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Sequent Calculi for Choice Logics

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Abstract
Choice logics constitute a family of propositional logics and are used for the representation of preferences, with especially *qualitative choice logic* (QCL) being an established formalism with numerous applications in artificial intelligence. While computational properties and applications of choice logics have been studied in the literature, only few results are known about the proof-theoretic aspects of their use. We propose a sound and complete sequent calculus for preferred model entailment in QCL, where a formula F is entailed by a QCL-theory T if F is true in all preferred models of T. The calculus is based on labeled sequent and refutation calculi, and can be easily adapted for different purposes. For instance, using the calculus as a cornerstone, calculi for other choice logics such as *conjunctive choice logic* (CCL) and *lexicographic choice logic* (LCL) can be obtained in a straightforward way.

Keywords Sequent calculus · Choice logics · Preferences · Non-monotonic logics · Refutation systems · Antisequents · Preferred Model Entailment

1 Introduction

Choice logics are propositional logics for the representation of preferences between different options [1]. These logics add new connectives to classical propositional logic that allow for the formalization of ranked options. A prominent example is *qualitative choice logic* (QCL) [2], which adds the connective *ordered disjunction* $\vec{\times}$ to classical propositional logic. Intuitively, $A \vec{\times} B$ means that if possible A, but if A is not possible than at least B. The semantics of a choice logic induce a preference ordering over the models of a formula.

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Let us make this more concrete with the help of a small example: say that we want to choose ice cream flavors. The following QCL-formula expresses that we want strawberry and either hazelnut or chocolate, but preferably hazelnut. Moreover, we do not want hazelnut and chocolate together.

$$\mathit{strawberry} \wedge (\mathit{hazelnut} \vec{\times} \mathit{chocolate}) \wedge \neg(\mathit{hazelnut} \wedge \mathit{chocolate}).$$

The models of the above formula according to QCL-semantics are $M_1 = \{\mathit{strawberry}, \mathit{hazelnut}\}$ and $M_2 = \{\mathit{strawberry}, \mathit{chocolate}\}$, with M_1 being preferred to M_2.

As choice logics are well suited for preference handling, they have a multitude of applications in AI such as logic programming [3], alert correlation [4], or database querying [5]. Recently, it has been suggested that choice logics can be used for preference learning [6], with the problem of antibiotics recommendations chosen as a particular use case [7]. But while computational properties and applications of choice logics have been studied in the literature, only few results are known about the proof-theoretic aspects of their use. In particular, no proof system capable of deriving valid sentences containing choice operators has been described yet. In this paper we propose a sound and complete calculus for preferred model entailment in QCL that can easily be generalized to other choice logics.

Entailment in choice logics is non-monotonic: conclusions that have been drawn might not be derivable in light of new information. For instance, let us say that, in our ice cream example above, we learn that hazelnut is not available. Then M_2 is the only model of the updated formula and we can no longer conclude that hazelnut is contained in all preferred models. It is therefore not surprising that choice logics are related to other non-monotonic formalisms. For example, it is known [2] that QCL can capture propositional circumscription [8] and that, if additional symbols in the language are admitted, circumscription can be used to generate models corresponding to the inclusion-preferred QCL models up to the additional atoms. We do not intend to use this translation of our choice logic formulas (or sequents) in order to employ an existing calculus for circumscription, for instance [9].

Instead, we define calculi in sequent format directly for choice logics, which are different from existing non-monotonic logics in the way non-monotonicity is introduced. Specifically, the non-standard part of our logics is a new logical connective which is fully embedded in the logical language. For this reason, calculi for choice logics also differ from most other calculi for non-monotonic logics: our calculi do not use non-standard inference rules as in default logic [10], modal operators expressing consistency or belief as in autoepistemic logic [11], or predicates whose extensions are minimized as in circumscription. However, one method that can also be applied to choice logics is the use of a refutation calculus (also known as rejection or antisequent calculus) axiomatising invalid formulas, i.e., non-theorems [12–15]. Refutation calculi were successfully employed for entailment in non-monotonic logics [9, 16]. Specifically, by combining a refutation calculus with an appropriate sequent calculus, elegant proof systems for the central non-monotonic formalisms such as default logic, autoepistemic logic, and circumscription were obtained.

Another aspect of choice logics semantics we must account for is their similarity to many-valued logics. Specifically, interpretations ascribe a natural number called satisfaction degree to choice logic formulas. Preferred models of a formula are then those models with the least degree. There are several kinds of sequent calculus systems for many-valued logics, where the representation as a hypersequent calculus [16, 17] plays a prominent role. However, there are crucial differences between choice logics and many-valued logics in the usual sense. Firstly, choice logic interpretations are classical, i.e., they set propositional variables to either true or false. Secondly, non-classical satisfaction degrees only arise when choice connectives, e.g. ordered disjunction in QCL, occur in a formula. Thirdly, when applying a choice connective $\circ$ to two formulas A and B, the degree of $A \circ B$ does not only depend on the degrees of A and B, but also on the maximum degrees that A and B can possibly assume. Therefore, techniques used in proof systems for conventional many-valued logics cannot be applied directly to choice logics.

In [18] a sequent calculus based system for reasoning with contrary-to-duty obligations was introduced, where a non-classical connective was defined to capture the notion of reparational obligation, which is in force only when a violation of a norm occurs. This is related to the ordered disjunction in QCL, however, based on the intended use in [18] the system was defined only for the occurrence of the new connective on the right side of the sequent sign. We aim for a proof system for reasoning with choice logic operators, and to deduce formulas from choice logic formulas. Thus, we need a calculus with left and right inference rules.

To obtain such a calculus we combine the idea of a refutation calculus with methods developed for multi-valued logics. First, we develop a (monotonic) sequent calculus for reasoning about satisfaction degrees using a labeled calculus, a method developed for (finite) many-valued logics [19–21]. Secondly, we define a labeled refutation calculus for reasoning about invalidity in terms of satisfaction degrees. Finally, we join both calculi to obtain a sequent calculus for the non-monotonic entailment of QCL. To this end, we introduce a new, non-monotonic inference rule that has sequents of the two labeled calculi as premises and formalizes degree minimization.

The rest of this paper is organized as follows. In the next section we present the basic notions of choice logics and introduce the most prominent choice logic QCL, as well as CCL (*conjunctive choice logic*) [22] and LCL (*lexicographic choice logic*) [1]. In Section 3 we develop a labeled sequent calculus for propositional logic extended by the QCL connective $\vec{\times}$. This calculus is shown to be sound and complete and already can be used to derive interesting sentences containing choice operators. In Section 4 we extend the previously defined sequent calculus with an appropriate refutation calculus and non-monotonic reasoning, to capture entailment in QCL. The developed methodology for QCL can be extended to other choice logics as well. In particular we show in Sect. 5 how the calculi can be adapted for CCL and LCL.

Note that this is an extended version of a paper published at IJCAR 2022 [23]. Newly added in this iteration are our calculi for the minmax and inclusion-based preferred model semantics (cf. Definitions 17 and 19, as well as Theorem 6), our calculus for LCL (Sect. 5.2), and a brief outline on how to obtain calculi for choice logics with multiple choice connectives (Sect. 5.3). The soundness of the cut-rule in our labeled calculi is now explicitly shown (cf. Propositions 2 and 5). Moreover, additional examples (Examples 3, 6, 11,12,13) and more detailed explanations have been added throughout the text.

2 Choice Logics

First, we formally define the notion of choice logics in accordance with the choice logic framework of [1, 24] before giving concrete examples in the form of QCL, CCL, and LCL. Finally, we define preferred model entailment.

2.1 Syntax and Semantics

Definition 1 Let $\mathcal{U}$ denote the (countably infinite) set of propositional variables (also called atoms). The set of choice connectives $\mathcal{C}_{\mathcal{L}}$ of a choice logic $\mathcal{L}$ is a finite set of symbols such that $\mathcal{C}_{\mathcal{L}} \cap \{\neg, \wedge, \vee\} = \emptyset$. The set $\mathcal{F}_{\mathcal{L}}$ of formulas of $\mathcal{L}$ is defined inductively as follows:

1. if $a \in \mathcal{U}$, then $a \in \mathcal{F}_{\mathcal{L}}$;
2. if $F \in \mathcal{F}_{\mathcal{L}}$, then $(\neg F) \in \mathcal{F}_{\mathcal{L}}$;
3. if $F, G \in \mathcal{F}_{\mathcal{L}}$, then $(F \circ G) \in \mathcal{F}_{\mathcal{L}}$ for $\circ \in (\{\wedge, \vee\} \cup \mathcal{C}_{\mathcal{L}})$.

For instance, in QCL the set of choice connectives is $\mathcal{C}_{\text{QCL}} = \{\vec{\times}\}$. An example of a QCL-formula is $((a \vec{\times} c) \wedge (b \vec{\times} c)) \in \mathcal{F}_{\text{QCL}}$. Formulas that do not contain a choice connective are referred to as classical formulas.

The semantics of a choice logic is given by two functions, satisfaction degree and optionality. The satisfaction degree of a formula given an interpretation is either a natural number or ∞. The lower this degree, the more preferable the interpretation. The optionality of a formula describes the maximum finite satisfaction degree that this formula can be ascribed, and is used to penalize non-satisfaction.

Definition 2 The optionality of a choice connective $\circ \in \mathcal{C}_{\mathcal{L}}$ in a choice logic $\mathcal{L}$ is given by a function $opt_{\mathcal{L}}^{\circ} \colon \mathbb{N}^2 \to \mathbb{N}$ such that $opt_{\mathcal{L}}^{\circ}(k, \ell) \leq (k+1) \cdot (\ell+1)$ for all $k, \ell \in \mathbb{N}$. The optionality of an $\mathcal{L}$-formula is given via $opt_{\mathcal{L}} \colon \mathcal{F}_{\mathcal{L}} \to \mathbb{N}$ with

1. $opt_{\mathcal{L}}(a) = 1$, for every $a \in \mathcal{U}$;
2. $opt_{\mathcal{L}}(\neg F) = 1$;
3. $opt_{\mathcal{L}}(F \wedge G) = max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$;
4. $opt_{\mathcal{L}}(F \vee G) = max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$;
5. $opt_{\mathcal{L}}(F \circ G) = opt_{\mathcal{L}}^{\circ}(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$, $\circ \in \mathcal{C}_{\mathcal{L}}$.

The optionality of a classical formula is always 1. Note that, for any choice connective $\circ$, the optionality of $F \circ G$ is bounded such that $opt_{\mathcal{L}}(F \circ G) \leq (opt_{\mathcal{L}}(F)+1) \cdot (opt_{\mathcal{L}}(G)+1)$. The reason for this is that there are $opt_{\mathcal{L}}(F)$ many finite degrees that could be ascribed to F, plus the infinite degree ∞. Likewise for G. Thus, there are at most $(opt_{\mathcal{L}}(F)+1) \cdot (opt_{\mathcal{L}}(G)+1)$ possibilities in which the degrees of F and G can be combined.

In this paper, an interpretation $\mathcal{I}$ is a set of propositional variables, i.e., $\mathcal{I} \subseteq \mathcal{U}$. A variable x is true under $\mathcal{I}$ iff $x \in \mathcal{I}$, and false under $\mathcal{I}$ iff $x \notin \mathcal{I}$. Regarding the domain of satisfaction degrees we write $\overline{\mathbb{N}}$ for $(\mathbb{N} \cup \{\infty\})$.

Definition 3 The satisfaction degree of a choice connective $\circ \in \mathcal{C}_{\mathcal{L}}$ in a choice logic $\mathcal{L}$ is given by a function $deg_{\mathcal{L}}^{\circ} \colon \mathbb{N}^2 \times \overline{\mathbb{N}}^2 \to \overline{\mathbb{N}}$ such that $deg_{\mathcal{L}}^{\circ}(k, \ell, m, n) \leq opt_{\mathcal{L}}^{\circ}(k, \ell)$ or $deg_{\mathcal{L}}^{\circ}(k, \ell, m, n) = \infty$ for all $k, \ell \in \mathbb{N}$ and all $m, n \in \overline{\mathbb{N}}$. The satisfaction degree of an $\mathcal{L}$-formula under an interpretation $\mathcal{I} \subseteq \mathcal{U}$ is given via $deg_{\mathcal{L}} \colon 2^{\mathcal{U}} \times \mathcal{F}_{\mathcal{L}} \to \overline{\mathbb{N}}$ with

1. $deg_{\mathcal{L}}(\mathcal{I}, a) = 1$ if $a \in \mathcal{I}$, $deg_{\mathcal{L}}(\mathcal{I}, a) = \infty$ otherwise for every $a \in \mathcal{U}$;
2. $deg_{\mathcal{L}}(\mathcal{I}, \neg F) = 1$ if $deg_{\mathcal{L}}(\mathcal{I}, F) = \infty$, $deg_{\mathcal{L}}(\mathcal{I}, \neg F) = \infty$ otherwise;
3. $deg_{\mathcal{L}}(\mathcal{I}, F \wedge G) = max(deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$;
4. $deg_{\mathcal{L}}(\mathcal{I}, F \vee G) = min(deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$;
5. $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = deg_{\mathcal{L}}^{\circ}(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G), deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$, $\circ \in \mathcal{C}_{\mathcal{L}}$.

Note that, by definition, either $deg_{\mathcal{L}}(\mathcal{I}, F) \leq opt_{\mathcal{L}}(F)$ or $deg_{\mathcal{L}}(\mathcal{I}, F) = \infty$ for all $\mathcal{L}$-formulas F, as intended.

We sometimes use the alternative notation $\mathcal{I} \models_m^{\mathcal{L}} F$ for $deg_{\mathcal{L}}(\mathcal{I}, F) = m$. If $m < \infty$, we say that $\mathcal{I}$ satisfies F (to a finite degree), and if $m = \infty$, then $\mathcal{I}$ does not satisfy F. If F is a classical formula, then $\mathcal{I} \models_1^{\mathcal{L}} F \iff \mathcal{I} \models F$ and $\mathcal{I} \models_\infty^{\mathcal{L}} F \iff \mathcal{I} \not\models F$. The symbol $\bot$ is shorthand for the formula $(a \wedge \neg a)$, where a can be any variable. We have $opt_{\mathcal{L}}(\bot) = 1$ and $deg_{\mathcal{L}}(\mathcal{I}, \bot) = \infty$ for any interpretation $\mathcal{I}$ in every choice logic.

The models of a choice logic formula are the interpretations that satisfy the formula, and the preferred model are the models that satisfy the formula to a minimal degree.

