> the *ante rem* structuralist holds that, say, the natural number structure and the Euclidean space structure exist objectively, independent of the mathematician, her form of life, and so forth, and also independent of whether the structures are exemplified in the non-mathematical realm. That is what makes them *ante rem*.

This is certainly unlike Dedekind's "free" human creations.[14] Notice the characteristic insistence on *absolute* independence from the mathematician, "her form of life, and so forth," which is what leads this philosophical line into heavyweight forms of platonism.[15] I will argue that this move is not only unconvincing, but also *unnecessary* to ground the relevant independence and objectivity.

[13] Also Cantor with his "immanent" reality of mathematical objects (and disregard of "transient" or metaphysical considerations, see Cantor (1883, Sect. 8).

[14] For an interpretation of Dedekind's idea, along Kantian lines that emphasize the productivity and autonomy of the understanding, see Ferreirós & Lassalle-Casanave (2022).

[15] Linnebo defines "Mathematical platonism" as the conjunction of three theses: *Existence*: There are mathematical objects; *Abstractness*: Mathematical objects are abstract; *Independence*: Mathematical objects are independent of intelligent agents and their language, thought, and practices (Linnebo, 2013, sec. 1).

Many authors have presented a rather different understanding of structuralism and its philosophical impact. Call this second interpretation *r*-structuralism, where *r* stands for relational. Their viewpoint is often intuitive and less elaborate than the previous one, and promotes the idea that mathematical structuralism actually reduces the platonistic implications of mathematics.

Let me present an early example that I find highly relevant, not only for the early date but also because of its author. Already in 1825, Gauss wrote that "mathematics is, in the most general sense, the science of relations, insofar as one abstracts from any content of the relations;"[16] this was left unpublished, but it can be interpreted to point the way towards a structuralist understanding. Gauss did publish in an influential paper the following:

> The mathematician abstracts entirely from the quality of the objects and the content of their relations; he just occupies himself with counting and comparing their relations to each other. (Gauss, 1831, 175-176)

It is well known that Poincaré expressed similar ideas many years later, in *Science and Hypothesis* (1902) and other places: the mathematician does not study objects, but relations between the objects; what is important is the relations considered, the objects can be replaced at will. Interestingly, the context of his statement was a discussion of Dedekind's work, in particular his ideas about the continuum and the real numbers (see Poincaré, 1902, 20).

Gauss's pronouncement implies that, unlike physics or chemistry, mathematics is not devoted to the study of some particular kind or kinds of objects. The mathematician compares relations and considers their interconnections, and in the process he (or she) abstracts entirely from the nature of the relata and even the content of the relations, paying attention only to formal features. We are left with an extremely abstract science that finds application (potentially at least) in any possible area of human experience: relations and interrelations can be found in any field. The mathematician relies on her own peculiar objects (e.g. complex numbers) to develop the analysis, but "mathematics is, in the most general sense, the science of relations". Could it be that the mathematical objects make "no substantial demand on the world", above and beyond the presence of relations?[17]

When Dedekind characterizes the natural number system, in § 6 of *Was sind und was sollen die Zahlen?* (1888, 809), he requires that we "disregard entirely the peculiar nature of the elements" (of whatever *simply infinite system* is being taken as a basis), retaining only that those elements are distinct, and that we "take into account only the relations to one another in which they are placed by the ordering mapping" (the *successor* function). This is a very explicit early example of the structuralist viewpoint, especially because Dedekind underscores the isomorphism of all simply infinite systems, the fact that the same "relations or laws" are valid for each and every one of them. Notice also that the emphasis is wholly on a *system of relations*, regardless of the nature of the relata and the concrete content of the relations. We are on similar grounds as with Gauss, and the implication seems to be, once again, that the objects of mathematics come in an "easy" way, free from metaphysical implications or presuppositions.[18]

[16] In Gauss (1917, 396).

[17] The phrase is from the introduction to Linnebo (2018), a work that can be linked with this line of thought. See also Thomasson (2014) on the idea of 'easy ontology'.

[18] Incidentally, Dedekind's idea (1888, 791) that all of pure mathematics is based "solely" on the notion of a mapping or Abbildung (representation, correspondence, functional relation) seems to clearly point in the direction of relationalism. On this topic, see Ferreirós (2017).

Numbers enter into scientific thinking as essential means to express and describe certain relations, patterns and structures. If one wanted to be specific about the metaphysical counterpart of numbers and number relations, the answer is not some 'objects' in the world, but some kinds of relations, more or less complex patterns, or relational interconnections. To this, of course, the mathematical ideal picture of the natural number structure adds the important element of idealization, insofar as it disregards feasibility and considers the structure as actually or potentially infinite (see Sect. 5)

Let us come back to recent work. If I understand his views correctly, also Hellman is motivated by seeing structuralism as a perspective on mathematics that is primarily conceptual and displaces interest from objects to relational systems. He writes:

> … it is characteristic of a thoroughgoing structuralism to treat even these [non-algebraic, monomorphic][19] systems as like the more "abstract" ones, in that the "objects" involved serve only to mark "positions" in a relational system; and the "axioms" governing these objects are thought of not as asserting definite truths, but as defining a type of structure of mathematical interest. (Hellman, 2005, 536)

Similar considerations are easy to find in the writings of almost all philosophical structuralists, the main differences being due to secondary considerations, which guide the choice of a preferred theoretical approach. I mean considerations about semantics, realism or anti-realism, about grounds for objectivity, modal considerations, and so on.

The question is how best to articulate these vague ideas that are shared by all, and how to navigate the details of the theoretical account. Assuming that we are interested in a relationalist (not a platonist) understanding of structuralism, my thesis will be that conceptualism is the best approach. In Sect. 4 we shall see its outlines and some of the reasons why it has not been articulated and proposed before.

A form of "conceptual structuralism" was proposed by Sol Feferman (2009; 2014), who contends that the basic objects of mathematics "exist only as thought-objects or mental conceptions," though their source lies ultimately in everyday practices. Feferman was, by his own admission, a philosopher "by temperament" and his ideas on this topic seem to have been elaborated over decades, by considering many different inputs. His first presentation of glimpses of such a view was in a 1977 paper given at Columbia University, which however remained unpublished.[20] The basic conceptions of mathematics are "of certain kinds of relatively simple ideal-world pictures," and Feferman insists that such basic conceptions "are communicated and understood prior to any axiomatics, indeed prior to any systematic logical development" (Feferman, 2014, 4–5).[21] Does his lapse into "mental conceptions" throw us into psychologism and relativism? Does it compromise the independence and objectivity of math?

