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Vertex stabilizers of locally *s*-arc transitive graphs of pushing up type

John van Bon[1] · **Chris Parker**[2]

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Abstract

Suppose that Δ is a thick, locally finite and locally s-arc transitive G-graph with $s \geq 4$. For a vertex z in Δ, let G_z be the stabilizer of z and $G_z^{[1]}$ the kernel of the action of G_z on the neighbours of z. We say Δ is of pushing up type provided there exist a prime p and a 1-arc (x, y) such that $C_{G_z}(O_p(G_z^{[1]})) \leq O_p(G_z^{[1]})$ for $z \in \{x, y\}$ and $O_p(G_x^{[1]}) \leq O_p(G_y^{[1]})$. We show that if Δ is of pushing up type, then $O_p(G_x^{[1]})$ is elementary abelian and $G_x/G_x^{[1]} \cong X$ with $\mathrm{PSL}_2(p^a) \leq X \leq \mathrm{P\Gamma L}_2(p^a)$.

Keywords Locally s-arc transitive graphs · Group amalgams

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1 Introduction

In this article, we consider graphs Δ that are connected, undirected and without loops or multiple edges. The vertex set of Δ is denoted by $V\Delta$, and the edge set is $E\Delta$. A *G-graph* is a graph Δ together with a subgroup $G \leq \mathrm{Aut}(\Delta)$. An *$s$-arc* emanating from $x_0 \in V\Delta$ is a path $(x_0, x_1, \ldots, x_s)$ with $x_{i-1} \neq x_{i+1}$ for $1 \leq i \leq s-1$. Denote by G_z the stabilizer of a vertex $z \in V\Delta$.

A G-graph Δ is

- *Thick* if the valency at each vertex is at least 3;
- *Locally finite* if for each $z \in V\Delta$, G_z is a finite group;
- *Locally s-arc transitive* if for every vertex $z \in V\Delta$, G_z is transitive on the set of s-arcs emanating from z.

✉ John van Bon
jozef.vanbon@unical.it

Chris Parker
c.w.parker@bham.ac.uk

1 Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Italy

2 School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK

This paper is part of ongoing research aimed at determining all vertex stabilizer amalgams for thick, locally finite and locally s-arc transitive G-graphs for $s \geq 4$. Throughout this introduction, Δ represents a thick, locally finite G-graph.

It is easy to see that if Δ is a locally s-arc transitive G-graph with $s \geq 1$, then G is transitive on $E\Delta$ and thus G has at most two orbits on $V\Delta$. If $s \geq 2$, then, for $x \in V\Delta$, we also know that G_x acts 2-transitively on $\Delta(x) = \{v \mid \{x, v\} \in E\Delta\}$. For (x_1, x_2) a 1-arc in Δ, the triple $(G_{x_1}, G_{x_2}; G_{x_1} \cap G_{x_2})$ is called the *vertex stabilizer amalgam* of Δ with respect to the 1-arc (x_1, x_2). When we study s-arc transitive G-graphs with $s \geq 1$, it is impossible to determine Δ. The best we can hope for is a description of the vertex stabilizer amalgam, and the best of all this will be described up to isomorphism of the amalgam.

For a vertex $z \in V\Delta$, G_z acts on $\Delta(z)$. The kernel of this action is denoted by $G_z^{[1]}$, and $G_z^{\Delta(z)}$ is the permutation group $G_z/G_z^{[1]}$. A locally finite and locally G-graph Δ is of

- *Local characteristic* p, if there exists a prime p such that

$$C_{G_z}(O_p(G_z^{[1]})) \leq O_p(G_z^{[1]}), \text{ for all } z \in V\Delta;$$

- *Pushing up type* with respect to the 1-arc (x, y) and the prime p, if Δ is of local characteristic p and

$$O_p(G_x^{[1]}) \leq O_p(G_y^{[1]}).$$

Assume that (x, y) is a 1-arc and set $G_{x,y} = G_x \cap G_y$. One consequence of the local characteristic p property is that $O_p(G_x^{[1]})$ and $O_p(G_y^{[1]})$ are non-trivial. In particular, this means that G_x and G_y are rather large and have potentially complicated structure. Notice that if K is a subgroup of $G_{x,y}$, and K is normalized by both G_x and G_y, then K fixes every vertex of Δ and is consequently trivial. Hence, if Δ is of pushing up type with respect to (x, y), then, as $O_p(G_x^{[1]})$ is non-trivial, we learn that Δ is not of pushing up type with respect to (y, x). Hence, in these circumstances, G has two orbits on $V\Delta$.

The generic examples of thick, locally finite and locally s-arc transitive G-graphs Δ with $s \geq 4$ have vertex stabilizer amalgams which are weak BN-pairs [6]. One of the achievements in [1, Theorem 1] is the proof that, for $s \geq 6$, the generic examples are the only examples. In particular, [1, Corollary 1] remarks that $s \leq 9$ for any such G-graph Δ. In [5], examples of G-graphs which are of pushing up type with $s = 5$ have been constructed via amalgams in $\mathrm{Sym}(p^{2a})$. Thus, the vertex stabilizer amalgam of Δ may not to be a weak BN-pair when $s \leq 5$. To determine the G-graphs Δ which are thick, locally finite and locally s-arc transitive with $4 \leq s \leq 5$, as in [1], we consider three distinct cases:

- Δ is not of local characteristic p.
- Δ is of local characteristic p but not of pushing up type.
- Δ is of pushing up type.

In the first case, [2, Theorem 1] shows that $s = 5$ and the vertex stabilizer amalgams are either isomorphic to certain subamalgams of the vertex stabilizer amalgam of the

G-graph for $p = 2$ constructed in [7], or isomorphic to the amalgam of two maximal p-local subgroups of $\mathrm{Aut}({}^3\mathrm{D}_4(2))$ (for $p = 7$) or of $\mathrm{Aut}(\mathrm{J}_2)$ (for $p = 5$).

We expect that the second possibility yields weak BN-pairs. The configurations appearing in option three are the subject of this article together with its companions [3, 4] which are in preparation.

Assume from now on that Δ is of pushing up type with respect to (x, y) and the prime p. For $z \in V\Delta$, set $Q_z = O_p(G_z^{[1]})$ and

$$L_z = \langle Q_u \mid u \in \Delta(z)\rangle Q_z.$$

The main result of this paper is as follows.

Theorem 1.1 *Suppose that $s \geq 4$ and Δ is a thick, locally finite, locally s-arc transitive G-graph of pushing up type with respect to the 1-arc (x, y) and the prime p. Then p is odd and the following hold*

(a) $G_x^{\Delta(x)} \cong X$ where $\mathrm{PSL}_2(p^a) \leq X \leq \mathrm{P\Gamma L}_2(p^a)$ and $\Delta(x)$ has size $p^a + 1$ and can be identified with the projective line for X;

(b) $L_x/Q_x \cong \mathrm{SL}_2(p^a)$, $O^p(L_x) \cong \mathrm{ASL}_2(p^a)'$ and Q_x is an elementary abelian p-group.

In [3, 4], van Bon establishes the isomorphism types of the vertex stabilizer amalgams appearing in the conclusion of Theorem 1.1 and so it completes the determination of G-graphs of pushing up type with $s \geq 4$. This then extends [1, Lemma 7.7] which can be interpreted to say that if Δ is of pushing up type, then $s \leq 5$.