Definition 4 Let $\mathcal{L}$ be a choice logic, $\mathcal{I}$ an interpretation, and F an $\mathcal{L}$-formula. $\mathcal{I}$ is a model of F, written as $\mathcal{I} \in Mod_{\mathcal{L}}(F)$, if $deg_{\mathcal{L}}(\mathcal{I}, F) < \infty$. $\mathcal{I}$ is a preferred model of F, written as $\mathcal{I} \in Prf_{\mathcal{L}}(F)$, if $\mathcal{I} \in Mod_{\mathcal{L}}(F)$ and $deg_{\mathcal{L}}(\mathcal{I}, F) \leq deg_{\mathcal{L}}(\mathcal{J}, F)$ for all other interpretations $\mathcal{J}$.

We also require the notion of classical counterparts for choice connectives and choice logic formulas.

Definition 5 Let $\mathcal{L}$ be a choice logic. The classical counterpart of a choice connective $\circ \in \mathcal{C}_{\mathcal{L}}$ is the classical binary connective $\circledast$ such that, for all atoms a and b, we have $deg_{\mathcal{L}}(\mathcal{I}, a \circ b) < \infty \iff \mathcal{I} \models a \circledast b$. The classical counterpart of an $\mathcal{L}$-formula F is denoted as $cp(F)$ and is obtained by replacing all occurrences of choice connectives in F by their classical counterparts.

Every choice connective has exactly one classical binary connective as its classical counterpart [24, Proposition 22]. For example, the classical counterpart of ordered disjunction $\vec{\times}$ is regular disjunction $\vee$, and the classical counterpart of the QCL-formula $F = (a \vec{\times} b) \vee c$ is $cp(F) = (a \vee b) \vee c$.

A natural property of choice logics considered in the literature is that choice connectives can be replaced by their classical counterpart without affecting satisfiability, meaning that $deg_{\mathcal{L}}(\mathcal{I}, F) < \infty \iff \mathcal{I} \models cp(F)$ holds for all $\mathcal{L}$-formulas F [24, Proposition 23].

2.2 Prominent Choice Logics

So far we introduced choice logics in a quite abstract way. We now introduce three particular instantiations, namely QCL [2], the first and most prominent choice logic in the literature, CCL [22], which introduces a connective $\vec{\odot}$ called ordered conjunction in place of QCL's ordered disjunction, and LCL [1], which replaces ordered disjunction with a lexicographic operator.

Definition 6 QCL is the choice logic such that $\mathcal{C}_{\text{QCL}} = \{\vec{\times}\}$, and, if $k = opt_{\text{QCL}}(F)$, $\ell = opt_{\text{QCL}}(G)$, $m = deg_{\text{QCL}}(\mathcal{I}, F)$, and $n = deg_{\text{QCL}}(\mathcal{I}, G)$, then

$$opt_{\text{QCL}}(F \vec{\times} G) = opt_{\text{QCL}}^{\vec{\times}}(k, \ell) = k + \ell, \text{ and}$$

$$deg_{\text{QCL}}(\mathcal{I}, F \vec{\times} G) = deg_{\text{QCL}}^{\vec{\times}}(k, \ell, m, n) = \begin{cases} m & \text{if } m < \infty; \\ n + k & \text{if } m = \infty, n < \infty; \\ \infty & \text{otherwise.} \end{cases}$$

In the above definition, we can see how optionality is used to penalize non-satisfaction: given a QCL-formula $F \vec{\times} G$ and an interpretation $\mathcal{I}$, if $\mathcal{I}$ satisfies F (to some finite degree), then $deg_{\text{QCL}}(\mathcal{I}, F \vec{\times} G) = deg_{\text{QCL}}(\mathcal{I}, F) \leq opt_{\text{QCL}}(F)$; if $\mathcal{I}$ does not satisfy F, then $deg_{\text{QCL}}(\mathcal{I}, F \vec{\times} G) = opt_{\text{QCL}}(F) + deg_{\text{QCL}}(\mathcal{I}, G) > opt_{\text{QCL}}(F)$. Therefore, interpretations

Table 1 The classical connectives $\wedge$, $\vee$ and the choice connectives $\vec{\times}$ (QCL), $\vec{\odot}$ (CCL), and $\vec{\diamond}$ (LCL), applied to atoms. Taken from [1]

$\mathcal{I}$	$a \wedge b$	$a \vee b$	$a \vec{\times} b$	$a \vec{\odot} b$	$a \vec{\diamond} b$
$\emptyset$	∞	∞	∞	∞	∞
$\{b\}$	∞	1	2	∞	3
$\{a\}$	∞	1	1	2	2
$\{a, b\}$	1	1	1	1	1

that satisfy F result in a lower degree, i.e., are more preferable, compared to interpretations that do not satisfy F. Table 1 shows how ordered disjunction behaves when applied to atoms. The following example highlights how classical conjunction interacts with ordered disjunction.

Example 1 Consider the QCL-formula $F = (a \vec{\times} c) \wedge (b \vec{\times} c)$. As mentioned already, the classical counterpart of $\vec{\times}$ is $\vee$, i.e., $cp(F) = (a \vee c) \wedge (b \vee c)$. Thus, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\} \in Mod_{\mathrm{QCL}}(F)$. Of these models, $\{a, b\}$ and $\{a, b, c\}$ satisfy F to a degree of 1 while $\{c\}$, $\{a, c\}$, and $\{b, c\}$ satisfy F to a degree of 2. Therefore, $\{a, b\}$, $\{a, b, c\} \in Prf_{\mathrm{QCL}}(F)$.

Next, we define CCL. Note that we follow the revised definition of CCL [1], which differs from the initial specification.[1] Intuitively, given a CCL-formula $F \vec{\odot} G$ it is best to satisfy both F and G, but also acceptable to satisfy only F. For instance, when buying a new car, one might insist that the car has cruise control (*cruise*), while preferring configurations that additionally feature a lane assistant (*lane*). This can be formalized in CCL as the formula $cruise \vec{\odot} lane$.

Definition 7 CCL is the choice logic such that $\mathcal{C}_{\mathrm{CCL}} = \{\vec{\odot}\}$, and, if $k = opt_{\mathrm{CCL}}(F)$, $\ell = opt_{\mathrm{CCL}}(G)$, $m = deg_{\mathrm{CCL}}(\mathcal{I}, F)$, and $n = deg_{\mathrm{CCL}}(\mathcal{I}, G)$, then

$$opt_{\mathrm{CCL}}(F \vec{\odot} G) = k + \ell, \text{ and}$$

$$deg_{\mathrm{CCL}}(\mathcal{I}, F \vec{\odot} G) = \begin{cases} n & \text{if } m = 1, n < \infty; \\ m + \ell & \text{if } m < \infty \text{ and } (m > 1 \text{ or } n = \infty); \\ \infty & \text{otherwise.} \end{cases}$$

Example 2 Consider the CCL-formula $G = (a \vec{\odot} c) \wedge (b \vec{\odot} c)$. Note that the classical counterpart of $\vec{\odot}$ is the first projection, i.e., $cp(G) = a \wedge b$. Thus, $\{a, b\}$, $\{a, b, c\} \in Mod_{\mathrm{CCL}}(G)$. Of these models, $\{a, b, c\}$ satisfies G to a degree of 1 while $\{a, b\}$ satisfies G to a degree of 2. Therefore, $\{a, b, c\} \in Prf_{\mathrm{CCL}}(G)$.

The last choice logic we consider, LCL, employs a more fine-grained type of preference: given a LCL-formula $F \vec{\diamond} G$, it is best to satisfy F and G, second-best to satisfy only F, and third-best to satisfy only G.

Definition 8 LCL is the choice logic such that $\mathcal{C}_{\mathrm{LCL}} = \{\vec{\diamond}\}$, and, if $k = opt_{\mathrm{LCL}}(F)$, $\ell = opt_{\mathrm{LCL}}(G)$, $m = deg_{\mathrm{LCL}}(\mathcal{I}, F)$, and $n = deg_{\mathrm{LCL}}(\mathcal{I}, G)$, then

$$opt_{\mathrm{LCL}}(F \vec{\diamond} G) = (k + 1) \cdot (\ell + 1) - 1, \text{ and}$$

[1] It seems that, under the initial definition of CCL, $a \vec{\odot} b$ is always ascribed a degree of 1 or ∞, i.e., non-classical degrees cannot be obtained (cf. Definition 8 in [22]).

$$deg_{\mathrm{LCL}}(\mathcal{I}, F\vec{\diamond}G) = \begin{cases} (m-1)\cdot\ell + n & \text{if } m < \infty, n < \infty; \\ k\cdot\ell + m & \text{if } m < \infty, n = \infty; \\ k\cdot\ell + k + n & \text{if } m = \infty, n < \infty; \\ \infty & \text{otherwise.} \end{cases}$$

Example 3 Consider the LCL-formula $H = (a\vec{\diamond}c) \wedge (b\vec{\diamond}c)$. Just as in the case of QCL, the classical counterpart of $\vec{\diamond}$ is $\vee$, i.e., $cp(H) = (a \vee c) \wedge (b \vee c)$. Thus, $\{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \in Mod_{\mathrm{LCL}}(H)$. Of these models, $\{a, b, c\}$ satisfies H to a degree of 1, $\{a, b\}$ satisfies H to a degree of 2, and $\{c\}, \{a, c\}, \{b, c\}$ satisfy H to a degree of 3. Therefore, $\{a, b, c\} \in Prf_{\mathrm{LCL}}(H)$.

2.3 Preferred Model Entailment

If $\mathcal{L}$ is a choice logic, then a set of $\mathcal{L}$-formulas is called an $\mathcal{L}$-theory. An $\mathcal{L}$-theory T entails a classical formula F, written as $T \mid\!\sim F$, if F is true in all preferred models of T. However, we first need to define what the preferred models of a choice logic theory are. There are several approaches for this. In the original QCL paper [2], a lexicographic and an inclusion-based approach were introduced. A simpler but less expressive minmax semantics was introduced later on [1].

Definition 9 Let $\mathcal{L}$ be a choice logic, $\mathcal{I}$ an interpretation, and T an $\mathcal{L}$-theory. $\mathcal{I}$ is a model of T, written as $\mathcal{I} \in Mod_{\mathcal{L}}(T)$, if $deg_{\mathcal{L}}(\mathcal{I}, F) < \infty$ for all $F \in T$. Moreover, $\mathcal{I}^k_{\mathcal{L}}(T)$ denotes the set of formulas in T satisfied to a degree of k by $\mathcal{I}$, i.e., $\mathcal{I}^k_{\mathcal{L}}(T) = \{F \in T \mid deg_{\mathcal{L}}(\mathcal{I}, F) = k\}$.

- $\mathcal{I}$ is a lexicographically preferred model of T, written as $\mathcal{I} \in Prf^{lex}_{\mathcal{L}}(T)$, iff $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and if there is no $\mathcal{J} \in Mod_{\mathcal{L}}(T)$ such that, for some $k \in \mathbb{N}$ and all $l < k$, $|\mathcal{I}^k_{\mathcal{L}}(T)| < |\mathcal{J}^k_{\mathcal{L}}(T)|$ and $|\mathcal{I}^l_{\mathcal{L}}(T)| = |\mathcal{J}^l_{\mathcal{L}}(T)|$ holds.
- $\mathcal{I}$ is an inclusion-based preferred model of T, written as $\mathcal{I} \in Prf^{inc}_{\mathcal{L}}(T)$, iff $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and if there is no $\mathcal{J} \in Mod_{\mathcal{L}}(T)$ such that, for some $k \in \mathbb{N}$ and all $l < k$, $\mathcal{I}^k_{\mathcal{L}}(T) \subset \mathcal{J}^k_{\mathcal{L}}(T)$ and $\mathcal{I}^l_{\mathcal{L}}(T) = \mathcal{J}^l_{\mathcal{L}}(T)$ holds.
- $\mathcal{I}$ is a minmax preferred model of T, written as $\mathcal{I} \in Prf^{mm}_{\mathcal{L}}(T)$, iff $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and if there is no $\mathcal{J} \in Mod_{\mathcal{L}}(T)$ such that $max\{deg_{\mathcal{L}}(\mathcal{I}, F) \mid F \in T\} > max\{deg_{\mathcal{L}}(\mathcal{J}, F) \mid F \in T\}$.

Intuitively, the lexicographic and inclusion-based approaches choose those models as preferred models that satisfy as many formulas in the theory to a degree of 1 as possible. If there is a tie between two interpretations with regards to degree 1, then it is determined which interpretation satisfies more formulas to a degree of 2, and so forth. The differences between the two approaches is how the phrase 'as many degrees as possible' is understood: either in terms of cardinality (lexicographic approach) or in terms of subset-maximization (inclusion-based approach). As for the minmax semantics, a finite theory $T = \{A_1, \ldots, A_n\}$ can straightforwardly be seen as the conjunction of its formulas, i.e., $T = A_1 \wedge \cdots \wedge A_n$. We now formally define preferred model entailment:

Definition 10 Let $\mathcal{L}$ be a choice logic, T an $\mathcal{L}$-theory, S a classical theory, and $\sigma \in \{lex, inc, mm\}$. $T \mid\!\sim^{\sigma}_{\mathcal{L}} S$ iff for all $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T)$ there is $F \in S$ such that $\mathcal{I} \models F$. We overload this notation and instead of $T \mid\!\sim^{\sigma}_{\mathcal{L}} \{F\}$ we also write $T \mid\!\sim^{\sigma}_{\mathcal{L}} F$.

Example 4 Consider the QCL-theory $T = \{\neg(a \wedge b), a \vec{\times} c, b \vec{\times} c\}$. Then $\{c\}, \{a, c\}, \{b, c\} \in Mod_{\text{QCL}}(T)$. Note that, because of $\neg(a \wedge b)$, a model of T cannot satisfy both $a \vec{\times} c$ and $b \vec{\times} c$ to a degree of 1. Specifically,

$$\{a, c\}^1_{\text{QCL}}(T) = \{\neg(a \wedge b), a \vec{\times} c\} \text{ and } \{a, c\}^2_{\text{QCL}}(T) = \{b \vec{\times} c\},$$
$$\{b, c\}^1_{\text{QCL}}(T) = \{\neg(a \wedge b), b \vec{\times} c\} \text{ and } \{b, c\}^2_{\text{QCL}}(T) = \{a \vec{\times} c\},$$
$$\{c\}^1_{\text{QCL}}(T) = \{\neg(a \wedge b)\} \text{ and } \{c\}^2_{\text{QCL}}(T) = \{a \vec{\times} c, b \vec{\times} c\}.$$

Thus, $\{a, c\}, \{b, c\} \in Prf^{lex}_{\text{QCL}}(T)$ but $\{c\} \notin Prf^{lex}_{\text{QCL}}(T)$. It can be concluded that $T \mid\!\sim^{lex}_{\text{QCL}} c \wedge (a \vee b)$. However, $T \not\mid\!\sim^{lex}_{\text{QCL}} a$ and $T \not\mid\!\sim^{lex}_{\text{QCL}} b$. Analogously for $\mid\!\sim^{inc}_{\text{QCL}}$.