This kind of conceptual structuralism is clearly in line with the second interpretation, *r*-structuralism. Thus Feferman is explicit in rejecting any form of heavyweight platonism,

[19] On monomorphic (categorically determined) structures, see Sect. 5 below.

[20] The title was 'Mathematics as objective subjectivity', see the FOM entry mentioned below; later he talked about 'intersubjectivity'. I believe that Feferman's thinking was influenced by mainstream ideas concerning structuralism, by philosophers such as Tait and others, but also (and strongly) by reflections on the *practice* of mathematics inspired by constructivist authors such as Weyl, Kreisel, etc.

[21] Interested readers should consult the *ten theses* that Feferman proposes, in both papers mentioned in the main text; albeit very interesting and suggestive, I find them too cursory to provide a solid understanding of his approach.

saying that his viewpoint "is an ontologically non-realist philosophy of mathematics" (Feferman, 2014, 4). But essentially the same standpoint can be presented without a plain rejection of abstract objects. In the next section I argue that structuralism does not require a rejection of the reality of mathematical objects altogether, although it rejects heavyweight platonism.

3 Logical Objects

The "mental conceptions" of mathematics are better described as thought-objects [*Gedankendinge*], an expression employed by Hilbert, the crucial point being that such logical objects can be *described and specified* by theoretical means. E. g., the natural numbers can be described or characterized by means of the Dedekind-Peano axioms in weak second-order logic, and the set theoretic universe (or universes) by the Zermelo-Fraenkel axioms in first-order logic.

We have said that numbers enter scientific thinking as essential means to express and describe certain relations, sometimes complex patterns of interrelations. How come, then, that mathematical language features numbers as objects?

Reification or hypostasis is a basic logico-linguistic phenomenon, and I venture to say that we should not ascribe a profound metaphysical significance to it. Whenever we formulate a theory about some subject matter (whether it is massive bodies or real numbers), the natural way is to refer to the relevant 'things' and their properties and interrelations, using the framework of basic first-order logic. In doing so, we come to talk about objects (like number π), we predicate of them, deal with relations or operations between them, quantify on them, and so forth. Object talk is admissible within any theory, but it lacks deep content—it is closer to surface grammar.

Think of the case when we are elaborating a theory of relations (as Peirce, Frege or Russell were). Is a relation the same, metaphysically speaking, as an object? One would say no,[22] but despite this, when formulating the theory we shall refer to relations as 'things', we shall discuss their properties (is it symmetric?) and interrelations (the composite of two relations), we shall quantify (for any relation there is the inverse), and in doing so we shall be using the framework of basic first-order logic. It has been proposed that we may talk about a notion of *logical object* (Parsons, 2009), that requires nothing more than the above, predication and quantification in a first-order logical framework.[23] Hence there is not just one kind of objects, and logical objects must be kept separate from additional connotations involved in the notion of a physical object (actual or *wirklich* in the sense of physically acting, or in naive language "really existing").

There is a long tradition of admitting the reality of abstract objects, without implying that they "exist" in anything like the physical sense of existence. This tradition has been

[22] In his well-known papers 'Function and concept' and 'On concept and object', Frege denied this in the most emphatic way. But the fact that we talk about "the concept *horse*" and the like, in apparent reference to an object, gave him philosophical trouble (hence his famous phrase "the concept *horse* is not a concept").

[23] A referee asked for clarifications, since I am interpreting the idea of *Gedankendinge* (Dedekind, Hilbert) in terms of logical objects, but also emphasizing first-order logic. One can say this: second-order logic can also be considered, as long as we refrain from adopting the "full" set-theoretic semantics (sometimes called "standard" semantics); that suffices e.g. for Dedekind's work on numbers. The reinterpretation of Hilbert's *Gedankendinge* I am proposing requires some adjustment, clearly, but our aim here is not exegetic but philosophical.

revitalized by authors such as Parsons and Tait,[24] but it begins with the founders of modern logic. Peirce distinguished the *reality* of logical or mathematical entities from what he called *existence*, the latter meaning "reacting with other like things in the environment;" Frege distinguished objectivity from *Wirklichkeit*, "actuality" (i.e., to act physically or to produce effects which may cause sense-perceptions).[25] The existence of mathematical objects is "ideal existence," as Hilbert and Zermelo said,[26] and with that adjective they seemed to aim at the same distinction (formulated differently by Frege and Peirce).

It is a subtle philosophical matter whether we choose to speak, like Feferman and others, of a "non-realist" philosophy (notice that in Peircean terminology one should write "non-existential"), or else of the "reality" of mathematical objects interpreted along lightweight lines, as explained e.g. by Parsons (2009).[27] The difference may be less than it appears at first sight, and in any case our choice will be in need of clarification. I prefer to follow the line of Parsons, but in doing so I believe we are not introducing an essential difference with the position of Feferman.

This whole discussion should remind the reader of recent contributions to the literature on ontology and philosophy of mathematics, above all Linnebo's *Thin Objects* (2018). In fact, what I call logical objects is substantially the same as his 'thin objects', although his work on abstractionism should be considered a particular way of presenting a more general idea. The bonus, of course, is the great precision and clarity of Linnebo's work. Notice that relations do the really substantial work in Linnebo's abstractionist introductions of thin objects, hence one can argue that his abstractionist position is a form of structuralism. (He starts in each case from a basic domain, say $D_0 = \{a, b\}$, but the nature of the entities in question is irrelevant, the only relevant thing is that they can bear certain kinds of relations, on which basis higher domains are introduced; we can always replace D_0 by another domain D'_0 which may be, let us say, a set of two apples, $\{apple_1, apple_2\}$.)

To try to reduce ambiguities, which are quite inevitable due to the overuse of the key terms in this discussion, I will be following Peirce's terminology. *Existence* implies physicality (in some sense), *reality* in the parlance of this paper does not (at places, I will still employ 'existence' to avoid excessive departure from common usage, but then I will use scare quotes). You may dislike that terminological choice: if so, all you need to do is exchange one word for the other. I reserve the word *platonism* for its heavyweight form (in agreement with Linnebo, 2013), while the lightweight version is called *realism*. If lightweight realism makes sense, as I believe it does, then one can be a realist in truth value without needing a platonic realm of 'heavy' things to sustain that. Remember that the *Gedanken-dinge* of Hilbert and Dedekind are 'light' things, 'thin' objects, in which case we may talk about logical objects.

