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Möbius function of the subgroup lattice of a finite group and Euler characteristic

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Abstract
The Möbius function of the subgroup lattice of a finite group has been introduced by Hall and applied to investigate several questions. In this paper, we consider the Möbius function defined on an order ideal related to the lattice of the subgroups of an irreducible subgroup G of the general linear group $\mathrm{GL}(n, q)$ acting on the n-dimensional vector space $V = \mathbb{F}_q^n$, where $\mathbb{F}_q$ is the finite field with q elements. We find a relation between this function and the Euler characteristic of two simplicial complexes Δ_1 and Δ_2, the former raising from the lattice of the subspaces of V, the latter from the subgroup lattice of G.

Keywords Möbius function · Subgroup lattice · Linear groups · Euler characteristic · Simplicial complexes

Mathematics Subject Classification 20B25 · 20D60 · 05E16 · 05E45

1 Introduction

In this paper, motivated by recent and less recent results about the Möbius function μ for the subgroup lattice $\mathcal{L}(G)$ of a finite group G, we give a result which relates the Möbius function for a subgroup G of $\mathrm{GL}(n, q)$ to two simplicial complexes: one

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defined from the lattice of the subspaces fixed by a reducible subgroup $H \leq G$ and the second from the lattice $\mathcal{L}(G)$ of the subgroups of G.

We will introduce in the Preliminaries all the definitions and details useful for reading the paper.

In his PhD thesis [8], Shareshian considers the problem of computing $\mu(1, G)$ for several finite classical groups G; the idea is to approximate $\mu(1, G)$ through a *good function* $f_{G,n,p}(u, 1)$, such that:

$$\mu(1, G(n,\ p^u)) = f_{G,n,p}(u, 1) + \sum_{K \in \mathcal{C}_9} \mu(1, K)\,. \tag{1}$$

Here, $G = G(n,\ p^u)$ denotes a family of finite classical groups with the same defining classical form, which act in a natural way on the vector space V of finite dimension n over the finite field $\mathbb{F}_q$ of order $q = p^u$. If $\mathcal{C}_1, \ldots, \mathcal{C}_8, \mathcal{C}_9$ are the Aschbacher classes of maximal subgroups of a finite classical group (see [4]), $\mathcal{C}_9$ is the class of almost-simple groups not belonging to the first 8 classes of "geometric type" and the function $f_{G,n,p}(u, 1)$ provides an estimate of $\mu(1, G)$ with respect to the contributions given by the subgroups of G which belong to the classes $\mathcal{C}_i$, for $i \in \{1, \ldots, 8\}$.

Actually, Shareshian's approach focuses on the first class $\mathcal{C}_1(G)$, that is, the class of reducible subgroups of G.

In particular, the reducible subgroups of G contribute to $f_{G,n,p}(u, 1)$ through the computation of the Möbius function of

$$\widehat{\mathcal{I}}_1(G) := \{K \leq G \mid K \leq M \text{ for some } M \in \mathcal{C}_1(G)\} \cup \{G\}\,,$$

which is obtained by adjoining the maximum G to the order ideal

$$\mathcal{I}_1(G) := \{K \leq G \mid K \leq M \text{ for some } M \in \mathcal{C}_1(G)\},$$

that is,

$$\mu_{\widehat{\mathcal{I}}_1(G)}(1, G) = -\sum_{K \in \mathcal{I}_1(G)} \mu(1, K)$$

and

$$\mu(1, G) = \mu_{\widehat{\mathcal{I}}_1(G)}(1, G) - \sum_{\substack{K < G \\ K \notin \mathcal{I}_1(G)}} \mu(1, K)\,. \tag{2}$$

In this paper, we will consider irreducible subgroups G of the general linear group $\mathrm{GL}(n, q)$, that is, groups of linear automorphisms of a vector space V of dimension n over the finite field $\mathbb{F}_q$ with q elements, which fix no non-trivial subspace of V. In this hypothesis, we will take a reducible subgroup H of G (that is, H fixes some proper subspace of V) and we will work on the analogue of $\mu_{\widehat{\mathcal{I}}_1(G)}(1, G)$, namely $\mu_{\widehat{\mathcal{I}}(G,H)}(H, G)$, so that

$$\mu(H,G) = \mu_{\widehat{\mathcal{I}}(G,H)}(H,G) - \sum_{\substack{K \notin \mathcal{I}(G,H) \\ H \leq K < G}} \mu(H,K) \tag{3}$$

(see Sects. 2 and 3 for all precise definitions).