Regarding $\mid\!\sim^{mm}_{\text{QCL}}$, note that all three models of T have the same minmax degree, namely 2. Thus, $\{a, c\}, \{b, c\}, \{c\} \in Prf^{lex}_{\text{QCL}}(T)$, which means that $T \not\mid\!\sim^{mm}_{\text{QCL}} c \wedge (a \vee b)$.

It is easy to see that preferred model entailment is non-monotonic. For example, $\{a \vec{\times} b\} \mid\!\sim^{\sigma}_{\text{QCL}} a$ but $\{a \vec{\times} b, \neg a\} \not\mid\!\sim^{\sigma}_{\text{QCL}} a$ for all $\sigma \in \{lex, inc, mm\}$. Previously it has been examined [1, 2] whether preferred model entailment for QCL and other choice logics satisfies semantical properties for non-monotonic entailment laid out by Kraus, Lehmann, and Magidor [25, 26]. It was found that, given finite theories, $\mid\!\sim^{\sigma}_{\mathcal{L}}$ satisfies cautious monotonicity and cumulative transitivity for all of $\sigma \in \{lex, inc, mm\}$. These properties are widely considered to be among the basic properties that any defeasible entailment relation should satisfy [25]. Moreover, $\mid\!\sim^{lex}_{\mathcal{L}}$ and $\mid\!\sim^{mm}_{\mathcal{L}}$ satisfy rational monotonicity, an even stronger and often desired property of non-monotonic entailment relations [25, 26], while $\mid\!\sim^{inc}_{\text{QCL}}$ does not.

3 The Sequent Calculus L[QCL]

Towards the development of a calculus for preferred model entailment, we first propose a labeled calculus for reasoning about the satisfaction degrees of QCL formulas in sequent format and prove its soundness and completeness. An advantage of the sequent calculus format is the possibility to have symmetrical left and right rules for all connectives, in particular for the choice connectives. This is in contrast to the representation of ordered disjunction in the calculus for deontic logic [18], in which only right-hand side rules are considered.

As the calculus will be concerned with satisfaction degrees rather than preferred models, entailment will be defined in terms of satisfaction degrees. To this end, the formulas occurring in the sequents of the calculus will be labeled.

Definition 11 (*labeled* QCL-*formulas*) Let A be a QCL-formula and $k \in \mathbb{N}$, then $(A)_k$ is a labeled QCL-formula. The labeled QCL-formula $(A)_k$ is satisfied by those interpretations that satisfy A to a degree of k.

Instead of labeling formulas with degree ∞ we will use the negated formula, i.e., instead of $(A)_\infty$ we will use $(\neg A)_1$. Observe that $(A)_k$ for $opt_{\mathcal{L}}(A) > k$ can never have a model. We will deal with such formulas by replacing them with $(\bot)_1$. For classical formulas, we may write A for $(A)_1$.

Definition 12 (*labeled* QCL-*sequents*) Let $(A_1)_{k_1}, \ldots, (A_m)_{k_m}$ and $(B_1)_{l_1}, \ldots, (B_n)_{l_n}$ be labeled QCL-formulas. Then

$$(A_1)_{k_1}, \ldots, (A_m)_{k_m} \vdash (B_1)_{l_1}, \ldots, (B_n)_{l_n}$$

is a labeled QCL-sequent.

$(A_1)_{k_1}, \ldots, (A_m)_{k_m} \vdash (B_1)_{l_1}, \ldots, (B_n)_{l_n}$ is valid if and only if every interpretation that satisfies all labeled QCL-formulas $(A_1)_{k_1}, \ldots, (A_m)_{k_m}$ to the degree specified by the label also satisfies at least one labeled QCL-formula out of $(B_1)_{l_1}, \ldots, (B_n)_{l_n}$ to the degree specified by the label.

In contrast to preferred model entail, the entailment in terms of satisfaction degrees, as defined above, is monotonic.

Frequently we will write $(A)_{<k}$ as shorthand for the sequence of labeled QCL-formulas $(A)_1, \ldots, (A)_{k-1}$ and $(A)_{>k}$ for the sequence of labeled QCL-formulas $(A)_{k+1}, \ldots, (A)_{opt_{\mathrm{QCL}}(A)}, (\neg A)_1$. Moreover, $\langle \Gamma, (A)_i \vdash \Delta \rangle_{i<k}$ will denote the sequence of labeled QCL-sequents

$$\Gamma, (A)_1 \vdash \Delta \; \ldots \; \Gamma, (A)_{k-1} \vdash \Delta.$$

Analogously, $\langle \Gamma, (A)_i \vdash \Delta \rangle_{i>k}$ will denote the sequence of labeled QCL-sequents

$$\Gamma, (A)_{k+1} \vdash \Delta \; \ldots \; \Gamma, (A)_{opt_{\mathrm{QCL}}(A)} \vdash \Delta \quad \Gamma, (\neg A)_1 \vdash \Delta.$$

Below we define the sequent calculus **L**[QCL] over labeled QCL-sequents. In addition to introducing inference rules for the choice connective $\vec{\times}$ we have to modify the inference rules for conjunction and disjunction of propositional **LK**. We first state the calculus, and then explain the intuition behind the rules.

Definition 13 (**L**[QCL]) The axioms of **L**[QCL] are labeled QCL-sequents $\Gamma \vdash \Delta$ such that $\bot \in \Gamma$ or such that $p \in \Gamma$ and $p \in \Delta$ for some atom p. The inference rules are given below. Whenever a labeled QCL-formula $(F)_k$ appears in the conclusion of an inference rule (except for the *dol*- and *dor*-rules) it holds that $k \leq opt_{\mathcal{L}}(F)$.
The rules for the classical connectives are

$$\frac{\Gamma \vdash (cp(A))_1, \Delta}{\Gamma, (\neg A)_1 \vdash \Delta}\neg l \qquad \frac{\Gamma, (cp(A))_1 \vdash \Delta}{\Gamma \vdash (\neg A)_1, \Delta}\neg r$$

$$\frac{\Gamma, (A)_k \vdash (B)_{<k}, \Delta \qquad \Gamma, (B)_k \vdash (A)_{<k}, \Delta}{\Gamma, (A \vee B)_k \vdash \Delta}\vee l$$

$$\frac{\langle \Gamma, (A)_i \vdash \Delta \rangle_{i<k} \quad \langle \Gamma, (B)_i \vdash \Delta \rangle_{i<k} \quad \Gamma \vdash (A)_k, (B)_k, \Delta}{\Gamma \vdash (A \vee B)_k, \Delta}\vee r$$

$$\frac{\Gamma, (A)_k \vdash (B)_{>k}, \Delta \quad \Gamma, (B)_k \vdash (A)_{>k}, \Delta}{\Gamma, (A \wedge B)_k \vdash \Delta}\wedge l$$

$$\frac{\langle \Gamma, (A)_i \vdash \Delta \rangle_{i>k} \quad \langle \Gamma, (B)_i \vdash \Delta \rangle_{i>k} \quad \Gamma \vdash (A)_k, (B)_k, \Delta}{\Gamma \vdash (A \wedge B)_k, \Delta}\wedge r$$

The rules for ordered disjunction are:

$$\frac{\Gamma, (A)_k \vdash \Delta}{\Gamma, (A \vec{\times} B)_k \vdash \Delta}\vec{\times} l_1 \qquad \frac{\Gamma, (B)_l, (\neg A)_1 \vdash \Delta}{\Gamma, (A \vec{\times} B)_{opt_{\mathrm{QCL}}(A)+l} \vdash \Delta}\vec{\times} l_2$$

$$\frac{\Gamma \vdash (A)_k, \Delta}{\Gamma \vdash (A \vec{\times} B)_k, \Delta}\vec{\times} r_1 \qquad \frac{\Gamma \vdash (\neg A)_1, \Delta \qquad \Gamma \vdash (B)_l, \Delta}{\Gamma \vdash (A \vec{\times} B)_{opt_{\mathrm{QCL}}(A)+l}, \Delta}\vec{\times} r_2$$

where $k \leq opt_{\mathcal{L}}(A)$ and $l \leq opt_{\mathcal{L}}(B)$.

The degree overflow rules[2] are:

$$\frac{\Gamma, \bot \vdash \Delta}{\Gamma, (A)_{opt_{\mathrm{QCL}}(A)+k} \vdash \Delta} dol \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash (A)_{opt_{\mathrm{QCL}}(A)+k}, \Delta} dor$$

where $k \in \mathbb{N}$.

The rules for negation are analogous to propositional **LK**. Note that we replace A by its classical counterpart $cp(A)$. This reflects the fact that negation in choice logics erases all information about preferences, and that we therefore are only interested in the classical satisfaction of A.

The idea behind the $\vee$-left rule is that a model M of the labeled QCL-formula $(A)_k$ is only a model of the labeled QCL-formula $(A \vee B)_k$ if there is no $l < k$ s.t. M is a model of $(B)_l$. Therefore, every model of $(A \vee B)_k$ is a model of Δ if and only if

- every model of $(A)_k$ is a model of Δ or of some $(B)_l$ with $l < k$,
- every model of $(B)_k$ is a model of Δ or of some $(A)_l$ with $l < k$.

Essentially, the same idea works for $\wedge$-left but with $l > k$. For the $\vee$-right rule, in order for every model of Γ to be a model of $(A \vee B)_k$, every model of Γ must either be a model of $(A)_k$ or of $(B)_k$ and no model of Γ can be a model of $(A)_l$ for $l < k$, i.e., $\Gamma, (A)_l \vdash \bot$. Similarly for $\wedge$-right.

Observe that, in case we are dealing with classical formulas only, the modified inference rules for $\wedge$ and $\vee$ are equivalent to the inference rules for $\wedge$ and $\vee$ of propositional **LK** without structural rules [27, Section 3.5]. Consider the $\vee$-left rule in **L**[QCL]: if A and B are classical, and $k = 1$, the rule equals the $\vee$-left rule of propositional **LK**, as $(A)_{<1}$ is empty. Similarly, the $\vee$-right rule in **L**[QCL] equals the $\vee$-right rule in propositional **LK**, because $\langle \Gamma, (A)_i \vdash \Delta \rangle_{i<1}$ is empty. Moreover, as $(A)_{>1} = \neg A$ for a classical formula A, the $\wedge$-left rule of **L**[QCL] is equivalent to the $\wedge$-left rule of propositional **LK** if A and B are classical formulas and $k = 1$ (but splits the proof-tree unnecessarily). Analogously for $\wedge$-right, as $\langle \Gamma, (A)_i \vdash \Delta \rangle_{i>1}$ equals $\Gamma, \neg A \vdash \Delta$ if A is classical. Therefore, this is equivalent to the $\wedge$-right rule of propositional **LK** if A and B are classical formulas and $k = 1$ (but adds an unnecessary third condition $\Gamma \vdash A, B, \Delta$).

The rules for ordered disjunction follow straightforwardly from QCL-semantics. If $A \vec{\times} B$ is satisfied to a degree $k \leq opt_{\mathrm{QCL}}(A)$, then we know that A must be satisfied to a degree of k. If $A \vec{\times} B$ is satisfied to a finite degree higher than $opt_{\mathrm{QCL}}(A)$, then we know that B is satisfied but A is not.

The intuition behind the degree overflow rules *dol* and *dor* is that we sometimes need to fix sequences in which a labeled QCL-formula F is assigned a label k with $opt_{\mathrm{QCL}}(F) < k < \infty$. This can happen after applying the rules for conjunction/disjunction. For instance, consider the $\wedge l$-rule: as the premise we have $\Gamma, (A \wedge B)_k \vdash \Delta$ with $k \leq opt_{\mathrm{QCL}}(A \wedge B)$. Recall that the optionality of this conjunct is defined as $opt_{\mathrm{QCL}}(A \wedge B) = max(opt_{\mathrm{QCL}}(F), opt_{\mathrm{QCL}}(G))$. Thus, it may be the case that, for example, $opt_{\mathrm{QCL}}(A) < k$. The $\wedge l$ rule, however, will introduce the premise $\Gamma, (A)_k \vdash (B)_{>k}, \Delta$. Since $(A)_k$ can never be satisfied, as $opt_{\mathrm{QCL}}(A) < k < \infty$, we have to apply the *dol*-rule which replaces $(A)_k$ by $\bot$.

We now provide some examples for valid derivations in **L**[QCL] before showing soundness and completeness.

[2] *dol*/*dor* stands for degree overflow left/right.

Example 5 The following is an $\mathbf{L}$[QCL]-proof of a valid sequent.[3]

$$\dfrac{\dfrac{\dfrac{\dfrac{b \lor c, \neg a, b \vdash a \land b, a \land c, b}{b \lor c, (a \vec{\times} b)_2 \vdash a \land b, a \land c, b}\,\vec{\times} l_2}{(a \vec{\times} b)_2 \vdash \neg(b \vec{\times} c), a \land b, a \land c, b}\,\neg r \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ a \lor b, \neg b, c \vdash a \land b, a \land c, b\end{matrix}}{a \lor b, (b \vec{\times} c)_2 \vdash a \land b, a \land c, b}\,\vec{\times} l_2}{(b \vec{\times} c)_2 \vdash \neg(a \vec{\times} b), a \land b, a \land c, b}\,\neg r}{((a \vec{\times} b) \land (b \vec{\times} c))_2 \vdash a \land b, a \land c, b}\,\land l}{\neg(a \land b), ((a \vec{\times} b) \land (b \vec{\times} c))_2 \vdash a \land c, b}\,\neg l$$

Example 6 The following end-sequent is similar to the end-sequent of Example 5, but with the exception that $(a \vec{\times} b) \land (b \vec{\times} c)$ is assigned a label of 1. However, $((a \vec{\times} b) \land (b \vec{\times} c))_1$ is unsatisfiable in view of $\neg(a \land b)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ b \lor c, a \vdash \neg b, a \land b, \bot\end{matrix} \quad \begin{matrix}\vdots\\ b \lor c, a \vdash c, a \land b, \bot\end{matrix}}{b \lor c, a \vdash (b \vec{\times} c)_2, a \land b, \bot}\,\vec{\times} r_2}{b \lor c, (a \vec{\times} b)_1 \vdash (b \vec{\times} c)_2, a \land b, \bot}\,\vec{\times} l_1}{(a \vec{\times} b)_1 \vdash (b \vec{\times} c)_2, \neg(b \vec{\times} c), a \land b, \bot}\,\neg r \qquad (\varphi)}{((a \vec{\times} b) \land (b \vec{\times} c))_1 \vdash a \land b, \bot}\,\land l}{\neg(a \land b), ((a \vec{\times} b) \land (b \vec{\times} c))_1 \vdash \bot}\,\neg l$$

where φ is

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ a \lor b, b \vdash \neg a, a \land b, \bot\end{matrix} \quad a \lor b, b \vdash b, a \land b, \bot}{a \lor b, b \vdash (a \vec{\times} b)_2, a \land b, \bot}\,\vec{\times} r_2}{a \lor b, (b \vec{\times} c)_1 \vdash (a \vec{\times} b)_2, a \land b, \bot}\,\vec{\times} l_1}{(b \vec{\times} c)_1 \vdash (a \vec{\times} b)_2, \neg(a \vec{\times} b), a \land b, \bot}\,\neg r$$

Example 7 The following proof shows how the $\land r$-rule can introduce more than three premises.

$$\dfrac{\dfrac{a, b \vdash a}{a, b, \neg a \vdash}\,\neg l \quad \dfrac{\dfrac{a, b, c \vdash b}{a, b, c, \neg b \vdash}\,\neg l}{a, b, (b \vec{\times} c)_2 \vdash}\,\vec{\times} l_2 \quad \dfrac{\dfrac{a, b \vdash b, c}{a, b \vdash b \lor c}\,\lor r}{a, b, \neg(b \vec{\times} c) \vdash}\,\neg l \qquad a, b \vdash a, (b \vec{\times} c)_1}{a, b \vdash (a \land (b \vec{\times} c))_1}\,\land r$$

Theorem 1 $\mathbf{L}$[QCL] *is sound and complete.*

Proof ((*Soundness of*) $\mathbf{L}$[QCL]) We have to prove that all rules of $\mathbf{L}$[QCL] are sound.

