Journal of Automated Reasoning (2022) 66:639–666
https://doi.org/10.1007/s10817-022-09630-6

Local is Best: Efficient Reductions to Modal Logic K

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Received: 2 October 2021 / Accepted: 7 April 2022 / Published online: 23 May 2022

Abstract
We present novel reductions of extensions of the basic modal logic K with axioms B, D, T, 4 and 5 to Separated Normal Form with Sets of Modal Levels SNF_{sml}. The reductions typically result in smaller formulae than the reductions by Kracht. The reductions to SNF_{sml} combined with a reduction to SNF_{ml} allow us to use the local reasoning of the prover $\mathrm{K_SP}$ to determine the satisfiability of modal formulae in the considered logics. We show experimentally that the combination of our reductions with the prover $\mathrm{K_SP}$ performs well when compared with a specialised resolution calculus for these logics, the built-in reductions of the first-order prover SPASS, and the higher-order logic prover LEO-III.

Keywords Modal logics · Theorem proving · Resolution method

1 Introduction

The main motivation for reducing problems in one logic (the source logic) to 'equivalent' problems in another formalism is to exploit results and tools for that formalism to solve theoretical or practical problems in the source logic. For propositional modal logics, this approach has been researched extensively for reductions of the satisfiability problem in

C. Dixon was partially supported by the EPSRC funded RAI Hubs FAIR-SPACE (EP/R026092/1) and RAIN (EP/R026084/1), and the EPSRC funded programme Grant S4 (EP/N007565/1).

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these logics to the satisfiability problem in 'stronger' logics such as first-order logic [14, 26], second-order theory of n successors [7], simple type theory [4], and regular grammar logics [25].

An alternative approach is to reduce propositional modal logics to a 'weaker' logic, in particular, the basic modal logic K. For extensions of K with one of the axioms B, D, alt_1, T, and $\mathsf{4}$, Kracht [17] defines reduction functions of their global and local satisfiability problem to the corresponding problem in K and proves their correctness. He also defines a reduction function for $\mathsf{K5}$, the extension of K with $\mathsf{5}$, to $\mathsf{K4}$, but this reduction is incorrect as not all theorems of $\mathsf{K4}$ are theorems of $\mathsf{K5}$. Several features of Kracht's approach are relevant to our work. First, as is not uncommon in modal logic, he uses $\Box$ as the only modal operator occurring in modal formulae and the modal operator $\Diamond$ is expressed as $\neg\Box\neg$. This negatively impacts the size of the resulting formulae as the reduction functions cannot treat the modal operators $\Box$ and $\Diamond$ differently (as we do here). Second, the basic idea underlying his reduction functions is the following: given a modal formula φ, generate sufficiently many instances Δ of a modal axiom Λ so that φ is $\mathsf{K}\Lambda$-satisfiable iff $\varphi \wedge \Delta$ is K-satisfiable. Third, Kracht is concerned with preservation of the computational complexity of the satisfiability problem under consideration, as well as the preservation of other theoretical properties. For instance, the local satisfiability problem in the modal logics covered by Kracht is PSPACE-complete. So, it is sufficient to ensure that Δ is polynomial in size with respect to φ. As Kracht himself concludes, his method offers a uniform way of transferring results about one modal logic to another, but may not be as useful for practical applications.

In [21, 23], we have introduced a new normal form for basic multi-modal logic, called Separated Normal Form with Modal Levels, SNF_{ml}, which uses labeled modal clauses. These labels refer to the level within a tree Kripke structure at which a modal clause holds. This can be seen as a compromise between approaches that label formulae with worlds at unspecified level [1, 3] and approaches that label formulae with paths [6, 30]. A combination of a normal form transformation for modal formulae and a resolution-based calculus for labeled modal clauses can then be used to decide local and global satisfiability in basic modal logic. In [22, 24], we have presented $\mathrm{K_SP}$, an implementation of that calculus, together with an experimental evaluation that indicates that $\mathrm{K_SP}$ performs well if propositional variables are evenly spread across a wide range of modal levels within the formulae one wants to decide.

A feature of SNF_{ml} is its use of additional propositional symbols as 'surrogates' for subformulae of a modal formula φ. In the following, we take advantage of the availability of those surrogates to provide a novel transformation from extensions of K with a single one of the axioms B, D, T, $\mathsf{4}$ and $\mathsf{5}$ to SNF_{ml}. Another novel aspect is that we modify the normal form so that it uses sets of modal levels as labels instead of a single modal level. In K, we only need a definition of a surrogate at the modal level at which the corresponding subformula occurs in φ. But in KB, KT, $\mathsf{K4}$ and $\mathsf{K5}$, we need a definition at every reachable modal level, of which there can be many. We call the resulting normal form, *Separated Normal Form with Sets of Modal Levels*, SNF_{sml}.

The structure of the paper is as follows. In Sect. 2, we recap common concepts of propositional modal logic including its syntax and semantics. Section 3 defines SNF_{sml}. Section 4 defines the reductions of K, KB, KD, KT, $\mathsf{K4}$ and $\mathsf{K5}$ to SNF_{sml}; correctness results are given in Sect. 5. Sect. 6 shows a reduction from SNF_{sml} to SNF_{ml} and its correctness; the reduction is needed to evaluate our result via $\mathrm{K_SP}$. Related work is discussed in Sect. 7. In Sect. 8, we compare the performance of a combination of our reductions and the modal-layered resolution calculus implemented in $\mathrm{K_SP}$ with resolution calculi specifically designed for the logics under consideration and with translation-based approaches built into the first-order theorem prover SPASS and the higher-order logic prover LEO-III. Section 9 provides concluding remarks and future work.

This paper is an extended and revised version of [29]. We provide correctness proofs for our reductions for K4 and K5 that were not included in [29]. Section 6 is new and not only defines a satisfiability preserving transformation from SNF_{sml} (with infinite sets of labels) to SNF_{ml}, but also proves its correctness via a simulation of Massacci's Single Step Tableaux (SST) calculus [18] for K4 and K5 using the modal-layered resolution calculus for SNF_{ml} clauses [21]. For K5, we also establish a bound on the length of prefixes in the SST calculus that preserves refutation completeness of the calculus without need for a loop check.

2 Preliminaries

The language of modal logic is an extension of the language of propositional logic with unary modal operators $\Box$ and $\Diamond$. More precisely, given a denumerable set of *propositional symbols*, $P = \{p, p_0, q, q_0, t, t_0, \ldots\}$ as well as propositional *constants* **true** and **false**, *modal formulae* are inductively defined as follows: constants and propositional symbols are modal formulae. If φ and ψ are modal formulae, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $\Box\varphi$, and $\Diamond\varphi$. We also assume that $\wedge$ and $\vee$ are associative and commutative operators and consider, e.g., $(p \vee (q \vee r))$ and $(r \vee (q \vee p))$ to be identical formulae. We often omit parentheses if this does not cause confusion. By $\mathsf{var}(\varphi)$, we denote the set of all propositional symbols occurring in φ. This function straightforwardly extends to finite sets of modal formulae. A *modal axiom (schema)* is a modal formula ψ representing the set of all instances of ψ.

A *literal* is either a propositional symbol or its negation; the set of literals is denoted by L_P. We denote by $\neg l$ the *complement* of the literal $l \in L_P$, that is, $\neg l$ denotes $\neg p$ if l is $p \in P$, and $\neg l$ denotes p if l is the literal $\neg p$. A *modal literal* is either $\Box l$ or $\Diamond l$, where $l \in L_P$.

A *(normal) modal logic* is a set of modal formulae which includes all propositional tautologies, the axiom schema $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, called the *axiom* K, is closed under modus ponens (if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$ then $\vdash \psi$) and the rule of necessitation (if $\vdash \varphi$ then $\vdash \Box\varphi$). K is the weakest modal logic, that is, the logic given by the smallest set of modal formulae constituting a normal modal logic. By $\mathsf{K}\Sigma$, we denote an *extensions* of K by a set Σ of axioms.

The standard semantics of modal logics is the *Kripke semantics* or *possible world semantics*. A *Kripke frame* F is an ordered pair $\langle W, R\rangle$ where W is a non-empty set of *worlds* and R is a binary (accessibility) relation over W. A *Kripke structure* M over P is an ordered pair $\langle F, V\rangle$ where F is a Kripke frame and the *valuation* V is a function mapping each propositional symbol in P to a subset $V(p)$ of W. We say $M = \langle F, V\rangle$ is *based on the frame* F. A *rooted Kripke structure* is an ordered pair $\langle M, w_0\rangle$ with $w_0 \in W$. In the following, we write $\langle W, R, V\rangle$ and $\langle W, R, V, w_0\rangle$ instead of $\langle\langle W, R\rangle, V\rangle$ and $\langle\langle\langle W, R\rangle, V\rangle, w_0\rangle$, respectively.

Satisfaction (or truth) of a formula at a world w of a Kripke structure $M = \langle W, R, V\rangle$ is inductively defined by:

Table 1 Modal axioms and relational frame properties

Name	Axiom	Frame property	
D	$\Box\varphi \rightarrow \Diamond\varphi$	Serial	$\forall v \exists w. vRw$
T	$\Box\varphi \rightarrow \varphi$	Reflexive	$\forall w. wRw$
B	$\varphi \rightarrow \Box\Diamond\varphi$	Symmetric	$\forall vw. vRw \rightarrow wRv$
4	$\Box\varphi \rightarrow \Box\Box\varphi$	Transitive	$\forall uvw. (uRv \wedge vRw) \rightarrow uRw$
5	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$	Euclidean	$\forall uvw. (uRv \wedge uRw) \rightarrow vRw$

$$
\begin{array}{ll}
\langle M, w\rangle \models \textbf{true}; & \langle M, w\rangle \not\models \textbf{false}; \\
\langle M, w\rangle \models p & \text{iff } w \in V(p), \text{ where } p \in P; \\
\langle M, w\rangle \models \neg\varphi & \text{iff } \langle M, w\rangle \not\models \varphi; \\
\langle M, w\rangle \models (\varphi \wedge \psi) & \text{iff } \langle M, w\rangle \models \varphi \text{ and } \langle M, w\rangle \models \psi; \\
\langle M, w\rangle \models (\varphi \vee \psi) & \text{iff } \langle M, w\rangle \models \varphi \text{ or } \langle M, w\rangle \models \psi; \\
\langle M, w\rangle \models (\varphi \rightarrow \psi) & \text{iff } \langle M, w\rangle \models \neg\varphi \text{ or } \langle M, w\rangle \models \psi; \\
\langle M, w\rangle \models \Box\varphi & \text{iff for every } v,\ wRv \text{ implies } \langle M, v\rangle \models \varphi; \\
\langle M, w\rangle \models \Diamond\varphi & \text{iff there is } v,\ wRv \text{ and } \langle M, v\rangle \models \varphi.
\end{array}
$$

If $\langle M, w\rangle \models \varphi$ holds then M is a *model* of φ, φ is *true at* w *in* M and M *satisfies* φ. A modal formula φ is *satisfiable* iff there exists a Kripke structure M and a world w in M such that $\langle M, w\rangle \models \varphi$.

In the following, we are interested in extensions of K with one of the axiom schemata shown in Table 1. Each of these axiom schemata defines a class of Kripke frames where the accessibility relation R satisfies the first-order property stated in the table.

Given a normal modal logic L with corresponding class of frames $\mathfrak{F}$, we say a modal formula φ is *L-satisfiable* iff there exists a frame $F \in \mathfrak{F}$, a valuation V and a world $w_0 \in F$ such that $\langle F, V, w_0\rangle \models \varphi$.

A *path rooted at* w *of length* k, $k \geq 0$, in a frame $F = \langle W, R\rangle$ is a sequence $\vec{w} = (w_0, w_1, \ldots, w_k)$ where for every i, $1 \leq i \leq k$, $w_{i-1} R w_i$. We say that the path $(w_0, w_1, \ldots, w_k)$ *connects* w_0 *and* w_k. For a path $\vec{w} = (w_0, \ldots, w_k)$ and world w_{k+1} with $w_k R w_{k+1}$, $\vec{w} \circ w_{k+1}$ denotes the path $(w_0, \ldots, w_k, w_{k+1})$. A path (w_0) of length 0 is identified with its root w_0. We denote the set of all paths rooted at a world w_0 in F by $\vec{F}[w_0]$ and the set of all paths by $\vec{F}$. The function $\mathsf{trm} : \vec{F} \rightarrow W$ maps every path $\vec{w} = (w_0, \ldots, w_k)$ to its terminal world w_k while the function $\mathsf{len} : \vec{F} \rightarrow \mathbb{N}$ maps every path $\vec{w} = (w_0, w_1, \ldots, w_k)$ to its length k.

A rooted Kripke structure $M = \langle W, R, V, w_0\rangle$ is a *rooted tree Kripke structure* iff R is a tree, that is, a directed acyclic connected graph where each node has at most one predecessor, with *root* w_0. It is a *rooted tree Kripke model* of a modal formula φ iff $\langle W, R, V, w_0\rangle \models \varphi$. In a rooted tree Kripke structure with root w_0 for every world $w_k \in W$ there is exactly one path $\vec{w}$ connecting w_0 and w_k; the *modal level of* w_k (*in* M), *denoted by* $\mathsf{ml}_M(w_k)$, is given by $\mathsf{len}(\vec{w})$.

Let $F = \langle W, R\rangle$ be a Kripke frame with $w \in W$. The *unraveling* $F^u[w]$ *of* F *at* w is the frame $\langle \vec{W}, \vec{R}\rangle$ where:

- $\vec{W} = \vec{F}[w]$ is the set of all paths rooted at w in F;
- for all $\vec{v}, \vec{w} \in \vec{W}$, if $\vec{w} = \vec{v} \circ w$ for some $w \in W$, then $\vec{v} \vec{R} \vec{w}$.

Let $F = \langle W, R\rangle$ and $F' = \langle W', R'\rangle$ be two Kripke frames. A function $f : W \mapsto W'$ is a *p-morphism* (or a *bounded morphism*) from F to F' if the following holds:

Table 2 Rewriting rules for simplification

$\varphi \wedge \varphi \Rightarrow \varphi$	$\varphi \wedge \neg\varphi \Rightarrow \mathbf{false}$	$\Box\mathbf{true} \Rightarrow \mathbf{true}$	$\neg\mathbf{true} \Rightarrow \mathbf{false}$	$\neg\neg\varphi \Rightarrow \varphi$
$\varphi \vee \varphi \Rightarrow \varphi$	$\varphi \vee \neg\varphi \Rightarrow \mathbf{true}$	$\Diamond\mathbf{false} \Rightarrow \mathbf{false}$	$\neg\mathbf{false} \Rightarrow \mathbf{true}$	
$\varphi \wedge \mathbf{true} \Rightarrow \varphi$	$\varphi \wedge \mathbf{false} \Rightarrow \mathbf{false}$	$\varphi \vee \mathbf{false} \Rightarrow \varphi$	$\varphi \vee \mathbf{true} \Rightarrow \mathbf{true}$	

- if $v R w$, then $f(v) R' f(w)$.
- if $f(u) R' w$, then there exists $v \in W$ such that $f(v) = w$ and $u R v$.

Analogously for Kripke models. For $F = \langle W, R \rangle$, $M = \langle F, V, w_0 \rangle$, and $M' = \langle F^u[w_0], V', (w_0) \rangle$ the function trm is a p-morphism from M' to M.

When considering satisfiability, the following holds (see, [12]):

Theorem 1 *Let φ be a modal formula. Then φ is* K*-satisfiable iff there is a finite rooted tree Kripke structure $M = \langle F, V, w_0 \rangle$ such that $\langle M, w_0 \rangle \models \varphi$.*

For the normal form transformation presented in the next section we assume that any modal formula φ has been simplified by exhaustively applying the rewrite rules in Table 2 and is in Negation Normal Form (NNF), that is, a formula where only propositional symbols are allowed in the scope of negations. We say that such a formula is in *simplified NNF*.

3 Layered Normal Form with Sets of Levels SNF_{sml}

In [21], we have introduced a novel clausal normal form, called *Separated Normal Form with Modal Levels*, SNF_{ml}, whose language extends that of the basic modal logic K with labels for modal levels. Clauses in SNF_{ml} have one of the following forms:

$$ml : \bigvee_{b=1}^{r} l_b \qquad ml : l' \rightarrow \Box l \qquad ml : l' \rightarrow \Diamond l$$

where $ml \in \mathbb{N} \cup \{\star\}$ and l, l', l_b are propositional literals with $1 \leq b \leq r$, $r \in \mathbb{N}$. Clauses $\star : \psi$ are *global clauses*.