The subject of this paper is somehow motivated by the following conjecture:

Conjecture 1.1 (Mann, [6]) *Let G be a PFG group and μ the Möbius function on the lattice of open subgroups of G. Then, $|\mu(H,G)|$ is bounded by a polynomial function in the index $|G : H|$ and the number of subgroups of G of index m with $\mu(H,G) \neq 0$ grows at most polynomially in m.*

Indeed, although the problem is still open in its general setting, it was reduced by Lucchini in [5] to the study of similar growth conditions for finite almost-simple groups.

The following theorem is the main result of the present paper:

Theorem 4.5 *Consider a vector space V of finite dimension over $\mathbb{F}_q$. Let G be an irreducible subgroup of* $\mathrm{GL}(V)$ *and $H \leq G$. Then,*

$$-\mu_{\widehat{\mathcal{I}}(G,H)}(H,G) = \sum_{E \in \Psi'(G,H)} (-1)^{|E|} = \sum_{X \in \Psi(G,H)} (-1)^{|X|} = -\tilde{\chi}(\Delta_1) = -\tilde{\chi}(\Delta_2). \tag{4}$$

$\widehat{\mathcal{I}}(G,H)$, $\Psi'(G,H)$, $\Psi(G,H)$, Δ_i, and $\tilde{\chi}(\Delta_i)$ (for $i = 1, 2$) are defined in Sect. 3, also for irreducible subgroups H. This will allow us to avoid the restriction to only reducible subgroups H of G in the statement of the Theorem.

In the final section, we will use Theorem 4.5 to deal with $\mu(H,G)$ in some particular case. In [1], Theorem 4.5 is used to attack Conjecture 1.1 for some class of subgroups H of linear and projective groups.

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2 Preliminaries

In this paper, all the groups and sets are finite.

For main results about posets and lattices, we refer to [9]. Here, we just recall some basic fact, useful for reading the paper.

Definition 2.1 Let $\mathcal{P}$ be a finite poset. The **Möbius function** associated with $\mathcal{P}$ is the map $\mu_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ satisfying

$$\mu_{\mathcal{P}}(x,y) = 0 \quad \text{unless} \quad x \leq y,$$

and defined recursively for $x \leq y$ by

$$\mu_{\mathcal{P}}(x,x) = 1 \quad \text{and} \quad \sum_{x \leq t \leq y} \mu_{\mathcal{P}}(x,t) = 0 \ \text{ if } \ x < y\,. \tag{5}$$

Notation *If $\mathcal{P}$ is the subgroup lattice $\mathcal{L}(G)$ of G, we will write $\mu(H, K)$ instead of $\mu_{\mathcal{L}(G)}(H, K)$.*

Definition 2.2 An (abstract) **simplicial complex** Δ on a vertex set T is a collection Δ of subsets of T satisfying the two following conditions:

- if $t \in T$, then $\{t\} \in \Delta$;
- if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

An element $F \in \Delta$ is called a face of Δ, and the dimension of F is defined to be $|F| - 1$. In particular, the empty set $\emptyset$ is a face of Δ (provided $\Delta \neq \emptyset$) of dimension -1.

Definition 2.3 The **Euler characteristic** $\chi(\Delta)$ of the simplicial complex Δ is so defined:

$$\chi(\Delta) := \sum_{i\geq 0}(-1)^i \mathcal{F}_i \;=\; \mathcal{F}_0 - \mathcal{F}_1 + \mathcal{F}_2 - \mathcal{F}_3 + \dots$$

where $\mathcal{F}_i$ denotes the number of the i-faces that is of the faces of dimension i, in Δ.