- For the axioms this is clearly the case.
- $(\neg r)$ and $(\neg l)$: follows from the fact that $deg_{\text{QCL}}(\mathcal{I}, F) < \infty \iff \mathcal{I} \models cp(F)$ for all QCL-formulas F.

[3] Note that, once we reach sequents containing only classical formulas, we do not continue the proof. However, it can be verified that the classical sequents on the left and right branch are provable in this case. Moreover, given a labeled QCL-formula $(A)_1$ with a label of 1, the label is often omitted for readability.

- $(\lor l)$: Assume that the conclusion of the rule is not valid, i.e., there is a model M of Γ and $(A \lor B)_k$ that is not a model of Δ. Then, M satisfies either A or B to degree k and neither to a degree smaller than k. Assume M satisfies A to a degree of k, the other case is symmetric. Then M is a model of Γ and $(A)_k$ but, by assumption, neither of Δ nor of $(B)_j$ for any $j < k$. Hence, at least one of the premises is not valid.
- $(\land l)$: Analogous to the proof for $(\lor l)$: assume that the conclusion of the rule is not valid, i.e., that there is a model M of Γ and $(A \land B)_k$ that is not a model of Δ. Then, M satisfies either A or B to degree k and neither to a degree higher than k. Assume M satisfies A to a degree of k, the other case is symmetric. Then M is a model of Γ and $(A)_k$ but, by assumption, neither of Δ nor of $(\neg B)_1$ or $(B)_j$ for any $j > k$. Hence, at least one of the premises is not valid.
- $(\lor r)$: Assume there is a model M of Γ that is not a model of Δ or of $(A \lor B)_k$. There are two possible cases why M is not a model of $(A \lor B)_k$:

  1. M satisfies neither A nor B to degree k. But in this case the premise $\Gamma \vdash (A)_k, (B)_k, \Delta$ is not valid as M is also not a model of Δ by assumption.
  2. M satisfies either A or B to a degree smaller than k. Assume that M satisfies A to degree $j < k$ (the other case is symmetric). Then the premise $\Gamma, (A)_j \vdash \Delta$ is not valid. Indeed, M is a model of Γ and $(A)_j$ but not of Δ.

- $(\land r)$: Analogous to the proof for $(\lor r)$: assume that the conclusion of the rule is not valid, i.e. there is a model M of Γ that is not a model of Δ or of $(A \land B)_k$. There are two possible cases why M is not a model of $(A \land B)_k$:

  1. M satisfies neither A nor B to degree k. However, then the premise $\Gamma \vdash (A)_k, (B)_k, \Delta$ is not valid as M is also not a model of Δ by assumption.
  2. M satisfies either A or B to a degree j higher than k. By symmetry, it suffices to consider the case that M satisfies A to a degree j higher than k. Then either $k < j \leq opt_{\text{QCL}}(A)$ or $j = \infty$. If $k < j \leq opt_{\text{QCL}}(A)$ the premise $\Gamma, (A)_j \vdash \Delta$ is not valid. If $k = \infty$ the premise $\Gamma, (\neg A)_1 \vdash \Delta$ is not valid.

- $(\vec{\times} l_1)$ and $(\vec{\times} r_1)$: follows from the fact that $(A)_k$ has the same models as $(A \vec{\times} B)_k$ for $k \leq opt_{\mathcal{L}}(A)$.
- $(\vec{\times} l_2)$: Assume the conclusion of the rule is not valid and let M be the model witnessing this. Then M is a model of $(A \vec{\times} B)_{opt_{QCL}(A)+l}$. By definition, M satisfies B to degree l and is not a model of A. However, then it is also a model of Γ, $(B)_l$ and $(\neg A)_1$, which means that the premise is not valid.
- $(\vec{\times} r_2)$. Assume that both premises are valid, i.e., every model of Γ is either a model of Δ or of $(\neg A)_1$ and $(B)_l$ with $l \leq opt_{\mathcal{L}}(B)$. Now, by definition, any model that is not a model of A (and hence a model of $(\neg A)_1$) and of $(B)_l$ satisfies $A \vec{\times} B$ to degree $opt_{QCL}(A) + l$. Therefore, every model of Γ is either a model of Δ or of $(A \vec{\times} B)_{opt_{QCL}(A)+l}$, which means that the conclusion of the rule is valid.
- (dol): $\Gamma, \bot$ has no models, i.e., the premise $\Gamma, \bot \vdash \Delta$ is valid. Indeed, the sequent $\Gamma, \bot \vdash \Delta$ is an axiom in our system. Crucially, the sequent $\Gamma, (A)_{opt_{\text{QCL}}(A)+k}$ has no models as well since A cannot be satisfied to a degree m with $opt_{\mathcal{L}}(A) < m < \infty$.
- (dor) is clearly sound. □

Proof (*Completeness of* **L**[QCL]) To prove completeness, we observe that for any sequent, we can decompose every formula into atomic and hence classical formulas by applying the rules of **L**[QCL]. Moreover, we observe that if all formulas are classical and labeled with 1, then all inference rules reduce to the inference rules of the classical propositional calculus without structural rules [27, Section 3.5], which is known to be complete. Therefore, we know that a sequent containing only classical formulas is valid if and only if it is provable. It remains to show that the rules of **L**[QCL] preserve validity when read "upwards".

- (dol): Assume that a sequent of the form $\Gamma, (A)_{opt_{\text{QCL}}(A)+k} \vdash \Delta$ with $k \in \mathbb{N}$ is valid. Since $\Gamma, \bot$ has no models, $\Gamma, \bot \vdash \Delta$ is valid.
- (dor): Assume that a sequent $\Gamma \vdash (A)_{opt_{\text{QCL}}(A)+k}, \Delta$ is valid. $(A)_{opt_{\text{QCL}}(A)+k}$ cannot be satisfied, i.e., $\Gamma \vdash \Delta$ is valid.
- $(\neg r)$ and $(\neg l)$: Assume that a sequent of the form $\Gamma \vdash (\neg A)_1, \Delta$ is valid. Then every model of Γ is either a model of $(\neg A)_1$ or of Δ. In other words, every model of Γ that is not a model of $(\neg A)_1$ (i.e., is model of $cp(A)$) is a model of Δ. Therefore, every interpretation that is a model of both Γ and $cp(A)$ must be a model of Δ. It follows that $\Gamma, cp(A) \vdash \Delta$ is valid. Similarly for $\Gamma, (\neg A)_1 \vdash \Delta$.
- $(\lor l)$ and $(\land l)$: Assume that a sequent of the form $\Gamma, (A \lor B)_k \vdash \Delta$ is valid, with $k \leq opt_{\mathcal{L}}(A \lor B)$. We claim that then both $\Gamma, (A)_k \vdash (B)_{<k}, \Delta$ and $\Gamma, (B)_k \vdash (A)_{<k}, \Delta$ are valid. Assume to the contrary that $\Gamma, (A)_k \vdash (B)_{<k}, \Delta$ is not valid (the other case is symmetric). Then, there is a model M of Γ and $(A)_k$ that is neither a model of $(B)_{<k}$ nor of Δ. But then M is also a model of Γ and $(A \lor B)_k$, but not of Δ, which contradicts the assumption that $\Gamma, (A \lor B)_k \vdash \Delta$ is valid. Therefore, both $\Gamma, (A)_k \vdash (B)_{<k}, \Delta$ and $\Gamma, (B)_k \vdash (A)_{<k}, \Delta$ are valid. Similarly for a sequent of the form $\Gamma, (A \land B)_k \vdash \Delta$.
- $(\lor r)$ and $(\land r)$: Assume that a sequent of the form $\Gamma \vdash (A \lor B)_k, \Delta$ is valid, with $k \leq opt_{\mathcal{L}}(A \lor B)$. We claim that then for all $i < k$ the sequents $\Gamma, (A)_i \vdash \Delta$ and $\Gamma, (B)_i \vdash \Delta$ and $\Gamma \vdash (A)_k, (B)_k, \Delta$ are valid. Assume by contradiction that there is an $i < k$ s.t. $\Gamma, (A)_i \vdash \Delta$ is not valid. Then, there is a model M of Γ and $(A)_i$ that is not a model of Δ. However, then M is a model of Γ but neither of Δ nor of $(A \lor B)_k$ (as M satisfies $A \lor B$ to degree $i \neq k$), which contradicts our assumption that $\Gamma \vdash (A \lor B)_k, \Delta$ is valid. The case that there is an $i < k$ s.t. $\Gamma, (B)_i \vdash \Delta$ is not valid is symmetric. Finally, we assume that $\Gamma \vdash (A)_k, (B)_k, \Delta$ is not valid. Then, there is a model M of Γ that is not a model of $(A)_k, (B)_k$ or Δ. Then, M is model of Γ but neither of Δ nor of $(A \lor B)_k$, contradicting our assumption. Therefore, all sequents listed above must be valid. Similarly for a sequent of the form $\Gamma \vdash (A \land B)_k, \Delta$.
- $(\vec{\times} l_1)$ and $(\vec{\times} r_1)$: Assume that a sequent of the form $\Gamma, (A \vec{\times} B)_k \vdash \Delta$ with $k \leq opt_{\text{QCL}}(A)$ is valid. Then $\Gamma, (A)_k \vdash \Delta$ is also valid since $(A \vec{\times} B)_k$ and $(A)_k$ have the same models if $k \leq opt_{\text{QCL}}(A)$. Analogously for sequents of the form $\Gamma \vdash (A \vec{\times} B)_k, \Delta$.
- $(\vec{\times} l_2)$: Assume a sequent of the form $\Gamma, (A \vec{\times} B)_{opt_{\text{QCL}}(A)+l} \vdash \Delta$ is valid, with $l \leq opt_{\mathcal{L}}(B)$. We claim that the sequent $\Gamma, (B)_l, \neg A \vdash \Delta$ is then also valid. Indeed, if M is a model of Γ, $(B)_l$ and $\neg A$, then it is also a model of Γ and $(A \vec{\times} B)_{opt_{\text{QCL}}(A)+l}$. Hence, by assumption, M must be a model of Δ.
- $(\vec{\times} r_2)$: Assume that a sequent of the form $\Gamma \vdash (A \vec{\times} B)_{opt_{QCL}(A)+l}, \Delta$ is valid, with $l \leq opt_{\mathcal{L}}(B)$. We claim that then also the sequents $\Gamma \vdash \neg A, \Delta$ and $\Gamma \vdash (B)_l, \Delta$ are valid. Assume by contradiction that the first sequent is not valid. This means that there is a model M of Γ that is not a model of neither $\neg A$ nor of Δ. However, then M is a model of A and therefore satisfies $A \vec{\times} B$ to a degree smaller than $opt_{\text{QCL}}(A)$. This contradicts our assumption that $\Gamma \vdash (A \vec{\times} B)_{opt_{\text{QCL}}(A)+l}, \Delta$ is valid. Assume now that the second sequent is not valid, i.e., that there is a model M of Γ that is neither a model of $(B)_l$ nor of Δ. Then, M cannot be a model of $(A \vec{\times} B)_{opt_{\text{QCL}}(A)+l}$ and we again have a contradiction to the assumption. □

A cut rule has not been introduced for **L**[QCL] so far. However, it is easy to see that **L**[QCL] is cut-admissible.

$$\frac{\Gamma \vdash (A)_k, \Delta \quad \Gamma', (A)_k \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, cut$$

Proposition 2 *The cut-rule is sound.*

Proof Assume $\Gamma \vdash (A)_k, \Delta$ and $\Gamma', (A)_k \vdash \Delta'$ are valid. Let M be some model of Γ, Γ'. M must satisfy some formula in $(A)_k, \Delta$. If M satisfies $(A)_k$ then M satisfies both Γ' and $(A)_k$ and thus also some formula in Δ'. In any case, M satisfies some formula in Δ, Δ'. □

We do not prove an effective cut-elimination theorem in the sense of Gentzen, i.e. by providing an algorithm for the elimination of cut inferences in a derivation. But since we do not use a cut rule when proving the completeness of **L**[QCL] (cf. Theorem 1), we obtain a cut-elimination theorem for free.

Another aspect of our calculus that should be mentioned is that, although **L**[QCL] is cut-free, we do not have the subformula property. This is especially obvious when looking at the rules for negation, where we use the classical counterpart $cp(A)$ of QCL-formulas. For example, $\neg(a \vec{\times} b)$ in the conclusion of the $\neg$-left rule becomes $cp(a \vec{\times} b) = a \vee b$ in the premise.