Given a rooted tree Kripke structure M, the satisfiability of an SNF_{ml} clause is defined as follows:

- $M \models \star : \varphi$
  iff $\langle M, w \rangle \models \varphi$ for every $w \in M$;
- $M \models ml : \varphi$ iff $\langle M, w \rangle \models \varphi$ for every world w with $\mathsf{ml}_M(w) = ml$.

The label $\star$ only occurs in the clausal normal form of a modal formula φ if we consider the problem whether there exists a Kripke structure M such that φ is true at all worlds of M. For satisfiability of φ all labels will be from a finite subset of $\mathbb{N}$. In this case the labels can be seen as a compromise between a normal form where even for local satisfiability all clauses are global and a normal form that uses paths to constrain a clause to just a subset of all worlds at a particular modal level.

A feature of our reductions is that the same formula $\bigvee_{b=1}^{r} l_b$, $l' \rightarrow \Box l$, or $l' \rightarrow \Diamond l$ may have to hold at several levels, possibly even an infinite number of levels. It therefore makes sense to label such formulae not with just a single level but a set of levels. We call this normal form *Separated Normal Form with Sets of Modal Levels*, SNF_{sml}. Informally, the labels in SNF_{sml} state that a formula is satisfied at all worlds in a given set of modal levels, instead of a single modal level as in SNF_{ml}. We write $S : \varphi$, where S is a set of natural numbers, to denote that a formula φ is true at all modal levels $ml \in S$. We write $\star : \varphi$ instead of $\mathbb{N} : \varphi$.

Formally, given a rooted tree Kripke structures $M = \langle W, R, V, w_0 \rangle$ and a set of modal levels S, by $M[S]$, we denote the set of worlds that are at a modal level in S, that is, $M[S] = \{w \in W \mid \mathsf{ml}_M(w) \in S\}$. The satisfaction of labeled formulae in M is then defined as follows:

$$M \models S : \varphi \text{ iff for every world } w \in M[S], \text{ we have } \langle M, w \rangle \models \varphi.$$

If $M \models S : \varphi$, then we say that $S : \varphi$ holds in M or is true in M. For a set Φ of labeled formulae, $M \models \Phi$ iff $M \models S : \varphi$ for every $S : \varphi$ in Φ, and we say Φ is K-*satisfiable*.

Note that if $S = \emptyset$, then $M \models S : \varphi$ trivially holds. Also, S : **false** with $0 \notin S$, is not in itself unsatisfiable, a Kripke structure M can satisfy S : **false** if it has no worlds w with $\mathsf{ml}_M(w) \in S$. On the other hand, S : **false** with $0 \in S$ is unsatisfiable as a rooted tree Kripke structure always has a world with modal level 0.

A labeled modal formula is then an SNF_{sml} clause iff it is of one of the following forms:

- Literal clause $\quad S : \bigvee_{b=1}^{r} l_b$
- Positive modal clause $\quad S : l' \to \Box l$
- Negative modal clause $\quad S : l' \to \Diamond l$

where $S \subseteq \mathbb{N}$ and l, l', l_b are propositional literals with $1 \leq b \leq r$, $r \in \mathbb{N}$. Positive and negative modal clauses are together known as *modal clauses*. We regard a literal clause as a set of literals, that is, two clauses are the same if they contain the same set of literals.

4 Reductions of Extensions of K with a Single Axiom to SNF_{sml}

In the following, we assume that the set P of propositional symbols is partitioned into two infinite sets Q and T such that Q contains the original propositional symbols and T surrogate symbols t_ψ and supplementary propositional symbols. In particular, for every modal formula ψ, we have $\mathsf{var}(\psi) \subset Q$ and there exists a propositional symbol $t_\psi \in T$ uniquely associated with ψ.

We introduce some notation that will be used in the following. Let $S^+ = \{l + 1 \in \mathbb{N} \mid l \in S\}$, $S^- = \{l - 1 \in \mathbb{N} \mid l \in S\}$, and $S^{\geq} = \{n \mid n \geq \mathsf{min}(S)\}$, where $\mathsf{min}(S)$ is the least element in S. Note that the restriction of the elements being in $\mathbb{N}$ implies that S^- cannot contain negative numbers.

Given a modal formula φ in simplified NNF and $L \in \{\mathsf{K}, \mathsf{KB}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}, \mathsf{K5}\}$, then we can obtain a set Φ_L of clauses in SNF_{sml} such that φ is L-satisfiable iff Φ_L is K-satisfiable as $\Phi_L = \{\{0\} : t_\varphi\} \cup \rho_L(\{0\} : t_\varphi \to \varphi)$, where ρ_L is defined as follows:

$$\begin{aligned}
\rho_L(S : t \to \mathbf{true}) &= \emptyset \\
\rho_L(S : t \to \mathbf{false}) &= \{S : \neg t\} \\
\rho_L(S : t \to (\psi_1 \land \psi_2)) &= \{S : \neg t \lor \eta(\psi_1), S : \neg t \lor \eta(\psi_2)\} \cup \delta_L(S, \psi_1) \cup \delta_L(S, \psi_2) \\
\rho_L(S : t \to \psi) &= \{S : \neg t \lor \psi\} \\
&\quad\ \text{if } \psi \text{ is a disjunction of literals} \\
\rho_L(S : t \to (\psi_1 \lor \psi_2)) &= \{S : \neg t \lor \eta(\psi_1) \lor \eta(\psi_2)\} \cup \delta_L(S, \psi_1) \cup \delta_L(S, \psi_2) \\
&\quad\ \text{if } (\psi_1 \lor \psi_2) \text{ is not a disjunction of literals} \\
\rho_L(S : t \to \Diamond\psi) &= \{S : t \to \Diamond\eta(\psi)\} \cup \delta_L(S^+, \psi) \\
\rho_L(S : t \to \Box\psi) &= P_L(S : t \to \Box\psi) \cup \Delta_L(S : t \to \Box\psi)
\end{aligned}$$

Table 3 Transformation of $\Box$-formulae in modal logic $L \in \{\mathsf{K}, \mathsf{KT}, \mathsf{KD}, \mathsf{KB}, \mathsf{K4}, \mathsf{K5}\}$

L	Axiom	$P_L(S : t_{\Box\psi} \to \Box\psi)$	$\Delta_L(S : t_{\Box\psi} \to \Box\psi)$
K		$S : t_{\Box\psi} \to \Box\eta(\psi)$	$\delta_L(S^+, \psi)$
KT	$\Box\psi \to \psi$	$S : t_{\Box\psi} \to \Box\eta(\psi), S : \neg t_{\Box\psi} \vee \eta(\psi)$	$\delta_L(S \cup S^+, \psi)$
KD	$\Box\psi \to \Diamond\psi$	$S : t_{\Box\psi} \to \Box\eta(\psi), S : t_{\Box\psi} \to \Diamond\eta(\psi)$	$\delta_L(S^+, \psi)$
KB	$\psi \to \Box\Diamond\psi$	$S : t_{\Box\psi} \to \Box\eta(\psi), S^- : \eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}$, $S^- : t_{\Box\neg t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$	$\delta_L(S^- \cup S^+, \psi)$
K4	$\Box\psi \to \Box\Box\psi$	$S^{\geq} : t_{\Box\psi} \to \Box\eta(\psi), S^{\geq} : t_{\Box\psi} \to \Box t_{\Box\psi}$	$\delta_L((S^+)^{\geq}, \psi)$
K5	$\Diamond\varphi \to \Box\Diamond\varphi$	$\star : t_{\Box\psi} \to \Box\eta(\psi)$, $\star : \neg t_{\Diamond t_{\Box\psi}} \vee t_{\Box\psi}, \star : t_{\Diamond t_{\Box\psi}} \to \Diamond t_{\Box\psi}$, $\star : \neg t_{\Diamond t_{\Box\psi}} \to \Box\neg t_{\Box\psi}, \star : t_{\Diamond t_{\Box\psi}} \to \Box t_{\Diamond t_{\Box\psi}}$	$\delta_L(\star, \psi)$

where η and δ_L are defined as follows:

$$\eta(\psi) = \begin{cases} \psi, & \text{if } \psi \text{ is a literal} \\ t_\psi, & \text{otherwise} \end{cases} \qquad \delta_L(S, \psi) = \begin{cases} \emptyset, & \text{if } \psi \text{ is a literal} \\ \rho_L(S : t_\psi \to \psi), & \text{otherwise} \end{cases}$$

and functions P_L, Δ_L are defined as shown in Table 3.

The function η maps a propositional literal ψ to itself while it maps every other modal formula ψ to a new propositional symbol $t_\psi \in T$ uniquely associated with ψ. We call t_ψ the *surrogate* of ψ or simply a surrogate. The functions P_{KB} and P_{K5} introduce additional propositional symbols, called *supplementary propositional symbols*, $t_{\Box\neg t_{\Box\psi}} \in T$ and $t_{\Diamond t_{\Box\psi}} \in T$, respectively, that do not correspond to subformulae of the formula we are transforming.

For P_{KT}, P_{KD} and P_{K4} the additional clauses $S : \neg t_{\Box\psi} \vee \eta(\psi)$, $S : t_{\Box\psi} \to \Diamond\eta(\psi)$ and $S : t_{\Box\psi} \to \Box t_{\Box\psi}$, respectively, are directly based on the axiom schemata. Intuitively, P_{KB} is based on the following consideration: take a world w in a Kripke structure M with a symmetric accessibility relation R. If there exists a world v with $w R v$ such that $\langle M, v\rangle \models \Box\psi$, then $\langle M, w\rangle \models \psi$. Now, take the contrapositive of that statement: If $\langle M, w\rangle \not\models \psi$, then for every world v with wRv, $\langle M, v\rangle \not\models \Box\psi$. Equivalently, $\langle M, w\rangle \models \psi$ or $\langle M, w\rangle \models \Box\neg\Box\psi$. This is expressed by the formula $\eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}$. For P_{K5}, the formula $t_{\Diamond t_{\Box\psi}} \to \Box t_{\Diamond t_{\Box\psi}}$ expresses an instance of axiom schema 5, $\Diamond\varphi \to \Box\Diamond\varphi$, with $\varphi = \Box\psi$, i.e., $\Diamond\Box\psi \to \Box\Diamond\Box\psi$. The contrapositive of axiom schema 5 is $\Diamond\Box\varphi \to \Box\varphi$, equivalent to $\neg\Diamond\Box\varphi \vee \Box\varphi$. For $\varphi = \psi$ this is expressed by the formula $\neg t_{\Diamond t_{\Box\psi}} \vee t_{\Box\psi}$. For the formula $\neg t_{\Diamond t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$, consider $\neg\Diamond\Box\psi$. By duality of $\Box$ and $\Diamond$, this is equivalent to $\neg\neg\Box\neg\Box\psi$ and $\Box\neg\Box\psi$. So, $\neg\Diamond\Box\psi \to \Box\neg\Box\psi$ in every normal modal logic, not only K5. The remaining labeled formulae introduced by P_{KB} and P_{K5} ensure that supplementary propositional symbols are defined.

To simplify presentation in the following, we define a function η_f as follows:

$$\eta_f(\varphi_1 \wedge \varphi_2) = \eta(\varphi_1) \wedge \eta(\varphi_2) \qquad \eta_f(\varphi_1 \vee \varphi_2) = \eta(\varphi_1) \vee \eta(\varphi_2)$$
$$\eta_f(\Box\varphi) = \Box\eta(\varphi) \qquad \eta_f(\Diamond\varphi) = \Diamond\eta(\varphi)$$

and we treat the two clauses $S : \neg t_{\psi_1\wedge\psi_2} \vee \eta(\psi_1)$ and $S : \neg t_{\psi_1\wedge\psi_2} \vee \eta(\psi_2)$ resulting from the normal form transformation of $\psi_1 \wedge \psi_2$ as a single 'clause' $S : \neg t_{\psi_1\wedge\psi_2} \vee \eta_f(\psi_1 \wedge \psi_2)$. We also interchangeably write $S : \neg t_{\Box\psi} \vee \eta_f(\Box\psi)$ for $S : t_{\Box\psi} \to \eta_f(\Box\psi)$ and, analogously, $S : \neg t_{\Diamond\psi} \vee \eta_f(\Diamond\psi)$ for $S : t_{\Diamond\psi} \to \eta_f(\Diamond\psi)$. We then call any clause of the form $S : \neg t_\psi \vee \eta_f(\psi)$ a *definitional clause*.

Definition 1 Let Φ be a set of SNF_{sml} clauses. We say $t_\psi \in T$ *occurs at level ml in* Φ iff either

(a) there exists a clause $S : \vartheta$ in Φ with $ml \in S$ such that ϑ is a propositional formula and t_ψ occurs positively in ϑ, or
(b) there exists a clause $S : t_{\Box\psi} \rightarrow \Box t_\psi$ in Φ with $ml - 1 \in S$, or
(c) there exists a clause $S : t_{\Diamond\psi} \rightarrow \Diamond t_\psi$ in Φ with $ml - 1 \in S$.

Definition 2 Let Φ be a set of SNF_{sml} clauses. Then Φ is *definition-complete* iff for every $t_\psi \in T$ and every level ml, if t_ψ occurs at level ml in Φ then either (i) $t_\psi = t_{\mathbf{true}}$, or (ii) there exists a clause $S : \neg t_\psi \vee \eta_f(\psi)$ in Φ with $ml \in S$.

Example 1 Consider the formula $\varphi = \Diamond q \wedge \Diamond\Diamond(\Box(p \wedge \Diamond\Diamond\neg p) \wedge \Diamond q)$ in the modal logic K4. Then $\{\{0\} : t_\varphi\} \cup \rho_{\mathsf{K4}}(\{0\} : t_\varphi \rightarrow \varphi)$ consists of the following clauses.

(1)	$\{0\} : t_\varphi$	(6)	$\{2\} : \neg t_{\psi_3} \vee t_{\Box\psi_4}$	(11)	$\{3\}^{\geq} : \neg t_{\psi_4} \vee p$
(2)	$\{0\} : \neg t_\varphi \vee t_{\Diamond q}$	(7)	$\{2\} : \neg t_{\psi_3} \vee t_{\Diamond q}$	(12)	$\{3\}^{\geq} : \neg t_{\psi_4} \vee t_{\Diamond\Diamond\neg p}$
(3)	$\{0\} : \neg t_\varphi \vee t_{\psi_1}$	(8)	$\{0,2\} : t_{\Diamond q} \rightarrow \Diamond q$	(13)	$\{3\}^{\geq} : t_{\Diamond\Diamond\neg p} \rightarrow \Diamond t_{\Diamond\neg p}$
(4)	$\{0\} : t_{\psi_1} \rightarrow \Diamond t_{\psi_2}$	(9)	$\{2\}^{\geq} : t_{\Box\psi_4} \rightarrow \Box t_{\psi_4}$	(14)	$\{4\}^{\geq} : t_{\Diamond\neg p} \rightarrow \Diamond\neg p$
(5)	$\{1\} : t_{\psi_2} \rightarrow \Diamond t_{\psi_3}$	(10)	$\{2\}^{\geq} : t_{\Box\psi_4} \rightarrow \Box t_{\Box\psi_4}$		

where $\psi_4 = p \wedge \Diamond\Diamond\neg p$, $\psi_3 = \Box\psi_4 \wedge \Diamond q$, $\psi_2 = \Diamond\psi_3$, and $\psi_1 = \Diamond\psi_2$. All propositional symbols of the form t_ψ are surrogate symbols for the respective ψ formulae, which are subformulae of φ. All clauses except for Clause (10) would also be present in $\{\{0\} : t_\varphi\} \cup \rho_{\mathsf{K}}(\{0\} : t_\varphi \rightarrow \varphi)$ but Clauses (9), (11) to (14) would be labeled with singleton sets $\{n\}$ instead of infinite sets $\{n\}^{\geq}$. Clauses (2) to (9) and (11) to (14) are definitional clauses. Clause (8) is an example of how sets of levels allow for a single definition of a clause appearing at different modal levels. Clause (10) is specific to K4, and all clauses from (9) onwards have a set of levels of the form $\{n\}^{\geq}$, which means that they hold at all levels greater than or equal to n.

