Journal of Algebraic Combinatorics
https://doi.org/10.1007/s10801-024-01318-x

An ultimately periodic chain in the integral Lie ring of partitions

Riccardo Aragona[1] · **Roberto Civino**[1] · **Norberto Gavioli**[1]

Received: 14 March 2023 / Accepted: 23 March 2024

Abstract

Given an integer n, we introduce the integral Lie ring of partitions with bounded maximal part, whose elements are in one-to-one correspondence to integer partitions with parts in $\{1, 2, \ldots, n-1\}$. Starting from an abelian subring, we recursively define a chain of idealizers and we prove that the sequence of ranks of consecutive terms in the chain is ultimately periodic. Moreover, we show that its growth depends of the partial sum of the partial sum of the sequence counting the number of partitions. This work generalizes our previous recent work on the same topic, devoted to the modular case where partitions were allowed to have a bounded number of repetitions of parts in a ring of coefficients of positive characteristic.

Keywords Integer partitions · Normalizer chain · Idealizer chain · Lie rings

Mathematics Subject Classification 17B70 · 17B60 · 20D20 · 05A17

1 Introduction and preliminaries

Given an integer $n \geq 3$, we defined in a recent work a Lie ring structure on the set of partitions with parts in $\{1, 2, \ldots, n-1\}$ and where each part is allowed to have at most $m-1$ repetitions, for some given $m > 2$. Within the acquired structure, here denoted

All the authors are members of INdAM-GNSAGA (Italy). R. Civino is funded by the Centre of excellence ExEMERGE at University of L'Aquila, with which also the other authors collaborate.

✉ Roberto Civino
roberto.civino@univaq.it

Riccardo Aragona
riccardo.aragona@univaq.it

Norberto Gavioli
norberto.gavioli@univaq.it

[1] DISIM , Università degli Studi dell'Aquila, via Vetoio, 67100 Coppito, AQ, Italy

Published online: 09 April 2024

as $\mathfrak{L}_m(n)$, we recursively defined a chain of idealizers $(\mathfrak{N}_i)_{i\geq -1}$ originating from an abelian subring $\mathcal{T}$. We proved that the rank of $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ as free $\mathbb{Z}_m$-module is q_{i+1}, where q_i is the partial sum of sequence counting the number of partitions of i into at least two parts, each allowed to be repeated at most $m-1$ times [1]. The objective was to tackle the unresolved problem posed by Aragona et al. [3] of computing the growth of a chain of normalizers in a Sylow m-subgroup starting from an elementary abelian regular subgroup $T < \mathrm{Sym}(m^n)$, when m is an odd prime. Indeed, $\mathfrak{L}_m(n)$ was constructed as the iterated wreath product of Lie rings of rank one [1] and, when m is prime, this corresponds exactly to the construction of the graded Lie algebra associated with the lower central series of a Sylow m-subgroup of $\mathrm{Sym}(m^n)$ [7]. In this construction, the abelian subring $\mathfrak{N}_{-1} = \mathfrak{T}$ at the base of the idealizer chain corresponds to the elementary abelian regular subgroup T at the base of normalizer chain. It is important to stress that the combinatorial equality mentioned above is valid only for the first $n-2$ terms of the chain, and the problem of understanding the general behavior of the chain is out of reach at the time of writing.

In this work, we address a similar problem in the case of characteristic zero, i.e., in a Lie ring $\mathfrak{L}(n)$ with integer coefficients of partitions with parts in $\{1, 2, \ldots, n-1\}$. The main combinatorial difference between the two settings is that now each part is allowed to have an unbounded number of repetitions. We show that this significantly affects how the idealizer chain grows. In particular, we prove here that the sequence of consecutive quotient ranks in the idealizer chain depends on the *second partial sum* of the sequence of the integer partitions and is (ultimately) periodic.

We conclude this section by introducing the construction of the partition Lie ring with integer coefficients as a Lie subring of the Witt algebra and by showing some preliminary properties.