A conceptual structuralist can therefore accept a form of lightweight realism, defined as the conjunction of three theses[28]: 1. *Reality*: there are mathematical objects, though they

[24] This form of lightweight platonism which I believe to agree with previous proposals by Tait (2005) and Parsons (2009), follows on the footsteps of Dedekind, Hilbert, Zermelo, Carnap, Quine.

[25] Frege (1893, xix). Peirce (1902, 375) Following Peirce, the quantifier $\exists x$ should be read as "there are" but not as "there exist"—that is to say, the basic logical operator indicates reality in the broad logical sense; claims of existence in the strict sense would involve extra information about actual *physical* reality via experimental data or the like.

[26] Zermelo (1930, 43), Hilbert's "ideal elements" in (1926).

[27] This is reminiscent of Maddy's (2011) and the way she oscillates between arrealism and thin realism.

[28] Compare with Linnebo's (2013) description of platonism, cited in a footnote above; the crucial difference is in condition 3.

are not analogous to physical objects; 2. *Abstractness*: mathematical objects are abstract; 3. *Objectivity*: mathematical objects, though not independent of intelligent agents, are independent of mental processes—of anyone's particular mental processes—, they are objective insofar as they are strongly intersubjective.

We shall have to clarify this last point. What is the foundation for such claims of objectivity? And *how objective* are the relevant (abstract) objects? Is objectivity an all-or-nothing aspect, or does it come in degrees?[29]

In order to analyze this crucial aspect, what becomes key is the theory in question, each time, and the grounds for its admission. Take the case of natural number arithmetic: we know this theory *with certainty*, which implies that the reality of natural numbers is as solid as the reality of truth and falsehood (see Sect. 5.2). We form a basic conception of numbers already thanks to counting practices, and the Peano-Dedekind axioms are recognized as *truths* with respect to such numbers. There is not much comparable with the reality of natural numbers, even inside the domain of mathematical theories. By presenting things this way, I hope, it becomes clear that all this has nothing to do with a Platonic Heaven.

A more interesting question is: How real and objective are the real numbers? My answer would be: Not like the naturals, because the corresponding theory is more complex, less certain. Why so? Precisely because it rests on hypothetical postulates of a kind that is not to be found in elementary arithmetic; intended here are axiomatic assumptions such as the continuity or completeness of **R** (see Sect. 5).

In the view that I defend, it makes sense to analyze the grounds for admission of a theory and to insist that different theories may stand on more or less solid ground. And, *pace* Quine, I interpret this to mean that not all our objects "are there" in just the same way. We have the right to make a difference between natural numbers and quarks, or between numbers and topologically complete spaces. It may all be myth-making as Quine said, but some myths are more solid than others, more closely connected with our basic practices and experiences. Precisely because, on that account, mathematical theories cannot be compared to fictional narratives, the position I am delineating should not be labelled a 'fictionalism'.

4 Assumptions Behind Conceptual Structuralism

Feferman contends that the basic objects of mathematics exist only as thought-objects, though their source lies ultimately in everyday practices. The basic conceptions are "relatively simple ideal-world pictures" communicated and understood prior to any axiomatics, indeed prior to any systematic logical development. How are we to understand the pre-theoretical and even pre-logical ingredients that Feferman emphasizes?

The thought-objects and ideal-world pictures are *described and specified* by theoretical means: the natural numbers can be characterized by means of the Dedekind-Peano axioms in weak second-order logic, the real numbers can be specified by the Hilbert axioms. Such theoretical systems, and even more informal theories like the ones employed by mathematicians in earlier times, can be understood in a shared way that remains free from subjectivism or relativism.

[29] See also a companion paper, Ferreirós (forthcoming), where I compare mathematical ontology with social ontology. We cannot go into this topic here.

Once a structure or relational system is thus specified, mathematicians proceed to inquire into its properties, the connections between its elements, its links with other systems or "objects", etc. This process is naturally described by the mathematician as an exploration, as the *discovery* of the features of the system, which are independent of our intentions or desires. To understand this phenomenological aspect of mathematical work, one does not need platonism.

And yet, not all logical objects, not all structures are *fully* determined by their available descriptions.[30] The fact that something is our conceptual creation does not imply that it will be epistemically constrained in the sense that we have full cognitive command and are able to determine all its features and aspects. Why should this be an inborn condition of all our conceptual creations? Intuitionism was wrong to the extent that it made this assumption, if it made it at all. It may even be the case that some structures, sufficiently well specified as to investigate them deeply and take them as basic to math, may not be fully *determinable* even in principle.

The history of philosophy teaches us that a conceptualist position is harder to formulate and maintain than its more extreme neighbors, platonism and nominalism, but simplicity is not the only criterion here. In order to understand the conceptual nature of mathematics, and to obtain an adequate account of the peculiar objectivity of mathematical knowledge, one needs to get into an analysis of knowledge shared by communities of agents in a strongly intersubjective way.

And in order to do so, one has to analyze the cognitive roots of human knowledge, the shaping of our shared conceptions, how they are not necessarily subjective,[31] how they depend on everyday practices, how they depend on symbols and symbolic practices. One has to deal with the question how logic and mathematics elaborate on the vernacular language and on pre-theoretical notions, such as the general common-sense ideas of order, succession, collection, relation, rule and operation; or the general idea of property and the basic meaning of the logical connectives. Indeed, if we formulate them avoiding certain mathematical idealizations (so that, e.g., a *collection* is not a set—an abstract object, and a *succession* is not actually infinite), all those general ideas are quite easily understood and accepted by an average agent, by which I mean a human being of average cognitive capacities.

The conception of an infinite structure of the natural numbers may be acknowledged as a human thought-product, but one can also understand that its source lies in everyday practices, and ultimately in the structure of the world. For our patterns of action are, in the end, just part of the structure of the world—thus the idea that numbers are human conceptions does not make numbers unreal. The natural numbers reflect structural-relational facts about experience, objective facts. The view that something is a conception, emphasis on conceptualism, does not imply that it is not based on experience or that it is disconnected from the real world. That may only seem so to adherents of old forms of dualism.

This kind of approach to mathematical knowledge is agent-based, indeed I contend that, if we are going to defend a form of *conceptual* structuralism, then agent-dependence is

[30] See Feferman (2009) and (2014).