Theorem 2 *Let $L \in \{\mathsf{K}, \mathsf{KB}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}, \mathsf{K5}\}$. Then $\Phi_L = \{\{0\} : t_\varphi\} \cup \rho_L(\{0\} : t_\varphi \rightarrow \varphi)$ is definition-complete.*

Proof By induction over the computation of Φ_L. It is straightforward to see that the transformation of labeled formulae $S : t \rightarrow (\psi_1 \wedge \psi_2)$ and $S : t \rightarrow (\psi_1 \vee \psi_2)$ only introduces surrogates at levels in S and Δ_L then adds definitional clauses for those surrogates. The transformation of a labeled formula $S : t_{\Diamond\psi} \rightarrow \Diamond\psi$ may introduce a surrogate at levels in S^+ and $\delta_L(S^+, \psi)$ then adds definitional clauses for those surrogates. The transformation of a labeled formula $S : t_{\Box\psi} \rightarrow \Box\psi$ depends on the logic L. We can see that for every level at which a new surrogate occurs in $P_L(S : t_{\Box\psi} \rightarrow \Box\psi)$, then $\Delta_L(S : t_{\Box\psi} \rightarrow \Box\psi)$ contains a definitional clause for it at that level. Where a definitional clause introduced in the transformation has the form $S : t_{\mathbf{true}} \rightarrow \mathbf{true}$ it will at some point be eliminated, but this is compatible with our notion of definition-completeness. $\square$

5 Correctness

5.1 Common Properties

Lemma 1 *Let φ be modal formula. Let $\Phi = \{\{0\} : t_\varphi\} \cup \rho_{\mathsf{K}}(\{0\} : t_\varphi \rightarrow \varphi)$ and $\Phi_L = \{\{0\} : t_\varphi\} \cup \rho_L(\{0\} : t_\varphi \rightarrow \varphi)$, for $L \in \{\mathsf{KB}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}, \mathsf{K5}\}$. Then $\Phi \subseteq \Phi_L$.*

Proof By definition of ρ_{K} and ρ_L, anything obtained via ρ_{K} is also obtained via ρ_L. Therefore, $\Phi \subseteq \Phi_L$. $\square$

Lemma 2 *Let $M = \langle W, R, V, w_0\rangle$ be a rooted Kripke structure. Let $\langle \vec{W}, \vec{R}\rangle$ be the unraveling of $\langle W, R\rangle$ at w_0. Let $\vec{M} = \langle \vec{W}, \vec{R}, \vec{V_\Sigma}, (w_0)\rangle$ where $\vec{V_\Sigma}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ for every propositional symbol $p \in Q$.*

Then for every modal formula ψ over Q and for every world $\vec{w} \in \vec{W}$, $\langle \vec{M}, \vec{w}\rangle \models \psi$ iff $\langle M, \mathsf{trm}(\vec{w})\rangle \models \psi$.

In contrast to similar results in the literature, see, e.g., [5, Propositions 2.14 and 2.15], we allow $\vec{V_\Sigma}(p)$ to differ from $V(p)$ for propositional symbols not in Q. This then allows us to freely define $\vec{V_\Sigma}(t_\psi)$ for $t_\psi \in T$.

Lemma 3 *Let $M = \langle W, R, V, w_0\rangle$ be a rooted Kripke structure. Let $\langle \vec{W}, \vec{R}\rangle$ be the unraveling of $\langle W, R\rangle$ at w_0. Let $\vec{M} = \langle \vec{W}, \vec{R}, \vec{V_\Sigma}, (w_0)\rangle$ be a Kripke structure such that*

- *$\vec{V_\Sigma}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ for every propositional symbol $p \in Q$, and*
- *$\vec{V_\Sigma}(t_\psi) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}, \vec{w}\rangle \models \psi\}$ for every $t_\psi \in T$.*

Then for every $t_\psi \in T$ and every world $\vec{w} \in \vec{W}$, $\langle \vec{M}, \vec{w}\rangle \models t_\psi$ iff $\langle \vec{M}, \vec{w}\rangle \models \psi$ iff $\langle M, \mathsf{trm}(\vec{w})\rangle \models \psi$.

Proof Let $\vec{w}$ be a world in $\vec{W}$. By Lemma 2 for every formula ψ over Q, $\langle \vec{M}, \vec{w}\rangle \models \psi$ iff $\langle M, \mathsf{trm}(\vec{w})\rangle \models \psi$. By definition of $\vec{V_\Sigma}$, for every $t_\psi \in T$, $\langle \vec{M}, \vec{w}\rangle \models t_\psi$ iff $\langle \vec{M}, \vec{w}\rangle \models \psi$. So, $\langle \vec{M}, \vec{w}\rangle \models t_\psi$ iff $\langle \vec{M}, \vec{w}\rangle \models \psi$ iff $\langle M, \mathsf{trm}(\vec{w})\rangle \models \psi$. $\square$

Lemma 4 *Let Φ be a set of definitional clauses such that every t_ψ occurring Φ is an element of T and all other propositional symbols occurring in Φ are in Q. Let $M = \langle W, R, V, w_0\rangle$ be a rooted Kripke structure. Let $\langle \vec{W}, \vec{R}\rangle$ be the unraveling of $\langle W, R\rangle$ at w_0. Let $\vec{M} = \langle \vec{W}, \vec{R}, \vec{V_\Sigma}, (w_0)\rangle$ be a Kripke structure such that*

- *$\vec{V_\Sigma}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ for every propositional symbol $p \in Q$, and*
- *$\vec{V_\Sigma}(t_\psi) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}, \vec{w}\rangle \models \psi\}$ for every surrogate $t_\psi \in T \cap \mathsf{var}(\Phi)$.*

Then $\vec{M} \models \Phi$.

Proof Let $T(Q, \Phi)$ be the set of all modal formulae over Q such that $\eta(\psi) \in \mathsf{var}(\Phi)$. Let $\psi \in T(Q, \Phi)$. If ψ is not literal, then $\eta(\psi) = t_\psi$ for some propositional symbol $t_\psi \in \mathsf{var}(\Phi) \setminus Q$. By Lemma 3, for every world $\vec{w} \in \vec{W}$, $\langle \vec{M}, \vec{w}\rangle \models \psi$ iff $\langle \vec{M}, \vec{w}\rangle \models t_\psi$ iff $\langle \vec{M}, \vec{w}\rangle \models \eta(\psi)$. If ψ is propositional symbol $p \in Q$, then $\eta(\psi) = \psi$ and, trivially, for every world $\vec{w} \in \vec{W}$, $\langle \vec{M}, \vec{w}\rangle \models \psi$ iff $\langle \vec{M}, \vec{w}\rangle \models \eta(\psi)$. Overall, (15) if $\psi \in T(Q, \Phi)$ and $\vec{w} \in \vec{W}$, then $\langle \vec{M}, \vec{w}\rangle \models \psi$ iff $\langle \vec{M}, \vec{w}\rangle \models \eta(\psi)$.

Let $S : \psi'$ be a clause in Φ. We show that $\vec{M} \models S : \psi'$. Depending on the form of $S : \psi'$ as stated in the lemma, we can distinguish the following cases:
Case (a): Let $\vec{w} \in \vec{M}[S]$ with $\langle \vec{M}, \vec{w}\rangle \models t_{\psi_1 \wedge \psi_2}$. By Lemma 3, this implies $\langle \vec{M}, \vec{w}\rangle \models \psi_1 \wedge \psi_2$. Since $\psi_1, \psi_2 \in T(Q, \Phi)$, by Property (15), $\langle \vec{M}, \vec{w}\rangle \models \eta(\psi_1) \wedge \eta(\psi_2)$. This implies $\langle \vec{M}, \vec{w}\rangle \models \neg t_{\psi_1 \wedge \psi_2} \vee \eta(\psi_i)$. Thus, $\vec{M} \models S : \neg t_{\psi_1 \wedge \psi_2} \vee \eta(\psi_i)$.
Case (b): Let $\vec{w} \in \vec{M}[S]$ with $\langle \vec{M}, \vec{w}\rangle \models t_{\psi_1 \vee \psi_2}$. By Lemma 3, this implies $\langle \vec{M}, \vec{w}\rangle \models \psi_1 \vee \psi_2$. Since $\psi_1, \psi_2 \in T(Q, \Phi)$, by Property (15), $\langle \vec{M}, \vec{w}\rangle \models \eta(\psi_1) \vee \eta(\psi_2)$. This implies $\langle \vec{M}, \vec{w}\rangle \models \neg t_{\psi_1 \vee \psi_2} \vee \eta(\psi_1) \vee \eta(\psi_2)$. Thus, $\vec{M} \models S : \neg t_{\psi_1 \vee \psi_2} \vee \eta(\psi_1) \vee \eta(\psi_2)$.
Case (c): disjunction of literals all of which are in $\mathsf{var}(\varphi)$. This case can be proven in analogy to Case (b).

Case (d): Let $\vec{w} \in \vec{M}[S]$ with $\langle \vec{M}, \vec{w} \rangle \models t_{\Box\psi}$. By Lemma 3, this implies $\langle \vec{M}, \vec{w} \rangle \models \Box\psi$. That means for every $\vec{v} \in \vec{W}$, if $\vec{w} R \vec{v}$ then $\langle \vec{M}, \vec{v} \rangle \models \psi$. Since $\psi \in T(Q, \Phi)$, by Property (15), this implies for every $\vec{v} \in \vec{W}$, if $\vec{w} R \vec{v}$ then $\langle \vec{M}, \vec{v} \rangle \models \eta(\psi)$. By the semantics of $\Box$, then $\langle \vec{M}, \vec{w} \rangle \models \Box\eta(\psi)$. Thus, $\vec{M} \models S : t_{\Box\psi} \to \Box\eta(\psi)$.
Case (e): Let $\vec{w} \in \vec{M}[S]$ with $\langle \vec{M}, \vec{w} \rangle \models t_{\Diamond\psi}$. By Lemma 3, this implies $\langle \vec{M}, \vec{w} \rangle \models \Diamond\psi$. That means there exists $\vec{v} \in \vec{W}$ such that $\vec{w} R \vec{v}$ and $\langle \vec{M}, \vec{v} \rangle \models \psi$. Since $\psi \in T(Q, \Phi)$, by Property (15), this implies $\langle \vec{M}, \vec{v} \rangle \models \eta(\psi)$. By the semantics of $\Diamond$, then $\langle \vec{M}, \vec{w} \rangle \models \Diamond\eta(\psi)$. Thus, $\vec{M} \models S : t_{\Diamond\psi} \to \Diamond\eta(\psi)$.
This covers all possible forms that clauses in Φ can take and we conclude that $\vec{M} \models \Phi$. □

Lemma 5 *Let φ be a L-satisfiable modal formula in simplified NNF where L is a normal modal logic and let $\Phi = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \to \varphi)$. Let $M = \langle W, R, V, w_0 \rangle$ be a rooted* K *model of φ. Let $\langle \vec{W}, \vec{R} \rangle$ be the unraveling of $\langle W, R \rangle$ at w_0. Let $\vec{M} = \langle \vec{W}, \vec{R}, \vec{V}, (w_0) \rangle$ be a Kripke structure such that*

- $\vec{V}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ *for every propositional symbol $p \in \mathsf{var}(\varphi)$, and*
- $\vec{V}(t_\psi) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}, \vec{w} \rangle \models \psi\}$ *for every surrogate $t_\psi \in T \cap \mathsf{var}(\Phi)$.*

Then $\vec{M} \models \Phi$.

Proof Each clause $S : \psi'$ in Φ except for $\{0\} : t_\varphi$ is a definitional clause $S : \neg t_\psi \vee \eta_f(\psi)$ with $t_\psi \in T$. By Lemma 4, $\vec{M} \models S : \psi'$.

Now consider $\{0\} : t_\varphi$. As $\langle M, w_0 \rangle \models \varphi$ and $(w_0) \in \vec{M}[0]$, by Lemma 2, we obtain $\langle \vec{M}, (w_0) \rangle \models \varphi$. By Lemma 3, $\langle \vec{M}, (w_0) \rangle \models \varphi$ implies $\langle \vec{M}, (w_0) \rangle \models t_\varphi$. Thus, $\vec{M} \models \{0\} : t_\varphi$.

This covers all possible forms that clauses in Φ can take and we can conclude that $\vec{M} \models \Phi$. □

Lemma 6 *Let φ be a modal formula in simplified NNF. Let $\Phi_K = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \to \varphi)$. Let Φ with $\Phi_K \subseteq \Phi$ be a definition-complete set of SNF_{sml} clauses, let $M = \langle W, R, V, w_0 \rangle$ be a tree* K *model of Φ and let $M' = \langle W, R', V, w_0 \rangle$ be such that*

(6a) *$R \subseteq R'$;*
(6b) *for every modal clause $S : t_{\Box\psi} \to \Box\eta(\psi)$ in Φ and every world $w \in M[S]$, $\langle M', w \rangle \models t_{\Box\psi} \to \Box\eta(\psi)$;*
(6c) *for every modal clause $S : t_{\Box\psi} \to \Box t_\psi$ in Φ and all worlds $v, w \in W$, if (i) $w \in M[S]$ and (ii) $w\ R'\ v$ then (iii) there exists a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ with $v \in M[S']$.*

Then $\langle M', w_0 \rangle \models \varphi$.

Proof As M is a model of Φ, (16) for every clause $S : \psi$ in Φ and every world $w \in M[S]$, $\langle M, w \rangle \models \psi$. Also, as both M and M' use the same valuation V and the same set of worlds W, (17) for every propositional literal l and every world $w \in W$, $\langle M, w \rangle \models l$ iff $\langle M', w \rangle \models l$.

We prove by structural induction on subformulae ϑ of φ that (18) if $\eta(\vartheta) = t_\vartheta$ for surrogate t_ϑ, $S : \neg t_\vartheta \vee \eta_f(\vartheta) \in \Phi$, and $w \in M[S]$ with $\langle M', w \rangle \models t_\vartheta$ then $\langle M', w \rangle \models \vartheta$.

In the base cases, we have to consider subformulae ϑ that are conjunctions or disjunctions of literals over propositional symbols in $\mathsf{var}(\varphi)$.

Case (1): Let ϑ be of the form $\psi_1 \wedge \psi_2$, where ψ_1 and ψ_2 are literals over $\mathsf{var}(\varphi)$, with $\eta(\psi_1 \wedge \psi_2) = t_{\psi_1 \wedge \psi_2}$. As Φ is definition-complete, there exist literal clauses $S : \neg t_{\psi_1 \wedge \psi_2} \vee \eta(\psi_1)$ and $S : \neg t_{\psi_1 \wedge \psi_2} \vee \eta(\psi_2)$ in Φ_K such that $\eta(\psi_1) = \psi_2$ and $\eta(\psi_2) = \psi_2$ are literals over $\mathsf{var}(\varphi)$. Assume $w \in M[S]$ with $\langle M', w \rangle \models t_{\psi_1 \wedge \psi_2}$.