The organization of this paper is as follows. In Sect. 2, we derive general properties of vertex stabilizer amalgams for Δ of pushing up type. In Sect. 3, we consider the possibility that $G_x^{\Delta(x)}$ is a projective linear group of degree at least 3; the main result of the section is Proposition 3.1 which asserts that $F^*(G_x/G_x^{[1]}) \ncong \mathrm{PSL}_n(p^a)$ with $n \geq 3$. The strategy followed to obtain the conclusion of Proposition 3.1 uses the results of Sect. 2 and is similar in flavour to [2] which exploits Zsigmondy primes. This fails to eliminate the possibility that $F^*(G_x/G_x^{[1]}) \cong \mathrm{PSL}_3(2)$, however, and so here we call upon a pushing up result [10] which allows us to compare non-central chief factors of the vertex stabilizers. This eventually leads to the elimination of this last case as well. Finally, in Sect. 4, we recall that the action of $G_x^{\Delta(x)}$ on $\Delta(x)$ is 2-transitive, and so with the help of the classification of finite 2-transitive groups and Proposition 3.1, we see that L_x is a rank 1 Lie type group or is of regular type. This is precisely the situation handled in Sect. 3 of [1]. After application of these results to our case, we are left in a situation where we can follow steps 1–9 of the proof of [1, Lemma 7.7] word for word to obtain the theorem.

Throughout this paper, we assume the following hypothesis:

Main Hypothesis *The G-graph Δ is thick, locally finite, locally s-arc transitive with $s \geq 4$ and, in addition, is of pushing up type with respect to the 1-arc (x, y) and prime p.*

The notation used in the paper is standard in the theory of (locally) s-arc transitive G-graphs and given in Sect. 2. Our group theoretic notation follows [9].

2 Preliminaries

In this section, we prove some properties of thick, locally finite and locally s-arc transitive G-graphs. We assume the Main Hypothesis, though some results also hold under weaker assumptions. First we fix the notation used throughout the article.

Notation 2.1 *Let $d(\cdot,\cdot)$ represent the standard distance function on Δ. For $u \in V\Delta$, (u, v) a 1-arc in Δ, $\Theta \subseteq V\Delta$ and $i \geq 1$,*

$$
\begin{aligned}
\Delta^i(u) &= \{v \in V\Delta \mid d(u, v) \leq i\};\\
\Delta(u) &= \Delta^1(u)\backslash\{u\};\\
q_u &= |\Delta(u)| - 1;\\
G_u &= \{g \in G \mid u^g = u\};\\
G_\Theta &= \bigcap_{\theta\in\Theta} G_\theta;\\
G_u^{[i]} &= G_{\Delta^i(u)};\\
G_\Theta^{[1]} &= \bigcap_{\theta\in\Theta} G_\theta^{[1]};\\
Q_u &= O_p(G_u^{[1]});\\
Z_u &= \Omega_1(Z(Q_u));\\
C_u &= \langle G_v^{[2]} \mid v \in \Delta(u)\rangle Q_u;\\
L_u &= \langle Q_v \mid v \in \Delta(u)\rangle Q_u;\\
L_{u,v} &= G_{u,v} \cap L_u;
\end{aligned}
$$

and (w, x, y, z) is a fixed 3-arc where (x, y) is an arc for which Δ is of pushing up type.

Notice that we do not know that $L_{u,v} = L_{v,u}$, and so the order of the vertices on the arc is important for the definition of $L_{u,v}$. We recall the fundamental properties of the G-graph Δ:

$$
\begin{aligned}
C_{G_u}(Q_u) \leq Q_u \text{ for } u \in \{x, y\};\ \text{and}\\
Q_x \leq Q_y.
\end{aligned}
$$

An essential tool when studying such G-graphs is given by the following lemma:

Lemma 2.2 *Assume that $R \leq G_{x,y}$, $N_{G_x}(R)$ is transitive on $\Delta(x)$ and $N_{G_y}(R)$ is transitive on $\Delta(y)$. Then $R = 1$.*

Proof See [9, 10.3.3]. □

Because Δ has pushing up type, $Q_x \leq Q_y$. In fact, $Q_x < Q_y$ as otherwise $Q_x = Q_y = 1$ by Lemma 2.2, and then, Δ is not of local characteristic p. One consequence of this observation, as mentioned in introduction, is that G has two orbits on Δ.

We present some elementary properties of the subgroups defined in Notation 2.1.

Lemma 2.3 *Let $N_x = O^p(L_x)$, and $\widetilde{G}_x = G_x/Q_x$. Then*

(i) $[G_x^{[1]}, L_x] \leq Q_x$;
(ii) $N_x \not\leq G_x^{[1]}$;
(iii) $L_x = N_x Q_y$ *and* $G_x = N_x G_{x,y}$;
(iv) *either* $\widetilde{N_x}$ *is quasisimple or an r-group, r a prime with $r \neq p$;*
(v) $G_x^{[1]} \cap L_x = \Phi(\widetilde{L}_x) = Z(\widetilde{L}_x)$;
(vi) *if $\widetilde{N_x}$ is an r-group, then $G_x^{[1]} \cap L_x = \widetilde{N_x}' = \Phi(\widetilde{N_x})$, and G_x acts transitively on the non-trivial elements of $N_x^{\Delta(x)}$.*

Proof The first statement follows from [1, Lemma 5.1 (a)] while the remainder of the statements can be found in [1, Lemma 5.2]. □

One important and frequently used consequence of Lemma 2.3 (iii) is that L_x operates transitively on $\Delta(x)$.

Lemma 2.4 *The following hold:*

(i) $G_{x,y}^{[1]} = G_x^{[2]}$, $G_y^{[2]} = G_y^{[3]}$ *and* $Q_x = O_p(G_x^{[2]})$;
(ii) $G_x^{[1]} \cap G_z = G_x^{[2]}$.

In particular, $G_x^{[1]}$ induces a semi-regular group on $\Delta(y)\backslash\{x\}$, $|G_x^{[1]}G_y^{[1]}/G_y^{[1]}|$ divides q_y and $|G_{x,y,z}| = |G_{x,y,z}^{\Delta(x)}||G_x^{[2]}|$.

Proof (i). By Lemma 2.3 (i) and (iii), $[L_x, G_x^{[1]}] \leq Q_x$ and L_x acts transitively on $\Delta(x)$. Hence,

$$[L_x, G_{x,y}^{[1]}] \leq [L_x, G_x^{[1]}] \leq Q_x \leq G_{x,y}^{[1]}.$$

Thus, L_x normalizes $G_{x,y}^{[1]}$. The transitivity of L_x on $\Delta(x)$ now yields $G_{x,y}^{[1]} = G_x^{[2]}$. In particular, $G_y^{[2]} \leq G_{u,y}^{[1]} = G_u^{[2]}$ for any $u \in \Delta(y)$. It follows that $G_y^{[2]} = G_y^{[3]}$. Since $Q_x \leq Q_v \leq G_v^{[1]}$ for all $v \in \Delta(x)$, we also have $Q_x \trianglelefteq G_x^{[2]}$. Hence, $Q_x = O_p(G_x^{[2]})$.
(ii). By Lemma 2.3 (i) and, as the G-graph is pushing up type,

$$[L_x, G_x^{[1]} \cap G_z] \leq [L_x, G_x^{[1]}] \leq Q_x \leq G_x^{[1]} \cap Q_y \leq G_x^{[1]} \cap G_z.$$

Thus, L_x normalizes $G_x^{[1]} \cap G_z$. Pick $g \in L_x$ with $y^g \neq y$, and put $y' = y^g$ and $z' = z^g$. Then $G_x^{[1]} \cap G_z = G_x^{[1]} \cap G_{z'}$. Since $s \geq 4$, $G_{z',y',x,y}$ acts transitively on $\Delta(y)\backslash\{x\}$ and it also normalizes $G_x^{[1]} \cap G_{z'} = G_x^{[1]} \cap G_z$. Therefore, $G_x^{[1]} \cap G_z \leq G_y^{[1]}$. Hence, from (i),

$$G_x^{[2]} \leq G_x^{[1]} \cap G_z \leq G_{x,y}^{[1]} = G_x^{[2]}.$$

Now (ii) follows.