[31] The topic is intimately linked with a properly philosophical discussion of traditional but ungrounded assumptions, in particular about the "subjectivity" of the "mental". I reserve myself a detailed discussion for another occasion, but let me say here that I am essentially in agreement with pragmatists like Putnam. The dichotomy subjective/objective has traditionally (and naively) been aligned with mind/matter or mind/nature—but this stands in need of reconsideration.

inevitable. The conceptions in question are developed and shared by human enquirers and one can hardly claim "full independence" from human agents; that makes no sense. This agent-dependence is probably the reason why forms of platonism and nominalism have often been proposed, while conceptualism remains little explored. But one should not worry. There is enough of a basis to argue that conceptual structuralism avoids any dangerous psychologism or subjectivism, that the objectivity of mathematical results and developments can be saved.

For these reasons, the standpoint of conceptual structuralism may seem to lie closer to constructivism than to the usual assumptions of model-theoretic philosophy of language, truth and logic. Constructivists have always been concerned to understand the shaping of mathematical knowledge from the activities of agents—activities such as proving and constructing, interpreted concretely and not through the lens of idealized mathematical models. But conceptual structuralism can also be adopted by those who merely want to interpret classical mathematics; nothing in this viewpoint forces you to share the criticisms voiced by intuitionists or predicativists.

The key ingredient in this argument must be an account of the objectivity of mathematical results based on their shared theoretical descriptions as understood and elaborated by human agents.[32] Crucial to conceptual structuralism is to view the objectivity of mathematics not as a consequence of the independent existence of abstract objects, but rather the opposite: we are justified in assuming the reality of mathematical objects as a result of the development of objectively established theoretical frameworks. *Objectivity comes first*, logical objects only second. This was, arguably, the idea behind Hilbert's celebrated principle that mathematical existence is nothing but axiomatic consistency.

That may be largely in accord with Feferman's intuitive idea:

> The objectivity of mathematics lies in its stability and coherence under repeated communication, critical scrutiny and expansion by many individuals often working independently of each other, but on a common cultural basis. Incoherent concepts, or ones that fail to withstand critical examination or lead to conflicting conclusions are eventually filtered out from mathematics. The objectivity of mathematics is a special case of intersubjective objectivity. (Feferman, 2014, 5)

However, Feferman's discussion of this key issue of objectivity was too cursory, the topic requires further elaboration. This is the aim of Sects. 5 and 6.

The more traditional philosopher of mathematics may perhaps be surprised when confronting this way of posing the questions. Some logicians seem to assume that they must frame their analysis in minimalist terms. Perhaps they imagine themselves living in a world where there is nothing but natural language, formal languages, and abstract objects (numbers, structures). As if one should not presuppose anything more—in particular *not* embodied human agents. We (like Feferman) emphasize the pre-logical, pre-mathematical elements that emerge in human agents as part of what is called their mental life. Nowadays, in the context of both naturalistic philosophy of science and practice-oriented analyses of mathematics, this option should not seem surprising.

[32] The need for such arguments is of course avoided by authors who postulate a realm of *sui generis* mathematical structures existing independently of human forms of life or culture. The price is that such a move seems unconvincing, or at least raises as many problems as it solves: famously we lack an account of how knowledge of them is secured, and we lack an account of such independent 'being' that may square with contemporary scientific or philosophical views (a relevant example being pragmatism).

I shall not try to delve deeper into such issues here. Conceptual structuralism calls for interaction with careful studies of the cognitive roots of human knowledge, very especially the roots of our basic conceptions of number, time, and space (intimately tied with mathematical knowledge). In a practice-oriented and agent-based approach, one assumes given embodied agents with practical abilities, and with linguistic abilities, living among physical or other natural objects. Practical abilities include our competence to handle measuring rods and clocks and other tools—think e.g. of a microscope—without which the experimental practices on which scientific knowledge depends would be impossible. And we must emphasize that human cognition is a more complex affair than what current cognitive science typically covers, that our knowledge builds crucially on explicit representations such as number-words, diagrams, maps, and algebraic symbols.

For our purposes we do not need to include more than that.[33] There is nothing mysterious there, except of course if you consider it your goal to explain such practical abilities on the basis of the fundamental theories of physics. But such a foundationalist goal would be misplaced.

5 Conceptual Structuralism, Hypotheses, and Objectivity

The conceptual work of mathematics implies to study and clarify relations, relational systems, and their interrelations (iteratively going up). This may sound complex but is meant exactly. Consider the following examples, of increasing complexity:

- An ordering relation, e.g. total order, as an example of the first level (merely a relation);
- A relational system such as an ordered field, e.g. a number-field with an ordering relation (like $\mathbf{Q}$);
- Interrelations between fields such as algebraic closure, or group structures associated with fields (in Galois theory); at level three, we have relations between structures.[34]

Interrelations between heterogeneous structures—such as groups and fields in Galois theory, Lie groups and Lie algebras, or algebraic varieties and sheaves in algebraic geometry—are particularly important in the modern practice of structuralist mathematics.

Thus the subject matter of mathematics properly speaking is not objects, but relations and structures. Theories of 'objects' are perfectly all right as we have seen, but they are not primary—they are the tools employed to study structures: relations among relations, relations among structures, and so forth. In fact, reification may just be a feature of human psychology: instead of keeping track of a very complex network of relations at different levels, we prefer to assume given certain abstract objects.

[33] Although it may be relevant to consider the practical abilities of using pens to write on paper, or keyboards to write on a computer, since they underlie our symbolic practices. Notice too that this presupposes complex abilities having to do with perception, e.g., to perceive differently shaped letters as tokens of the same type.

[34] Category theory takes this third level as a basic ground, and iterates from there.

The key point is, furthermore, that mathematics establishes results about *hypothetical states of affairs*,[35] theorems about hypothesized structures (described by axioms which, as Riemann and Poincaré realized, can be regarded as *hypotheses*). There are two qualifications to be added, namely that part of mathematics is not hypothetical—what I call 'elementary' math—and that the hypothesized structures are designed to fit with the elementary ones.

5.1 Hypotheses

Some mathematical structures are implementations of content extracted from our dealings with the world, with a measure of idealization, as is the case with basic arithmetic or even with basic group theory; some are extrapolations from world phenomena with a more serious degree of hypotheticalness, as with the real numbers **R** or real functions f: $\mathbf{R} \to \mathbf{R}$ or continuous groups. While in the first case there is almost no hypothetical component (except for the idealization involved in disregarding feasibility), structures of the second kind incorporate assumptions that constitute strong hypotheses—this is the case in particular with *continuity*.[36] And there are yet further levels, as some structures are further iterations based on extrapolation from the previous structures, looking for higher-order closure. This remark can be applied e.g. to the set-theoretic universe **V**, or to categories, but we shall not enter into deep waters here.