By Property (17), $\langle M, w\rangle \models t_{\psi_1\wedge\psi_2}$. By Property (16), $\langle M, w\rangle \models \neg t_{\psi_1\wedge\psi_2} \vee \psi_i$ for $1 \leq i \leq 2$. We therefore have $\langle M, w\rangle \models \psi_i$ for $1 \leq i \leq 2$. By Property (17), this implies $\langle M', w\rangle \models \psi_i$ for $1 \leq i \leq 2$. Thus, $\langle M', w\rangle \models \psi_1 \wedge \psi_2$ by the semantics of conjunction.
Case (2): Let ϑ be a disjunction of literals over $\mathsf{var}(\varphi)$ with $\eta(\vartheta) = t_\vartheta$. As Φ is definition-complete, there exists a clause $S : \neg t_\vartheta \vee \vartheta$ in Φ_K. Assume $w \in M[S]$ with $\langle M', w\rangle \models t_\vartheta$. By Property (17), $\langle M, w\rangle \models t_\vartheta$. By Property (16), $\langle M, w\rangle \models \neg t_\vartheta \vee \vartheta$. By the semantics of disjunction, there is a propositional literal l in ϑ such that $\langle M, w\rangle \models l$. By Property (17), $\langle M', w\rangle \models l$. Thus, $\langle M', w\rangle \models \vartheta$ by the semantics of disjunction.
In the induction step, we consider subformulae ϑ of φ under the induction hypothesis that Property (18) holds for all proper subformulae of ϑ.
Case (3): Let ϑ be of the form $\psi_1 \wedge \psi_2$ with $\eta(\psi_1 \wedge \psi_2) = t_{\psi_1\wedge\psi_2}$. As Φ is definition-complete, there exist clauses $S : \neg t_{\psi_1\wedge\psi_2} \vee \eta(\psi_1)$ and $S : \neg t_{\psi_1\wedge\psi_2} \vee \eta(\psi_2)$ in Φ. Assume $w \in M[S]$ with $\langle M', w\rangle \models t_{\psi_1\wedge\psi_2}$. By Property (17), $\langle M, w\rangle \models t_{\psi_1\wedge\psi_2}$. By Property (16), $\langle M, w\rangle \models \neg t_{\psi_1\wedge\psi_2} \vee \eta(\psi_i)$ for $1 \leq i \leq 2$. We therefore have $\langle M, w\rangle \models \eta(\psi_i)$, for $1 \leq i \leq 2$. As $\eta(\psi_1)$ and $\eta(\psi_2)$ are literals, by Property (17) and the semantics of conjunction, $\langle M', w\rangle \models \eta(\psi_1) \wedge \eta(\psi_2)$. *Case (3-a):* If ψ_i, $1 \leq i \leq 2$, is a literal, then $\eta(\psi_i) = \psi_i$ and we immediately have $\langle M', w\rangle \models \psi_i$. *Case (3-b):* If ψ_i, $1 \leq i \leq 2$, is not a literal, then $\eta(\psi_i) = t_{\psi_i}$. Since Φ is definition-complete, there must be a clause $S' : \neg t_{\psi_i} \vee \eta_f(\psi_i)$ in Φ_K with $w \in M[S']$. Then, by induction hypothesis, $\langle M', w\rangle \models t_{\psi_i}$ implies $\langle M', w\rangle \models \psi_i$. Taking both cases together $\langle M', w\rangle \models \psi_1 \wedge \psi_2$.
Case(4): Let ϑ be of the form $\psi_1 \vee \psi_2$ where ϑ is not a disjunction of literals and $\eta(\vartheta) = t_{\psi_1\vee\psi_2}$. As Φ is definition-complete, there exists a clause $S : \neg t_{\psi_1\vee\psi_2} \vee \eta(\psi_1) \vee \eta(\psi_2)$ in Φ. Assume $w \in M[S]$ with $\langle M', w\rangle \models t_{\psi_1\vee\psi_2}$. By Property (17), $\langle M, w\rangle \models t_{\psi_1\vee\psi_2}$. By Property (16), $\langle M, w\rangle \models \neg t_{\psi_1\vee\psi_2} \vee \eta(\psi_1) \vee \eta(\psi_2)$. By the semantics of disjunction, $\langle M, w\rangle \models \eta(\psi_1)$ or $\langle M, w\rangle \models \eta(\psi_2)$.
As $\eta(\psi_1)$ and $\eta(\psi_2)$ are literals, by Property (17), $\langle M', w\rangle \models \eta(\psi_1)$ or $\langle M', w\rangle \models \eta(\psi_2)$. In analogy to Case (3-a): and Case (3-b): above we can show that $\langle M', w\rangle \models \psi_i$ for $i = 1$ or $i = 2$. This implies $\langle M', w\rangle \models \psi_1 \vee \psi_2$.
Case (5): Let ϑ be of the form $\Box\psi$ with $\eta(\Box\psi) = t_{\Box\psi}$. As Φ is definition-complete, there exists a clause $S : t_{\Box\psi} \rightarrow \Box\eta(\psi)$ in Φ. Assume $w \in M[S]$ (Condition (6c)-i) with $\langle M', w\rangle \models t_{\Box\psi}$. By Assumption (6b), $\langle M', w\rangle \models t_{\Box\psi} \rightarrow \Box\eta(\psi)$. By semantics of implication, $\langle M', w\rangle \models \Box\eta(\psi)$. Also, $\langle M', w\rangle \models t_{\Box\psi}$ implies $\langle M, w\rangle \models t_{\Box\psi}$. As $w \in M[S]$ and M is a model of Φ, $\langle M, w\rangle \models t_{\Box\psi} \rightarrow \Box\eta(\psi)$ and by the semantics of implication, $\langle M, w\rangle \models \Box\eta(\psi)$. Let $v \in W$ with $w\ R'\ v$ (Condition (6c)-ii). As $\langle M', w\rangle \models \Box\eta(\psi)$, $\langle M', v\rangle \models \eta(\psi)$. *Case (5-a):* If ψ is a literal, then $\eta(\psi) = \psi$ and we immediately have $\langle M', v\rangle \models \psi$ and as v was an arbitrary R'-successor of w, $\langle M', w\rangle \models \Box\psi$. *Case (5-b):* If ψ is not a literal, then $\eta(\psi) = t_\psi$. As Conditions (6c)-i and (6c)-ii hold, by Assumption (6c)-iii there exists a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ with $v \in M[S']$. Then, by induction hypothesis, $\langle M', v\rangle \models t_\psi$ implies $\langle M', v\rangle \models \psi$. By semantics of $\Box$ we again obtain $\langle M', w\rangle \models \Box\psi$.
Case (6): Let ϑ be of the form $\Diamond\psi$ with $\eta(\Diamond\psi) = t_{\Diamond\psi}$. As Φ is definition-complete, there exists a clause $S : t_{\Diamond\psi} \rightarrow \Diamond\eta(\psi)$ in Φ. Assume $w \in M[S]$ with $\langle M', w\rangle \models t_{\Diamond\psi}$. By Property (17), $\langle M, w\rangle \models t_{\Diamond\psi}$. By Assumption (16), $\langle M, w\rangle \models t_{\Box\psi} \rightarrow \Diamond\eta(\psi)$. By semantics of implication $\langle M, w\rangle \models \Diamond\eta(\psi)$. That means there exists $v \in W$ with $w\ R\ v$ and $\langle M, v\rangle \models \eta(\psi)$. As $\eta(\psi)$ is a literal, by Property (17), $\langle M', v\rangle \models \eta(\psi)$. As $R \subseteq R'$, $w\ R\ v$ implies $w\ R'\ v$. So there exists $v \in W$ with $w\ R'\ v$ and $\langle M', v\rangle \models \eta(\psi)$ which means $\langle M', w\rangle \models \Diamond\eta(\psi)$. *Case (6-a):* If ψ is a literal, then $\eta(\psi) = \psi$ and we immediately have $\langle M', w\rangle \models \Diamond\psi$. *Case (5-b):* If ψ is not a literal, then $\eta(\psi) = t_\psi$. Since Φ is definition-complete, there must be a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ with $v \in M[S']$. Then,

by induction hypothesis, $\langle M', v\rangle \models t_\psi$ implies $\langle M', v\rangle \models \psi$. By semantics of $\Diamond$, we then obtain $\langle M', w\rangle \models \Diamond\psi$. □

Lemma 7 *Let φ be a modal formula in simplified NNF. Let $\Phi_K = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \to \varphi)$. Let Φ with $\Phi_K \subseteq \Phi$ be a definition-complete set of SNF_{sml} clauses, let $M = \langle W, R, V, w_0\rangle$ be a rooted tree* K *model of Φ. Then $\langle M, w_0\rangle \models \varphi$.*

Proof It is sufficient to show that if we take $M' = M$, then the Kripke structures M and M' satisfy the three preconditions of Lemma 6:

- Condition (6a) trivially holds as both models have the same accessibility relation.
- For Condition (6b) let $S : t_{\Box\psi} \to \Box\eta(\psi)$ be a modal clause in Φ and $w \in M[S]$. Then (i) as M is a model of Φ, $M \models S : t_{\Box\psi} \to \Box\eta(\psi)$; (ii) as $w \in M[S]$, by definition of $\models$, $\langle M, w\rangle \models t_{\Box\psi} \to \Box\eta(\psi)$; (iii) as $M' = M$, $\langle M, w\rangle \models t_{\Box\psi} \to \Box\eta(\psi)$ implies $\langle M', w\rangle \models t_{\Box\psi} \to \Box\eta(\psi)$.
- For Condition (6c) let $S : t_{\Box\psi} \to \Box t_\psi$ in Φ, $w, v \in W$, $\mathsf{ml}_M(w) = ml \in S$ (i.e., $w \in M[S]$), and $w\, R\, v$. As M is a tree model, $\mathsf{ml}_M(v) = ml + 1$. The surrogate t_ψ occurs at level $ml + 1$ in Φ. As Φ is definition-complete by assumption, there exists a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ with $ml + 1 \in S'$. Thus, Condition (6c) holds.

By Lemma 6, $\langle M', w_0\rangle \models \varphi$. □

5.2 Basic Modal Logic K

Corollary 1 *Let φ be a modal formula in simplified NNF. Let $\Phi_K = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \to \varphi)$. Let $M = \langle W, R, V, w\rangle$ be a rooted Kripke model such that $\langle M, w\rangle \models \varphi$. Let $\langle \vec{W}, \vec{R}\rangle$ be the unraveling of $\langle W, R\rangle$ at w_0. Let $\vec{M} = \langle \vec{W}, \vec{R}, \vec{V}, w_0\rangle$ be a Kripke structure such that*

- *$\vec{V}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ for every propositional symbol $p \in \mathsf{var}(\varphi)$, and*
- *$\vec{V}(t_\psi) = \{\vec{w} \in W \mid \langle \vec{M}, \vec{w}\rangle \models \psi\}$ for every surrogate $t_\psi \in \mathsf{var}(\Phi_K) \setminus \mathsf{var}(\varphi)$.*

Then $\vec{M} \models \Phi_K$.

Proof Follows from Lemma 5 for logic $L =$ K. □

Theorem 3 *Let φ be a modal formula in simplified NNF. Let $\Phi_K = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \to \varphi)$. If Φ_K is* K*-satisfiable, then φ is* K*-satisfiable.*

Proof Let $M = \langle W, R, V, w_0\rangle$ be a tree K model of Φ_K. Φ_K is definition-complete by Theorem 2, and, by Lemma 7, it follows that $\langle M, w_0\rangle \models \varphi$.

Correctness proofs for the reductions for KD and KT are straightforward. In the remainder of the section, we consider the reductions for KB, K4 and K5.

5.3 Modal Logic KB

See [29] for the proofs of the following theorems.

Theorem 4 *Let φ be a modal formula in simplified NNF. Let $\Phi_B = \{\{0\} : t_\varphi\} \cup \rho_{KB}(\{0\} : t_\varphi \to \varphi)$. If φ is* KB*-satisfiable, then Φ_B is* K*-satisfiable.*

Theorem 5 *Let φ be a modal formula in simplified NNF. Let $\Phi_B = \{\{0\} : t_\varphi\} \cup \rho_{KB}(\{0\} : t_\varphi \to \varphi)$. If Φ_B is* K*-satisfiable, then φ is* KB*-satisfiable.*

5.4 Modal Logic K4

Theorem 6 *Let φ be a modal formula in simplified NNF. Let $\Phi_4 = \{\{0\} : t_\varphi\} \cup \rho_{K4}(\{0\} : t_\varphi \rightarrow \varphi)$. If φ is* K4*-satisfiable, then Φ_4 is* K*-satisfiable.*

Proof Let $M = \langle W, R, V, w_0\rangle$ be a rooted model of φ with $\langle M, w_0\rangle \models \varphi$ and transitive relationship R.

Let $\langle \vec{W}, \vec{R}\rangle$ be the unraveling of $\langle W, R\rangle$ at w_0. The function trm is a p-morphism from $\langle \vec{W}, \vec{R}\rangle$ to $\langle W, R\rangle$. Let $\vec{M}_4 = \langle \vec{W}, \vec{R}, \vec{V}_4, (w_0)\rangle$ where

- $\vec{V}_4(p) = \vec{V}(p)$ for every propositional symbol $p \in \mathsf{var}(\varphi)$,
- $\vec{V}_4(t_\psi) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}_4, \vec{w}\rangle \models \psi\}$ for every surrogate $t_\psi \in \mathsf{var}(\Phi_4) \setminus \mathsf{var}(\varphi)$ introduced by ρ_{K4}.

We show that all clauses in Φ_4 hold in $\vec{M}_4$.

Let $\Phi = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \rightarrow \varphi)$. By Lemma 1, $\Phi \subseteq \Phi_4$, and by Lemma 5, $\vec{M}_4 \models \Phi$.

All definitional clauses in $\Phi_4 \setminus \Phi$ are true in $\vec{M}_4$ by Lemma 4. It remains to consider clauses of the form (19) $S' : t_{\Box\psi} \rightarrow \Box t_{\Box\psi}$. Let $\vec{w} \in \vec{W}[S']$ with $\langle \vec{M}_4, \vec{w}\rangle \models t_{\Box\psi}$. Then $\vec{w} \in \vec{V}_4(t_{\Box\psi})$ and by definition of $\vec{V}_4$, $\langle \vec{M}_4, \vec{w}\rangle \models \Box\psi$. Let $\vec{u} \in \vec{W}$ such that $\vec{w}\vec{R}\vec{u}$. By Lemma 2, $\langle M, \mathsf{trm}(\vec{w})\rangle \models \Box\psi$. Since trm is a p-morphism $\mathsf{trm}(\vec{w})\, R\, \mathsf{trm}(\vec{u})$. As $\Box$ is a K4 modality, $\langle M, \mathsf{trm}(\vec{u})\rangle \models \Box\psi$ and by Lemma 2, $\langle \vec{M}_4, \vec{u}\rangle \models \Box\psi$. By definition of $\vec{V}_4$, $\vec{u} \in \vec{V}_4(t_{\Box\psi})$ and $\langle \vec{M}_4, \vec{u}\rangle \models t_{\Box\psi}$. As $\vec{u} \in \vec{W}$ was an arbitrary world with $\vec{w}\vec{R}\vec{u}$, we have $\langle \vec{M}_4, \vec{w}\rangle \models \Box t_{\Box\psi}$. Thus, Clause (19) holds in $\vec{M}_4$. □

Theorem 7 *Let φ be a modal formula. Let $\Phi_4 = \{\{0\} : t_\varphi\} \cup \rho_{K4}(\{0\} : t_\varphi \rightarrow \varphi)$. If Φ_4 is* K*-satisfiable, then φ is* K4*-satisfiable.*

Proof Let $M = \langle W, R, V, w_0\rangle$ be a rooted tree K model of Φ_4. Let $M^4 = \langle W, R^4, V^4, w_0\rangle$ be a Kripke structure such that

(a) R^4 is the transitive closure of R, that is, R^4 is the smallest relation on W such that $R \subseteq R^4$ and for every u, v, w, $u R^4 v$ and $v R^4 w$ implies $u R^4 w$;
(b) $V^4(p) = V(p)$ for every propositional symbol.

Let $\Phi = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \rightarrow \varphi)$. We show that M^4 satisfies the three preconditions of Lemma 6. By Lemma 6 this in turn implies that $M^4 \models \varphi$.