Moreover, note that we introduced no structural rules (i.e., weakening or contraction) in **L**[QCL], as they are not needed for completeness of the calculus. It is easy to see, however, that weakening and contraction are sound in this setting. Thus, if desired, one could extend **L**[QCL] with the following rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma, (A)_k \vdash \Delta}\, wl \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash (A)_k, \Delta}\, wr$$

$$\frac{\Gamma, (A)_k, (A)_k \vdash \Delta}{\Gamma, (A)_k \vdash \Delta}\, cl \qquad \frac{\Gamma \vdash (A)_k, (A)_k, \Delta}{\Gamma \vdash (A)_k, \Delta}\, cr$$

Towards Preferred Model Entailment While we believe that **L**[QCL] is interesting in its own right, the question of how this calculus can be used to obtain a calculus for preferred model entailment arises. Essentially, an inference rule has to be added that allows for the transition from standard to preferred model inferences. As a first approach we consider theories $\Gamma \cup \{A\}$ with Γ consisting only of classical formulas and A being a QCL-formula. In this simple case, preferred models of $\Gamma \cup \{A\}$ are those models of $\Gamma \cup \{A\}$ that satisfy A to the smallest possible degree. We call the resulting calculus $\mathbf{L}[\mathrm{QCL}]^{naive}_{\mid\sim}$.

Definition 14 ($\mathbf{L}[\mathrm{QCL}]^{naive}_{\mid\sim}$) The labeled sequent calculus $\mathbf{L}[\mathrm{QCL}]^{naive}_{\mid\sim}$ is **L**[QCL] extended by the inference rule

$$\frac{\langle \Gamma, (A)_i \vdash \bot \rangle_{i<k} \quad \Gamma, (A)_k \vdash \Delta}{\Gamma, A \mid\sim^{lex}_{\mathrm{QCL}} \Delta}\ \mid\sim_{naive}$$

Intuitively, the inference rule $\mid\sim_{naive}$ states that, if there are no interpretations that satisfy Γ while also satisfying A to a degree lower than k, and if Δ follows from all models of $\Gamma, (A)_k$, then Δ is entailed by the preferred models of $\Gamma \cup \{A\}$. However, it can be shown that $\mathbf{L}[\mathrm{QCL}]^{naive}_{\mid\sim}$ is unsound.

Proposition 3 $\mathbf{L}[\mathrm{QCL}]^{naive}_{\mid\sim}$ *is unsound.*

Proof Consider the invalid entailment $\neg a, a \vec{\times} b \mid\!\sim^{lex}_{\text{QCL}} a$, which is derivable in $\mathbf{L}[\text{QCL}]^{naive}_{\mid\sim}$ as follows:

$$\dfrac{\dfrac{\neg a, a \vdash a}{\neg a, (a \vec{\times} b)_1 \vdash a}\,\vec{\times} l_1}{\neg a, a \vec{\times} b \mid\!\sim^{lex}_{\text{QCL}} a}\ \mid\!\sim_{naive}$$

□

Thus, an extension of $\mathbf{L}[\text{QCL}]$ by $\mid\!\sim_{naive}$ does not yield the desired calculus, not even in this restricted setting where we consider only a single non-classical formula A. What is missing is an assertion that $\Gamma, (A)_k$ is satisfiable. Unfortunately, this cannot be formulated in $\mathbf{L}[\text{QCL}]$. A way of addressing this problem is to make use of a refutation calculus, as has been done for other non-monotonic logics [9].

4 A Calculus for Preferred Model Entailment

To obtain a calculus for preferred model entailment, we first need to introduce a refutation calculus, which we call $\mathbf{L}[\text{QCL}]^-$. In the literature, such a rejection method for first-order logic with equality was first introduced in [14] and proved complete w.r.t. finite model theory. The refutation calculus $\mathbf{L}[\text{QCL}]^-$ used in this work is based on a simpler rejection method for propositional logic defined in [9]. Using $\mathbf{L}[\text{QCL}]^-$, we prove that $(A)_k$ is satisfiable by deriving the antisequent $(A)_k \nvdash \bot$.

Definition 15 (*labeled* QCL-*antisequents*) $\Gamma \nvdash \Delta$ is a labeled QCL-antisequent if and only if $\Gamma \vdash \Delta$ is a labeled QCL-sequent. $\Gamma \nvdash \Delta$ is valid if and only if $\Gamma \vdash \Delta$ is not valid, i.e., if at least one model that satisfies all labeled QCL-formulas in Γ to the degree specified by the label satisfies no labeled QCL- formula in Δ to the degree specified by the label.

The rules for $\mathbf{L}[\text{QCL}]^-$ can be derived from the rules of $\mathbf{L}[\text{QCL}]$. For example, consider the antisequent $\Gamma, (A \vee B)_k \nvdash \Delta$. To show that this antisequent is valid, we must show that the corresponding sequent $\Gamma, (A \vee B)_k \vdash \Delta$ is not valid. This in turn means that we must show that at least one of the two premises $\Gamma, (A)_k \vdash (B)_{<k}, \Delta$ and $\Gamma, (B)_k \vdash (A)_{<k}, \Delta$ introduced by the $\vee l$-rule are not valid. In other words, we must show that either the antisequent $\Gamma, (A)_k \nvdash (B)_{<k}, \Delta$ or the antisequent $\Gamma, (B)_k \nvdash (A)_{<k}, \Delta$ is valid. We therefore introduce two rules:

$$\dfrac{\Gamma, (A)_k \nvdash (B)_{<k}, \Delta}{\Gamma, (A \vee B)_k \nvdash \Delta} \nvdash \vee l_1 \qquad \dfrac{\Gamma, (B)_k \nvdash (A)_{<k}, \Delta}{\Gamma, (A \vee B)_k \nvdash \Delta} \nvdash \vee l_2$$

Which one of these two rules should be applied must be guessed, i.e., we trade the branching of $\mathbf{L}[\text{QCL}]$ for non-determinism.

The rules for other connectives are derived similarly. Binary rules are translated into two rules; one inference rule per premise. $(\vee r)$ in $\mathbf{L}[\text{QCL}]$ has an unbounded number of premises, but due to the rules' structure it can be translated into three inference rules. Similarly for $(\wedge r)$, but we need to introduce two extra rules for the case that either A or B is not satisfied.

Moreover, regarding the degree overflow rules we introduce a right-hand side rule, but no left-hand side rule. The reason for this is that the antisequent $\Gamma, (A)_{opt_{\text{QCL}}(A)+k} \nvdash \Delta$ is always invalid, i.e., a left-hand side degree overflow rule could never be used in the derivation of a valid antisequent.

Definition 16 ($\mathbf{L}[\text{QCL}]^-$) The axioms of $\mathbf{L}[\text{QCL}]^-$ are labeled QCL-antisequents of the form $\Gamma \nvdash \Delta$, where Γ and Δ are disjoint sets of atoms and $\bot \notin \Gamma$. The inference rules of

L[QCL]$^-$ are given below. Whenever a labeled QCL-formula $(F)_k$ appears in the conclusion of an inference rule (except for the $\nvdash$ *dor*-rule) it holds that $k \leq opt_{\mathcal{L}}(F)$.

The rules for the classical connectives are:

$$\frac{\Gamma, (cp(A))_1 \nvdash \Delta}{\Gamma \nvdash (\neg A)_1, \Delta} \nvdash \neg r \qquad \frac{\Gamma \nvdash (cp(A))_1, \Delta}{\Gamma, (\neg A)_1 \nvdash \Delta} \nvdash \neg l$$

$$\frac{\Gamma, (A)_k \nvdash (B)_{<k}, \Delta}{\Gamma, (A \vee B)_k \nvdash \Delta} \nvdash \vee l_1 \qquad \frac{\Gamma, (B)_k \nvdash (A)_{<k}, \Delta}{\Gamma, (A \vee B)_k \nvdash \Delta} \nvdash \vee l_2$$

$$\frac{\Gamma, (A)_i \nvdash \Delta}{\Gamma \nvdash (A \vee B)_k, \Delta} \nvdash \vee r_1 \qquad \frac{\Gamma, (B)_i \nvdash \Delta}{\Gamma \nvdash (A \vee B)_k, \Delta} \nvdash \vee r_2$$

where $i < k$;

$$\frac{\Gamma \nvdash (A)_k, (B)_k, \Delta}{\Gamma \nvdash (A \vee B)_k, \Delta} \nvdash \vee r_3$$

$$\frac{\Gamma, (A)_k \nvdash (B)_{>k}, \Delta}{\Gamma, (A \wedge B)_k \nvdash \Delta} \nvdash \wedge l_1 \qquad \frac{\Gamma, (B)_k \nvdash (A)_{>k}, \Delta}{\Gamma, (A \wedge B)_k \nvdash \Delta} \nvdash \wedge l_2$$

$$\frac{\Gamma, (A)_i \nvdash \Delta}{\Gamma \nvdash (A \wedge B)_k, \Delta} \nvdash \wedge r_1 \qquad \frac{\Gamma, (\neg A)_1 \nvdash \Delta}{\Gamma \nvdash (A \wedge B)_k, \Delta} \nvdash \wedge r_2$$

where $k < i \leq opt_{\text{QCL}}(A)$;

$$\frac{\Gamma, (B)_i \nvdash \Delta}{\Gamma \nvdash (A \wedge B)_k, \Delta} \nvdash \wedge r_3 \qquad \frac{\Gamma, (\neg B)_1 \nvdash \Delta}{\Gamma \nvdash (A \wedge B)_k, \Delta} \nvdash \wedge r_4$$

where $k < i \leq opt_{\text{QCL}}(B)$;

$$\frac{\Gamma \nvdash (A)_k, (B)_k, \Delta}{\Gamma \nvdash (A \wedge B)_k, \Delta} \nvdash \wedge r_5$$

The rules for ordered disjunction are:

$$\frac{\Gamma, (A)_k \nvdash \Delta}{\Gamma, (A \vec{\times} B)_k \nvdash \Delta} \nvdash \vec{\times} l_1 \qquad \frac{\Gamma, (B)_l, (\neg A)_1 \nvdash \Delta}{\Gamma, (A \vec{\times} B)_{opt_{QCL}(A)+l} \nvdash \Delta} \nvdash \vec{\times} l_2$$

$$\frac{\Gamma \nvdash (A)_k, \Delta}{\Gamma \nvdash (A \vec{\times} B)_k, \Delta} \nvdash \vec{\times} r_1 \qquad \frac{\Gamma \nvdash (\neg A)_1, \Delta}{\Gamma \nvdash (A \vec{\times} B)_{opt_{\text{QCL}}(A)+l}, \Delta} \nvdash \vec{\times} r_2$$

$$\frac{\Gamma \nvdash (B)_l, \Delta}{\Gamma \nvdash (A \vec{\times} B)_{opt_{\text{QCL}}(A)+l}, \Delta} \nvdash \vec{\times} r_3$$

where $k \leq opt_{\mathcal{L}}(A)$ and $l \leq opt_{\mathcal{L}}(B)$.

The degree overflow rule is:

$$\frac{\Gamma \nvdash \Delta}{\Gamma \nvdash (A)_{opt_{\text{QCL}}(A)+k}, \Delta} \nvdash dor$$

where $k \in \mathbb{N}$.

As already mentioned, instead of branching over several premises it must be guessed non-deterministicly which rule to apply next in **L**[QCL]$^-$, e.g., whether to apply $\wedge l_1$ or $\wedge l_2$. Moreover, in the proof of an antisequent $\Gamma \nvdash \Delta$ the axiom we encounter directly witnesses a counter example for the corresponding sequent $\Gamma \vdash \Delta$. These differences between **L**[QCL]$^-$ and **L**[QCL] reflect the duality between validity and satisfiability in classical logic.

Example 8 The following derivation is related to Example 5 and shows that $\neg(a \wedge b)$, $((a \vec{\times} b) \wedge (b \vec{\times} c))_2$ is satisfiable.

$$
\begin{array}{cl}
a, a, c \nvdash b, b, \bot & \\ \hline
a, c, \neg b \nvdash b, \bot & \nvdash \neg l \\ \hline
a, c \nvdash a \wedge b, b, \bot & \nvdash \wedge r_4 \\ \hline
a, c, \neg b \nvdash a \wedge b, \bot & \nvdash \neg l \\ \hline
(a \vee b), c, \neg b \nvdash a \wedge b, \bot & \nvdash \vee l_1 \\ \hline
(a \vee b), (b \vec{\times} c)_2 \nvdash a \wedge b, \bot & \nvdash \vec{\times} l_2 \\ \hline
(b \vec{\times} c)_2 \nvdash \neg(a \vec{\times} b), a \wedge b, \bot & \nvdash \neg r \\ \hline
((a \vec{\times} b) \wedge (b \vec{\times} c))_2 \nvdash a \wedge b, \bot & \nvdash \wedge l_2 \\ \hline
\neg(a \wedge b), ((a \vec{\times} b) \wedge (b \vec{\times} c))_2 \nvdash \bot & \nvdash \neg l
\end{array}
$$

The interpretation $\{a, c\}$ witnesses the axiom $a, a, c \nvdash b, b, \bot$ and also the final antisequent $\neg(a \wedge b), ((a \vec{\times} b) \wedge (b \vec{\times} c))_2 \nvdash \bot$.

Theorem 4 $\mathbf{L}[\text{QCL}]^-$ *is sound and complete.*

Proof (*Soundness of* $\mathbf{L}[\text{QCL}]^-$) It is easy to see that, for each rule, the same model witnessing the validity of the premise also witnesses the validity of the conclusion. We exemplify this on hand of the $\nvdash \vee l_1$-rule: assume $\Gamma, (A)_k \nvdash (B)_{<k}, \Delta$ is valid. Then there exists a model M of $\Gamma, (A)_k$. which does not satisfy B to a degree lower than k, and does not satisfy any formula in Δ. Thus, M satisfies $\Gamma, (A \vee B)_k$ and hence $\Gamma, (A \vee B)_k \nvdash \Delta$ is valid. □

Proof (*Completeness of* $\mathbf{L}[\text{QCL}]^-$) Analogously to $\mathbf{L}[\text{QCL}]$, in $\mathbf{L}[\text{QCL}]^-$ we can decompose the formulas of any antisequent into atomic formulas by applying the rules of $\mathbf{L}[\text{QCL}]^-$. Thus, it again suffices to show that the rules of $\mathbf{L}[\text{QCL}]^-$ preserve validity when read "upwards". Assume $\Gamma \nvdash \Delta$ is valid, i.e. $\Gamma \vdash \Delta$ is not valid. There must be a rule in $\mathbf{L}[\text{QCL}]$ for which $\Gamma \vdash \Delta$ is the conclusion. This rule cannot be the *dol*-rule, since both the premise and conclusion of this rule are always valid. By the soundness of $\mathbf{L}[\text{QCL}]$, the fact that $\Gamma \vdash \Delta$ is not valid implies that at least one of the premises $\Gamma^* \vdash \Delta^*$ of the rule is not valid. However, then $\Gamma^* \nvdash \Delta^*$ is valid. Now, by the construction of $\mathbf{L}[\text{QCL}]^-$, there is a rule that allows us to derive $\Gamma \nvdash \Delta$ from $\Gamma^* \nvdash \Delta^*$. □

Observe that we have not introduced a cut rule for $\mathbf{L}[\text{QCL}]^-$. Indeed, a counterpart of the cut rule would not be sound. One possibility is to introduce a contrapositive of cut as described in [9].

$$
\frac{\Gamma \nvdash \Delta \quad \Gamma, (A)_k \vdash \Delta}{\Gamma \nvdash (A)_k, \Delta}\, cut2
$$

Proposition 5 *The cut2-rule is sound.*

Proof Assume $\Gamma \nvdash \Delta$ and $\Gamma, (A)_k \vdash \Delta$ are valid. Then there is a model M of Γ that does not satisfy any formula in Δ. This further means that M does not satisfy $(A)_k$, otherwise $\Gamma, (A)_k \vdash \Delta$ would not be valid. Thus, $\Gamma \nvdash (A)_k, \Delta$ is valid. □

It is easy to see that, as in $\mathbf{L}[\text{QCL}]$, contraction is also sound in $\mathbf{L}[\text{QCL}]^-$. In contrast to $\mathbf{L}[\text{QCL}]$, however, weakening is not sound in $\mathbf{L}[\text{QCL}]^-$. With left weakening we could, e.g., derive $a, b \nvdash b$ (which is not valid) from $a \nvdash b$ (which is valid). Likewise, with right weakening we could derive $a \nvdash a, b$ from $a \nvdash b$.