Finally, as $(G_x^{[1]} \cap G_z)^{\Delta(y)} = 1$ by (ii), $G_x^{[1]}$ induces a group which acts semi-regularly on $\Delta(y)\backslash\{x\}$. Therefore, we also have $|G_x^{[1]}G_y^{[1]}/G_y^{[1]}|$ divides $\Delta(y) - 1 = q_y$. To see that $|G_{x,y,z}| = |G_{x,y,z}^{\Delta(x)}||G_x^{[2]}|$, just observe that (ii) gives

$$G_x^{[2]} \leq G_x^{[1]} \cap G_{x,y,z} \leq G_x^{[1]} \cap G_z = G_x^{[2]}.$$

□

Lemma 2.5 *The following hold:*

(i) $G_y^{[2]}$ *and* $G_x^{[1]} \cap G_z^{[1]}$ *are* p*-groups;*
(ii) $G_x^{[1]} \cap G_z^{[1]} = G_x^{[2]} \cap G_z^{[2]} = G_{x,y,z}^{[1]} = Q_x \cap Q_z$*;*
(iii) $Q_y G_x^{[1]} \cap G_z^{[2]} = Q_z$*;*
(iv) $Q_y G_x^{[2]} \cap Q_y G_z^{[2]} = Q_y$*;*
(v) either $G_x^{[2]}/Q_x$ *is abelian or* $G_x^{[1]} = G_x^{[2]}$*;*
(vi) $\mathrm{lcm}(q_x, q_y)$ *divides* $|G_{w,x,y,z}|$ *and* $\pi(G_{x,y}) = \pi(G_{w,x,y,z})$.

Proof (i). Let $R \in \mathrm{Syl}_r(G_x^{[1]})$, with $r \neq p$. Then, Lemma 2.3 (i) yields $[R, L_x] \leq [G_x^{[1]}, L_x] \leq Q_x$. Hence, L_x normalizes $Q_x R$ and acts on $Q_x R$ by conjugation. Therefore, the Frattini argument gives $L_x = Q_x N_{L_x}(R)$. In particular, $N_{L_x}(R)$ is transitive on $\Delta(x)$. Now using Lemma 2.3 (i) we get $[R, N_{L_x}(R)] \leq [G_x^{[1]}, L_x] \cap R \leq Q_x \cap R = 1$. So for all $R_0 \leq R$, $C_{L_x}(R_0) \geq C_{L_x}(R) = N_{L_x}(R)$ is transitive on $\Delta(x)$.

Let $T \in \mathrm{Syl}_t(G_y^{[2]})$, with $t \neq p$. By the Frattini argument $G_y = N_{G_y}(T)G_y^{[2]}$. Hence, $N_{G_y}(T)$ is transitive on $\Delta(y)$. Since $T \leq G_x^{[1]}$, we have $C_{L_x}(T)$ is transitive on $\Delta(x)$. Thus, T is normalized by both $N_{G_y}(T)$ and $C_{L_x}(T)$ and so $T = 1$ by Lemma 2.2. We conclude that $G_y^{[2]}$ is a p-group as claimed.

Suppose that $T \in \mathrm{Syl}_t(G_x^{[1]} \cap G_z^{[1]})$, with $t \neq p$. Since $s \geq 4$, there exists a $c \in C_{L_x}(T)$ with $y^c \neq y$ and $z^c \neq z$. Indeed, that we can choose c with $y^c \neq y$ follows from the transitivity of G_x on $\Delta(x)$. We claim for such a $c \in C_{L_x}(T)$, $z^c \neq z$. In the counter case, (x, y, z, y^c, x) is a circuit of length 4 and G_{x,y,z,y^c} is transitive on $\Delta(y^c)\backslash\{z\}$ as $s \geq 4$. As $x \in \Delta(y^c)$, we infer that $|\Delta(y^c)| = 2$, contrary to Δ being thick.

Let $\gamma = (z^c, y^c, x, y)$ and $N = G_x^{[1]} \cap G_{z^c}^{[1]}$. Now $T = T^c \in \mathrm{Syl}_t(N)$ and $N \trianglelefteq G_\gamma$. Then, by Lemma 2.4 (ii), $N \leq G_x^{[2]} \leq G_y^{[1]}$. Since $s \geq 4$, G_γ is transitive on $\Delta(y)\backslash\{x\}$. By the Frattini argument $G_\gamma = N_{G_\gamma}(T)N$. Thus, $N_{G_\gamma}(T)$ is transitive on $\Delta(y)\backslash\{x\}$ since $N \leq G_y^{[1]}$. As $T \leq G_z^{[1]}$, $T \leq G_u^{[1]}$, for all $u \in \Delta(y)$. But then, $T \leq G_y^{[2]}$ which is a p-group and thus $T = 1$. This completes the proof of (i).

(ii). From Lemma 2.4 (ii), we have $G_x^{[1]} \cap G_z = G_x^{[2]}$ and $G_x \cap G_z^{[1]} = G_z^{[2]}$, and thus,

$$G_x^{[2]} \cap G_z^{[2]} \leq G_{x,y,z}^{[1]} \leq G_x^{[1]} \cap G_z^{[1]} \leq G_x^{[2]} \cap G_z^{[2]}$$

which gives $G_x^{[1]} \cap G_z^{[1]} = G_{x,y,z}^{[1]} = G_x^{[2]} \cap G_z^{[2]}$.