- Condition (6a) holds as $R \subseteq R^4$.
- For Condition (6b) let (20) $S' : t_{\Box\psi} \rightarrow \Box\eta(\psi)$ be a modal clause in Φ_4. Φ_4 also contains the clause (21) $S' : t_{\Box\psi} \rightarrow \Box t_{\Box\psi}$. By definition of ρ_{K4}, for all $n \geq \min(S')$, $n \in S'$. Let $w \in M[S']$ such that $\langle M^4, w\rangle \models t_{\Box\psi}$. By Clause (20), $\langle M^4, w\rangle \models \Box\eta(\psi)$ should hold. Assume $\langle M^4, w\rangle \not\models \Box\eta(\psi)$, that is, there exists $v \in W$ with $w R^4 v$ and $\langle M^4, v\rangle \not\models \eta(\psi)$. As $V = V^4$, $\langle M, w\rangle \models t_{\Box\psi}$ and by Clause (20) which is true in M, $\langle M, w\rangle \models \Box\eta(\psi)$. Thus, for every world $u \in W$, if wRu then $\langle M, u\rangle \models \eta(\psi)$. $\eta(\psi)$ is either a propositional symbol or its negation. As $V = V^4$, $\langle M, u\rangle \models \eta(\psi)$ iff $\langle M^4, u\rangle \models \eta(\psi)$. That means wRv cannot hold. Consequently, $w R^4 v$ was introduced by the closure operation on R. This in turn implies that there exist $v_0, \ldots, v_m, m > 1$, such that $v_0 = u, v_m = v$, for every $i, 1 \leq i \leq m$, $v_{i-1} R v_i$ holds, $\langle M^4, v_0\rangle \models t_{\Box\psi}$, and $\langle M^4, v_m\rangle \not\models \eta(\psi)$. Note that (22) for every $i, 0 \leq i \leq m$, $v_i \in M[S']$. Since $\langle M, v_0\rangle \models t_{\Box\psi}$, using Clause (21) and Property (22), by induction, we can show that for every i, $0 \leq i \leq m$, $\langle M, v_i\rangle \models t_{\Box\psi}$. From $\langle M, v_{m-1}\rangle \models t_{\Box\psi}$, $v_{m-1} \in M[S']$ and Clause (20), we then obtain $\langle M, v_m\rangle \models \eta(\psi)$. As $V = V^4$, $\langle M, v_m\rangle \models \eta(\psi)$ iff $\langle M^4, v_m\rangle \models \eta(\psi)$. This contradicts $\langle M^4, v_m\rangle \not\models \eta(\psi)$.

- For Condition (6c) let (23) $S : t_{\Box\psi} \rightarrow \Box t_\psi$ be in Φ_4, $v, w \in W$, $\mathsf{ml}_M(w) = ml \in S$ (i.e., $w \in M[S]$) and $w R^4 v$. We need to show that there exists a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ_4 with $v \in M[S']$.
  As in the previous case, $w R^4 v$ means that there exist $v_0, \dots, v_m$, $m > 1$, such that $v_0 = w$, $v_m = v$, and for every i, $1 \leq i \leq m$, $v_{i-1} R v_i$ holds. Then $\mathsf{ml}_M(v) = \mathsf{ml}_M(v_m) = \mathsf{ml}_M(v_0) + m = \mathsf{ml}_M(w) + m$.
  By definition of ρ_{K4}, for all $n \geq \min(S)$, $n \in S$. That means Φ_4 contains t_ψ at every level $ml' \geq \min(S) \geq ml$. This includes $ml'=\mathsf{ml}_M(w) + m=\mathsf{ml}_M(v)$. By Theorem 2, Φ_4 is definition-complete and therefore there exists a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ_4 with $ml' \in S'$ and $v \in M[S']$. $\Box$

5.5 Modal Logic K5

Theorem 8 *Let φ be a modal formula. Let $\Phi_5 = \{\{0\} : t_\varphi\} \cup \rho_{K5}(\{0\} : t_\varphi \rightarrow \varphi)$. If φ is* K5-*satisfiable, then Φ_5 is* K-*satisfiable.*

Proof Let $M = \langle W, R, V, w_0\rangle$ be a model of φ with $\langle M, w_0\rangle \models \varphi$ and Euclidean relation R.

Let $\langle \vec{W}, \vec{R}\rangle$ be the unraveling of $\langle W, R\rangle$ at w_0. The function trm is a p-morphism from $\langle \vec{W}, \vec{R}\rangle$ to $\langle W, R\rangle$. Let $\vec{M}_5 = \langle \vec{W}, \vec{R}, \vec{V}_5, (w_0)\rangle$ where

- $\vec{V}_5(p) = \vec{V}(p)$ for every propositional symbol $p \in \mathsf{var}(\varphi)$,
- $\vec{V}_5(t_\psi) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}_5, \vec{w}\rangle \models \psi\}$ for every surrogate $t_\psi \in \mathsf{var}(\Phi_5) \setminus \mathsf{var}(\varphi)$ introduced by rewriting, and
- $\vec{V}_5(t_{\Diamond t_{\Box\psi}}) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}_5, \vec{w}\rangle \models \Diamond\Box\psi\}$ for every supplementary propositional symbol $t_{\Diamond t_{\Box\psi}} \in \mathsf{var}(\Phi_5) \setminus \mathsf{var}(\varphi)$ introduced by rewriting.

We show that all clauses in Φ_5 hold in $\vec{M}_5$.

Let $\Phi = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \rightarrow \varphi)$. By Lemma 1, $\Phi \subseteq \Phi_5$, and by Lemma 5, $\vec{M}_5 \models \Phi$. All definitional clauses in $\Phi_5 \setminus \Phi$ are true in $\vec{M}_5$ by Lemma 4.

Next consider clauses of the form

$$(24) \quad \star : t_{\Diamond t_{\Box\psi}} \rightarrow \Diamond t_{\Box\psi} \qquad (26) \quad \star : t_{\Diamond t_{\Box\psi}} \rightarrow t_{\Box\psi}$$

$$(25) \quad \star : \neg t_{\Diamond t_{\Box\psi}} \rightarrow \Box\neg t_{\Box\psi} \qquad (27) \quad \star : t_{\Diamond t_{\Box\psi}} \rightarrow \Box t_{\Diamond t_{\Box\psi}}$$

involving supplementary propositional symbols. These are not in Φ. Let $\vec{w} \in \vec{W}$ such that $\langle \vec{M}_5, \vec{w}\rangle \models t_{\Diamond t_{\Box\psi}}$. Then $\vec{w} \in \vec{V}_5(t_{\Diamond t_{\Box\psi}})$ and by definition of $\vec{V}_5$, $\langle \vec{M}_5, \vec{w}\rangle \models \Diamond\Box\psi$. By semantics of $\Diamond$, there exists $\vec{u} \in W$ such that $\langle \vec{M}_5, \vec{u}\rangle \models \Box\psi$. By definition of $\vec{V}_5$, $\vec{u} \in \vec{V}_5(t_{\Box\psi})$ and so $\langle \vec{M}_5, \vec{u}\rangle \models t_{\Box\psi}$. By semantics of $\Diamond$, $\langle \vec{M}_5, \vec{w}\rangle \models \Diamond t_{\Box\psi}$. Thus, Clause (24) holds in $\vec{M}_5$.

By Lemma 2, $\langle \vec{M}_5, \vec{w}\rangle \models \Diamond\Box\psi$ iff $\langle M, \mathsf{trm}(\vec{w})\rangle \models \Diamond\Box\psi$. As $\Diamond$ and $\Box$ are K5 modal operators, $\langle M, \mathsf{trm}(\vec{w})\rangle \models \Diamond\Box\psi$ implies $\langle M, \mathsf{trm}(\vec{w})\rangle \models \Box\psi$. By Lemma 2, $\langle \vec{M}_5, \vec{w}\rangle \models \Box\psi$. By definition of $\vec{V}_5$, $\vec{w} \in \vec{V}_5(t_{\Box\psi})$ and $\langle \vec{M}_5, \vec{w}\rangle \models t_{\Box\psi}$. So, Clause (26) holds in $\vec{M}_5$.

As we have seen, if $\langle \vec{M}_5, \vec{w}\rangle \models t_{\Diamond t_{\Box\psi}}$ then $\langle M, \mathsf{trm}(\vec{w})\rangle \models \Diamond\Box\psi$ As $\Diamond$ and $\Box$ are K5 modal operators, this implies $\langle M, \mathsf{trm}(\vec{w})\rangle \models \Box\Diamond\Box\psi$. By Lemma 2, $\langle M, \mathsf{trm}(\vec{w})\rangle \models \Box\Diamond\Box\psi$ iff $\langle \vec{M}_5, \vec{w}\rangle \models \Box\Diamond\Box\psi$. By the semantics of $\Box$, for every $\vec{u} \in \vec{V}_5$, if $\vec{w}\vec{R}\vec{u}$ then $\langle \vec{M}_5, \vec{u}\rangle \models \Diamond\Box\psi$. By definition of $\vec{V}_5$, $\vec{u} \in \vec{V}_5(t_{\Diamond t_{\Box\psi}})$. Again, by semantics of $\Box$, $\langle \vec{M}_5, \vec{w}\rangle \models \Box t_{\Diamond t_{\Box\psi}}$. Thus, Clause (27) holds in $\vec{M}_5$.

For Clause (25), we have to consider a world $\vec{w} \in \vec{W}$ such that $\langle \vec{M}_5, \vec{w}\rangle \models \neg t_{\Diamond t_{\Box\psi}}$. Then $\vec{w} \notin \vec{V}_5(t_{\Diamond t_{\Box\psi}})$ and by definition of $\vec{V}_5$, $\langle \vec{M}_5, \vec{w}\rangle \not\models \Diamond\Box\psi$. By the semantics of $\Diamond$,

$\langle \vec{M}_5, \vec{w}\rangle \models \Box\neg\Box\psi$. By the semantics of $\Box$, for every $\vec{u} \in \vec{W}$, $\langle \vec{M}_5, \vec{u}\rangle \models \neg\Box\psi$, that is, $\langle \vec{M}_5, \vec{u}\rangle \not\models \Box\psi$. By the definition of $\vec{V}_5$, $\vec{u} \notin \vec{V}_5(t_{\Box\psi})$ and therefore $\langle \vec{M}_5, \vec{u}\rangle \not\models t_{\Box\psi}$ and $\langle \vec{M}_5, \vec{u}\rangle \models \neg t_{\Box\psi}$. By the semantics of $\Box$, $\langle \vec{M}_5, \vec{w}\rangle \models \Box\neg t_{\Box\psi}$. So, we have $\langle \vec{M}_5, \vec{w}\rangle \models \neg t_{\Diamond t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$. Thus, Clause (25) holds in $\vec{M}_5$. $\Box$

Theorem 9 *Let φ be a modal formula. Let $\Phi_5 = \{\{0\} : t_\varphi\} \cup \rho_{K5}(\{0\} : t_\varphi \to \varphi)$. If Φ_5 is* K*-satisfiable, then φ is* K5*-satisfiable.*

Proof Let $M = \langle W, R, V, w_0\rangle$ be a rooted tree K model of Φ_5. Let $M^5 = \langle W, R^5, V^5, w_0\rangle$ be a Kripke structure such that

(a) R^5 is the Euclidean closure of R, that is, R^5 is the smallest relation on W such that $R \subseteq R^5$ and for every u, v, w, $u R^5 v$ and $u R^5 w$ implies $v R^5 w$;
(b) $V^5(p) = V(p)$ for every propositional symbol.

Let $\Phi = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \to \varphi)$. We show that M^5 satisfies the three preconditions of Lemma 6. By Lemma 6 this in turn implies that $M^5 \models \varphi$.

- Condition (6a) holds as $R \subseteq R^5$.
- For Condition (6b) let (28) $\star : t_{\Box\psi} \to \Box\eta(\psi)$ be a modal clause in Φ_5. Then Φ_5 also contains the clauses

(29)	$\star : t_{\Diamond t_{\Box\psi}} \to t_{\Box\psi}$	(31)	$\star : t_{\Diamond t_{\Box\psi}} \to \Diamond t_{\Box\psi}$
(30)	$\star : \neg t_{\Diamond t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$	(32)	$\star : t_{\Diamond t_{\Box\psi}} \to \Box t_{\Diamond t_{\Box\psi}}$

  Let $w \in W$ such that $\langle M^5, w\rangle \models t_{\Box\psi}$. By Clause (28), $\langle M^5, w\rangle \models \Box\eta(\psi)$ should hold. Assume $\langle M^5, w\rangle \not\models \Box\eta(\psi)$, that is, there exists $v \in W$ with $w R^5 v$ and $\langle M^5, v\rangle \not\models \eta(\psi)$. As $V^5 = V$, $\langle M^5, w\rangle \models t_{\Box\psi}$ implies $\langle M, w\rangle \models t_{\Box\psi}$ and since Clause (28) is true in M, $\langle M, w\rangle \models \Box\eta(\psi)$. By semantics of $\Box$, for every world $u \in W$ if wRu then $\langle M, u\rangle \models \eta(\psi)$. As $\eta(\psi)$ is a propositional literal and $V^5 = V$ $\langle M, u\rangle \models \eta(\psi)$ implies $\langle M^5, u\rangle \models \eta(\psi)$. Thus, v must be a world such that $w\,R\,v$ does not hold. So, $w\,R^5\,v$ was introduced by the closure operation on R. This in turn means that there exist $u, v_0, \dots, v_m, w'_0, \dots, w'_n, \in W$,
  $m, n \geq 1$, such that $v_0 = u$, $v_m = v$, $w'_0 = u$, $w'_n = w$, for every i, $1 \leq i \leq m$, $v_{i-1}\,R\,v_i$ holds, for every j, $1 \leq j \leq n$, $w'_{j-1}\,R\,w'_j$ holds, $\langle M, w_n\rangle \models t_{\Box\psi}$, $\langle M, v_m\rangle \not\models \eta(\psi)$, and $w_n\,R\,v_m$ does not hold. W.l.o.g. let the sequences be such that $m + n$ is minimal among all the sequences we could choose.
  Because $v_{m-1}\,R\,v_m$ and $\langle M, v_m\rangle \not\models \eta(\psi)$, we have (33) $\langle M, v_{m-1}\rangle \not\models \Box\eta(\psi)$. By Clause (28), $\langle M, v_{m-1}\rangle \models t_{\Box\psi} \to \Box\eta(\psi)$ and with Property (33) we then obtain (34) $\langle M, v_{m-1}\rangle \models \neg t_{\Box\psi}$. Clause (29) $\star : t_{\Diamond t_{\Box\psi}} \to t_{\Box\psi}$ implies that $\langle M, v_{m-1}\rangle \models \neg t_{\Diamond t_{\Box\psi}} \vee t_{\Box\psi}$ and with Property (34), we have $\langle M, v_{m-1}\rangle \models \neg t_{\Diamond t_{\Box\psi}}$. By Clause (32), for every i, $0 \leq i \leq m$, $\langle M, v_i\rangle \models \neg t_{\Diamond t_{\Box\psi}} \vee \Box t_{\Diamond t_{\Box\psi}}$, which allows us, by induction, to establish that for every i, $0 \leq i \leq m-1$, $\langle M, v_i\rangle \models \neg t_{\Diamond t_{\Box\psi}}$. Since $v_0 = u = w'_0$, $\langle M, w'_0\rangle \models \neg t_{\Diamond t_{\Box\psi}}$.
  Because $w'_{n-1} R w'_n$ and $\langle M, w_n\rangle \models t_{\Box\psi}$, we have $\langle M, w'_{n-1}\rangle \models \Diamond t_{\Box\psi}$. Clause (30) $\star : \neg t_{\Diamond t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$ implies $\langle M, w'_{n-1}\rangle \models t_{\Diamond t_{\Box\psi}} \vee \Box\neg t_{\Box\psi}$ and with $\langle M, w'_{n-1}\rangle \models \neg\Box\neg t_{\Box\psi}$ we obtain $\langle M, w'_{n-1}\rangle \models t_{\Diamond t_{\Box\psi}}$. Clause (29) $\star : t_{\Diamond t_{\Box\psi}} \to t_{\Box\psi}$ then gives us $\langle M, w'_{n-1}\rangle \models t_{\Box\psi}$. Using Clauses (29) and (30), we can then inductively show that for every j, $0 \leq j \leq n-1$, $\langle M, w'_j\rangle \models t_{\Diamond t_{\Box\psi}}$. Consequently, $\langle M, w'_0\rangle \models t_{\Diamond t_{\Box\psi}}$ holds, contradicting $\langle M, w'_0\rangle \models \neg t_{\Diamond t_{\Box\psi}}$.

– For Condition (6c) let (35) $\star : t_{\Box\psi} \rightarrow \Box t_\psi$ be in Φ_5, $v, w \in W$, $\mathsf{ml}_M(w) = ml \in S$ (i.e., $w \in M[S]$) and $w R^5 v$. We need to show that there exists a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ_5 with $v \in M[S']$. This is straightforward here as by definition of ρ_{K5} the definitional clause for t_ψ in Φ_5 has the form $\star : \neg t_\psi \vee \eta_f(\psi)$ and $v \in M[\star]$ trivially holds. □

5.6 Summary

Theorem 10 *Let φ be a modal formula in simplified NNF, $L \in \{\mathsf{K}, \mathsf{KB}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}, \mathsf{K5}\}$, and $\Phi_L = \{\{0\} : t_\varphi\} \cup \rho_L(\{0\} : t_\varphi \rightarrow \varphi)$. Then φ is* L*-satisfiable iff Φ_L is* K*-satisfiable.*

Proof Follows from Corollary 1, [21, Theorem 5.2], Theorems 4, 5, 6, 7, 8, 9 for the logics KB, K4, K5, and corresponding results for the logics KD and KT.