The central non-algebraic structures, which encapsulate the core 'existential' assumptions of mathematics, namely the natural-number system **N**, the real-number system **R**, and the cumulative hierarchy **V** of set theory, are considered by authors such as Isaacson (2011, 26) to have been *fully captured*. They distinguish them from "*general*" structures, such as typically are the algebraic or topological structures, and they base this distinction on well-known categoricity results obtained within second-order logic. Yet these results are themselves hypothetical, exactly insofar as they presuppose the full semantics of second-order logic – and thus the thesis is contentious.[37]

According to Feferman (2014, 22), distinctions have to be made between those three cases, and I agree completely. He writes:

> The direct apprehension of these [basic structures] leads one to speak of truth in a structure in a way that may be accepted uncritically when the structure is such as the integers but *may* be put into question when the conception of the structure is less definite as in the case of the geometrical plane or the continuum, and *should* be put into question when it comes to the universe of sets.

This standpoint is based on careful logical analysis of the above-mentioned results.

[35] This happy expression is due to Peirce (1902, 141), following on the footsteps of Riemann, and in agreement with Poincaré and others. He explained that mathematicians mean by a 'hypothesis' "a proposition imagined to be strictly true of an ideal state of things" (1902, 137). See the paper by J. Carter in Reck & Schiemer, 2020.

[36] The relevant axiom can be formulated e.g. in terms of cuts (Dedekind), in terms of least upper bounds, or in terms of nested closed intervals (Bolzano-Weierstrass).

[37] The claim is only that the full semantics (sometimes called the 'standard' semantics) is hypothetical, to the extent that it presupposes arbitrary infinite subsets. The same cannot be said of weaker forms of second-order logic. See Ferreirós, 2018 and 2020.

The categoricity of the **N**-structure can be obtained in weak second-order logic and does not even depend on its impredicativity. Hence its fully determinate nature, so to speak. In practice, this is admitted by mathematicians and logicians who differ in their acceptance of some questionable foundational principles. The requirements for the categoricity of the **R**-structure are incomparably greater, as it does depend on *full* impredicative second-order logic. And the categoricity of the **V**-structure is, according to some, an illusion as it is contradicted by the myriad independence results in set theory; to put it more positively, it is merely *an ideal* that guides some important research projects in advanced set theory (while it is abandoned in other projects).[38]

5.2 Elementary Mathematics

In the practice of mathematics, one can identify a *plurality* of theoretical levels and forms of practice—with explicit interconnections among themselves and with pre-mathematical practices (see Ferreirós, 2016). Often, new theoretical strata are introduced in such a way that they are *constrained* by the previous strata with which they connect back—thus the first element needed to understand the objectivity of mathematical results is the *interplay* of practices and theoretical strata. But second, some ingredients of mathematical knowledge (what one may call 'elementary' mathematics) have such strong cognitive and practical roots that our knowledge of them is marked by certainty.

The obvious example is the natural number system as described by the Peano-Dedekind axioms—we *know* those axioms to be true of (counting) numbers. The argument is that our simplest conception of numbers is formed already in relation to counting (a basic, pre-mathematical practice), and the axioms are recognized to be true (see the details Ferreirós, 2016, ch. 7). Through counting we obtain the conception of an arbitrary natural number as the outcome of a given counting process; this corresponds (in mathematical language) to the conception of an arbitrary number as the last element of an initial segment of the number structure. And this makes the Dedekind-Peano axioms obvious. Obviously each number has a successor, clearly different numbers have different successors, the number series is unlimited, and obviously reasoning by mathematical induction is conclusive.

The peculiarities of this case are reflected in the fact that natural number arithmetic has not been a bone of contention in foundational studies: even those who disagree strongly about more advanced strata of mathematics are happy to admit PA as a theory. As Koellner put it (2009), there is no convincing case for pluralism with regard to first-order arithmetic, because "the clarity of our conception of the structure of the natural numbers," and our experience with that conception, make such a pluralism untenable.

Mathematical knowledge, in its elementary strata, is likely to be the best expression of the strength that shared experience and intersubjective agreements can attain. If you consider the practice of counting from a cognitive viewpoint, it is highly complex: it requires abilities of coordination, of categorization, of word production, that by no means are cognitively simple (see e.g. Carey, 2009 and Sect. 4). Yet most human beings have no great difficulty mastering that practice.[39]

[38] To exemplify both viewpoints in the views of leading experts, compare the ideas of Shelah (2003) and Woodin (2001).

[39] Also the conception of basic group theory can be recognized as elementary in the relevant sense, and arguably there is an 'elementary' geometry too—although it is an open question what, exactly, this basic geometry would include or exclude. Consider the seeming universality of simple symmetric shapes like the

It is noteworthy that, already at this level, epistemic constraint fails—i.e. there are true arithmetic propositions for which we lack evidence (Shapiro, 2007, 339). Failure of epistemic constraint is the first criterion of objectivity established by C. Wright in well known work that is the basis for Shapiro's discussion. I surmise that this is perfectly compatible with a conceptualist understanding of mathematical knowledge.

The crucial point here is that 'elementary' mathematics has such strong cognitive and practical roots as to be indubitable. This is the anchoring point for the rest of mathematics. Within the complex web of mathematical practices, the 'elementary' ones are accessible to an average human agent, providing basic shared knowledge and a key source of constraints. For, as mentioned before, new theoretical strata are often introduced in such a way that they are constrained by the previous strata—this applies especially to the central structures discussed in Sect. 5.1.

5.3 Advanced Mathematics

More advanced mathematical theories are built on the basis of hypothetical assumptions, and this makes it more difficult to understand their objectivity. Still, the interplay of mathematical theories and practices constrains the freedom of such hypotheses and often leads to unavoidable results. The real number structure is paradigmatic for this higher level of complexity.