1.1 Preliminaries on the Witt algebra

Let $\mathfrak{L}$ and $\mathfrak{H}$ be finite dimensional Lie algebras over a field K, and let $\mathrm{U}(\mathfrak{L})$ be the universal enveloping algebra of $\mathfrak{L}$. It is well known that $\mathrm{U}(\mathfrak{L})$ is a Hopf algebra (see Kassel [4, Proposition V.2.4]) and that its dual space $\mathrm{U}(\mathfrak{L})^* = \mathrm{Hom}_K(\mathfrak{L}, K)$ is an associative algebra on which $\mathfrak{L}$ acts by way of the coadjoint action as a derivation Lie algebra. In particular, the tensor product $\mathrm{U}(\mathfrak{L})^* \otimes_K \mathfrak{H}$ inherits from $\mathfrak{H}$ a Lie algebra structure on which $\mathfrak{L}$ acts as a Lie algebra of derivations via its action on $\mathrm{U}(\mathfrak{L})^*$. The standard wreath product $\mathfrak{H} \wr \mathfrak{L}$ is then defined to be the semidirect product $(\mathrm{U}(\mathfrak{L})^* \otimes_K \mathfrak{H}) \rtimes \mathfrak{L}$ [6]. In the special case, when $\mathfrak{L} = Ku$ is a one-dimensional Lie algebra and K is a field of characteristic 0, the algebra $\mathrm{U}(\mathfrak{L}) = K[u^{(n)}]_{n\geq 0}$ is the associative K-algebra of the divided powers $u^{(n)} = \frac{u^n}{n!}$, where $u^{(i)}u^{(j)} = \binom{i+j}{i}u^{(i+j)}$. Its dual $\mathrm{U}(\mathfrak{L})^*$ can be then identified with the formal power series ring $K[[x]]$ on which $\mathfrak{L}$ acts as an algebra of standard derivations, i.e., $u \cdot f(x) = \frac{\partial}{\partial x} f(x)$.

We recall (see Strade [8, Chapter 2]) that the n-th Witt Lie algebra is defined as

$$\mathrm{W}(n) := \left\{ \sum_{k=1}^{n} f_k \partial_k \mid f_k \in K[[x_1, \ldots, x_n]] \right\}$$

with bracket operation defined on its basis via

$$[f\partial_k, g\partial_h] := \left(f\frac{\partial}{\partial_k}g\right)\partial_h - \left(g\frac{\partial}{\partial_h}f\right)\partial_k.$$

The iterated (restricted) wreath product

$$(Ku)^{\wr n} := \underbrace{Ku \wr \cdots \wr Ku}_{n \text{ times}}$$

can be identified with the subalgebra of $\mathrm{W}(n)$ having the monomial elements $x_{i_1}^{\lambda_{i_1}} \cdots x_{i_k}^{\lambda_{i_h}} \partial_k$, where $1 \le i_1 < i_2 < \cdots < i_h < k \le n$, as a basis. A similar construction, with $K = \mathbb{F}_p$ and divided powers of exponent at most $p-1$, has been used to describe the graded Lie algebra associated with the lower central series to the iterated wreath product of n copies of the cyclic group of order p, i.e., the Sylow p-subgroup of the symmetric group $\mathrm{Sym}(p^n)$ [7].

1.2 The integral ring of partitions

In the present paper, we consider the *integral ring of partitions*, i.e., the Lie subring of the iterated wreath product $(\mathbb{Q}u)^{\wr n}$, generated over $\mathbb{Z}$ by the monomial elements described above. We now give a description of this ring starting from the construction of the *Lie ring over $\mathbb{Z}_m$ of partitions with bounded maximal part* [1], customizing it for $m = 0$. We advise the reader to also refer to the cited paper for further details and to the original problem of characterizing the chain of normalizers originating from a regular elementary abelian group of order 2^n in the Sylow 2-subgroup of the symmetric group [2, 3].

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a sequence of non-negative integers with finite support, i.e.,

$$\mathrm{wt}(\Lambda) = \sum_{i=1}^{\infty} i\lambda_i < \infty.$$

The sequence Λ defines a *partition* of $N = \mathrm{wt}(\Lambda)$. Each nonzero i is a *part* of the partition, and the integer λ_i is the multiplicity of the part i in Λ. The maximal part of Λ is the number $\max\bigl(\{i \mid \lambda_i \neq 0\}\bigr)$. The set of the partitions whose maximal part less than or equal to j is denoted by $\mathcal{P}(j)$.