6 From SNF$_{sml}$ to SNF$_{ml}$ Using Bounds

As K$_S$P does not support SNF$_{sml}$, in our evaluation of the effectiveness of the reductions defined in Sect. 4, we have used a transformation from SNF$_{sml}$ to SNF$_{ml}$. For KD, KT, KB such a transformation is straightforward as the sets of modal levels occurring in the normal form of modal formulae are all finite. Thus, instead of a single SNF$_{sml}$ clause $S : \neg t_\psi \vee \eta_f(\psi)$, we can use the finite set of SNF$_{ml}$ clauses $\{ml : \neg t_\psi \vee \eta_f(\psi) \mid ml \in S\}$.

However, for K4 and K5 the sets of modal levels labeling clauses are in general not finite. But just as in first-order clausal logic where for every unsatisfiable clause set, there exists a finite subset of its Herbrand expansion that is unsatisfiable, for every unsatisfiable set of SNF$_{sml}$ clauses Φ, there exists an unsatisfiable set of SNF$_{sml}$ clauses $\Phi' = \{S' : C \mid S : C \in \Phi\}$ such that all sets of modal levels S' are finite. The question is whether there is a computable function that can generate Φ' from Φ.

This is indeed the case. For K4, we can take advantage of a bound established by Massacci [18] on the length of prefixes in SST depending on the modal formula φ under consideration. We can use that bound to limit the maximal modal level occurring in a set of modal levels S labeling SNF$_{sml}$ clauses. As all such sets are finite we can straightforwardly use a finite set of SNF$_{ml}$ clauses instead. In order to establish that this approach preserves completeness, we show that for every closed tableaux for a modal formula φ in simplified NNF in the SST (SST) calculus with prefixes limited by Massacci's bound, we can construct a resolution refutation from a set of SNF$_{ml}$ clauses for φ where modal levels are subject to a corresponding bound. Completeness then follows from Massacci's result that the SST calculus with that bound is still refutationally complete.

Formally, the SST calculus uses prefixed formulae, that is, pairs $\sigma : \varphi$, where the prefix σ is a non-empty sequence of natural numbers and φ is a modal formula. Intuitively, σ "names" a world that satisfies φ. In the following, σ is a prefix, $\sigma_0.\sigma_1$ the concatenation of the sequence σ_0 with the sequence σ_1 and $\sigma.n$ the concatenation of σ with n. If $\sigma = n_1.n_2.\ldots.n_{k-1}.n_k$ is a prefix, the *length* of the prefix σ is k and is denoted by $|\sigma|$. For a logic L, an *L-tableau* $\mathcal{T}$ in the SST calculus is a (binary) tree where each node is labeled with a prefixed formula. Nodes other than the root node are labeled with a second prefixed formula, its *premise*, and with the name of the SST rule that was applied to the premise to obtain the formula labeling the node. An L-tableau $\mathcal{T}$ is an *L-tableau for the modal formula φ* if the root of $\mathcal{T}$ is labeled with $1 : \varphi$. An *L-branch $\mathcal{B}$* is a path from the root to a leaf while a *partial L-branch $\mathcal{B}$* is path from the root to some node in the tree. Given a partial L-branch $\mathcal{B} = (m_0, \ldots, m_k)$ and a path $\mathcal{P} = (n_1, \ldots, n_l)$ in $\mathcal{T}$ such that n_1 is a child of m_k, then $\mathcal{B} \circ \mathcal{P}$ denotes the partial

Table 4 Single Step Tableaux rules for K4 and K5

$\alpha : \dfrac{\sigma : (\varphi \wedge \psi)}{\sigma : \varphi \quad \sigma : \psi}$	$\beta : \dfrac{\sigma : (\varphi \vee \psi)}{\sigma : \varphi \mid \sigma : \psi}$	$\pi : \dfrac{\sigma : \Diamond\varphi}{\sigma.n : \varphi}$ with $\sigma.n$ new in the branch		
$K : \dfrac{\sigma : \Box\varphi}{\sigma.n : \varphi}$	$4 : \dfrac{\sigma : \Box\varphi}{\sigma.n : \Box\varphi}$	$4^R : \dfrac{\sigma.n : \Box\varphi}{\sigma : \Box\varphi}$	$4^D : \dfrac{\sigma.n : \Box\varphi}{\sigma.n.m : \Box\varphi}$	$Cxt^1 : \dfrac{1.n : \Box\varphi}{1 : \Box\Box\varphi}$

L-branch $\mathscr{B}' = (m_0, \ldots, m_k, n_1, \ldots, n_l)$. The SST rules for K4 consist of (α), (β), (π), (K) and (4) in Table 4 while the SST rules for K5 consist of (α), (β), (π), (K), (4^R), (4^D) and (Cxt^1). Nodes are added to a tableau $\mathscr{T}$ and labeled as follows: if the antecedent $\sigma : \psi$ of a SST rule (r) labels a node on a branch $\mathscr{B}$, then we extend the branch with nodes labeled with the consequents of the rule and each of those nodes is labeled with $\sigma : \psi$ as premise and (r) as the rule that was applied to create the nodes. Note that rules (K) and (4) can only be applied to a formula $\sigma : \Box\varphi$ if a prefix $\sigma.n$, introduced by an application of rule (π), is already present in a branch. Analogously, for rule 4^D. By a *systematic tableau construction*, we mean an application of the procedure in [11, p. 374] adapted to SST rules.

A prefixed formula $\sigma : \varphi$ is *in* a branch $\mathscr{B}$, denoted by $\sigma : \varphi \in \mathscr{B}$, if there is a node in $\mathscr{B}$ labeled with $\sigma : \varphi$. A prefix is *present* in a branch $\mathscr{B}$ if there is a prefixed formula in $\mathscr{B}$ with that prefix, and it is new if it is not present. A branch $\mathscr{B}$ is *closed* if there is a prefix σ such that either (i) σ : **false** is present in $\mathscr{B}$ or (ii) for some propositional symbol p, both $\sigma : p$ and $\sigma : \neg p$ are present in $\mathscr{B}$. A tableau is *closed* if every branch is closed. A prefixed formula $\sigma : \varphi$ is *reduced for rule* (r) *in* $\mathscr{B}$, if (r) has the form $\sigma : \varphi/\sigma' : \varphi'$ and $\sigma' : \varphi'$ is in $\mathscr{B}$; if (r) has the form $\sigma : \varphi/\sigma_1 : \varphi_1|\sigma_2 : \varphi_2$ and at least one of $\sigma_1 : \varphi_1$ and $\sigma_2 : \varphi_2$ is in $\mathscr{B}$.

By $\mathscr{B}|_\sigma^p$, $\mathscr{B}|_\sigma^m$, and $\mathscr{B}|_\sigma^a$, we denote the sets $\{l|l \in L_P, \sigma : l \in \mathscr{B}\}$, $\{\Diamond l|l \in L_P, \sigma : \Diamond l \in \mathscr{B}\} \cup \{\Box l|l \in L_P, \sigma : \Box l \in \mathscr{B}\}$, and $\mathscr{B}|_\sigma^p \cup \mathscr{B}|_\sigma^m$, respectively.

For a modal formula φ in simplified NNF let $d_\Diamond^\varphi$ be the maximal nesting of $\Diamond$-operators not under the scope of any $\Box$ operators in φ, $n_\Box^\varphi$ be the number of $\Box$-subformulae in φ, and $n_\Diamond^\varphi$ be the number of $\Diamond$-subformulae below $\Box$-operators in φ.

Theorem 11 *A systematic tableau construction of a* K4*-tableau for a modal formula φ in simplified NNF under the following Constraints (TC1) and (TC2)*

(TC1) a rule (r) is only applicable to a prefixed formula $\sigma : \psi$ in a branch $\mathscr{B}$ if the formula is not already reduced for (r) in $\mathscr{B}$;
(TC2) rule (π) is only applicable to prefixed formulae $\sigma : \Diamond\psi$ with $|\sigma| < 2 + d_\Diamond^\varphi + n_\Diamond^\varphi \times n_\Box^\varphi$

terminates in one of following states:

(1) all branches of the constructed tableau are closed and φ is K4*-unsatisfiable or*
(2) at least one branch $\mathscr{B}$ is not closed, no rule is still applicable to a labeled formula in $\mathscr{B}$, and φ is K4*-satisfiable.*

Proof Follows from Theorems 8.1 and 8.4 in [18]. Theorem 8.4 does not require that the tableau construction is systematic, but then allows for a third possible termination state, namely, that in every branch some rule is still applicable. The proof states explicitly that the construction only terminates in this state if it was not systematic. We assume a systematic construction and thereby exclude that third possibility. □

Theorem 11 allows rule (π) to be applied to a prefix σ of length $1 + d_\Diamond^\varphi + n_\Diamond^\varphi \times n_\Box^\varphi$, creating a prefix of length $2 + d_\Diamond^\varphi + n_\Diamond^\varphi \times n_\Box^\varphi$. No prefix of greater length can occur in a tableau.

Table 5 Inference rules of the Modal-Layered Resolution calculus

$$\mathsf{LRES}: \frac{\begin{array}{l} ml: D \lor l \\ ml: D' \lor \neg l \end{array}}{ml: D \lor D'} \qquad \mathsf{MRES}: \frac{\begin{array}{l} ml: l_1 \to \Box l \\ ml: l_2 \to \Diamond \neg l \end{array}}{ml: \neg l_1 \lor \neg l_2} \qquad \mathsf{GEN2}: \frac{\begin{array}{l} ml: l'_1 \to \Box l_1 \\ ml: l'_2 \to \Box \neg l_1 \\ ml: l'_3 \to \Diamond l_2 \end{array}}{ml: \neg l'_1 \lor \neg l'_2 \lor \neg l'_3}$$

$$\mathsf{GEN1}: \frac{\begin{array}{l} ml: l'_1 \to \Box \neg l_1 \\ \quad \vdots \\ ml: l'_m \to \Box \neg l_m \\ ml: l' \to \Diamond \neg l \\ ml+1: l_1 \lor \ldots \lor l_m \lor l \end{array}}{ml: \neg l'_1 \lor \ldots \lor \neg l'_m \lor \neg l'} \qquad \mathsf{GEN3}: \frac{\begin{array}{l} ml: l'_1 \to \Box \neg l_1 \\ \quad \vdots \\ ml: l'_m \to \Box \neg l_m \\ ml: l' \to \Diamond l \\ ml+1: l_1 \lor \ldots \lor l_m \end{array}}{ml: \neg l'_1 \lor \ldots \lor \neg l'_m \lor \neg l'}$$

For K5, Massacci did not provide a bound on the length of prefixes in his SST calculus that preserves refutational completeness. However, using the techniques that he applied to prove such a bound for K4, Theorem 12 establishes a bound for K5.

Theorem 12 *A systematic tableau construction of a* K5*-tableau for a modal formula φ in simplified NNF under the following Constraints (TC1) and (TC2)*

(TC1) a rule (r) is only applicable to a prefixed formula $\sigma : \psi$ in a branch $\mathscr{B}$ if the formula is not already reduced for (r) in $\mathscr{B}$;
(TC2) rule (π) is only applicable to prefixed formulae $\sigma : \Diamond\psi$ with $|\sigma| < 2 + d^{\varphi}_{\Diamond} + n^{\varphi}_{\Diamond}$

terminates in one of following conditions:

(1) all branches of the constructed tableau are closed and φ is K5*-unsatisfiable or*
(2) at least one branch $\mathscr{B}$ is not closed, no rule is still applicable to a labeled formula in $\mathscr{B}$, and φ is K5*-satisfiable.*

In order to establish a relationship between closed tableaux and resolution refutations of a set of SNF_{ml} clauses, we formally define the modal-layered resolution calculus. Table 5 shows the inference rules of the calculus restricted to labels occurring in the clauses produced by our reductions. For GEN1 and GEN3, if the modal clauses in the premises occur at the modal level ml, then the literal clause in the premises occurs at the modal level, $ml + 1$.

Let Φ be a set of SNF_{ml} clauses. A *(resolution) derivation from Φ* is a sequence of sets $\Phi_0, \Phi_1, \ldots$ where $\Phi_0 = \Phi$ and, for each $i > 0$, $\Phi_{i+1} = \Phi_i \cup \{D\}$, where $D \notin \Phi_i$ is the resolvent obtained from Φ_i by an application of one the inference rules to premises in Φ_i. A *(resolution) refutation of Φ* is a derivation $\Phi_0, \ldots, \Phi_k$, $k \in \mathbb{N}$, where $0 : \mathbf{false} \in \Phi_k$.

To map a set of SNF_{sml} clauses to a set of SNF_{ml} clauses, using a bound $n \in \mathbb{N}$ on the modal levels, we define a function db_n on clauses and sets of clauses in SNF_{sml} as follows:

$$\mathrm{db}_n(S : \varphi) = \{ml : \varphi \mid ml \in S \text{ and } ml \leq n\}$$
$$\mathrm{db}_n(\Phi) = \bigcup_{S:\varphi\in\Phi} \mathrm{db}_n(S : \varphi)$$

Note that prefixes in SST-tableaux have a minimal length of 1 while the minimal modal level in SNF_{ml} clauses is 0. So, a prefix of length n in a prefixed formula corresponds to a modal level $n - 1$ in an SNF_{ml} clause.

Theorem 13 *Let φ be a* K4*-unsatisfiable formula in simplified NNF. Let $hb^{\varphi}_{K4} = 2 + d^{\varphi}_{\Diamond} + n^{\varphi}_{\Diamond} \times n^{\varphi}_{\Box}$. Let $\Phi_4 = \mathrm{db}_{hb^{\varphi}_{K4}-1}(\{\{0\} : t_{\varphi}\} \cup \rho_{K4}(\{0\} : t_{\varphi} \rightarrow \varphi))$. Then there is a resolution refutation of Φ_4.*

Proof (Sketch) We inductively construct a closed K4-tableau $\mathscr{T}$ for φ as follows:

1. The root node of $\mathscr{T}$ is labeled with the prefixed formulae $1 : \varphi$.
2. While the tableau is not closed do:
   Let $\mathscr{B}$ be the left-most branch of $\mathscr{T}$ that is not closed yet, and let σ be the longest prefix of any prefixed formula in $\mathscr{B}$.
   (a) If rule (r), $r \in \{\alpha, \beta\}$, can be applied to a formula $\sigma : \psi$ in $\mathscr{B}$ such that $\sigma : \psi$ is not already reduced in $\mathscr{B}$, then extend $\mathscr{B}$ by applying (r) to $\sigma : \psi$;
   (b) If every formula $\sigma : \psi$ to which a rule (r), $r \in \{\alpha, \beta\}$, could be applied is already reduced in $\mathscr{B}$, then $\mathscr{B}|^{p}_{\sigma}$ must be a consistent set of propositional literals (otherwise $\mathscr{B}$ would be closed), $\mathscr{B}|^{m}_{\sigma}$ has the form $\{\Diamond\varphi_1, \ldots, \Diamond\varphi_m, \Box\psi_1, \ldots, \Box\psi_n\}$ with $m > 0$ and $n \geq 0$, and there exists at least one j, $1 \leq j \leq m$, such that $\{\Diamond\varphi_j, \Box\psi_1, \ldots, \Box\psi_n\}$ is K4-unsatisfiable. We pick exactly one such j. First, extend $\mathscr{B}$ by applying rule (π) to $\sigma : \Diamond\varphi_j$, adding a node labeled with $\sigma' : \varphi_j$, where $\sigma' = \sigma.n_j$ for some $\sigma.n_j$ that is new in $\mathscr{B}$. Second, extend $\mathscr{B}$ by applying rule (K) to $\sigma : \Box\psi_1$, ..., $\sigma : \Box\psi_n$, respectively, adding nodes labeled with $\sigma' : \psi_l$, $1 \leq l \leq n$. Third, extend $\mathscr{B}$ by applying rule (4) to $\sigma : \Box\psi_1$, ..., $\sigma : \Box\psi_n$, respectively, adding nodes labeled with $\sigma' : \Box\psi_l$, $1 \leq l \leq n$.