Since, by (i), $G_x^{[1]} \cap G_z^{[1]} = G_x^{[2]} \cap G_z^{[2]}$ is a p-group which is normalized by $G_x^{[2]}$ and $G_z^{[2]}$, Lemma 2.4 (i) implies $G_x^{[1]} \cap G_z^{[1]} \leq O_p(G_x^{[2]}) = Q_x$ and $G_x^{[1]} \cap G_z^{[1]} \leq O_p(G_z^{[2]}) = Q_z$. Hence

$$Q_x \cap Q_z \leq G_x^{[2]} \cap G_z^{[2]} \leq Q_x \cap Q_z$$

and this completes the proof of (ii).
(iii). Suppose that $R \leq Q_y G_x^{[1]} \cap G_z^{[2]}$ has p'-order. Since $G_x^{[1]}$ is normalized by Q_y, we have $R \leq G_x^{[1]}$. Hence, $R \leq G_x^{[1]} \cap G_z^{[2]}$ which by (i) is a p-group. Thus, $R = 1$ and $Q_y G_x^{[1]} \cap G_z^{[2]}$ is a p-group. Since $Q_z \leq Q_y G_x^{[1]} \cap G_z^{[2]} \trianglelefteq G_z^{[2]}$ and $O_p(G_z^{[2]}) = Q_z$, we have $Q_y G_x^{[1]} \cap G_z^{[2]} = Q_z$.
(iv). Using (iii) and the modular law, we have

$$Q_y G_x^{[2]} \cap Q_y G_z^{[2]} = Q_y(Q_y G_x^{[2]} \cap G_z^{[2]}) \leq Q_y(Q_y G_x^{[1]} \cap G_z^{[2]}) = Q_y Q_z = Q_y.$$

(v). Suppose $G_x^{[1]} \neq G_x^{[2]}$. Let $g \in G_x^{[1]} \setminus G_x^{[2]}$. Then, by Lemma 2.4 (ii), $z^g \neq z$. Then, $G_x^{[1]} G_z^{[2]} = (G_x^{[1]} G_z^{[2]})^g = G_x^{[1]} G_{z^g}^{[2]}$. Hence, using (ii) and the modular law, we have

$$\begin{aligned}
[G_z^{[2]}, G_z^{[2]}] &\leq [G_x^{[1]} G_z^{[2]}, G_x^{[1]} G_z^{[2]}] \cap G_z^{[2]} \leq [G_x^{[1]} G_z^{[2]}, G_x^{[1]} G_{z^g}^{[2]}] \cap G_z^{[2]} \\
&\leq G_x^{[1]} [G_z^{[2]}, G_{z^g}^{[2]}] \cap G_z^{[2]} \leq G_x^{[1]} (G_z^{[2]} \cap G_{z^g}^{[2]}) \cap G_z^{[2]} \\
&\leq G_x^{[1]} Q_z \cap G_z^{[2]} = Q_z.
\end{aligned}$$

It follows that $G_z^{[2]}/Q_z$ is abelian.
(vi). Since $s \geq 4$, $|G_{w,x,y,z}|$ is divisible by $\operatorname{lcm}(q_x, q_y)$. Let $t \in \pi(G_{x,y})$ and $h \in G_{x,y}$ have order t. If t divides $q_x q_y$, then t divides $|G_{w,x,y,z}|$. Suppose t does not divide $q_x q_y$. Then, $\langle h \rangle$ fixes a vertex in both $u' \in \Delta(x) \backslash \{y\}$ and $z' \in \Delta(y) \backslash \{x\}$. Thus, t divides $|G_{u',x,y,z'}| = |G_{w,x,y,z}|$ since $s \geq 4$. It follows that $\pi(G_{x,y}) \subseteq \pi(G_{w,x,y,z})$. The reverse inclusion is immediate since $G_{w,x,y,z}$ is a subgroup of $G_{x,y}$. ☐

Lemma 2.6 *If $G_x^{[2]}/Q_x$ is abelian of exponent t, then C_y/Q_y is abelian of exponent t and has order at least t^2.*

Proof For $u, v \in \Delta(y)$, $[G_u^{[2]}, G_v^{[2]}] \leq Q_u \cap Q_v \leq Q_y$ by Lemma 2.5 (ii) and the fact that $G_x^{[2]}/Q_x$ is abelian. Hence, C_y/Q_y is abelian and, as $G_x^{[2]} Q_y \cap G_u^{[2]} Q_y = Q_y$ by Lemma 2.5 (iv), the claim follows. ☐

The proof of the following lemma is based on an argument that can be found in [1, Lemma 4.8].

Lemma 2.7 *Let t be a prime dividing q_x. Then $G_x^{[2]}$ is a t'-group if and only if $G_x^{[1]}$ is a t'-group.*

Proof It suffices to prove that $G_x^{[1]}$ is a t'-group whenever $G_x^{[2]}$ is a t'-group. Assume that $G_x^{[2]}$ is a t'-group. Let (x, y, z, a) be a 3-arc and let $T \in \mathrm{Syl}_t(G_{x,y,z,a})$. Then, as

$s \geq 4$ and t divides q_x, T acts non-trivially on $\Delta(x)$. Hence, again as $s \geq 4$, T fixes no 4-arcs starting with a vertex in a^G. If $N_{G_y}(T) \not\leq G_z$, then there exists $g \in N_{G_y}(T) \backslash G_z$ such that $T = T^g \leq G^g_{x,y,z,a} = G_{x^g,y,z^g,a^g}$. Hence, T fixes the 4-arc (a, z, y, z^g, a^g), which is a contradiction. Therefore, $N_{G_y}(T) \leq G_z$. Let $X \in \mathrm{Syl}_t(G_x^{[1]})$ be normalized by T. Then, $N_X(T) \leq G_z$. Hence,

$$N_X(T) \leq X \cap G_z \leq G_x^{[1]} \cap G_z = G_x^{[2]}$$

by Lemma 2.4 (ii) and so $N_X(T) = 1$ as $G_x^{[2]}$ is a t'-group. It follows that $C_X(T) \leq N_X(T) = 1$, and so we conclude that $X = 1$. Hence, $G_x^{[1]}$ is a t'-group and this establishes the claim. $\square$

The next lemma, which is fundamental for the approach to our proof of Theorem 1.1, is the origin of the name pushing up type amalgam.

Lemma 2.8 *Suppose that K is a non-trivial characteristic subgroup of Q_y. Then, $N_{G_x}(K) = G_{x,y}$. In particular, no non-trivial characteristic subgroup of Q_y is normalized by L_x.*

Proof Suppose K is a non-trivial characteristic subgroup of Q_y. Then, K is normalized by G_y and so also by $G_{x,y}$. Since G_x acts 2-transitively on $\Delta(x)$, $G_{x,y}$ is a maximal subgroup of G_x. Hence, if $N_{G_x}(K) > G_{x,y}$, then K is normalized by G_x. But then K is normalized by G_x and G_y and so $K = 1$ by Lemma 2.2. $\square$

3 $L_x^{\Delta(x)}$ is not a projective linear group of degree at least 3

In this section, we intend to demonstrate

Proposition 3.1 *Suppose that $s \geq 4$ and Δ is a thick, locally finite, locally s-arc transitive G-graph of pushing up type with respect to (x, y) and the prime p. Then, for $n \geq 3$ and a a natural number, $F^*(G_x^{\Delta(x)}) \not\cong \mathrm{PSL}_n(p^a)$ acting on projective points.*

Throughout this section, we assume that the Main Hypothesis holds and

$$L_x^{\Delta(x)} \cong \mathrm{PSL}_n(q)$$

with $n \geq 3$, $q = p^a$ and $\Delta(x)$ corresponding to the points in projective $(n-1)$-space. We continue with the notation established in Notation 2.1.

Before we start on the proof, we record the following facts about projective linear groups.