We recall here the definition of an order ideal:

Definition 2.4 Let $(\mathcal{P}, \leq)$ be a poset. An **order ideal** of $\mathcal{P}$ is a subset $I \subseteq \mathcal{P}$ such that

$$\forall x \in I,\ \forall t \in \mathcal{P} \qquad t \leq x \;\Rightarrow\; t \in I\,. \tag{6}$$

In particular, if A is a subset of $\mathcal{P}$, then the set

$$\mathcal{P}_{\leq A} := \{s \in \mathcal{P} \mid s \leq a \text{ for some } a \in A\} \subseteq \mathcal{P}$$

is the order ideal of $\mathcal{P}$ generated by A.

Observe that a simplicial complex on a set of vertices T is just an order ideal of the Boolean algebra B_T that contains all one element subsets of T.

Definition 2.5 Let Δ be a simplicial complex, and $\chi(\Delta)$ be the Euler characteristic of Δ. The **reduced Euler characteristic** $\tilde{\chi}(\Delta)$ of Δ is defined by $\tilde{\chi}(\emptyset) = 0$, and $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$ if $\Delta \neq \emptyset$. See also Equation (3.22) in [9].

It is possible to build a simplicial complex from a poset $\mathcal{P}$ in the following way:

The **order complex** $\Delta(\mathcal{P})$ of $\mathcal{P}$ is defined as the simplicial complex whose vertices are the elements of $\mathcal{P}$ and whose k-dimensional faces are the chains $a_0 \prec a_1 \prec \cdots \prec a_k$ of length k of distinct elements $a_0, \dots, a_k \in \mathcal{P}$.

Now, denote by $\widehat{\mathcal{P}}$ the finite poset obtained from $\mathcal{P}$ by adjoining a least element $\hat{0}$ and a greatest element $\hat{1}$. The Möbius function $\mu_{\widehat{\mathcal{P}}}(\hat{0}, \hat{1})$ is related to $\tilde{\chi}(\Delta(\mathcal{P}))$ by a well-known result by Hall in [3] about the computation of $\mu_{\widehat{\mathcal{P}}}(\hat{0}, \hat{1})$ by means of the chains of even and odd length between $\hat{0}$ and $\hat{1}$.

Proposition 2.6 (see [9], Proposition 3.8.6) *Let $\mathcal{P}$ be a finite poset. Then,*

$$\mu_{\widehat{\mathcal{P}}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(\mathcal{P})).$$

3 The ideal $\mathcal{I}(G, H)$ and the complexes Δ_i in Theorem 4.5

In this section, and in the rest of the paper, G is an irreducible subgroup of the general linear group $\mathrm{GL}(n, q)$ over the vector space $V = \mathbb{F}_q^n$ of finite dimension n over the finite field $\mathbb{F}_q$ with q elements. We consider the natural action of G on the set of subspaces of V.

We define the order ideal $\mathcal{I}(G, H)$ and the simplicial complexes Δ_i (for $i = 1, 2$) that we consider in Theorem 4.5. In Remark 2, we will explicitly observe that the two complexes Δ_i are not the order simplicial complex rising from $\mathcal{I}(G, H)$.

Given a subgroup G of $\mathrm{GL}(n, q)$ and a subgroup H of G, put

$$\mathcal{L}(G)_{\geq H} := \{K \leq G \mid H \leq K\}$$

and

$$\mathcal{C}(G, H) := \{\,\mathrm{stab}_G(W) \mid 0 < W < V\,,\ H \subseteq \mathrm{stab}_G(W)\,\}.$$

Definition 3.1 The **reducible subgroup ideal** in $\mathcal{L}(G)_{\geq H}$ is the order ideal of $\mathcal{L}(G)_{\geq H}$ generated by $\mathcal{C}(G, H)$. Namely,

$$\mathcal{I}(G, H) = \{K \leq G \mid H \leq K \leq M \text{ for some } M \in \mathcal{C}(G, H)\}\,.$$

Remark 1 If H is reducible, that is, H fixes some non-trivial subspace W of V, then $H \in \mathcal{I}(G, H)$. Otherwise, if H is irreducible, $H \notin \mathcal{I}(G, H)$ and $\mathcal{I}(G, H)$ is empty. The subspaces fixed by H are said to be H-invariant.