We can prove that it is indeed possible to construct a closed K4-tableau in the manner described above. Then, according to Theorem 11, the construction will terminate with a closed tableau that only contains prefixes σ with $|\sigma| \leq hb^{\varphi}_{K4}$.

Case (a): Assume there are no applications of rule (π) in the construction of $\mathscr{T}$. As only an application of rule (π) would introduce a new prefix in a tableau derivation, the only prefix occurring in the tableaux is 1 with $|1| = 1 \leq 2 \leq hb^{\varphi}_{K4}$.
As rule (π) was not used, only rules (α) and (β) have been used. We can prove that the propositional formula $\bar{\varphi}$ obtained from φ by replacing all subformulae of φ of the form $\Box\psi$ by $t_{\Box\psi}$ and all subformulae of the form $\Diamond\psi$ by $t_{\Diamond\psi}$ is unsatisfiable. Then $\bar{\Phi}_4 = \mathrm{db}_{hb^{\bar{\varphi}}_{K4}-1}(\{\{0\} : t_{\bar{\varphi}}\} \cup \rho_{K4}(\{0\} : t_{\bar{\varphi}} \rightarrow \bar{\varphi})) \subseteq \Phi_4$ and $\bar{\Phi}_4$ only contains literal clauses with label 0 independent of the bound $hb^{\bar{\varphi}}_{K4}$. As $\bar{\varphi}$ is unsatisfiable so must be $\bar{\Phi}_4$ and there must be a resolution refutation of $\bar{\Phi}_4$ using only the inference rule LRES due to the refutational completeness of LRES, for sets of literal clauses.
Case (b): Let N be the set of all nodes on $\mathscr{T}$ labeled with rule (π), i.e., each of those nodes was added by an application of rule (π).
Let $\mathfrak{B}$ be the set of all partial branches such that for every node n in N, $\mathfrak{B}$ contains a partial branch $(n_0, \ldots, n_k)$, $0 \leq k$, where n_0 is the root node of $\mathscr{T}$ and n is the successor node of n_k in $\mathscr{B}$. Each partial branch in $\mathfrak{B}$ represents a 'state' that a branch of $\mathscr{T}$ was in just before rule (π) was applied in our construction. We define a well-founded partial order $\prec$ on $\mathfrak{B}$ as $\mathscr{B} \prec \mathscr{B}'$ iff $\mathscr{B}$ is an extension of $\mathscr{B}'$.
We first show that for every $\mathscr{B} \in \mathfrak{B}$, we can derive a literal clause $ml : C_{\mathscr{B}}$ from Φ_4 that subsumes $ml : \neg t_{\Diamond\varphi_{\mathscr{B}}} \vee \bigvee(\{\neg t_{\Box\psi} \mid \Box\psi \in \mathscr{B}|^{m}_{\sigma_{\mathscr{B}}}\})$ where $ml = |\sigma_{\mathscr{B}}| - 1$. The proof proceeds by induction on $\langle \mathfrak{B}, \prec \rangle$ and the derivation of the literal clause $ml : C_{\mathscr{B}}$ involves the rules GEN1, GEN2 and GEN3.

Then consider the closed tableau $\mathscr{T}$ with root node n_φ. Let $\mathscr{T}'$ be the subtree of $\mathscr{T}$ with root node n_φ and containing only those nodes and branches formed by applications of rules (α) and (β) to n_φ and its descendants.
Each branch $\mathscr{B}$ of $\mathscr{T}'$ such that the propositional formula $\bigwedge \mathscr{B}|_1^p$ is satisfiable must be an element of $\mathfrak{B}$ with associated literal clause $0 : C_{\mathscr{B}}$. Let $\mathcal{C}$ be the set of all those clauses.
With $\bar{\varphi}$ and $\bar{\Phi}_4$ defined as in Case (a), we can then show that $\bar{\Phi}_4 \cup \mathcal{C}$ is unsatisfiable. As $\bar{\Phi}_4 \subseteq \Phi_4$ and all clauses in $\mathcal{C}$ are derivable from Φ_4, Φ_4 is unsatisfiable and there must exist a resolution refutation of it. □

In analogy, we can also prove a corresponding result for K5 with the bound established in Theorem 12.

Theorem 14 *Let φ be a* K5 *unsatisfiable formula in simplified NNF. Let $hb^\varphi_{K5} = 2 + d^\varphi_\Diamond + n^\varphi_\Diamond$. Let $\Phi_5 = \mathrm{db}_{hb^\varphi_{K5}-1}(\{\{0\} : t_\varphi\} \cup \rho_{K5}(\{0\} : t_\varphi \to \varphi))$. Then there is a resolution refutation of Φ_5.*

Example 2 Reconsider the K4-unsatisfiable formula $\varphi = \Diamond q \wedge \Diamond\Diamond(\Box(p \wedge \Diamond\Diamond\neg p) \wedge \Diamond q)$ from Example 1. We have $d^\varphi_\Diamond = 3$, $n^\varphi_\Diamond = 2$, and $n^\varphi_\Box = 1$. So, $hb^\varphi_{\mathsf{K4}} = 2 + d^\varphi_\Diamond + n^\varphi_\Diamond \times n^\varphi_\Box = 2 + 3 + 2 \times 1 = 7$. By Theorem 11 a systematic tableau construction of a K4-tableau for φ where rule (π) is only applicable to prefixed formulae $\sigma : \Diamond\psi$ with $|\sigma| < hb^\varphi_{\mathsf{K4}}$ should terminate with a closed tableau. Below is such a tableau.

	(36)	$1 : \Diamond q \wedge \Diamond\Diamond(\Box(p \wedge \Diamond\Diamond\neg p) \wedge \Diamond q)$
$[(\alpha);(\pi), 36]$	(37)	$1.1 : q$
$[(\alpha);(\pi), 36]$	(38)	$1.2 : \Diamond(\Box(p \wedge \Diamond\Diamond\neg p) \wedge \Diamond q)$
$[(\pi);(\alpha), 38]$	(39)	$1.2.1 : \Box(p \wedge \Diamond\Diamond\neg p)$
$[(\pi);(\alpha), 38]$	(40)	$1.2.1 : \Diamond q$
$[(\pi), 40]$	(41)	$1.2.1.1 : q$
$[(K);(\alpha), 39]$	(42)	$1.2.1.1 : p$
$[(K);(\alpha), 39]$	(43)	$1.2.1.1 : \Diamond\Diamond\neg p$
$[(4), 39]$	(44)	$1.2.1.1 : \Box(p \wedge \Diamond\Diamond\neg p)$
$[(\pi), 43]$	(45)	$1.2.1.1.1 : \Diamond\neg p$
$[(K);(\alpha), 44]$	(46)	$1.2.1.1.1 : p$
$[(K);(\alpha), 44]$	(47)	$1.2.1.1.1 : \Diamond\Diamond\neg p$
$[(4), 44]$	(48)	$1.2.1.1.1 : \Box(p \wedge \Diamond\Diamond\neg p)$
$[(\pi), 45]$	(49)	$1.2.1.1.1.1 : \neg p$
$[(K);(\alpha), 48]$	(50)	$1.2.1.1.1.1 : p$
$[(K);(\alpha), 48]$	(51)	$1.2.1.1.1.1 : \Diamond\Diamond\neg p$
		clash between (49) and (50)

Note that the bound is not reached in this particular tableau. The bound is a worst case, and tableaux requiring such a bound exist for the input formula φ.

From the resulting clauses of $\Phi_\varphi = \{\{0\} : t_\varphi\} \cup \rho_{\mathsf{K4}}(\{0\} : t_\varphi \to \varphi)$ in Example 1 and $\mathrm{db}_{hb^\varphi_{\mathsf{K4}}-1} = \mathrm{db}_6(\Phi_\varphi)$, the set of clauses in SNF_{ml} of φ is as follows, where ψ subformulae are defined as in Example 1.

(52) $0 : t_\varphi$
(53) $0 : \neg t_\varphi \vee t_{\Diamond q}$
(54) $0 : \neg t_\varphi \vee t_{\psi_1}$
(55) $0 : t_{\psi_1} \rightarrow \Diamond t_{\psi_2}$
(56) $0 : t_{\Diamond q} \rightarrow \Diamond q$
(57) $1 : t_{\psi_2} \rightarrow \Diamond t_{\psi_3}$
(58) $2 : \neg t_{\psi_3} \vee t_{\Box \psi_4}$
(59) $2 : \neg t_{\psi_3} \vee t_{\Diamond q}$
(60) $2 : t_{\Diamond q} \rightarrow \Diamond q$
(61) $2 : t_{\Box \psi_4} \rightarrow \Box t_{\psi_4}$
(62) $2 : t_{\Box \psi_4} \rightarrow \Box t_{\Box \psi_4}$
(63) $3 : t_{\Box \psi_4} \rightarrow \Box t_{\psi_4}$
(64) $3 : t_{\Box \psi_4} \rightarrow \Box t_{\Box \psi_4}$
(65) $3 : \neg t_{\psi_4} \vee p$
(66) $3 : \neg t_{\psi_4} \vee t_{\Diamond \Diamond \neg p}$
(67) $3 : t_{\Diamond \Diamond \neg p} \rightarrow \Diamond t_{\Diamond \neg p}$
(68) $4 : t_{\Box \psi_4} \rightarrow \Box t_{\psi_4}$
(69) $4 : t_{\Box \psi_4} \rightarrow \Box t_{\Box \psi_4}$
(70) $4 : \neg t_{\psi_4} \vee p$
(71) $4 : \neg t_{\psi_4} \vee t_{\Diamond \Diamond \neg p}$
(72) $4 : t_{\Diamond \Diamond \neg p} \rightarrow \Diamond t_{\Diamond \neg p}$
(73) $4 : t_{\Diamond \neg p} \rightarrow \Diamond \neg p$
(74) $5 : t_{\Box \psi_4} \rightarrow \Box t_{\psi_4}$
(75) $5 : t_{\Box \psi_4} \rightarrow \Box t_{\Box \psi_4}$
(76) $5 : \neg t_{\psi_4} \vee p$
(77) $5 : \neg t_{\psi_4} \vee t_{\Diamond \Diamond \neg p}$
(78) $5 : t_{\Diamond \Diamond \neg p} \rightarrow \Diamond t_{\Diamond \neg p}$
(79) $5 : t_{\Diamond \neg p} \rightarrow \Diamond \neg p$
(80) $6 : t_{\Box \psi_4} \rightarrow \Box t_{\psi_4}$
(81) $6 : t_{\Box \psi_4} \rightarrow \Box t_{\Box \psi_4}$
(82) $6 : \neg t_{\psi_4} \vee p$
(83) $6 : \neg t_{\psi_4} \vee t_{\Diamond \Diamond \neg p}$
(84) $6 : t_{\Diamond \Diamond \neg p} \rightarrow \Diamond t_{\Diamond \neg p}$
(85) $6 : t_{\Diamond \neg p} \rightarrow \Diamond \neg p$

A refutation for this set of clauses is the following.

[GEN1,68,73,76]	(86)	$4 : \neg t_{\Box \psi_4} \vee \neg t_{\Diamond \neg p}$
[GEN1,64,67,86]	(87)	$3 : \neg t_{\Box \psi_4} \vee \neg t_{\Diamond \Diamond \neg p}$
[LRES,66,87]	(88)	$3 : \neg t_{\Box \psi_4} \vee \neg t_{\psi_4}$
[GEN1,60,61,62,88]	(89)	$2 : \neg t_{\Diamond q} \vee \neg t_{\Box \psi_4}$
[LRES,59,89]	(90)	$2 : \neg t_{\psi_3} \vee \neg t_{\Box \psi_4}$
[LRES,58,90]	(91)	$2 : \neg t_{\psi_3}$
[GEN1,57,91]	(92)	$1 : \neg t_{\psi_2}$
[GEN1,55,92]	(93)	$0 : \neg t_{\psi_1}$
[LRES,54,93]	(94)	$0 : \neg t_\varphi$
[LRES,52,94]	(95)	$0 : \bot$

Note that the refutation uses $t_{\Box\psi_4} \rightarrow \Box t_{\psi_4}$ twice, in the form of clauses (61) and (63), corresponding to the two applications of (4) in the tableau. Note also that the maximal level of a clause involved in the refutation is 5 (Clause (76)) and therefore equal to the length of the longest prefix occurring in the tableau, 1.2.1.1.1.1, minus one.

7 Comparison With Related Work

The approaches most closely related to ours are Kracht's reductions of normal modal logics to basic modal logic [16, 17], the global modal resolution calculus [20], and Schmidt and Hustadt's axiomatic translation principle for translations of normal modal logics to first-order logic [31].

The first significant difference to our approach is that Kracht's reductions and the axiomatic translation exclude the modal operator $\Diamond$ from the language and only consider the modal operator $\Box$. In order to present Kracht's approach, we need some additional notions. Let $\mathsf{sf}(\varphi)$, $\mathsf{dg}(\varphi)$, and $|S|$ denote the set of all subformulae of φ, the maximum nesting of modal operators in φ, and the cardinality of the set S, respectively. Let $\Diamond^0\psi = \Box^0\psi = \Box^{<1}\psi = \psi$, $\Box^{<n+1}\psi = (\psi \wedge \Box\Box^{<n}\psi)$, $\Box^{n+1}\psi = \Box\Box^n\psi$, and $\Diamond^{n+1}\psi = \Diamond\Diamond^n\psi$. We can then define a reduction function ρ_L^{K} for a normal modal logic L in $\{\mathsf{KB}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}\}$ as follows:

$$\rho_L^{K}(\varphi) = \begin{cases} \varphi \wedge \Box^{<|\mathsf{sf}(\varphi)|+1} P_{K4}^{\mathsf{K}}(\varphi), & \text{for } L = \mathsf{K4} \\ \varphi \wedge \Box^{<\mathsf{dg}(\varphi)+1} P_L^{\mathsf{K}}(\varphi) & \text{otherwise} \end{cases}$$

where $P_{KB}^{\mathsf{K}}(\varphi) = \{\neg\psi \rightarrow \Box\neg\Box\psi \mid \Box\psi \in \mathsf{sf}(\varphi)\}$ $P_{KD}^{\mathsf{K}}(\varphi) = \{\neg\Box\,\mathbf{false}\}$
$P_{K4}^{\mathsf{K}}(\varphi) = \{\Box\psi \rightarrow \Box\Box\psi \mid \Box\psi \in \mathsf{sf}(\varphi)\}$ $P_{KT}^{\mathsf{K}}(\varphi) = \{\Box\psi \rightarrow \psi \mid \Box\psi \in \mathsf{sf}(\varphi)\}$

Table 6 Inference rules in [20] for K5 (EUC1 and EUC2)

[EUC1]	$\square^*(l_1 \to \neg\square\neg l)$	[EUC2]	$\square^*(l \to \square l_2)$
$\square^*(\textbf{true} \to \neg l_1 \vee t_{\Diamond l})$	$\square^*(t_{\Diamond l} \to \neg\square\neg l)$	$\square^*(t_{\Diamond l} \to \square l_2)$	$\square^*(t_{\Diamond l} \to \neg\square\neg l)$
$\square^*(\neg t_{\Diamond l} \to \square\neg l)$	$\square^*(t_{\Diamond l} \to \square t_{\Diamond l})$	$\square^*(\neg t_{\Diamond l} \to \square\neg l)$	$\square^*(t_{\Diamond l} \to \square t_{\Diamond l})$

Kracht shows that φ is L-satisfiable iff $\rho_L^{\mathsf{K}}(\varphi)$ is K-satisfiable. There are three differences to our approach. First, $P_L^{\mathsf{K}}(\varphi)$ will include an axiom instance for every occurrence of a subformula $\neg\square\psi$, equivalent to $\Diamond\neg\psi$, in φ. In contrast, our approach requires no logic specific treatment of such subformulae. Second, the use of $\square^{<n}P_L^{\mathsf{K}}(\varphi)$ in ρ_L^{K} means that the axiom instance is available at every modal level. This means, for example, that for $\vartheta_1 = \Diamond^{100}(\neg p \wedge \square p)$, the formula $\rho_{KT}^{\mathsf{K}}(\vartheta_1)$ contains the axiom instance $\square p \to p$ over 100 times, although it is only required at the level at which $\square p$ occurs. Third, this is further compounded if the formula ψ in $\square\psi$ is itself a complex formula. We try to avoid that by using a surrogate propositional symbol t_ψ instead, but this will only have a positive effect if the definitional clauses for t_ψ do not have to be repeated.