We are now ready to combine $\mathbf{L}[\text{QCL}]$ and $\mathbf{L}[\text{QCL}]^-$ by defining an inference rule that allows us to go from labeled QCL-sequents to non-monotonic inferences. We first consider preferred model entailment under minmax semantics (recall Definitions 9, 10).

Definition 17 ($\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\!\sim}$) The axioms of $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\!\sim}$ are either labeled QCL-sequents of the form $(p)_1 \vdash (p)_1$ with p being an atom, or labeled QCL-antisequents of the form $\Gamma \nvdash \Delta$, where Γ and Δ are disjoint sets of atoms and $\bot \notin \Gamma$. The inference rules of $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\!\sim}$ are the inference rules of $\mathbf{L}[\mathrm{QCL}]$ and $\mathbf{L}[\mathrm{QCL}]^{-}$, extended by

$$\frac{\langle \Gamma, (A_1 \wedge \cdots \wedge A_n)_i \vdash \bot \rangle_{i<k} \quad \Gamma, (A_1 \wedge \cdots \wedge A_n)_k \nvdash \bot \quad \Gamma, (A_1 \wedge \cdots \wedge A_n)_k \vdash \Delta}{\Gamma, A_1, \ldots, A_n \mid\!\sim^{mm}_{\mathrm{QCL}} \Delta} \ \mid\!\sim_{mm}$$

and

$$\frac{\Gamma, cp(A_1), \ldots, cp(A_n) \vdash \bot}{\Gamma, A_1, \ldots, A_n \mid\!\sim^{mm}_{\mathrm{QCL}} \Delta} \ \mid\!\sim_{unsat}$$

where Γ consists only of classical formulas and every A_j with $1 \leq j \leq n$ is a QCL-formula.

Observe that the premises of the new rules $\mid\!\sim_{mm}$ and $\mid\!\sim_{unsat}$ are QCL-sequents and QCL-antisequents, while the conclusion is a $\mid\!\sim_{mm}$ sequent. Consequently, any proof in $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\!\sim}$ can contain only one application of the new rules, in the very last step of the proof. The $\mid\!\sim_{mm}$-rule makes use of the fact that, under minmax semantics, a QCL-theory $T = \{A_1, \ldots, A_n\}$ is semantically equivalent to $A_1 \wedge \cdots \wedge A_n$. The rule can be explained as follows: first, an optimal degree k is guessed. The premises $\langle \Gamma, (A_1 \wedge \cdots \wedge A_n)_i \vdash \bot \rangle_{i<k}$ along with $\Gamma, (A_1 \wedge \cdots \wedge A_n)_k \nvdash \bot$ ensure that models satisfying $A_1 \wedge \cdots \wedge A_n$ to a degree of k are preferred, while the premise $\Gamma, (A_1 \wedge \cdots \wedge A_n)_k \vdash \Delta$ ensures that Δ is entailed by those preferred models. The rule $\mid\!\sim_{unsat}$ is needed in case a theory is classically unsatisfiable.

Example 9 The valid entailment $\neg(a \wedge b), (a \vec{\times} b), (b \vec{\times} c) \mid\!\sim^{mm}_{\mathrm{QCL}} a \wedge c, b$ is provable in $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\!\sim}$ by choosing $k = 2$. Let $\Gamma = \neg(a \wedge b)$ and $\Delta = a \wedge c, b$.

$$\frac{(\varphi_1) \quad \overset{(\varphi_2)}{\Gamma, ((a \vec{\times} b) \wedge (b \vec{\times} c))_2 \nvdash \bot} \quad \overset{(\varphi_3)}{\Gamma, ((a \vec{\times} b) \wedge (b \vec{\times} c))_2 \vdash \Delta}}{\Gamma, (a \vec{\times} b), (b \vec{\times} c) \mid\!\sim^{mm}_{\mathrm{QCL}} \Delta} \ \mid\!\sim_{mm}$$

where (φ_1) is the derivation

$$\frac{\dfrac{\dfrac{\dfrac{\overset{\vdots}{\Gamma, b \vee c, a, \vdash \neg b, \bot} \quad \overset{\vdots}{\Gamma, b \vee c, a \vdash c, \bot}}{\Gamma, b \vee c, a \vdash (b \vec{\times} c)_2, \bot} \ \vec{\times} r_2}{\Gamma, b \vee c, (a \vec{\times} b)_1 \vdash (b \vec{\times} c)_2, \bot} \ \vec{\times} l_1}{\Gamma, (a \vec{\times} b)_1 \vdash (b \vec{\times} c)_2, \neg(b \vec{\times} c), \bot} \ \neg r \quad \dfrac{\dfrac{\overset{\vdots}{\Gamma, a \vee b, b \vdash (a \vec{\times} b)_2, \bot}}{\Gamma, a \vee b, (b \vec{\times} c)_1 \vdash (a \vec{\times} b)_2, \bot} \ \vec{\times} l_1}{\Gamma, (b \vec{\times} c)_1 \vdash (a \vec{\times} b)_2, \neg(a \vec{\times} b), \bot} \ \neg r}{\Gamma, ((a \vec{\times} b) \wedge (b \vec{\times} c))_1 \vdash \bot} \ \wedge l$$

Note that (φ_2) is the $\mathbf{L}[\mathrm{QCL}]^{-}$-proof from Example 8 and (φ_3) is the $\mathbf{L}[\mathrm{QCL}]$-proof from Example 5.

Theorem 6 $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\!\sim}$ *is sound and complete.*

Proof (*Soundness of* $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\!\sim}$) Consider first the $\mid\!\sim_{mm}$-rule and assume that all premises are derivable. By the soundness of $\mathbf{L}[\mathrm{QCL}]$ and $\mathbf{L}[\mathrm{QCL}]^{-}$ they are also valid. From the first set of premises $\langle \Gamma, (A_1 \wedge \cdots \wedge A_n)_i \vdash \bot \rangle_{i<k}$ we can conclude that if there is some model M of Γ that satisfies $A_1 \wedge \cdots \wedge A_n$ to a degree of k, then $M \in Prf^{mm}_{\mathrm{QCL}}(\Gamma \cup \{A_1, \ldots, A_k\})$. The premise $\Gamma, (A_1 \wedge \cdots \wedge A_n)_k \nvdash \bot$ ensures that there is such a model M. By the last premise $\Gamma, (A_1 \wedge \cdots \wedge A_n)_k \vdash \Delta$, we can conclude that all models of $\Gamma \cup \{A_1, \ldots, A_k\}$ that are equally as preferred as M, i.e., all $M' \in Prf^{mm}_{\mathrm{QCL}}(\Gamma \cup \{A_1, \ldots, A_k\})$, satisfy at least one formula in Δ. Therefore, $\Gamma, A_1, \ldots, A_k \mid\!\sim^{mm}_{\mathrm{QCL}} \Delta$ is valid.

Now consider the $\mid\!\sim_{unsat}$-rule and assume that $\Gamma, cp(A_1), \ldots, cp(A_k) \vdash \bot$ is derivable and therefore valid. Then $\Gamma \cup cp(A_1), \ldots, cp(A_k)$ has no models. Since in general we have $deg_{\mathrm{QCL}}(\mathcal{I}, F) < \infty \iff \mathcal{I} \models cp(F)$, also $\Gamma \cup \{A_1, \ldots, A_k\}$ has no models and thus no preferred models. Then $\Gamma, A_1, \ldots, A_k \mid\!\sim^{mm}_{\mathrm{QCL}} \Delta$ is valid. □

Proof (*Completeness of* $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\sim}$) Assume that $\Gamma, A_1, \ldots, A_k \mid\!\sim^{mm}_{\mathrm{QCL}} \Delta$ is valid. If $\Gamma \cup \{A_1, \ldots, A_k\}$ is unsatisfiable then $\Gamma, cp(A_1), \ldots, cp(A_k) \vdash \bot$ is valid, i.e., we can apply the $\mid\!\sim_{unsat}$-rule.

Now consider the case that $\Gamma \cup \{A_1, \ldots, A_k\}$ is satisfiable and assume that some preferred model M of $\Gamma \cup \{A_1, \ldots, A_k\}$ satisfies $A_1 \wedge \cdots \wedge A_n$ to a degree of k. Then, we claim that all premises of the rule are valid and, by the completeness of $\mathbf{L}[\mathrm{QCL}]$ and $\mathbf{L}[\mathrm{QCL}]^{-}$, also derivable.

Assume by contradiction that one of the premises is not valid. First, consider the case that $\Gamma, (A_1 \wedge \cdots \wedge A_n)_i \vdash \bot$ is not valid for some $i < k$. Then there is a model M' of Γ that satisfies $A_1 \wedge \cdots \wedge A_n$ to a degree of $i < k$. However, this contradicts the assumption that M is a preferred model of $\Gamma \cup \{A_1, \ldots, A_k\}$.

Next, assume that $\Gamma, (A_1 \wedge \cdots \wedge A_n)_k \nvdash \bot$ is not valid. However, M satisfies $(A_1 \wedge \cdots \wedge A_n)_k$ and does not satisfy $\bot$. Contradiction.

Finally, we assume that $\Gamma, (A_1 \wedge \cdots \wedge A_n)_k \vdash \Delta$ is not valid. Then, there is a model M' of Γ that satisfies $A_1 \wedge \cdots \wedge A_n$ to a degree of k but does not satisfy any formula in Δ. But M' is a preferred model of $\Gamma \cup \{A_1, \ldots, A_k\}$, which contradicts $\Gamma, A_1, \ldots, A_k \mid\!\sim^{mm}_{\mathrm{QCL}} \Delta$ being valid. □

To obtain a calculus for preferred model entailment under lexicographic semantics, we adapt the $\mid\!\sim_{mm}$-rule of $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\sim}$.

Definition 18 ($\mathbf{L}[\mathrm{QCL}]^{lex}_{\mid\sim}$) Let $\leq_l$ be the order on vectors in $\mathbb{N}^k$ defined by

- $\vec{v} <_l \vec{w}$ if there is some $n \in \mathbb{N}$ such that $\vec{v}$ has more entries of value n and for all $1 \leq m < n$ both vectors have the same number of entries of value m.
- $\vec{v} =_l \vec{w}$ if, for all $n \in \mathbb{N}$, $\vec{v}$ and $\vec{w}$ have the same number of entries of value n.

The axioms and inference rules of $\mathbf{L}[\mathrm{QCL}]^{lex}_{\mid\sim}$ are the same as those of $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\sim}$, except that $\mid\!\sim_{mm}$ is replaced by

$$\frac{\langle \Gamma, (A_1)_{w_1}, \ldots, (A_k)_{w_k} \vdash \bot \rangle_{\vec{w} <_l \vec{v}} \quad \Gamma, (A_1)_{v_1}, \ldots, (A_k)_{v_k} \nvdash \bot \quad \langle \Gamma, (A_1)_{w_1}, \ldots, (A_k)_{w_k} \vdash \Delta \rangle_{\vec{w} =_l \vec{v}}}{\Gamma, A_1, \ldots, A_k \mid\!\sim^{lex}_{\mathrm{QCL}} \Delta} \mid\!\sim_{lex}$$

where $\vec{v}, \vec{w} \in \mathbb{N}^k$, Γ consists only of classical formulas and every A_j with $1 \leq j \leq k$ is a QCL-formula.

Instead of guessing the degree of preferred models as in $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\sim}$, we now guess a "degree-profile" (in form of the vector $\vec{v}$) of at least one preferred model of $\Gamma \cup \{A_1, \ldots, A_k\}$. The rule $\mid\!\sim_{lex}$ can be explained as follows: The premises shown in the left branch confirm that our guess is indeed optimal, i.e., that $\Gamma, (A_1)_{w_1}, \ldots, (A_k)_{w_k}$ cannot be satisfied if $\vec{w}$ is better than $\vec{v}$ with respect to the *lex*-semantics. The center premise ensures that our degree-profile is satisfiable. The right premise ensures that *all* preferred models, meaning all models with a degree profile $\vec{w}$ as good as $\vec{v}$ with respect to the *lex*-semantics, satisfy at least one formula in Δ. Note that, as for $\mathbf{L}[\mathrm{QCL}]^{mm}_{\mid\sim}$, any proof in $\mathbf{L}[\mathrm{QCL}]^{lex}_{\mid\sim}$ can contain only one application of the new rules, in the very last step of the proof. Let us provide a small example before showing soundness and completeness of $\mathbf{L}[\mathrm{QCL}]^{lex}_{\mid\sim}$.