Lemma 3.2 *Assume that $n \geq 3$, p is a prime, $q = p^k$ and $\mathrm{PSL}_n(q) \trianglelefteq H \leq \mathrm{P\Gamma L}_n(q)$ acting on the projective space PV. Let $u, v \in PV$ be distinct points. The following statements hold:*

(i) $|H/H_v| - 1 = q\left(\frac{q^{n-1}-1}{q-1}\right)$.
(ii) *There exists a unique* $E \trianglelefteq H_v$ *such that* $O_p(E) = O_p(H_v)$, $E/O_p(E) \cong \mathrm{SL}_{n-1}(q)$ *and* $|H_v/E|$ *divides* $(q-1)k$. *Moreover,* $O_p(E)$ *is a natural module for* $E/O_p(E)$.
(iii) *Either* $n = 3$ *and* $|H_{u,v}/O_p(H_{u,v})|$ *divides* $(q-1)^2k$, *or there exists* $O_p(H_{u,v}) \trianglelefteq F \trianglelefteq H_{u,v}$ *such that* $F/O_p(F) \cong \mathrm{SL}_{n-2}(q)$ *and* $|H_{u,v}/F|$ *divides* $(q-1)^2k$.
(iv) *Let* N *and* E *be subgroups of* H_v *with* $O_p(H_v) = O_p(N) = O_p(E)$ *and* $E/O_p(E) \cong \mathrm{SL}_{n-1}(q)$. *Suppose* E *normalizes* N. *Then, one of the following holds:*

   (a) $E \le N$;
   (b) $N/O_p(H_v)$ *cyclic,* $[N, E] \le O_p(H_v)$ *and* $|N/O_p(H_v)|$ *divides* $(q-1)$;
   (c) $E' \le N < E$, $n = 3$ *and either*
      (ci) $q = p = 2$ *and* $N/O_2(H_v) \cong \mathrm{C}_3$; *or*
      (cii) $q = p = 3$ *and* $N/O_3(H_v) \cong \mathrm{Q}_8$.

Proof $(i) - (iii)$. Straight forward.
(iv). For a subgroup $X \le H_v$, we will denote with $\overline{X}$ its image in $H_v/O_p(H_v)$. Since E normalizes N, we have $[\overline{N}, \overline{E}] \trianglelefteq \overline{E}$. Hence either $\overline{E} \le [\overline{N}, \overline{E}]$, $[\overline{N}, \overline{E}] \le Z(\overline{E})$, or $n = 3 = q$ and $[\overline{N}, \overline{E}] = [\overline{E}, \overline{E}] \cong Q_8$ or $n = 3$, $q = 2$ and $[\overline{N}, \overline{E}] = [\overline{E}, \overline{E}] \cong C_3$. In the second case, the Three Subgroup Lemma gives $[\overline{N}, [\overline{E}, \overline{E}]] = 1$. It follows that $\overline{N} \le C_{\overline{H_v}}([\overline{E}, \overline{E}])$. Thus $[\overline{N}, \overline{E}] = 1$ too and $\overline{N}$ is a cyclic group whose order divides $q-1$. In the third case, $\overline{E} \le \overline{N}$ or $\overline{N} \cong \mathrm{Q}_8$. In the fourth case, $\overline{E} \le \overline{N}$ or $\overline{N} \cong \mathrm{C}_3$. □

We begin the proof of Proposition 3.1 with a lemma which restricts the structure of $G_x^{[2]}/Q_x$.

Lemma 3.3 *One of the following holds:*

(i) $G_x^{[2]}/Q_x \cong \mathrm{C}_t$ *with* t *dividing* $q-1$; *or*
(ii) $G_x^{\Delta(x)} \cong \mathrm{PSL}_3(2)$ *and* $G_x^{[2]}/Q_x \cong \mathrm{C}_3$.

Proof Let $P = G_{x,y}$ and put $\overline{P} = P/G_x^{[1]}Q_y$. Then, as $G_x^{[1]}Q_y = G_x^{[1]}O_p(P)$, Lemma 3.2 implies $\mathrm{SL}_{n-1}(q) \trianglelefteq \overline{P} \le \Gamma\mathrm{L}_{n-1}(q)$. Let $\overline{E} \le \overline{P}$ with $\overline{E} \cong \mathrm{SL}_{n-1}(q)$. Recall that $C_y \le G_y^{[1]}$ and C_y is normal in G_y. Hence, $\overline{C}_y$ is a normal subgroup of $\overline{P}$.

Let $u \in \Delta(y)\backslash\{x\}$ with $z \ne u$. Then, by Lemma 2.5 (iv), $[G_z^{[2]}, G_u^{[2]}] \le Q_y$. It follows that $Q_yG_z^{[2]}$ is normal in C_y. Since $\overline{G_z^{[2]}}$ is normal in $\overline{C_y}$, $\overline{G_z^{[2]}}$ is subnormal in $\overline{P}$. If $\overline{G_z^{[2]}}$ does not centralize $\overline{E}$, then $\overline{E}' \le \overline{G_z^{[2]}}$ by Lemma 3.2 (iii) and Lemma 2.6. Since this is true for all $z \in \Delta(y)\backslash\{x\}$ and $\overline{[G_z^{[2]}, G_u^{[2]}]} = 1$, this is impossible unless $(n, q) = (3, 2)$ and $\overline{G_z^{[2]}}$ is cyclic of order 3. Hence, in the general case, $\overline{G_z^{[2]}}$ centralizes $\overline{E}$ and we conclude that $\overline{G_z^{[2]}}$ is cyclic of order t dividing $q-1$. We have demonstrated

$$\overline{G_z^{[2]}} \cong \begin{cases} \mathrm{C}_t & t \text{ divides } q-1 \\ \mathrm{C}_3 & (n, q) = (3, 2). \end{cases}$$

Finally, Lemma 2.5 (iii) yields

$$G_z^{[2]}/Q_z = G_z^{[2]}/(G_z^{[2]} \cap G_x^{[1]} Q_y) \cong G_z^{[2]} G_x^{[1]} Q_y / G_x^{[1]} Q_y = \overline{G_z^{[2]}}$$

and this completes the proof. □

Lemma 3.4 *If $F^*(G_x^{\Delta(x)}) \cong \mathrm{PSL}_n(q)$ with $n \geq 3$, then $F^*(G_x^{\Delta(x)}) \cong \mathrm{PSL}_3(2)$ and $G_x^{[2]}/Q_x \cong \mathrm{C}_3$.*

Proof Suppose that $F^*(G_x^{\Delta(x)}) \not\cong \mathrm{PSL}_3(2)$ with $G_x^{[2]}/Q_x \cong \mathrm{C}_3$. We first prove that
1°. $q^{n-1} = 2^6$ or $q^{n-1} - 1 = p^2 - 1$ and p is a Mersenne prime.

Suppose that t is a Zsigmondy prime for $((n-1)k, p)$. By [2, 3.8], t does not divide $(n-1)k$, and since $n - 1 \geq 2$, t does not divide $q - 1$. Therefore, as t divides $q_x = q(q^{n-1} - 1)/(q - 1)$, combining Lemmas 2.7 and 3.3 yields t does not divide $|G_x^{[1]}|$. Since t divides $|G_{x,y}|$, t divides $|G_{w,x,y}|$ by Lemma 2.5 (vi). Therefore, t divides $|G_{w,x,y}{}^{\Delta(x)}|$. However, Lemma 3.2 (iii) then implies t divides $|\mathrm{PSL}_{n-2}(q)|$, contrary to t being a Zsigmondy prime. Therefore, Zsigmondy's Theorem (see [2, 3.8]) implies that $q^{n-1} = 2^6$ or $q^{n-1} - 1 = p^2 - 1$ and $p = 2^r - 1$ is a Mersenne prime.
2°. We have $q^{n-1} \neq 2^6$.