The real number system **R** is not a simple counterpart of "the given" in nature or in some form of intuition, either pure or empirical. The principle of continuity or completeness is a hypothetical assumption and cannot be regarded as certain or necessary. It is often said that continuity is an intuitive property of the line, or that the reality of continuous motion is given to us in experience, but in fact our experiences with figures or with motion do not even suffice to ground the perfect denseness that is attributed to **Q**. This perfect denseness is thus an idealized property that is attributed in thought to the rational number system (to be precise, the property is that, whenever $q < r$, there is t such that $q < t$, $t < r$). Even more remote from experience, more hypothetical, is the completeness property attributed to **R**.[40]

Moreover, in light of mathematical and logical results obtained during the last hundred years, there is reason to doubt whether **R** is fully specified with the usual axiom systems. The set-theoretic structure **R** is categorical only relative to a background model of set theory.[41] Parallel considerations apply, all the more, to assumptions such as the notion of a totality of functions f: $\mathbf{R} \to \mathbf{R}$, essentially equivalent to the assumption of a powerset $\wp(\mathbf{R})$.

Yet, despite the hypothetical nature of such assumptions, the interconnections between them and previous theory (i.e., theoretical ingredients belonging to previous strata of knowledge) do *enforce* certain results. Easy examples are the non-denumerability of **R**,

Footnote 39 (continued)

circle and square, and the sophisticated results obtained on their basis (e.g. the Pythagorean theorem, developed independently in China and Greece, Ferreirós & García-Pérez, 2020).

40 For more on this topic see Ferreirós (2016), chs. 6 & 8. Let me add that Poincaré was in agreement with the basic twist of the idea as just described (1902, ch. 2), which is also in agreement with Riemann, Dedekind, Hilbert (see Dedekind's quotation in Sect. 1).

41 Regarding the background model as fixed by second-order quantification does not change this. Prominently, it is compatible with all our current knowledge that the Continuum Hypothesis may not be a definite mathematical problem (Feferman, 2011).

the existence of transcendental (not algebraic) real numbers, or the fact that there is no one-to-one correspondence between **R** and the set of all functions *f*: **R** → **R**. Cantor's non-denumerability result is an objective result, even if we grant that the conception of **R** and **N** as infinite sets is hypothetical. In particular, one proves a lemma establishing that no denumerable sequence can exhaust the real numbers; anyone who admits the conception of real-number decimal expansions will have to admit this lemma.[42] She may not accept that there is a well-defined set of all real numbers, and thus she will not see any real content in the sentence: 'The cardinality of the set of real numbers is greater than the cardinality of the set of naturals' (compare Brouwer, 1913). But she will agree on the fact that the real numbers cannot be exhausted by a denumerable sequence of them.

This is the kind of constraining, induced by the interplay of mathematical practices and strata, that I am arguing explains the objectivity and non-arbitrariness of mathematical developments—even across deep foundational disagreements. We introduce the set **R** by means of a hypothesis, but some of its properties are enforced and completely non-arbitrary.[43] Most of us just admit the hypothetical assumption, and the resulting "ideal-world picture" is quite unambiguous.

Consider also a key feature of the real number structure, namely that one must distinguish between algebraic and transcendental numbers. The existence of transcendental (i.e., not algebraic) real numbers can be established in more than one way. One of them is set-theoretic (a consequence of Cantor's lemma), but there is also Liouville's proof, based on the fact that algebraic numbers cannot be too well approximated by rational numbers. That is, Liouville proved that, if α is a root of a polynomial of degree n, then

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^n}$$

for all integers p, q and for a constant C which depends on the value of α. Knowing this property of algebraic numbers, it was not difficult for Liouville to *exhibit* numbers that in fact can be approximated by rationals extremely well, so that they *cannot be* algebraic numbers—must be transcendental. This was just a matter of offering examples of particular real-number expansions, perfectly constructive.

The failure of epistemic constraint is much stronger at this level than it was with arithmetic. Most transcendental numbers have never been named and will never be studied; the theory of transcendental numbers is, in all likelihood, full of 'blind spots'. But the main point is that, even admitting the uncertainties induced by the adoption of hypothetical assumptions, one still has remarkable intersubjective agreement. Many key results concerning the hypothetical structures are enforced, perfectly objective, and this underlies the reality we ascribe to the objects of those advanced theories. Hopefully this quick sketch will suffice to convince readers that indeed one has the ingredients to offer an account of the intersubjective objectivity of mathematical results, without the need for platonistic assumptions. Large parts of mathematics investigate into what must be the case in hypothetical states of things.

[42] For further details on this topic, see Ferreirós, 2016, chs. 8 & 9.

[43] There is more: although the Axioms of Infinity and Power Sets are two of the most characteristic hypothetical assumptions of modern math, their introduction as new hypotheses can be explained by reference to the web of mathematical practices around 1850. This claim is substantiated in Ferreirós, 2016.

6 Robust Intersubjectivity as Objectivity

Some authors have emphasized the role of "the imagination" in mathematics, arguing that the contents of our imagination can be communicated to others, the features of the imagination can be delineated and scrutinized; and under examination, what is private and subjective becomes public and objective.[44] In this way, mathematical conceptions would transcend the realm of the subjective and become *objectively* shared, communicated and confirmed. But the mathematician is trained in the ideal of objective thinking, mathematics is justly reputed to be the most sharply precise of all sciences. Therefore such philosophical statements may seem confusing—objective? intersubjective? or merely subjective? What is all that supposed to mean?

The intersubjectivity of mathematical structures has also been compared with the reality of social objects. One can adduce the examples of social realities that have the status of objective facts in the world, but are only facts by human agreement—things like money, property, governments, and marriages. It is true that such things exist only because "we believe them to exist" (or, as I would rather say, we join in the communal agreement that they are real), "yet many facts regarding these things are 'objective' facts in the sense that they are not a matter of preferences, evaluations, or moral attitudes" (Searle, 1995, 1).

The analogy between mathematical and social objects is illuminating, but I find it necessary to add that the objectivity of mathematics is different from even the most solid social facts. Consider e.g. marriage, an institution that—among other things—has to do with offspring, and with kinship relations between social groups. Defined broadly, marriage is considered a cultural universal, but the broad definition must include monogamous, polygamous and temporary forms of marriage (plus the recent issue of same-sex marriage). The enormous plurality and diversity of forms of marriage contrasts with the univocity of natural numbers.

I do not mean to deny that a great variety of counting systems have been devised in different cultures (using body parts, tallies, fingers and toes, or numerals), nor of course that many cultures lack means to express numbers beyond three or four. The key point, for my argument, is that counting systems underwriting a *precise* number concept (such as recursive systems of number-words or the famous count systems using body parts of Papua New Guinea) are essentially isomorphic. Abstractly described, they comply with the principles of Peano-Dedekind arithmetic.[45] This is where the reality of numbers comes from.

To put it otherwise: although there have been many cultures without a developed number concept, no culture has ever developed an alternative conception of (natural) number incommensurable with ours. This is very unlike the situation with social institutions.