Example 10 Consider the valid entailment $\neg(a \wedge b), (a \vec{\times} b), (b \vec{\times} c) \mathrel{|\!\sim}^{lex}_{\mathrm{QCL}} a \wedge c, b$ similar to Example 9. Let $\Gamma = \neg(a \wedge b)$ and $\Delta = a \wedge c, b$. Therefore, we can also write the entailment as $\Gamma, (a \vec{\times} b), (b \vec{\times} c) \mathrel{|\!\sim}^{lex}_{\mathrm{QCL}} \Delta$. Note that it is not possible to satisfy all labeled QCL-formulas on the left to a degree of 1. Rather, it is optimal to either satisfy $\Gamma, (a \vec{\times} b)_1, (b \vec{\times} c)_2$ or, alternatively, $\Gamma, (a \vec{\times} b)_2, (b \vec{\times} c)_1$. We choose $\vec{v} = (1, 2)$. Observe that $\vec{w} = (1, 1)$ is the only vector $\vec{w}$ such that $\vec{w} <_l \vec{v}$. Moreover, $(1, 2) =_l \vec{v}$ and $(2, 1) =_l \vec{v}$. Thus, we get

$$\dfrac{\begin{matrix}\vdots & \vdots & \vdots \\ \Gamma, (a \vec{\times} b)_1, (b \vec{\times} c)_1 \vdash \bot & \Gamma, (a \vec{\times} b)_1, (b \vec{\times} c)_2 \nvdash \bot & \Gamma, (a \vec{\times} b)_1, (b \vec{\times} c)_2 \vdash \Delta \ *\end{matrix}}{\Gamma, (a \vec{\times} b), (b \vec{\times} c) \mathrel{|\!\sim}^{lex}_{\mathrm{QCL}} \Delta} \mathrel{|\!\sim}_{lex}$$

where $*$ is

$$\begin{matrix}\vdots \\ \Gamma, (a \vec{\times} b)_2, (b \vec{\times} c)_1 \vdash \Delta\end{matrix}$$

It can be verified that indeed all branches are provable, but we do not show this explicitly here.

Theorem 7 $\mathbf{L}[\mathrm{QCL}]^{lex}_{|\sim}$ *is sound and complete.*

Proof *(Soundness of* $\mathbf{L}[\mathrm{QCL}]^{lex}_{|\sim}$*)* Consider the $\mathrel{|\!\sim}_{lex}$-rule and assume that all premises are derivable. By the soundness of $\mathbf{L}[\mathrm{QCL}]$ and $\mathbf{L}[\mathrm{QCL}]^-$ they are also valid. From the first set of premises $\langle \Gamma, (A_1)_{w_1}, \ldots, (A_k)_{w_k} \vdash \bot \rangle_{\vec{w} <_l \vec{v}}$ we can conclude that if there is some model M of Γ that satisfies A_i to a degree of v_i for all $1 \leq i \leq k$, then $M \in Prf^{lex}_{\mathrm{QCL}}(\Gamma \cup \{A_1, \ldots, A_k\})$. The premise $\Gamma, (A_1)_{v_1}, \ldots, (A_k)_{v_k} \nvdash \bot$ ensures that there is such a model M. By the last set of premises $\langle \Gamma, (A_1)_{w_1}, \ldots, (A_k)_{w_k} \vdash \Delta \rangle_{\vec{w} =_l \vec{v}}$ we can conclude that all models of $\Gamma \cup \{A_1, \ldots, A_k\}$ that are equally as preferred as M, i.e., all $M' \in Prf^{lex}_{\mathrm{QCL}}(\Gamma \cup \{A_1, \ldots, A_k\})$, satisfy at least one formula in Δ. Therefore, $\Gamma, A_1, \ldots, A_k \mathrel{|\!\sim}^{lex}_{\mathrm{QCL}} \Delta$ is valid. □

Proof (*Completeness of* $\mathbf{L}[\mathrm{QCL}]^{lex}_{|\sim}$) Assume that $\Gamma, A_1, \ldots, A_k \mathrel{|\!\sim}^{lex}_{\mathrm{QCL}} \Delta$ is valid. If $\Gamma \cup \{A_1, \ldots, A_k\}$ is unsatisfiable then $\Gamma, cp(A_1), \ldots, cp(A_k) \vdash \bot$ is valid, i.e., we can apply the $\mathrel{|\!\sim}_{unsat}$-rule. Now consider the case that $\Gamma \cup \{A_1, \ldots, A_k\}$ is satisfiable and assume that some preferred model M of $\Gamma \cup \{A_1, \ldots, A_k\}$ satisfies A_i to a degree of v_i for all $1 \leq i \leq k$. Then, we claim that all premises of the rule are valid and, by the completeness of $\mathbf{L}[\mathrm{QCL}]$ and $\mathbf{L}[\mathrm{QCL}]^-$, also derivable.

Assume by contradiction that one of the premises is not valid. First, consider the case that $\Gamma, (A_1)_{w_1}, \ldots, (A_k)_{w_k} \vdash \bot$ is not valid for some $\vec{w} <_l \vec{v}$. Then there is a model M' of Γ that satisfies A_i to a degree of w_i for all $1 \leq i \leq k$. However, this contradicts the assumption that M is a preferred model of $\Gamma \cup \{A_1, \ldots, A_k\}$.

Next, assume that $\Gamma, (A_1)_{v_1}, \ldots, (A_k)_{v_k} \nvdash \bot$ is not valid. However, M satisfies $\Gamma, (A_1)_{v_1}, \ldots, (A_k)_{v_k}$ and does not satisfy $\bot$. Contradiction.

Finally, we assume that $\Gamma, (A_1)_{w_1}, \ldots, (A_k)_{w_k} \vdash \Delta$ is not valid for some $\vec{w} =_l \vec{v}$. Then, there is a model M' of Γ that satisfies A_i to a degree of w_i for all $1 \leq i \leq k$ but does not satisfy any formula in Δ. But M' is a preferred model of $\Gamma \cup \{A_1, \ldots, A_k\}$, which contradicts $\Gamma, A_1, \ldots, A_k \mathrel{|\!\sim}^{lex}_{\mathrm{QCL}} \Delta$ being valid. □

Finally, a calculus for the inclusion-based approach of preferred model entailment can be obtained by simply adapting the way in which vectors over $\mathbb{N}^k$ are compared (cf. Definition 18).

Definition 19 ($\mathbf{L}[\mathrm{QCL}]^{inc}_{\mid\sim}$) The calculus $\mathbf{L}[\mathrm{QCL}]^{inc}_{\mid\sim}$ is defined analogously to $\mathbf{L}[\mathrm{QCL}]^{lex}_{\mid\sim}$ (cf. Definition 18) except that the order $\leq_l$ is replaced by the order $\leq_i$:

- $\vec{v} <_i \vec{w}$ if there is some $n \in \mathbb{N}$ such that every entry in $\vec{w}$ with value n also has value n in $\vec{v}$, there is an entry in $\vec{v}$ with value n that has a value higher than n in $\vec{w}$, and for all $1 \leq m < n$ both vectors have exactly the same entries with value m.
- $\vec{v} =_i \vec{w}$ if $\vec{v} \not<_i \vec{w}$ and $\vec{w} \not<_i \vec{v}$.

Soundness and completeness of $\mathbf{L}[\mathrm{QCL}]^{inc}_{\mid\sim}$ are analogous to that of $\mathbf{L}[\mathrm{QCL}]^{lex}_{\mid\sim}$ (cf. Theorem 7).

5 Beyond QCL

QCL was the first choice logic to be described [2], and applications concerned with QCL and ordered disjunction have been discussed in the literature [3–7]. For this reason, the main focus in this paper lies with QCL. However, as we have seen in Section 2, CCL with its ordered conjunction and LCL with its lexicographic operator show that interesting logics similar to QCL exist. We will now demonstrate that the calculi for QCL introduced in the previous sections can easily be adapted for other choice logics.

To introduce a labeled calculus for some logic $\mathcal{L}$ belonging to the choice logic framework of [1] (cf. Sect. 2.1) it suffices to replace the $\vec{\times}$-rules in $\mathbf{L}[\mathrm{QCL}]$ by appropriate rules for the choice connectives of $\mathcal{L}$. The rules for the classical connectives in $\mathbf{L}[\mathrm{QCL}]$ can be retained. Moreover, note that the inference rules for preferred model entailment (i.e., the rules $\mid\sim_{mm}$, $\mid\sim_{lex}$, $\mid\sim_{inc}$, $\mid\sim_{unsat}$ from Definitions 17, 18, 19) do not depend on any specific choice logic. Thus, once labeled calculi are developed for $\mathcal{L}$, the calculi for preferred model entailment follow immediately.

5.1 Calculi for CCL

First, we introduce $\mathbf{L}[\mathrm{CCL}]$ by defining rules for the choice connective $\vec{\odot}$ of CCL. Recall that $A \vec{\odot} B$ expresses that, if possible, both A and B should be satisfied, but if this is not possible then satisfying only A is also acceptable.

Definition 20 $\mathbf{L}[\mathrm{CCL}]$ is $\mathbf{L}[\mathrm{QCL}]$, except that the $\vec{\times}$-rules are replaced by the following $\vec{\odot}$-rules:

$$\frac{\Gamma, (A)_1, (B)_k \vdash \Delta}{\Gamma, (A \vec{\odot} B)_k \vdash \Delta}\vec{\odot}l_1 \qquad \frac{\Gamma, (A)_1, (\neg B)_1 \vdash \Delta}{\Gamma, (A \vec{\odot} B)_{opt_{\mathrm{CCL}}(B)+1} \vdash \Delta}\vec{\odot}l_2$$

$$\frac{\Gamma, (A)_l \vdash \Delta}{\Gamma, (A \vec{\odot} B)_{opt_{\mathrm{CCL}}(B)+l} \vdash \Delta}\vec{\odot}l_3$$

$$\frac{\Gamma \vdash (A)_1, \Delta \quad \Gamma \vdash (B)_k, \Delta}{\Gamma \vdash (A \vec{\odot} B)_k, \Delta}\vec{\odot}r_1 \qquad \frac{\Gamma \vdash (A)_1, \Delta \quad \Gamma \vdash (\neg B)_1, \Delta}{\Gamma \vdash (A \vec{\odot} B)_{opt_{\mathrm{CCL}}(B)+1}, \Delta}\vec{\odot}r_2$$

$$\frac{\Gamma \vdash (A)_l, \Delta}{\Gamma \vdash (A \vec{\odot} B)_{opt_{\mathrm{CCL}}(B)+l}, \Delta}\vec{\odot}r_3$$

where $k \leq opt_{\mathrm{CCL}}(B)$ and $1 < l \leq opt_{\mathrm{CCL}}(A)$.[4]

The $\vec{\odot}l_1$-rule takes care of the case in which A is optimally satisfied, and B is satisfied to some degree. In $\vec{\odot}l_2$ and $\vec{\odot}l_3$ the label m of $(A\vec{\odot}B)_m$ is higher than the optionality of B. If $m = opt_{\mathrm{CCL}}(B) + 1$ we know that B cannot be satisfied, and hence we need to apply $\vec{\odot}l_2$. If $m = opt_{\mathrm{CCL}}(B) + l$ with $l > 1$ then, by the semantics of CCL (cf. Definition 7), it must be that A is satisfied to a degree of l, regardless of whether B is satisfied or not.

Example 11 The following is a small $\mathbf{L}$[CCL]-proof of a valid sequent, showcasing the application of the $\vec{\odot}l_2$- and $\vec{\odot}l_3$-rules.

$$\dfrac{\dfrac{\vdots \atop \Gamma, (a)_1, (\neg b)_1 \vdash a \wedge \neg b}{\Gamma, (a\vec{\odot}b)_2 \vdash a \wedge \neg b}\,\vec{\odot}l_2}{((a\vec{\odot}b)\vec{\odot}c)_3 \vdash a \wedge \neg b}\,\vec{\odot}l_3$$

Theorem 8 $\mathbf{L}$[CCL] *is sound and complete.*

Proof (*Soundness of* $\mathbf{L}$[CCL]) We consider the newly introduced rules.

- For $\vec{\odot}l_1$, $\vec{\odot}l_2$, and $\vec{\odot}l_3$ this follows directly from the definition of CCL.
- $(\vec{\odot}r_1)$. Assume both premises are valid, i.e., every model of Γ is a model of Δ or of $(A)_1$ and $(B)_k$ with $k \leq opt_{\mathrm{CCL}}(B)$. By definition, any model that satisfies $(A)_1$ and $(B)_k$ satisfies $A\vec{\odot}B$ to degree k. Thus, every model of Γ is a model of Δ or of $(A\vec{\odot}B)_k$, which means the conclusion of the rule is valid.
- $(\vec{\odot}r_2)$. Assume both premises are valid, i.e., every model of Γ is either a model of Δ or of $(A)_1$ and $(\neg B)_1$. By definition, any model that satisfies $(A)_1$ and does not satisfy B (and hence satisfies $(\neg B)_1$) satisfies $A\vec{\odot}B$ to degree $opt_{\mathrm{CCL}}(B) + 1$.
- $(\vec{\odot}r_3)$. Assume that the premise is valid, i.e., every model of Γ is either a model of Δ or of $(A)_l$ with $1 < l \leq opt_{\mathrm{CCL}}(A)$. By definition, any model that satisfies $(A)_l$, regardless of what degree this model ascribes to B, satisfies $A\vec{\odot}B$ to degree $opt_{\mathrm{CCL}}(B) + l$. □

Proof (*Completeness of* $\mathbf{L}$[CCL]) We adapt the completeness proof of $\mathbf{L}$[QCL] (cf. Theorem 1).

- Assume that a sequent of the form $\Gamma, (A\vec{\odot}B)_k \vdash \Delta$ is valid, with $k \leq opt_{\mathrm{CCL}}(B)$. All models that satisfy $(A\vec{\odot}B)_k$ must satisfy A to a degree of 1 and B to a degree of k. Thus, $\Gamma, (A)_1, (B)_k \vdash \Delta$ is valid. Similarly for the cases $\Gamma, (A\vec{\odot}B)_{opt_{\mathrm{CCL}}(B)+1} \vdash \Delta$ and $\Gamma, (A\vec{\odot}B)_{opt_{\mathrm{CCL}}(B)+l} \vdash \Delta$ with $1 < l \leq opt_{\mathrm{CCL}}(A)$.
- Assume that a sequent of the form $\Gamma \vdash (A\vec{\odot}B)_k, \Delta$ is valid, with $k \leq opt_{\mathrm{CCL}}(B)$. We claim that then $\Gamma \vdash (A)_1, \Delta$ and $\Gamma \vdash (B)_k, \Delta$ are valid. Assume, for the sake of a contradiction, that the first sequent is not valid. This means that there is a model M of Γ that is neither a model of $(A)_1$ nor of Δ. However, then M satisfies $A\vec{\odot}B$ to a degree higher than $opt_{\mathrm{CCL}}(B)$. This contradicts the assumption that $\Gamma \vdash (A\vec{\odot}B)_k, \Delta$ is valid. Assume now that the second sequent is not valid, i.e., that there is a model M of Γ that is neither a model of $(B)_k$ nor of Δ. Then M cannot be a model of $(A\vec{\odot}B)_k$, contradicting the assumption. Similarly for the cases $\Gamma \vdash (A\vec{\odot}B)_{opt_{\mathrm{CCL}}(B)+1}, \Delta$ and $\Gamma \vdash (A\vec{\odot}B)_{opt_{\mathrm{CCL}}(B)+l}, \Delta$ with $1 < l \leq opt_{\mathrm{CCL}}(A)$.