Definition 3.2 If H is reducible, we set

$$\widehat{\mathcal{I}}(G, H) := \mathcal{I}(G, H) \cup \{G\}$$

by adjoining the maximum G to $\mathcal{I}(G, H)$, which has minimum H. Otherwise, if H is irreducible, we set $\widehat{\mathcal{I}}(G, H) := \{H, G\}$ by adjoining the minimum H and the maximum G to the empty poset $\emptyset$.

Remark 2 Here, we just note that the poset $\mathcal{I}(G, H)$ has already a minimum if H is reducible, so that $\mu_{\widehat{\mathcal{I}}(G,H)}(H, G)$ is not, in general, the reduced Euler characteristic of the order complex $\Delta(\mathcal{I}(G, H))$ of $\mathcal{I}(G, H)$.

To define the simplicial complexes Δ_i of Theorem 4.5, we begin with fixing some more notation. We denote by $\mathcal{S}(V, H)$ the lattice of H-invariant subspaces of V and define

$$\mathcal{S}(V, H)^* := \mathcal{S}(V, H) \setminus \{0, V\}.$$

Moreover, given an irreducible group $G \leq \mathrm{GL}(V)$, and $H \leq G$, we will consider the following three sets:

(a) $\Psi(G, H) := \{X \subseteq \mathcal{C}(G, H) \mid \bigcap_{M \in X} M \neq H\}$;

(b) $\Psi(G, H)^{\complement} := \{Y \subseteq \mathcal{C}(G, H) \mid \bigcap_{M \in Y} M = H\}$;

(c) $\Psi'(G, H) := \{E \subseteq \mathcal{S}(V, H)^* \mid \bigcap_{W \in E} \mathrm{stab}_G(W) \neq H\}$.

Observe that $\emptyset \in \Psi(G, H)$ and $\emptyset \in \Psi'(G, H)$, but $\emptyset \notin \Psi(G, H)^{\complement}$.

Remark 3 If H is an irreducible subgroup of G, then $\mathcal{S}(V, H)^* = \emptyset$ and $\Psi'(G, H) = \{\emptyset\} = \Psi(G, H)$. Since for irreducible H we have that $\widehat{\mathcal{I}}(G, H) = \{H, G\}$, then Theorem 4.5 is trivially verified in this case.

Definition 3.3 The simplicial complexes Δ_i of Theorem 4.5 are so defined:
Δ_1:

- The set of vertices T_1 is given by the subspaces $W \in \mathcal{S}(V, H)^*$ for which $H \neq \mathrm{stab}_G(W)$;
- the set of faces of Δ_1 is given by $\Psi'(G, H)$.

Δ_2:

- The set of vertices T_2 is given by the subgroups $M \in \mathcal{C}(G, H)$ such that $H \neq M$;
- the set of faces of Δ_2 is given by $\Psi(G, H)$.

We explicitly observe what happens in the special case when, for some proper non-trivial subspace W of V, the subgroup $H = \mathrm{stab}_G(W)$ is maximal with respect to the property of being a stabilizer of a proper non-trivial subspace of V in G. In this case, we note that, by definition, $T_1 = T_2 = \emptyset$ and the set of faces of Δ_1 and Δ_2 is $\{\emptyset\}$. Then, Theorem 4.5 is trivially verified.