The deeper reasons for this singularity of mathematical knowledge is the peculiar nature of its links with basic cognition and with basic human practices. Meant here are practices such as counting and measuring, where human beings interact with the world around them in ways that are enormously constrained. Mathematical knowledge (which is always in some way or another related with number and/or geometric forms) does not allow for the kind of plurality or relativity that we find in other cultural realms. A convincing explanation of this fact can hardly come from claims about the Platonic reality of abstract objects.

[44] See Feferman's post to FOM list, Jan 3, 1998.

[45] For more on this topic and a defense of the certainty of arithmetical knowledge, see Ferreirós, 2016, ch. 7.

After all, even if such objects exist, how could we know that our mathematical claims (axioms, theorems, problem-solutions) are true of them? One can easily imagine that the "true" system of real numbers, the one that exists independently of our forms of life, lacks the completeness property—and our claims about real numbers would be just false. How could we know? And, how could the absolute existence of things invisible to us rule out cultural relativities in the human claims?

The objectivity of mathematics is a special case of intersubjective objectivity, but it is *indeed so special and robust* as to deserve separate classification: a whole category of its own. There is simply nothing comparable to the solidity of the intersubjective objectivity of math, and thus it would deserve a special name. Whatever the name, the comparison between mathematical objects and social institutions or facts is only partly illuminating, and just as much confusing, perhaps.

7 Conclusion

The tension between platonism and structuralism has been resolved, I surmise, in a way that makes sense of the proposals of classical figures like Riemann, Dedekind, Hilbert and Noether. Mathematical work is first and foremost *conceptual* work, the study of relations and interrelations, that finds its current expression in structural methodologies (abstract structures, morphisms, categories). This way of understanding structuralism in mathematics captures some key insights not only of the mathematicians just mentioned, but also of philosophers such as Peirce—according to whom mathematics deals with "necessary conclusions" about "hypotetical states of things" —and Cassirer—who thinks that modern math is based on pure "functional concepts" whose presuppositions are given by the logic of relations, and that the objects of mathematics are "ideal objects whose whole content is exhausted in their mutual relations".[46]

Needless to say, it is not my intention to claim that the position outlined in the previous pages reflects in all details the ideas of Cassirer or Peirce, Hilbert or Riemann. On the contrary, there are points where it is quite obvious that significant differences of opinion or viewpoint can be highlighted. Perhaps the author who might come closer to my viewpoint is, arguably, C. S. Peirce—whose work nevertheless is sometimes puzzling, and difficult to interpret. The important idea is that the conceptual structuralism I have sketched incorporates some key insights of those classical figures.

The price to be paid, in the path to conceptual structuralism, is an explicit acknowledgement of the role of agents (and communities of agents) in the making of mathematical knowledge. This implies that mathematical structures are not completely independent of human mathematicians and their form of life—especially their cognitive abilities and the forms of culture enabling symbolic frameworks. Conceptual understanding cannot be found beyond the agents: the conceptual plane is found, rather, in the trading zone where agents elaborate ideas and formulas, thanks to their interactions with symbolic and theoretical frameworks, and exchange them with each other.

But we have given arguments to the effect that this in no way compromises the objectivity of mathematical results. Of course, some authors may find *intersubjective* reality

[46] Peirce, 1902, Cassirer, 1910. I refer again to the recent compilation Reck & Schiemer (2020) for details about these and other figures.

too weak, and try to get a much stronger form of objectivity by postulating a transcendent (fully independent) realm of structures. The prices to be paid along this course are excessive: mathematical knowledge becomes a mystery, the truth of our axioms and their relation to the 'real' structures becomes unfathomable.

A conceptual variant of structuralism has resources to make sense of the certainty of arithmetical knowledge, this being the strongest possible form of objectivity. Natural-number arithmetic presents us already with such a rich realm of truths, that epistemic constraint fails (Shapiro, 2007). This should not come as a surprise, as the conceptions we form by no means have to be fully surveyable.

On the other hand, the form of conceptual structuralism that we have proposed makes room for important *differences* between mathematical theories. In particular, advanced mathematics builds on hypothetical assumptions, hence it does not provide us with a certainty comparable to basic arithmetic. Yet even this is no obstacle for a robust form of objectivity.

Acknowledgments Thanks are due to María de Paz and two unkown reviewers, for their help with the paper. Related ideas were presented previously at conferences in Évora (Portugal) and Munich.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. Proyectos de Excelencia de la Junta de Andalucía, Project No. P12-HUM-1216. Spanish Ministry of Science, Project FFI2017-84524-P.

Declarations

Conflict of interest The author declares that he has no conflict of interest.

Human and Animal Rights This article does not contain any studies with human participants performed by the author.

References

Aleksandrov, P. (1936). In memory of Emmy Noether. In A. Dick (Ed.), *Emmy Noether 1882–1935* (pp. 153–179). Birkhäuser.

Awodey, S. (2004). An answer to hellman's question: Does category theory provide a framework for mathematical structuralism. *Philosophia Mathematica, 12*, 54–64.

Brouwer, L. E. J. (1913). Intuitionism and formalism. *Bulletin of the American Mathematical Society, 20*(2), 81–96.

Cantor, G. (1883). Grundlagen einer allgemeinen Mannigfaltigkeitslehre. *English in Ewald, 1996*, 878–920.

Carey, S. (2009). Where our number concepts come from. *Journal of Philosophy, 106*(4), 220–254.

Cassirer, E. (1910). *Substanzbegriff und Funktionsbegriff*. English translation *Substance and Function*. Chicago: Open Court, 1923.

Corry, L. (1996). *Modern Algebra and the rise of mathematical structures*. Birkhäuser.

Dedekind, R. (1872). Continuity and irrational numbers. *English in Ewald, 1996*, 766–779.

Dedekind, R. (1888). *Was sind und was sollen die Zahlen?* Braunschweig, Vieweg. *English in Ewald, 1996*, 787–833.

Ewald, W. (Ed.). (1996). *From Kant to Hilbert: A source book* (Vol. 2). Oxford University Press.