Assume that $q^{n-1} = 2^6$. Then, as $n \geq 3$, we have one of the following cases $F^*(G_x^{\Delta(x)}) \cong \mathrm{PSL}_3(8)$ with $q_x = 8(8+1)$, $\mathrm{PSL}_4(4)$ with $q_x = 4(4^2 + 4 + 1)$ or $\mathrm{PSL}_7(2)$ with $q_x = 2(2^6 - 1)$. Since $s \geq 4$, $G_{x,y}$ has order divisible by q_x^2. The first case $|G_x^{[1]}|$ is coprime to 3 by Lemmas 2.7 and 3.3. Therefore, $|G_{x,y}|_3 = |G_{x,y}^{\Delta(x)}|_3 = 3^3 < 3^4 = (q_x^2)_3$, which is a contradiction. Similarly, the second case is impossible as 7^2 does not divide $|G_{x,y}^{\Delta(x)}|$. Therefore, $F^*(G_x^{\Delta(x)}) \cong \mathrm{PSL}_7(2)$ and $G_x^{[2]} = Q_x$ is a 2-group by Lemma 3.3. Set $H = G_{w,x,y,z}$. Then, by Lemma 2.5 (v) $\pi(H) = \pi(G_{x,y}) \supseteq \{2, 3, 5, 7, 31\}$ and by Lemma 2.4 (ii) $H \cap G_x^{[1]} \leq G_z \cap G_x^{[1]}$ is a 2-group. Hence, $H^{\Delta(x)} \leq G_{w,x,y}^{\Delta(x)} \cong 2^{10} : \mathrm{SL}_5(2)$ and $\pi(H^{\Delta(x)}) \supseteq \{3, 5, 7, 31\}$. By [8], the maximal over-groups of a Singer cycle in $\mathrm{SL}_5(2)$ have order $5 \cdot 31 = 155$ and so we conclude that $G_{w,x,y}^{\Delta(x)} = H^{\Delta(x)} O_2(G_{w,x,y}^{\Delta(x)})$. In particular, we see that H has a quotient isomorphic to $\mathrm{SL}_5(2)$ and has order $2^\ell \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ for some ℓ. Since $s \geq 4$, H operates transitively on $\Delta(z) \setminus \{y\}$. Hence, for $a \in \Delta(z) \setminus \{y\}$, $|H : H_a| = q_x = 126$ and so $|H_a| = 2^{\ell-1} \cdot 5 \cdot 31$. Therefore,

$$2^{10}.5.31 \geq |H_a^{\Delta(x)} O_2(G_{w,x,y}^{\Delta(x)})/O_2(G_{w,x,y}^{\Delta(x)})| \geq 2^9 \cdot 5 \cdot 31 > 5 \cdot 31$$

and $H_a^{\Delta(x)} O_2(G_{w,x,y}^{\Delta(x)})/O_2(G_{w,x,y}^{\Delta(x)})$ contains a Singer cycle of $\mathrm{SL}_5(2)$, which is a contradiction.

Because of (**1**°) and (**2**°), it remains to exclude the possibility that $q^{n-1} - 1 = p^2 - 1$ with $p = 2^r - 1$ a Mersenne prime. Thus, $n = 3$, $q_x = p(p+1) = 2^r p$ and $p - 1 = 2(2^{r-1} - 1)$ is not divisible by 4.

By Lemma 2.4 (ii),

$$G_{w,x,y,z} \cap G_x^{[1]} = G_x^{[1]} \cap G_z = G_x^{[2]}$$

and by Lemma 3.3 $|G_x^{[2]}|$ divides $(p-1)|Q_x|$. By Lemma 3.2 $|G_{w,x,y,z}^{\Delta(x)}|$ divides $(p-1)^2p^2$. Hence $|G_{w,x,y,z}|$ divides $(p-1)^3p^2|Q_x|$. So, as 4 does not divide $p-1$, and $G_{w,x,y,z}$ acts transitively on $\Delta(z)\backslash\{y\}$, $q_z = q_x = 2^r p$ is not divisible by 16. Hence, $r \in \{2,3\}$. Furthermore, if $r=3$, then $|G_x^{[2]}|$ is even. We signal
3°. $q = p = 2^r - 1 \in \{3,7\}$ and $|G_x^{[2]}|$ is even if $q = 7$.

Suppose that $G_x^{[2]}$ has odd order. Then, $r=2$, $q=p=3$, $G_x = G_x^{[1]}L_x$, $L_x/Q_x \cong \mathrm{PSL}_3(3)$ and $q_x = 12$. By Lemma 2.7, $G_x^{[1]}$ also has odd order. Let $S \in \mathrm{Syl}_2(G_{w,x,y,z})$. Then, $S \cap G_x^{[1]} = 1$ and so $S \cong S^{\Delta(x)} \leq G_{w,x,y}^{\Delta(x)}$. Hence, S is elementary abelian and $|S| \leq 4$. Since $q_x = 12$ and q_x divides $|G_{w,x,y,z}|$ by Lemma 2.5 (v), we conclude that $|S| = 4$. Since $s \geq 4$, and $q_z = q_x = 12$, we now know that $|G_{w,x,y,z,a}|$ is odd for all $a \in \Delta(z) \setminus \{y\}$.

Let $T \in \mathrm{Syl}_2(G_{y,z})$ with $S \leq T$. Then, $T \cap G_z^{[1]} = 1$ and $T \cong \mathrm{SDih}(16)$ is semidihedral. Since S is elementary abelian of order 4, we have $Z(T) \leq S$. In particular, $Z(T)G_z^{[1]}/G_z^{[1]}$ acts by conjugation inverting each element of $Q_yG_z^{[1]}/G_z^{[1]}$. Observe that Q_y has 4 orbits of length 3 on $\Delta(z)\backslash\{y\}$ each of which is fixed by $Z(T)$. Hence, $Z(T)$ fixes a vertex $a \in \Delta(z)\backslash\{y\}$, contrary to $|G_{w,x,y,z,a}|$ being odd. This contradiction shows that $|G_x^{[2]}|$ is divisible by 2.

Assume that $|G_x^{[2]}|$ is even. Then, $q \in \{3,7\}$ by (**3**°). Lemma 3.3 states that $G_x^{[2]}/Q_x$ is cyclic, and so the Sylow 2-subgroups of $G_x^{[2]}/Q_x$ have order 2. By Lemma 2.6, C_y/Q_y is abelian and the Sylow 2-subgroups of C_y are elementary abelian and have order at least 4. As $C_y^{\Delta(x)}$ is normal in $G_{x,y}^{\Delta(x)}$ and $G_{x,y}^{\Delta(x)}/Q_y^{\Delta(x)}$ is isomorphic to a subgroup of $\mathrm{GL}_2(q)$ containing $\mathrm{SL}_2(q)$, Lemma 3.2(iv) yields that $C_y^{\Delta(x)}/Q_y^{\Delta(x)}$ is cyclic. Lemma 2.4 (ii) gives $G_x^{[1]} \cap C_y \leq G_x^{[1]} \cap G_y^{[1]} = G_x^{[2]}$, and thus, $C_y^{\Delta(x)}/Q_y^{\Delta(x)} \cong C_y/G_x^{[2]}Q_y$. We deduce that C_y/Q_y contains exactly three involutions. If $q_y > 2$, then there exists $u, v \in \Delta(y)$ such that $G_u^{[2]}Q_y/Q_y \cap G_v^{[2]}Q_y/Q_y$ has an involution and this contradicts Lemma 2.5 (iv). Hence, $q_y = 2$ and G_y acts transitively on the three involutions in C_y/Q_y. Let $S_y \in \mathrm{Syl}_2(C_y)$. Then, S_yQ_y is normalized by G_y, and thus, S_yQ_y has a unique G_y-conjugacy class of involutions.