4 Computing $\mu_{\widehat{\mathcal{I}}(G,H)}(H, G)$

In order to prove Theorem 4.5, we need Proposition 4.2 that gives a link between $\Psi(G, H)$ and $\Psi'(G, H)$ and also shows that the reduced Euler characteristics of the complexes Δ_i coincide. The proof of Proposition 4.2 follows at once from Lemma 4.1.

Lemma 4.1 *Let T be a subgroup of a finite group L acting on a finite set X, and let $X' \subseteq X$ be a subset such that $T \leq L_x$ for all $x \in X'$. (As usual, L_x denotes the stabilizer of x in L.) Set*

- $\mathcal{L} := \{L_x \mid x \in X'\}$*;*
- $\mathcal{R} := \{E \subseteq \mathcal{L} \mid \bigcap_{K \in E} K \neq T\}$*;*
- $\mathcal{S} := \{Q \subseteq X' \mid \bigcap_{x \in Q} L_x \neq T\}$*.*

Then,

$$\sum_{E \in \mathcal{R}} (-1)^{|E|} = \sum_{Q \in \mathcal{S}} (-1)^{|Q|}.$$

Proof If $Q \in \mathcal{S}$ and $E \in \mathcal{R}$, set

$$\mathcal{L}_Q = \{L_x \mid x \in Q\} \text{ and } \mathcal{S}_E = \{Q \in \mathcal{S} \mid E = \mathcal{L}_Q\}.$$

It is immediate to realize that

$$\mathcal{S} = \bigsqcup_{E \in \mathcal{R}} \mathcal{S}_E$$

is the disjoint union of all the $\mathcal{S}_E$. So, it suffices to show that for each $E \in \mathcal{R}$ the following identity is verified:

$$(-1)^{|E|} = \sum_{Q \in \mathcal{S}_E} (-1)^{|Q|} . \tag{7}$$

For $E = \emptyset$, the identity (7) is trivially true. Now, fix a non-empty $E \in \mathcal{R}$, and for each $K \in E$ define

$$X'_K = \{x \in X' \mid L_x = K\} \subseteq X'.$$

Let $Q \in \mathcal{S}_E$ and observe that Q can be represented as the following disjoint union:

$$Q = \bigsqcup_{K \in E} Q_K,$$

where $Q_K = \{x \in Q \mid L_x = K\} \subseteq X'_K$ and $Q_K \neq \emptyset$. (This property characterizes the elements Q of $\mathcal{S}_E$.) Since $\sum_{\emptyset \neq Q_K \subseteq X'_K} (-1)^{|Q_K|} = -1$ (see Remark 4), we get

$$\begin{aligned}\sum_{Q \in \mathcal{S}_E} (-1)^{|Q|} &= \sum_{Q \in \mathcal{S}_E} (-1)^{\sum_{K \in E} |Q_K|} = \prod_{K \in E} \left(\sum_{\emptyset \neq Q_K \subseteq X'_K} (-1)^{|Q_K|} \right) \\ &= \prod_{K \in E} (-1) = (-1)^{|E|}\end{aligned}$$

and we obtain the identity (7). □

Proposition 4.2 *Let V be a vector space of finite dimension over $\mathbb{F}_q$ and consider a subgroup $H \leq G \leq \mathrm{GL}(V)$. We have:*

$$\sum_{E \in \Psi'(G,H)} (-1)^{|E|} = \sum_{X \in \Psi(G,H)} (-1)^{|X|} . \tag{8}$$

Equivalently, $\tilde{\chi}(\Delta_1) = \tilde{\chi}(\Delta_2)$.

Proof Consider the natural action of G on the set of subspaces of V. By Lemma 4.1, the equality follows at once from the definitions of $\mathcal{S}(V, H)^*$, $\mathcal{C}(G, H)$, $\Psi(G, H)$ and $\Psi'(G, H)$.

With previous notation, we take $T = H$, $L = G$, $\mathcal{R} = \Psi(G, H)$, $\mathcal{S} = \Psi'(G, H)$. □

To prove Theorem 4.5, we need Proposition 4.4 which is achieved through Theorem 4.3 (Crosscut Theorem, see [9, Corollary 3.9.4]) and Remark 4.