Feferman, S. (2009). 'Conceptions of the continuum,' Intellectica 51:1 (2009), available for download in http://math.stanford.edu/~feferman/papers.html no. 85.
Feferman, S. (2011). Is the continuum hypothesis a definite mathematical problem? DRAFT 9/18/11 for: Exploring the Frontiers of Incompleteness (EFI) Project, Harvard 2011–2012. Available in http://logic.harvard.edu/EFI_Feferman_IsCHdefinite.pdf
Feferman, S. (2014). Logic, mathematics and conceptual structuralism. In R. Penelope (Ed.), *The metaphysics of logic.* University of Tasmania.
Ferreirós, J. (2007). *Labyrinth of thought: A history of set theory*. Birkhäuser.
Ferreirós, J. (2009). Hilbert, logicism, and mathematical existence. *Synthese*, 170(1): 33–70.
Ferreirós, J. (2016). *Mathematical Knowledge and the Interplay of Practices.*. Princeton: Princeton University Press.
Ferreirós, J. (2017). Dedekind's Map-theoretic Period. *Philosophia Mathematica,* 25 (3): 318–340.
Ferreirós, J. (2018). On the semantics of higher-order logic, in: *(Un-)Certainty and (In-)Exactness,* eds. G. Basti, F. Bertato. Rome: Aracne. 179–190.
Ferreirós, J. (2020). La herencia oscura del logicismo, *Metatheoria,* Vol. 10:2, 19–30.
Ferreirós, J. forthcoming. Degrees of Objectivity? *Mathemata*, and social objects. To appear in *Topoi*, special issue on mathematical and social ontology.
Ferreirós, J., & García-Pérez, M. (2020). On the nature of proto-geometry. *Philosophical Psychology, 33*(2), 181–205.
Ferreirós, J., A. Lassalle-Casanave. (2022). Dedekind and Wolffian Deductive Method. *Journal for General Philosophy of Science,* https://doi.org/10.1007/s10838-021-09578-y.
Frege, G. (1893). *Grundgesetze de Arithmetik* (Vol. 1, p. 1962). Hildesheim: Georg Olms.
Gauss, C. F. (1831). Selbstanzeige der Theoria residuorum biquadraticorum. *Nachrichten Göttingen*. English transl. in Ewald 1996, Vol. 1, pp. 306–313.
Gauss, C. F. (1917). *Werke*, vol. X-1, *Nachträge zur reinen Mathematik*. Leipzig: Teubner. Repr. Hildesheim, Olms, 1981.
Goldstein, C. (1989). Le métier des nombres aux 17^{e} et 19^{e} siècles. In M. Serres (Ed.), *Éléments d'Histoire des Sciences Paris, Bordas. English translation* (p. 1995). Blackwell.
Hellman, G. (2005). Structuralism. In S. Stewart (Ed.), *Oxford handbook of philosophy of mathematics and logic* (pp. 536–562). Oxford University Press.
Hilbert, D. (1926). On the infinite. In van Heijenoort, ed. *From Frege to Gödel: A source book*, 367–392.
Hilbert, D. (1927). The foundations of mathematics. In van Heijenoort, ed. *From Frege to Gödel: A source book*, 464–479
Hilbert, D. (1897). *Zahlbericht. English: The theory of algebraic number fields* (p. 1998). Springer.
Isaacson, D. (2011). The reality of mathematics and the case of set theory. In Z. Novak & A. Simonyi (Eds.), *Truth Reference and Realism* (pp. 1–75). Central European Univ Press.
Koellner, P. (2009). Truth in mathematics: The question of pluralism. In O. Bueno & Ø. Linnebo (Eds.), *New Waves in Philosophy of Mathematics* (pp. 80–116). Palgrave Macmillan.
Krömer, R. (2007). *Tool and objects: A history and philosophy of category theory*. Basel: Birkhäuser.
Laugwitz, D. (1996). *Bernhard Riemann, 1826–1866: Wendepunkte in der Auffassung der Mathematik*. Birkhäuser.
Linnebo, Ø. (2013). Platonism in the Philosophy of Mathematics, *The Stanford Encyclopedia of Philosophy,* Winter 2013, E. N. Zalta (ed.), https://plato.stanford.edu/.
Linnebo, Ø. (2018). *Thin Objects: An abstractionist account*. Oxford: Oxford University Press.
Mac Lane, S. (1996). Structure in mathematics. *Philosophia Mathematica, 4*, 174–183.
Marquis, J.P. (2020). "Category Theory", *The Stanford Encyclopedia of Philosophy* (Fall 2020 Edition), E. N. Zalta (ed.), https://plato.stanford.edu/archives/fall2020/entries/category-theory/.
Maddy, P. (2011). *Defending the Axioms*. Oxford: Oxford University Press.
Marquis, J. P. (2009). *From a geometrical point of view: a study in the history and philosophy of category theory*. Springer.
McLarty, C. (2006). Emmy Noether's "Set Theoretic" topology: From dedekind to the rise of functors. In J. Ferreirós & J. Gray (Eds.), *The architecture of modern mathematics: Essays in history and philosophy* (pp. 211–235). Oxford University Press.
Parsons, C. (2004). Structuralism and metaphysics. *Philosophical Quarterly, 54*(214), 56–77.
Parsons, C. (2009). *Mathematical thought and its objects*. Cambridge University Press.
Peirce, C. S. (1902). The nature of mathematics. In: *Philosophical Writings* (Dover, New York, 1955), p. 135–149 (date of section II, which is from the Minute Logic of 1902).
Poincaré, H. (1902), *La Science et l'Hypothèse*. Reprint of English version: Dover, New York, p 1952.
Reck, E. (2003). Dedekind's structuralism: An interpretation and partial defense. *Synthese, 137*, 369–419.
Reck, E., & Schiemer, G. (Eds.). (2020). *The prehistory of mathematical structuralism*. Oxford Univ Press.

Searle, J. (1995). *The construction of social reality*. Allen Lane.
Shapiro, S. (2008). Mathematical structuralism. *The internet encyclopedia of philosophy*, ISSN 2161–0002, http://www.iep.utm.edu/m-struct/.
Shapiro, S. (1997). *Philosophy of mathematics: Structure and ontology*. Oxford University Press.
Shapiro, S. (2007). The objectivity of mathematics. *Synthese, 156*, 337–381.
Shelah, S. (2003). Logical dreams. *Bulletin of the American Mathematical Society, 40*, 203–228.
Tait, W. (2005). *The Provenance of pure reason: Essays in the philosophy of mathematics and its history*. Oxford University Press.
Thomasson, A.L. (2014). *Ontology Made Easy*. New York: Oxford University Press.
Woodin, H. (2001). The continuum hypothesis, *Notices of the AMS*. 567. Jun/Jul 2001.
Zermelo, E. (1930). On boundary numbers and domains of sets. In Ewald 1996.