The global modal resolution (GMR) calculus operates on SNF_K clauses, that is, clauses of the form

$$\square^*(\textbf{start} \to \textstyle\bigvee_{b=1}^{r} l_b) \quad \square^*(\textbf{true} \to \textstyle\bigvee_{b=1}^{r} l_b) \qquad \square^*(l' \to \square l) \qquad \square^*(l' \to \neg\square l)$$

where l, l', l_b are propositional literals with $1 \le b \le r$, $r \in \mathbb{N}$, and $\square^*$ is the universal operator. The calculus has specific inference rules for normal modal logics such as KB, KD, KT, K4, K5. Table 6 shows the two additional rules for K5, the only logic for which there are rules for both $\square$ and $\neg\square\neg$, i.e., $\Diamond$. These inference rules can be seen to perform an 'on-the-fly' computation of a reduction. Note that the clauses produced by P_{K5} differ from those produced by GMR for K5. Implicitly, our results here also show that it should be possible to eliminate EUC1 from the GMR calculus.

For the axiomatic translation, we only present the function P_L^{RS} that computes the logic dependent first-order clausal formulae that are part of the overall translation.

$$\begin{aligned}
P_{\mathsf{KB}}^{\mathsf{RS}}(\square\psi) &= \{\forall x(\neg Q_{\square\psi}(y) \vee \neg R(x,y) \vee Q_\psi(x)) \mid \square\psi \in \mathsf{sf}(\varphi)\}\\
P_{\mathsf{KD}}^{\mathsf{RS}}(\square\psi) &= \{\forall x(\neg Q_{\square\psi}(x) \vee Q_{\neg\square\neg\psi}(x)) \mid \square\psi \in \mathsf{sf}(\varphi)\}\\
P_{\mathsf{KT}}^{\mathsf{RS}}(\square\psi) &= \{\forall x(\neg Q_{\square\psi}(x) \vee Q_\psi(x)) \mid \square\psi \in \mathsf{sf}(\varphi)\}\\
P_{\mathsf{K4}}^{\mathsf{RS}}(\square\psi) &= \{\forall xy(\neg Q_{\square\psi}(x) \vee \neg R(x,y) \vee Q_{\square\psi}(y)) \mid \square\psi \in \mathsf{sf}(\varphi)\}\\
P_{\mathsf{K5}}^{\mathsf{RS}}(\square\psi) &= \{\forall xy(\neg Q_{\square\psi}(y) \vee \neg R(x,y) \vee Q_{\square\psi}(x)),\\
&\qquad \forall xy(\neg Q_{\square\neg\square\psi}(y) \vee \neg R(x,y) \vee Q_{\square\neg\square\psi}(x)) \mid \square\psi \in \mathsf{sf}(\varphi)\}
\end{aligned}$$

Here the variables x and y range over worlds. The predicate symbols Q_ψ correspond to our surrogate symbols t_ψ. The clausal formulae used in the treatment of KT and K4 are translations of the SNF_{ml} clauses we use (or vice versa). KB and K5 are handled in a different way as the first-order clausal formulae refer directly the accessibility relation and can therefore more easily express the transfer of information to a predecessor world. The universal quantification over worlds also means that the constraints expressed by the formulae hold at all modal levels without the need of any repetition.

In Sect. 8, we will also use the relational and semi-functional translation of modal logics to first-order logic combined with structural transformation to clause normal form. In both approaches $\square\psi$ is translated as $\forall xy(\neg Q_{\square\psi}(x) \vee \neg R(x,y) \vee Q_\psi(y))$, while $\Diamond\psi$ becomes

$\forall x \exists y(\neg Q_{\Diamond\psi}(x) \vee (R(x, y) \wedge Q_{\psi}(y)))$ and $\forall x \exists \alpha(\neg Q_{\Diamond\psi}(x) \vee (\mathrm{def}(x) \wedge Q_{\psi}([x\alpha])))$ in the relational and semi-functional translation, respectively. Here, the variables x and y also range over worlds while α and β range over partial accessibility functions.

Then, depending on the modal logics, further formulae representing the semantic properties of the accessibility R are added. For the relational translation these will simply be the formulae in the fourth column of Table 1. The semi-functional translation uses collections of partial accessibility function in addition to the accessibility relation. A predicate def is used to represent on which worlds a partial accessibility function is defined. For each modal logic there is then again a background theory consisting of formulae over def and R that represents the properties of the underlying accessibility relation which is added to the translation of a formula. For example, for $\mathsf{K5}$ the background theory is: $\forall xy \forall \alpha\beta((\neg \mathrm{def}(x) \vee \mathrm{def}(y)) \wedge (\neg \mathrm{def}(\mathrm{w}_0) \vee R(\mathrm{w}_0, [\mathrm{w}_0\alpha])) \wedge (\neg \mathrm{def}(x) \vee \neg \mathrm{def}(y) \vee R([x\alpha], [y\beta])))$, where w_0 is a constant representing the root world in a rooted Kripke structure.

In [35], Sebastiani and Venscovi present an encoding of K into propositional logic. In this encoding, each propositional symbol produced by their reduction corresponds to labeled formulae in each possible world following closely the application of the tableau rules for K given in [18]. As noted by the authors, such encoding leads to an exponential blow-up in the size of a formula in the worst case (if P $\neq$ NP). In practice, however, their implemented tool K_m2SAT, combined with state of the art SAT solvers, performed well in most of the usual benchmarks. The InkreSAT prover [15] provides decision procedures for K, KT, $\mathsf{K4}$ and $\mathsf{S4}$, that is, a subset of the logics considered here. Their approach also reduces the satisfiability problem for a particular logic into the satisfiability problem for propositional logic. Differently from [35], the translation is interleaved with calls to the underlying SAT prover; the result from the SAT prover is then incrementally used to guide the translation. This helps with earlier simplification and better performance when compared with K_m2SAT. In the worst case, however, as with [35], the translation is exponential in the size of the input formula. In [24], we have compared InKreSAT and $\mathrm{K_SP}$, K_m2SAT does not appear to be publicly available anymore. The evaluation indicates InKreSAT has the second best performance when all the benchmarks were considered, but less impressive results on the LWB benchmark with the fifth best performance among the six competing tools.

8 Evaluation

For our evaluation, we have restricted ourselves to fully automatic provers with built-in support for all the six logics we have considered. By 'built-in support,' we mean the possibility to specify the logic either via command-line option or via a configuration option within an input file together with modal formula.

We have compared the performance of the following approaches: (i) the combination of our reductions with the modal-layered resolution (MLR) calculus for SNF_{ml} clauses [21], *R+MLR calculus* for short, implemented in the modal theorem prover $\mathrm{K_SP}$, with three different refinements for resolution inferences on labeled propositional clauses; (ii) the global modal resolution (GMR) calculus, also implemented in $\mathrm{K_SP}$, with three different refinements for resolution inferences on propositional clauses; (iii) the combinations of the relational and semi-functional translation of modal logics to first-order logic with ordered first-order resolution implemented in the first-order theorem prover SPASS; (iv) the higher-order logic prover LEO-III with **E** 2.6 as external reasoner.

In total this gives us nine different approaches to compare. The axiomatic translation is currently not implemented in SPASS. Other provers, such as LWB [13] and MleanCoP [27], do not have built-in support for the full range of logics considered here. LoTREC 2.0 [8] supports all the logics, but is not intended as automatic theorem prover.

K_SP [19] implements the reductions presented in Sect. 3, the transformation from SNF_{sml} to SNF_{ml} presented in Sect. 6, as well as a normal form transformation of modal formulae to sets of SNF_K clauses, required for the GMR calculus. It implements both the R+MLR and the GMR calculus. Resolution inferences between (labeled) propositional clauses can either be unrestricted (`cplain` option), restricted by an ordering (`cord` option), that is, clauses can only be resolved on their maximal literals with respect to an ordering chosen by the prover in such a way to preserve completeness, restricted to negative resolution (`cneg` option), that is, one of the premises in an inference has to be a negative clause, or restricted to positive resolution. We do not include the last option in our evaluation as it typically performs worse. K_SP also implements a range of simplification rules that are applied to modal formulae before their transformation to normal form. Of those, we have enabled pure literal elimination (`early_ple` option), simplification using the Box Normal Form [28] and Prenex Normal Form (`bnfsimp` and `prenex` options) [22]. For clause processing, unit resolution and pure elimination are enabled (`unit`, `lhs_unit`, and `ple` options).

SPASS 3.9 [38, 39] supports automated reasoning in extended modal logics, including all logics considered here, PDL-like modal logics as well as description logics. It includes eight different translations of modal logics to first-order logic. In our evaluation, we have used the relational translation and the semi-functional translation. For the local satisfiability problem in KB to $\mathsf{K5}$, for the relational translation, SPASS adds the first-order frame properties given in Table 1 while for the semi-functional translation, it adds the background theories devised by Nonnengart [26]. For the transformation to first-order clausal form, we have enabled renaming of quantified subformulae. The only inference rules used are ordered resolution and ordered factoring, the reduction rules used are condensing, backward subsumption, and forward subsumption. For the relational and semi-functional translation for K, KB, KD, and KT we thereby obtain a decision procedure, while for the other logics we do not. For $\mathsf{K4}$ and $\mathsf{K5}$, the fragment of first-order clausal logic corresponding to the semi-functional translation of modal formula and their background theories is decidable by ordered resolution with selection [32]. However, the non-trivial ordering and selection function required is not currently implemented in SPASS.

LEO-III [9, 36] makes use of a semantic embedding approach [10] to automatically transform modal formulae into corresponding HOL formulae. This embedding is most closely related to the relational translation in that it employs a representation of the worlds and accessibility relationship in Kripke frames and deals with modal logics other than the basic modal logic K by adding the corresponding frame properties. LEO-III implements extensional paramodulation for higher-order logic [37] but can also collaborate with external reasoners during proof search. In our evaluation, we have exclusively used **E** 2.6 [33, 34] as external reasoner.

For our evaluation, we have chosen the LWB basic modal logic benchmark collection [2], with 20 formulae in each of 18 parameterized classes. For K, all formulae in 9 classes are satisfiable while all formulae in the other 9 classes are unsatisfiable. In simplified NNF, 63% of modal operators are $\Box$ and 37% are $\Diamond$ operators. We have used the collection for each of the six logics. If a formula is unsatisfiable in K then it remains unsatisfiable in the other five logics, while the opposite is not true. As we move to logics other than K, it is also no longer the case that all formulae in a class have the same satisfiability status.

Table 7 shows the results of our evaluation. The first column lists the six logics. We then separate the 360 LWB benchmark formulae into satisfiable (S) and unsatisfiable (U) formulae with respect to each logic. This gives us 12 *categories*. The third column then indicates the

Table 7 Experimental results on LWB benchmark collection

L	*S*	Total	KSP (GMR, `cneg`)	KSP (GMR, `cord`)	KSP (GMR, `cplain`)	LEO-III	
K	S	180	112	139	93	0	
K	U	180	154	156	152	58	
KD	S	180	125	145	119	0	
KD	U	180	154	**156**	152	54	
KT	S	100	53	**69**	38	0	
KT	U	260	234	**238**	225	166	
KB	S	122	53	**81**	42	0	
KB	U	238	197	208	198	121	
K4	S	161	40	**60**	38	0	
K4	U	199	146	144	148	75	
K5	S	60	**17**	15	10	0	
K5	U	300	256	256	**261**	243	
All	S	803	400	**509**	340	0	
All	U	1357	1141	1158	1136	717	
L	*S*	Total	KSP (R+MLR, `cneg`)	KSP (R+MLR, `cord`)	KSP (R+MLR, `cplain`)	SPASS (semi-functional)	SPASS (relational)
K	S	180	142	**158**	138	92	97
K	U	180	**159**	158	156	134	122
KD	S	180	141	**156**	133	107	103
KD	U	180	155	**156**	155	136	130
KT	S	100	47	56	27	47	39
KT	U	260	231	**238**	222	222	199
KB	S	122	26	75	16	31	23
KB	U	238	207	**214**	201	159	169
K4	S	161	39	53	17	0	0
K4	U	199	**155**	132	153	109	35
K5	S	60	8	10	4	7	0
K5	U	300	255	247	247	255	124
All	S	803	403	508	335	284	262
All	U	1357	**1162**	1145	1134	1015	779

number of formulae in each. The last nine columns in the table show how many formulae within a category each of the approaches was able to solve with a time limit of 100 CPU seconds for each formula. In the last two lines of the table, we sum up the results for all logics. Benchmarking was performed on a PC with an AMD Ryzen 5 5600X CPU @ 4.60GHz max and 32GB main memory using Fedora release 34 as operating system. As we can see, the two best performing approaches are the GMR calculus with the ordered resolution refinement (`cord`) and the R+MLR calculus with the ordered resolution refinement, with the former performing slightly better. Each approach achieves the highest number of solved formulae in 3 categories and they are joint best on a further two categories. The GMR calculus is better on satisfiable formulae in almost all logics as it avoids the duplication of clauses introduced by the transformation from SNF_{sml} to SNF_{ml} required for the R+MLR calculus. On the other hand, the R+MLR calculus is better on most categories of unsatisfiable formulae. For SPASS, overall, we see a clear advantage of the semi-functional translation over the relational one, on both satisfiable and unsatisfiable formulae. LEO-III performs reasonably well on unsatisfiable formulae but cannot solve any of the satisfiable formulae. It is interesting to see that it performs better than SPASS with the relational translation on unsatisfiable K4 and K5 formulae. We put this down to the use of **E** as external prover. Out of 717 formulae solved by LEO-III, **E** provides the proof for 573 of those, including 267 out of 318 unsatisfiable K4 and K5 formulae solved. This shows that **E** solves considerably more of those formulae than SPASS with 159 formulae.

It is worth pointing out that the results for K_SP in Table 7 differ from those in [29]. First, improvements have been made to the implementation of the GMR calculus meaning it solves more formulae now. Second, in [29], we have used bounds for the reduction from SNF_{sml} to SNF_{ml} for K4 and K5 that were sufficient for the LWB benchmark formulae, but lower than the worst case bounds we established in Sect. 6. Here, we use the latter which results in fewer formulae being solved by the R+MLR calculus.

9 Conclusion and Future Work

We have presented new reductions of propositional modal logics KB, KD, KT, K4, K5 to Separated Normal Form with Sets of Modal Levels. We have shown experimentally that these reductions allow us to reason effectively in these logics.

The obvious next step is to consider extensions of the basic modal logic K with combinations of the axioms B, D, T, 4, and 5. Unfortunately, a simple combination of the reductions for each of the axioms is not sufficient to obtain a satisfiability-preserving reduction for the such modal logics. An example is the simple formula $\neg p \wedge \Diamond\Diamond\Box p$ which is KB4-unsatisfiable. If we define

$$\begin{aligned} P_{KB4}(S : t_{\Box\psi} \to \Box\psi) &= P_{KB}(S : t_{\Box\psi} \to \Box\psi) \cup P_{K4}(S : t_{\Box\psi} \to \Box\psi) \\ \Delta_{KB4}(S : t_{\Box\psi} \to \Box\psi) &= \delta_{KB4}(\star, \psi), \end{aligned}$$

that is, P_{KB4} is the union of P_{KB} and P_{K4}, then the clause set obtained from $\{\{0\} : t_0\} \cup \rho_{KB4}(\{0\} : t_0 \to \neg p \wedge \Diamond\Diamond\Box p)$ is K-satisfiable. The same issue also occurs in the axiomatic translation of modal logics to first-order logic where the translation for KB4 is not simply the combination of the translations for KB and K4 [31, Theorem 5.6]. We are currently exploring solutions to this problem.

Regarding practical applications, it would be advantageous to have an implementation of a calculus that operates directly SNF_{sml} clauses. This would greatly reduce the number of inference steps performed on satisfiable formulae and simplify proof search in general. Again, such an implementation is future work.

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