Remark 4 For every finite set A of cardinality $n > 0$, we have

$$\sum_{S\subseteq A}(-1)^{|S|} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k = (1-1)^n = 0.$$

Theorem 4.3 *(Crosscut Theorem) Let L be a finite lattice with minimum $\hat{0}$ and maximum $\hat{1}$, so that $\hat{0} \neq \hat{1}$. Let M be the set of all coatoms in L. Let $X \subseteq L$ be a subset such that $M \subseteq X$ and $\hat{1} \notin X$.*

Given $\mathcal{Y} := \{Y \subseteq X \mid Y \neq \emptyset$ and $\bigcap_{Y\in\mathcal{Y}} Y = \hat{0}\}$, the following equality holds:

$$\mu_L(\hat{0}, \hat{1}) = \sum_{Y\in\mathcal{Y}}(-1)^{|Y|}.$$

Proposition 4.4 *Let V be a vector space of finite dimension over $\mathbb{F}_q$. Let $H \leq G \leq \mathrm{GL}(V)$. Then, we have that*

$$\mu_{\widehat{\mathcal{I}}(G,H)}(H, G) = \sum_{Y\in\Psi(G,H)^{\complement}} (-1)^{|Y|}. \tag{9}$$

Proof We observe that $\widehat{\mathcal{I}}(G, H) \subseteq \mathcal{L}(G)_{\geq H}$ is a lattice because the join of two subgroups $K_1, K_2 \in \mathcal{I}(G, H)$ is either in $\mathcal{I}(G, H)$ or equal to G. The meet of $K_1, K_2 \in \mathcal{I}(G, H)$ is $K_1 \cap K_2$. Hence, $\widehat{\mathcal{I}}(G, H)$ is a finite lattice, whose set of coatoms is contained in $\mathcal{C}(G, H)$, and its maximum $G \notin \mathcal{C}(G, H)$, because G is assumed to be irreducible. Since

$$\Psi(G, H)^{\complement} = \{Y \subseteq \mathcal{C}(G, H) \mid Y \neq \emptyset \text{ and } \bigcap_{M\in Y} M = H\},$$

by Theorem 4.3 we immediately obtain (9). □

Now, observe that the disjoint union $\Psi(G, H) \cup \Psi(G, H)^{\complement}$ is the power set of $\mathcal{C}(G, H)$, so that by Remark 4 we have

$$\sum_{X\in\Psi(G,H)} (-1)^{|X|} + \sum_{Y\in\Psi(G,H)^{\complement}} (-1)^{|Y|} = 0. \tag{10}$$

If we put together Eqs. (8), (9), and (10), we get:

Theorem 4.5 *Let V be a vector space of finite dimension over $\mathbb{F}_q$. Let $H \leq G \leq \mathrm{GL}(V)$. Then,*

$$-\mu_{\widehat{\mathcal{I}}(G,H)}(H, G) = \sum_{E\in\Psi'(G,H)} (-1)^{|E|}. \tag{11}$$

Proof By (10), we have

$$\sum_{X \in \Psi(G,H)} (-1)^{|X|} = - \sum_{Y \in \Psi(G,H)^{\complement}} (-1)^{|Y|}.$$

Then, by Proposition 4.2 and Proposition 4.4,

$$\sum_{E \in \Psi'(G,H)} (-1)^{|E|} = \sum_{X \in \Psi(G,H)} (-1)^{|X|} = - \sum_{Y \in \Psi(G,H)^{\complement}} (-1)^{|Y|}$$

$$= -\mu_{\widehat{\mathcal{I}}(G,H)}(H, G)\,.$$

□

5 Final remark

Going back to Conjecture 1.1, we observe that the knowledge of

$$\sum_{E \in \Psi'(G,H)} (-1)^{|E|}$$

coming from (11) can be exploited to estimate the value $\mu(H, G)$ of the Möbius function μ of G for $H \leq G$, at least in some particular case. Here, we just give an example for a particular reducible H, taking $G = \mathrm{GL}(n, q)$.