The *power monomial* x^Λ, where Λ is a partition, is defined as $x^\Lambda = \prod_i x_i^{\lambda_i}$. Given a positive integer n and denoting by ∂_k the derivation given by the standard partial derivative with respect to x_k, where $1 \le k \le n$, we define by $\mathfrak{L}(n)$ the free $\mathbb{Z}$-module spanned by the basis

$$\mathcal{B} := \left\{x^\Lambda \partial_k \mid 1 \le k \le n \text{ and } \Lambda \in \mathcal{P}(k-1)\right\},$$

and we set $\mathcal{B}_u := \left\{x^\Lambda \partial_k \in \mathcal{B} \mid k = u\right\}$. The module $\mathfrak{L}(n)$ is endowed with a structure of Lie ring, where the Lie bracket is defined on the basis $\mathcal{B}$ by

$$\begin{aligned}\left[x^\Lambda \partial_k, x^\Theta \partial_j\right] &:= \partial_j(x^\Lambda)x^\Theta \partial_k - x^\Lambda \partial_k(x^\Theta)\partial_j \\ &= \begin{cases} \partial_j(x^\Lambda)x^\Theta \partial_k & \text{if } j < k, \\ -x^\Lambda \partial_k(x^\Theta)\partial_j & \text{if } j > k, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This operation is then extended to a Lie product on $\mathfrak{L}(n)$ by bilinearity, and the resulting structure $\mathfrak{L}(n)$ is called the *integral Lie ring of partitions with parts in* $\{1, 2, \dots, n-1\}$.

In the remainder of the paper, we deal with homogeneous subrings of $\mathfrak{L}(n)$, which are defined as follows.

Definition 1.1 A Lie subring $\mathfrak{H}$ of $\mathfrak{L}(n)$ is said to be *homogeneous* if it is the free $\mathbb{Z}$-module spanned by some subset $\mathcal{H}$ of $\mathcal{B}$.

The following result on homogeneous subrings can be proved as in the case of the modular ring $\mathfrak{L}_m(n)$ [1, Theorem 2.5]. Here and the remainder of the paper, if $\mathcal{H}$ is a subset of $\mathcal{B}$, then its *idealizer* is defined as

$$N_{\mathcal{B}}(\mathcal{H}) := \{b \in \mathcal{B} \mid [b, h] \in \mathbb{Z}\mathcal{H} \text{ for all } h \in \mathcal{H}\}.$$

Theorem 1.2 *Let $\mathfrak{H}$ be a homogeneous subring of $\mathfrak{L}$ having basis $\mathcal{H} \subseteq \mathcal{B}$. The idealizer of $\mathfrak{H}$ in $\mathfrak{L}(n)$ is the homogeneous subring of $\mathfrak{L}(n)$ spanned by $N_{\mathcal{B}}(\mathcal{H})$ as a free $\mathbb{Z}$-module.*

The chain of idealizers in $\mathfrak{L}_m(n)$ originated from the abelian homogeneous Lie subring $\mathfrak{T} = \langle \partial_1, \dots, \partial_n\rangle$ [1]. Analogously, here we deal with the idealizer chain defined in $\mathfrak{L}(n)$ starting from $\mathfrak{T}$ and defined as follows:

$$\mathfrak{N}_i = \begin{cases} \mathfrak{T} & i = -1, \\ N_{\mathfrak{L}(n)}(\mathfrak{T}) & i = 0, \\ N_{\mathfrak{L}(n)}(\mathfrak{N}_{i-1}) & i \geq 1.\end{cases} \tag{1}$$

1.3 Organization of the paper

The remainder of the paper is organized as follows: Sect. 2 is devoted to combinatoric aspects of $\mathfrak{L}(n)$: we introduce a chain of subsets $(\mathcal{N}_i)_{i\geq -1}$, and we show that the cardinalities of $\mathcal{N}_i \setminus \mathcal{N}_{i-1}$ depend, *up to periodicity*, on the second partial sum of the sequence of integer partitions (see Corollary 2.12 and 2.13). In Sect. 3, we show (see Theorem 3.4) that the free $\mathbb{Z}$-modules spanned by the sets $\mathcal{N}_i$s coincide with the idealizers of Eq. (1), yielding our main contribution of Corollary 3.6, which connects, up to periodicity, the rank of the free $\mathbb{Z}$-modules $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ with the second partial sum of the sequence of integer partitions.