Since $q_y = 2$, we have $O^{p'}(G_{x,y}) \leq G_y^{[1]}$ and so $O^{p'}(G_{x,y}) = O^{p'}(G_y^{[1]})$ is normal in G_y. Because $q = p \in \{3,7\}$, $O^{p'}(G_x)/Q_x \cong \mathrm{PSL}_3(p)$ or $\mathrm{SL}_3(p)$ and, as the Schur multiplier of $\mathrm{PSL}_2(p)$ has order 2, we get $O^{p'}(G_y^{[1]})/Q_x \cong \mathrm{ASL}_2(p)$. In particular, $G_x^{[2]} \cap O^{p'}(G_y^{[1]}) = Q_x$.

Let $a \in O^{p'}(G_y^{[1]})$ be an involution and set $T = C_y\langle a\rangle$. Then, Lemma 3.2 (iv) implies

$$[T^{\Delta(x)}, O^{p'}(G_y^{[1]})^{\Delta(x)}] = [a^{\Delta(x)}, O^{p'}(G_y^{[1]})^{\Delta(x)}][C_y^{\Delta(x)}, O^{p'}(G_y^{[1]})^{\Delta(x)}] \leq Q_y^{\Delta(x)}$$

and so $T \leq C_yG_x^{[1]}$, as $|C_y^{\Delta(x)}|$ is even. Hence, using Lemma 2.4 (i)

$$T = C_yG_x^{[1]} \cap T = C_y(G_x^{[1]} \cap T) \leq C_y(G_x^{[1]} \cap G_y^{[1]}) = C_yG_x^{[2]} = C_y.$$

Thus, $a \in C_y \cap O^{p'}(G_y^{[1]})$. Since $O^{p'}(G_y^{[1]})$ is normal in G_y, we deduce that $S_y \le O^{p'}(G_y^{[1]})$ which is impossible as $O^{p'}(G_y^{[1]}) \cap G_x^{[2]} = Q_x$. This completes the proof. □

Lemma 3.5 *Suppose that $G_x^{\Delta(x)} \cong \mathrm{PSL}_3(2)$. Pick $U \in \mathrm{Syl}_3(G_y)$ and set $D = U \cap C_y$ and $F = D \cap L_{x,y}$. Then*

(i) $q_x = 6$ and $q_y = 2$;
(ii) $|G_x^{[2]}/Q_x| = 3$;
(iii) D is elementary abelian of order 9, $C_y = DQ_y$ and $D \in \mathrm{Syl}_3(G_y^{[1]})$;
(iv) F has order 3, $Q_yF \trianglelefteq G_y$, $F = Z(U)$ and U is extraspecial of order 27.

Proof As $G_x^{\Delta(x)} \cong \mathrm{PSL}_3(2)$, we have $p = 2$, $q_x = 6$ and Lemma 3.4 implies $|G_x^{[2]}/Q_x| = 3$.

By Lemma 2.6, C_y/Q_y is an elementary abelian 3-group of rank at least 2. Since $G_x^{[1]} \cap G_y^{[1]} \le G_x^{[2]}$ and $G_{x,y}^{\Delta(x)} \cong \mathrm{Sym}(4)$, we deduce C_y/Q_y has order 9. Hence, D is elementary abelian of order 9, $C_y = DQ_y$ and $D \in \mathrm{Syl}_3(G_y^{[1]})$. This proves (iii).

We know FQ_y has index 3 in C_y and FQ_y is normalized by $G_{x,y}$. If $FQ_y = G_u^{[2]}Q_y$ for some $u \in \Delta(y)\backslash\{x\}$, then as $G_{x,y}$ is transitive on $\Delta(y) \setminus \{x\}$, we have $G_u^{[2]}Q_y = G_z^{[2]}Q_y$ for all $u \in \Delta(y) \setminus \{x\}$ contrary to Lemma 2.5 (iv). So of the four subgroups of index 3 in C_y, there are only three candidates for $G_u^{[2]}Q_y$ and so we conclude that $q_y = 2$ and this proves (i).

Because U acts transitively on $\Delta(y)$, U permutes the three subgroups of $\{G_u^{[2]}Q_y \mid u \in \Delta(y)\}$. In particular, as $D \in \mathrm{Syl}_3(G_y^{[1]})$ and $G_y/G_y^{[1]} \cong \mathrm{Sym}(3)$, we have U is non-abelian of order 27. As FQ_y is normalized by U, $F = Z(U)$. This concludes the proof. □

Lemma 3.6 *Assume that $G_x^{\Delta(x)} \cong \mathrm{PSL}_3(2)$. Then, L_x has either 1, 2, 3 or 6 non-central L_x chief factors, each of which is 3-dimensional.*

Proof We establish [10, Hypothesis] with $p = 2$ using boldface letters for the groups used in [10, Hypothesis]. So set $\mathbf{M} = L_x$, $\mathbf{E} = Q_x$, $\mathbf{B} \in \mathrm{Syl}_2(\mathbf{M})$ and $\mathbf{P}_1, \mathbf{P}_2 \le \mathbf{M}$ such that $\mathbf{P}_1 \cap \mathbf{P}_2 = \mathbf{B}$. To reassure ourselves, this means that $\mathbf{M}/\mathbf{E} = L_x/Q_x \cong \mathrm{PSL}_3(2)$, $\mathbf{B}/\mathbf{E} \cong \mathrm{Dih}(8)$ and $\mathbf{P}_1/\mathbf{E} \cong \mathbf{P}_2/\mathbf{E} \cong \mathrm{Sym}(4)$. We choose notation so that $\mathbf{P}_1 = L_{x,y}$. We have $\mathbf{P}_1 = \mathbf{P}_1^*$ and $\mathbf{P}_2 = \mathbf{P}_2^*$. This means that [10, Hypothesis (WBN)] is satisfied. So we take $\mathbf{S} = \mathbf{B}$ and $\mathbf{T} = Q_y$. By Lemma 2.8, [10, Hypothesis (P)] holds. Since $\mathbf{P}_1 = \mathbf{P}_1^*$ and $\mathbf{P}_2 = \mathbf{P}_2^*$, $O^{2'}(\mathbf{P}_1) = \mathbf{P}_1$ and $O^{2'}(\mathbf{P}_2) = \mathbf{P}_2$. Thus, setting

$$\mathbf{L} = \langle O^2(O^{2'}(\mathbf{P}_1^*)), O^2(O^{2'}(\mathbf{P}_2^*))\rangle = \langle O^2(\mathbf{P}_1^*), O^2(\mathbf{P}_2^*)\rangle$$

and remembering $\mathbf{E} \le \mathbf{T}$, we have

$$\mathbf{LE} = \mathbf{LT}.$$

So, as $\mathbf{E}$ is a 2-group, $\mathbf{E}/O_2(\mathbf{E})$ is a $2'$-group, and $\mathbf{E} \le \mathbf{F}O_2(\mathbf{E})$ for every subgroup $\mathbf{F} \le \mathbf{M}$ with

$$\mathbf{L} \le \mathbf{EF}.$$

Hence, [10, Hypothesis (A)] and [10, Hypothesis (B)] both hold.