Following the idea suggested by Shareshian in [8], one could write

$$\mu(H, G) = f_{G,n,q}(H) + \sum_{K \in \mathcal{C}_9} \mu(H, K)\,, \tag{12}$$

where $f_{G,n,q}(H)$ depends on the classes $\mathcal{C}_i(G, H)$, for $i = 1, \dots, 8$, in Aschbacher's classification. In some lucky case, H is not contained in maximal subgroups belonging to classes (C_i), $i \neq 1,\ 9$. This happens, for example, in the following case:

Let $V \simeq \mathbb{F}_q^n$ be a vector space of finite dimension n over $\mathbb{F}_q$ and fix the following basis of V :

$$\mathcal{E} := \{w_1, \dots, w_m,\ v_{m+1}, \dots, v_n\},$$

so that

$$V = \langle w_1, \dots, w_m \rangle \oplus \langle v_{m+1}, \dots, v_n \rangle$$

If $W = \langle w_1, \dots, w_m \rangle$, H is the subgroup of $\mathrm{GL}(n, q)$ acting as $\mathrm{GL}(m, q)$ on W and fixing all the elements $v_{m+1}, \dots, v_n$.

We do not give in this context the details of the proof of the following theorem, which needs many technical arguments. In [1], all the details are given.

Theorem 5.1 ([2]) *Let* $G = \mathrm{GL}(n, q)$*, and let* $H \leq G$ *be such that*

$$H = \mathrm{GL}(m, q) \oplus I_{n-m}.$$

Let $q = p$ *be an odd prime, and let the dimension n be prime. If* $n - m + 1$ *is prime, then*

$$\mu_{\widehat{\mathcal{I}}(G)}(H, G) = 0 = f_{G,n,p}(H)$$

so that

$$\mu(H, G) = \sum_{K \in \mathcal{C}_9(G,H)\,,\, H \subseteq K} \mu(H, K)\,.$$

In general, we do not have much information about the ninth class. Just to give an example, we considered the groups of low dimension studied by Schröder in her PhD thesis ([7]), and we saw that in dimension $n = 13$ also class $\mathcal{C}_9(G, H)$ is empty for $p > 5$. In this case, $\mu(H, G) = 0$.

All the details and data needed to prove Theorem 5.1 are contained in [2] and are available if requested.

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References

1. Di Gravina, L.: A closure operator on the subgroup lattice of $\mathrm{GL}(n, q)$ and $\mathrm{PGL}(n, q)$ in relation to the zeros of the Möbius function. J. Group Theory **27**, 275–296 (2024)
2. Di Gravina, L.: Some questions about the Möbius function of finite linear groups, Ph.D. Thesis, Università degli Studi di Milano-Bicocca (2022)
3. Hall, P.: The Eulerian functions of a group. Q. J. Math. **7**, 134–151 (1936)
4. Kleidman, P., Liebeck, M.: The Subgroup Structure of the Finite Classical Groups. Cambridge University Press, Cambridge (1990)
5. Lucchini, A.: On the subgroups with non-trivial Möbius number. J. Group Theory **13**, 589–600 (2010)
6. Mann, A.: A probabilistic zeta function for arithmetic groups. Internat. J. Algebra Comput. **15**, 1053–1059 (2005)
7. Schröder, A.K.: The maximal subgroups of the classical groups in dimension 13, 14 and 15, Ph.D. Thesis, University of St Andrews (2015)
8. Shareshian, J.: Combinatorial properties of subgroup lattices of finite groups, Ph.D. Thesis, Rutgers - The State University of New Jersey (1996)
9. Stanley, R.P.: Enumerative Combinatorics, Volume I, second ed., Cambridge University Press, Cambridge, 2012 (first ed. published by Wadsworth & Brooks/Cole, 1986)