□

[4] Note that the rules $\vec{\odot}l_2$ and $\vec{\odot}r_2$ are different from those provided in a previous iteration of this paper [23], where A was erroneously assigned a label of $l \leq opt_{\mathrm{CCL}}(A)$ instead of 1.

We do not define the refutation calculus $\mathbf{L}[\text{CCL}]^-$ here, but the necessary rules for $\vec{\odot}$ can be inferred from the $\vec{\odot}$-rules of $\mathbf{L}[\text{CCL}]$ in a similar way to how $\mathbf{L}[\text{QCL}]^-$ was derived from $\mathbf{L}[\text{QCL}]$: if a rule contains only a single premise then it suffices to replace the $\vdash$-symbol with the $\nvdash$-symbol; if a rule contains two premises then we introduce two rules in $\mathbf{L}[\text{CCL}]^-$, one for each premise. Once $\mathbf{L}[\text{CCL}]$ and $\mathbf{L}[\text{CCL}]^-$ are established, calculi for preferred model entailment follow immediately.

5.2 Calculi for LCL

Our methods can also be adapted for LCL (cf. Sect. 2), in which $A\vec{\diamond}B$ expresses that it is best to satisfy A and B, second best to satisfy only A, third best to satisfy only B, and unacceptable to satisfy neither.

Definition 21 $\mathbf{L}[\text{LCL}]$ is $\mathbf{L}[\text{QCL}]$, except that the $\vec{\times}$-rules are replaced by the following $\vec{\diamond}$-rules:

$$\frac{\Gamma, (A)_k, (B)_l \vdash \Delta}{\Gamma, (A\vec{\diamond}B)_{(k-1)\cdot opt_{\text{LCL}}(B)+l} \vdash \Delta}\vec{\diamond}l_1$$

$$\frac{\Gamma, (A)_k, (\neg B)_1 \vdash \Delta}{\Gamma, (A\vec{\diamond}B)_{opt_{\text{LCL}}(A)\cdot opt_{\text{LCL}}(B)+k} \vdash \Delta}\vec{\diamond}l_2$$

$$\frac{\Gamma, (\neg A)_1, (B)_l \vdash \Delta}{\Gamma, (A\vec{\diamond}B)_{opt_{\text{LCL}}(A)\cdot opt_{\text{LCL}}(B)+opt_{\text{LCL}}(A)+l} \vdash \Delta}\vec{\diamond}l_3$$

$$\frac{\Gamma \vdash (A)_k, \Delta \quad \Gamma \vdash (B)_l, \Delta}{\Gamma \vdash (A\vec{\diamond}B)_{(k-1)\cdot opt_{\text{LCL}}(B)+l}, \Delta}\vec{\diamond}r_1$$

$$\frac{\Gamma \vdash (A)_k, \Delta \quad \Gamma \vdash (\neg B)_1, \Delta}{\Gamma \vdash (A\vec{\diamond}B)_{opt_{\text{LCL}}(A)\cdot opt_{\text{LCL}}(B)+k}, \Delta}\vec{\diamond}r_2$$

$$\frac{\Gamma \vdash (\neg A)_1, \Delta \quad \Gamma \vdash (B)_l, \Delta}{\Gamma \vdash (A\vec{\diamond}B)_{opt_{\text{LCL}}(A)\cdot opt_{\text{LCL}}(B)+opt_{\text{LCL}}(A)+l}, \Delta}\vec{\diamond}r_3$$

where $k \leq opt_{\text{LCL}}(A)$ and $l \leq opt_{\text{LCL}}(B)$.

The labels used in the above rules might appear quite involved. However, finding the correct rule to apply given a labeled LCL-formula $(A\vec{\diamond}B)_m$ is actually a straightforward task: the values for $opt_{\text{LCL}}(A)$ and $opt_{\text{LCL}}(B)$ can be computed according to Definition 8. If $m \leq opt_{\text{LCL}}(A) \cdot opt_{\text{LCL}}(B)$ then the $\vec{\diamond}l_1$-rule must be applied. If $opt_{\text{LCL}}(A) \cdot opt_{\text{LCL}}(B) < m \leq opt_{\text{LCL}}(A) \cdot opt_{\text{LCL}}(B) + opt_{\text{LCL}}(A)$ then the $\vec{\diamond}l_2$-rule must be applied. If $opt_{\text{LCL}}(A) \cdot opt_{\text{LCL}}(B) + opt_{\text{LCL}}(A) < m \leq opt_{\text{LCL}}(A) \cdot opt_{\text{LCL}}(B) + opt_{\text{LCL}}(A) + opt_{\text{LCL}}(B)$ then the $\vec{\diamond}l_3$-rule must be applied.

Example 12 The following is a small $\mathbf{L}[\text{LCL}]$-proof of a valid sequent. Since we have a label of 2 in the end-sequent, and since $opt_{\text{LCL}}(a \vee b) = opt_{\text{LCL}}(b \vee c) = 1$, we know that the $\vec{\diamond}l_2$-rule must be applied.

$$\cfrac{\cfrac{\vdots \\ a \vee b \vdash b \vee c, a \wedge \neg b}{a \vee b, \neg(b \vee c) \vdash a \wedge \neg b}\neg l}{((a \vee b)\vec{\diamond}(b \vee c))_2 \vdash a \wedge \neg b}\vec{\diamond}l_2$$

Theorem 9 $\mathbf{L}[\text{LCL}]$ *is sound and complete.*

Proof (*Soundness of* **L**[LCL]) We consider the newly introduced rules.

- For $\vec{\diamond}l_1$, $\vec{\diamond}l_2$, and $\vec{\diamond}l_3$ this follows directly from the definition of LCL.
- $(\vec{\diamond}r_1)$. Assume both premises are valid, i.e., every model of Γ is a model of Δ or of $(A)_k$ and $(B)_l$ with $k \leq opt_{\mathrm{LCL}}(A)$ and $l \leq opt_{\mathrm{LCL}}(B)$. By definition, any model that satisfies $(A)_k$ and $(B)_l$ satisfies $A\vec{\diamond}B$ to degree $(k-1)\cdot opt_{\mathrm{LCL}}(B)+l$. Thus, every model of Γ is a model of Δ or of $(A\vec{\diamond}B)_{(k-1)\cdot opt_{\mathrm{LCL}}(B)+l}$, which means the conclusion of the rule is valid.
- $(\vec{\diamond}r_2)$. Assume both premises are valid, i.e., every model of Γ is either a model of Δ or of $(A)_k$ and $(\neg B)_1$ with $k \leq opt_{\mathrm{LCL}}(A)$. By definition, any model that satisfies $(A)_k$ and does not satisfy B (and hence satisfies $(\neg B)_1$) satisfies $A\vec{\diamond}B$ to degree $opt_{\mathrm{LCL}}(A)\cdot opt_{\mathrm{LCL}}(B)+k$.
- $(\vec{\diamond}r_3)$. Analogous to $(\vec{\diamond}r_2)$. □

Proof (*Completeness of* **L**[LCL]) We adapt the completeness proof of **L**[QCL] (cf. Theorem 1).

- Assume that a sequent of the form $\Gamma, (A\vec{\diamond}B)_m \vdash \Delta$ is valid, with $m = (k-1)\cdot opt_{\mathrm{LCL}}(B)+l$ such that $k \leq opt_{\mathrm{LCL}}(A)$ and $l \leq opt_{\mathrm{LCL}}(B)$. Now assume some model satisfies Γ, $(A)_k$, and $(B)_l$. Then M satisfies Γ and $(A\vec{\diamond}B)_m$, and, since $\Gamma, (A\vec{\diamond}B)_m \vdash \Delta$ is valid, M also satisfies Δ. Thus, $\Gamma, (A)_k, (B)_l \vdash \Delta$ is valid.
  The proofs for sequents of the form $\Gamma, (A\vec{\diamond}B)_{opt_{\mathrm{LCL}}(A)\cdot opt_{\mathrm{LCL}}(B)+k} \vdash \Delta$ and $\Gamma, (A\vec{\diamond}B)_{opt_{\mathrm{LCL}}(A)\cdot opt_{\mathrm{LCL}}(B)+opt_{\mathrm{LCL}}(A)+l} \vdash \Delta$ are analogous.
- Assume that a sequent of the form $\Gamma \vdash (A\vec{\diamond}B)_m, \Delta$ is valid, with $m = (k-1)\cdot opt_{\mathrm{LCL}}(B)+l$ such that $k \leq opt_{\mathrm{LCL}}(A)$ and $l \leq opt_{\mathrm{LCL}}(B)$. We claim that then $\Gamma \vdash (A)_k, \Delta$ and $\Gamma \vdash (B)_l, \Delta$ are valid. Assume, for the sake of a contradiction, that the first sequent is not valid. This means that there is a model M of Γ that is neither a model of $(A)_k$ nor of Δ. Following Definition 8, M must satisfy $A\vec{\diamond}B$ to some degree other than m. This contradicts the assumption that $\Gamma \vdash (A\vec{\diamond}B)_m, \Delta$ is valid. Assume now that the second sequent is not valid, i.e., that there is a model M of Γ that is neither a model of $(B)_l$ nor of Δ. Again, this means that M satisfies $A\vec{\diamond}B$ to some degree other than m, and this would contradict our assumption that $\Gamma \vdash (A\vec{\diamond}B)_m, \Delta$ is valid. Thus, both $\Gamma \vdash (A)_k, \Delta$ and $\Gamma \vdash (B)_k, \Delta$ are valid and $\Gamma \vdash (A\vec{\diamond}B)_m, \Delta$ is provable.
  The proofs for sequents of the form $\Gamma \vdash (A\vec{\diamond}B)_{opt_{\mathrm{LCL}}(A)\cdot opt_{\mathrm{LCL}}(B)+k}, \Delta$ and $\Gamma \vdash (A\vec{\diamond}B)_{opt_{\mathrm{LCL}}(A)\cdot opt_{\mathrm{LCL}}(B)+opt_{\mathrm{LCL}}(A)+l}, \Delta$ are analogous. □

Just as with CCL, the refutation calculus **L**[LCL]$^-$ can be obtained from **L**[LCL] by modifying the $\vec{\diamond}$-rules accordingly. Calculi for preferred model entailment then follow immediately.

5.3 Multiple Choice Connectives

Lastly, we want to point out that, according to the choice logic framework (cf. Sect. 2.1), choice logics can make use of more than one choice connective. Indeed, a combination of QCL and CCL into the so-called QCCL has been suggested previously [1]. QCCL is simply the choice logic containing the choice connectives $\mathcal{C}_{\mathrm{QCCL}} = \{\vec{\times}, \vec{\odot}\}$, with the optionality and satisfaction degree of $\vec{\times}$ (resp. $\vec{\odot}$) defined in the same way as in QCL (resp. CCL). A calculus for QCCL can be obtained simply by adding both the rules for $\vec{\times}$ and $\vec{\odot}$. We demonstrate this with a small example.

Example 13 The following is a proof of a valid sequent in QCCL. Note that we use lexicographic entailment, but one could also use the minmax or inclusion-based approaches instead. Since the formulas $(a\vec{\odot}b)$ and $(b\vec{\times}c)$ are jointly satisfiable to a degree of 1 we can simply guess the optimal degree-profile $(a\vec{\odot}b)_1, (b\vec{\times}c)_1$. Thus, we only have two branches in the $\mathrel{|\!\!\sim}_{lex}$-rule.

$$\dfrac{\dfrac{\dfrac{a, b, b \nvdash \bot}{a, b, (b\vec{\times}c)_1 \nvdash \bot}\;{\nvdash}\,\vec{\times}l_1}{(a\vec{\odot}b)_1, (b\vec{\times}c)_1 \nvdash \bot}\;{\nvdash}\,\vec{\odot}l_1 \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ a, b, b \vdash a \wedge b\end{array}}{a, b, (b\vec{\times}c)_1 \vdash a \wedge b}\;\vec{\times}l_1}{(a\vec{\odot}b)_1, (b\vec{\times}c)_1 \vdash a \wedge b}\;\vec{\odot}l_1}{(a\vec{\odot}b), (b\vec{\times}c) \mathrel{|\!\!\sim}^{lex}_{\mathrm{QCCL}} a \wedge b}\;\mathrel{|\!\!\sim}_{lex}$$

6 Conclusion

In this paper we introduce a sound and complete sequent calculus for preferred model entailment in QCL. This non-monotonic calculus is built on two calculi: a monotonic labeled sequent calculus and a corresponding refutation calculus.

Our systems are modular and can easily be adapted: on the one hand, calculi for choice logics other than QCL can be obtained by introducing suitable rules for the choice connectives of the new logic, as exemplified with our calculi for CCL and LCL; on the other hand, non-monotonic calculi for alternative preferred model semantics can be obtained by adapting the inference rule which transitions from preferred model entailment to the labeled calculi (e.g., the $\mathrel{|\!\!\sim}_{mm}$, $\mathrel{|\!\!\sim}_{lex}$ or $\mathrel{|\!\!\sim}_{inc}$-rule).

Our work contributes to the line of research on non-monotonic sequent calculi that make use of refutation systems [9, 16]. Our system is the first proof calculus for choice logics, which have been studied mainly from the viewpoint of their computational properties [1] and potential applications [3–7] so far.

Regarding future work, we aim to investigate the proof complexity of our calculi, and how this complexity might depend on which choice logic or preferred model semantics is considered.

Another interesting avenue for future work is to examine alternative semantics for languages using ordered disjunction or other choice connectives, and see whether our methods can be adapted to those approaches. We now give a brief overview over relevant work in this direction. In Prioritized QCL (PQCL) and QCL+ [28] ordered disjunction is defined in the same way as in QCL, but the classical connectives are given new semantics. As pointed out in previous work [1], both PQCL and QCL+ can be captured by the choice logic framework as fragments by allowing negations only in front of atoms. Another interesting paper is that of Maly and Woltran [29], in which the concept of satisfaction degrees is abandoned and the semantics rather 'directly' induces a partial order over models. The most recent reinterpretation of QCL that we are aware of is an approach [30, 31] using game theoretic semantics, with a special focus on providing an alternative negation for the language of QCL. A logic similar to LCL was proposed by Charalambidis et al. [32]. In contrast to LCL, their lexicographic logic uses lists of truth values to rank interpretations rather than satisfaction degrees. In the world of logic programming, recent works [33, 34] have suggested a new semantics for logic programs with ordered disjunction (LPODs) [3]. While the original semantics of LPODs uses satisfaction degrees as in QCL, the new approach uses a four-valued logic. Developing a calculus for LPODs might prove to be interesting since they contain two sources of non-monotonicity (logic programming itself as well as ordered disjunction).

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