Since [10, Hypothesis] holds, and since $\mathbf{M} = L_x$, we can conclude from [10, Theorem] that one of the cases (4), (5), (8), (11) or (13) of that theorem holds. In particular, we see that $\mathbf{E} = Q_x$ has either 1, 2, 3 or 6 non-central L_x chief factors, each of which is 3-dimensional. This proves the claim. ◻

Lemma 3.7 *(i) L_x has two non-central chief factors in Q_x, each of which is 3-dimensional.*
(ii) F has three non-central chief factors in Q_y (where F is as in Lemma 3.5).

Proof Consider the non-central chief factors of G_y in Q_y. If the chief factor is not centralized by $F = Z(U)$, then, as U is extraspecial of order 27, the chief factor has order a multiple of 2^6 and F acts fixed point freely. Thus, the number of F-chief factors is a multiple of 3 which we denote by $3f$. As G_y has characteristic p, $f \ge 1$.

From the perspective of L_x, we have Q_y/Q_x is a non-central F-chief factor, and, as each L_x non-central chief factor in Q_x is 3-dimensional by Lemma 3.6, FQ_x has one non-central chief factor for each L_x non-central chief factor. Thus, L_x has $3f - 1$ non-central chief factors in Q_x. Using Lemma 3.6, we deduce that L_x has 2 non-central chief factors in Q_x. Thus, (i) holds and (ii) follows from this. ◻

Recall the definition of Z_x and Z_y from Notation 2.1 and observe $Z_y \le Z_x$, since Δ is of local characteristic 2 and $Q_x \le Q_y$.

Lemma 3.8 *Suppose $G_x^{\Delta(x)} \cong \mathrm{PSL}_3(2)$. Then*

(i) $Q_y = Q_xQ_z$ and $Z_x \cap Z_z = Z_y$;
(ii) $[Z_x, F] \ne 1$.

Proof We continue the notation from Lemma 3.5.

(i). We know $FQ_y \le G_y^{[1]}$. Since F normalizes Q_z and $Q_z \ne Q_x$, we have $Q_xQ_z = Q_y$. Since Δ is of local characteristic p and $Q_y = Q_xQ_z$, we glean $Z(Q_y) \le Z(Q_x) \cap Z(Q_z) \le Z(Q_y)$.
(ii). Suppose $[Z_x, F] = 1$. Then $[Z_x, O^2(L_x)] = 1$ and $L_x = Q_xO^2(L_x)$. Since $O^2(L_x)$ acts transitively on $\Delta(x)$ and $Z_y \le Z_x$, we obtain $Z_y = 1$ from Lemma 2.2 and this is a contradiction. ◻

Lemma 3.9 *Suppose $G_x^{\Delta(x)} \cong \mathrm{PSL}_3(2)$. Then, $Z_z \not\le G_x^{[1]}$.*

Proof Again we use the notation started in Lemma 3.5. By Lemma 3.4, we know $G_x^{\Delta(x)} \cong \mathrm{PSL}_3(2)$. Set $V_y = \langle Z_u \mid u \in \Delta(y)\rangle$. Then, $V_y \le Q_y$ is normalized by G_y and F does not centralize $Z_x \le V_y$ by Lemma 3.8 (ii). Hence, U acts faithfully on V_y and so we conclude that F has at least three non-central chief factors in V_y. Since F has exactly three non-central chief factors in Q_y and Q_y/Q_x is such a factor, we conclude that $V_y \not\le G_x^{[1]}$ and this delivers the claim. ◻

Proof of Proposition 3.1 Lemma 3.4 gives $G_x^{\Delta(x)} \cong \mathrm{PSL}_3(2)$ and, by Lemma 3.9, $Z_z \not\leq Q_x$. As $F \leq G_y^{[1]}$ normalizes Z_z, $Q_y = Q_x Z_z = Z_x Q_z$. Since $Q_x \cap Z_z$ is centralized by $Z_x Q_z$, we deduce $Q_x \cap Z_z = Z_y$.

Let $u \in \Delta(y) \setminus \{x, z\}$. Since $s \geq 4$, we have $Z_z \not\leq Q_u$. We calculate using the fact that Q_x is the unique Sylow 2-subgroup of $G_x^{[2]}$ by Lemma 3.5 (ii)

$$[G_x^{[2]}, Z_z] \leq G_x^{[2]} \cap Z_z \leq Q_x \cap Z_z = Z_y$$

and

$$[G_u^{[2]}, Z_z] \leq Q_u \cap Z_z = Z_y.$$

It follows that

$$[F, Z_z] \leq [G_u^{[2]} G_x^{[2]}, Z_z] \leq Z_y \leq Q_x.$$

However, $Z_z Q_x / Q_x$ is not centralized by F, and so this is impossible. This contradiction completes the proof. □

4 The main theorem

Suppose that X is a 2-transitive group in its action on Ω. Then, [1, Lemma 2.2] (for example) yields that either there is a prime r such that $F^*(X)$ is a regular elementary abelian r-group, or $F^*(X)$ is a non-abelian simple group. In the first case, we say that X is of *regular type*, and in the second that X is of *simple type*. When X is of simple type, the description of $F^*(X)$ and Ω is conveniently presented in [1, Lemma 2.5] (this result requires the classification of the finite simple groups). Since $G_x^{\Delta(x)}$ acts 2-transitively on $\Delta(x)$ and as we also know that $1 \neq Q_y^{\Delta}(x) \trianglelefteq G_{x,y}^{[1]}$, Proposition 3.1 combined with [1, Lemma 2.5] yields

Lemma 4.1 *The group $F^*(G_x^{\Delta(x)})$ is either of regular type or is of simple type and is isomorphic to a rank 1 group of Lie type in characteristic p in its natural permutation representation (including* $\mathrm{Ree}(3)'$*).*

□

We can now move directly to the proof Theorem 1.1.

Proof of Theorem 1.1 Set $N = O^p(L_x)Q_x$. Then $O_p(N) = Q_x$ and $N = O^p(N)Q_x$. Define $S = Q_y$ and $\widetilde{G} = G_x/Q_x$. Then, with $L = L_x$, we have $L = NS$, $O_p(N) = O_p(L) = Q_x < Q_y = S$ and $C_S(Q_x) \leq C_G(Q_x) \leq Q_x$. Furthermore, [1, Hypothesis 3.3(b) and (c)] follows from Lemma 2.3(v) and Lemma 4.1, respectively. Because of Lemma 2.8 and [1, Lemma 3.8], we are in the same conclusions as [1, 7.7 steps **1**° and **2**°]. Following [1, 7.7 steps **3**° through **9**°] verbatim (being careful to note the role of x and y are reversed) yields $O^p(L_x) \cong \mathrm{AGL}_2(q)'$, $q = p^r$ with p odd, and Q_x elementary abelian. This completes the proof. □

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Declarations

Conflict of interest The authors have no conflict of interest to declare.

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