2 Combinatorics of the integral Lie ring of partitions

The methods crafted for positive characteristic scenarios are not well-suited for the integral Lie ring of partitions. This situation calls for the introduction of novel combinatorial tools that we are in the process of defining.

2.1 Levels of the basis elements

As we will show in the remainder of this paper, the combinatorial properties of $\mathfrak{L}(n)$ depend on the behavior of the functions defined below.

Definition 2.1 Let $i \geq -1$ be an integer and let $1 \leq r_i \leq n-1$ be such that $i \equiv r_i \bmod (n-1)$. Write

$$i = (h_i - 1)(n-1) + r_i,$$

where

$$h_i := \left\lfloor \frac{i-1}{n-1} \right\rfloor + 1.$$

The *weight-degree function* is defined as $\mathrm{wd}\colon \mathbb{Z}\mathcal{B} \to \mathbb{Z}$ by

$$\mathrm{wd}(c \cdot x^{\Lambda}\partial_k) := \mathrm{wt}(\Lambda) - \deg(x^{\Lambda}) + n - k,$$

and the *i-th level function* $\mathrm{lev}_i\colon \mathbb{Z}\mathcal{B} \to \mathbb{Z}$ by

$$\mathrm{lev}_i(c \cdot x^{\Lambda}\partial_k) := h_i\, \mathrm{wd}(x^{\Lambda}\partial_k) + \deg(x^{\Lambda}) - 1.$$

We now show that the weight-degree function is bounded.

Lemma 2.2 *Let $x^{\Lambda}\partial_k \in \mathcal{B}$. Then $\mathrm{wd}(x^{\Lambda}\partial_k) \geq n-k$ and $\mathrm{wd}(x^{\Lambda}\partial_k) = n-k$ if and only if $x^{\Lambda}\partial_k = x_1^{\lambda_1}\partial_k$. Moreover, assume that $\mathrm{lev}_i(x^{\Lambda}\partial_k) \leq i$ for some i. Then $\mathrm{wd}(x^{\Lambda}\partial_k) \leq n-1$ and $\mathrm{wd}(x^{\Lambda}\partial_k) = n-1$ if and only if $x^{\Lambda}\partial_k = \partial_1$.*

Proof Note that $\mathrm{wd}(x^{\Lambda}\partial_k) = \big(\mathrm{wt}(\Lambda) - \deg(x^{\Lambda})\big) + n - k > n - k$ unless $\mathrm{wt}(\Lambda) = \deg(x^{\Lambda})$, which is equivalent to $x^{\Lambda}\partial_k = x_1^{\lambda_1}\partial_k$.

Let now i be such that $\mathrm{lev}_i(x^{\Lambda}\partial_k) \leq i$ and assume $\mathrm{wd}(x^{\Lambda}\partial_k) \geq n-1$. Then

$$\begin{aligned} i &\geq \mathrm{lev}_i(x^{\Lambda}\partial_k) \\ &= h_i\, \mathrm{wd}(x^{\Lambda}\partial_k) + \deg(x^{\Lambda}) - 1 \\ &\geq h_i(n-1) + \deg(x^{\Lambda}) - 1 \\ &= (n-1) + (h_i - 1)(n-1) + r_i + \deg(x^{\Lambda}) - r_i - 1 \\ &= i + (n-1) - r_i + \deg(x^{\Lambda}) - 1. \end{aligned}$$

This implies that $(n-1) - r_i + \deg(x^\Lambda) - 1 \le 0$. Since $r_i \le n-1$, this is possible only if either $r_i = n-1$ and $\deg(x^\Lambda) \le 1$ or $r_i = n-2$ and $\deg(x^\Lambda) = 0$.

In the first case, we have either $k = 1$ and hence $x^\Lambda \partial_k = \partial_1$, as required, or $x^\Lambda = x_j^{\lambda_j}$, where $1 \le j < k$ and $\lambda_j \le 1$. This implies $n - 1 \le \mathrm{wd}(x^\Lambda \partial_k) = (j-1)\lambda_j + n - k \le j + n - k - 1$ and consequently $k \le j$, which contradicts $j < k$.

In the second case, we have $x^\Lambda \partial_k = \partial_k$ and $n - 1 \le \mathrm{wd}(x^\Lambda \partial_k) = n - k$. This leads again to $k = 1$ and $x^\Lambda \partial_k = \partial_1$. □

Proposition 2.3 *If $x^\Lambda \partial_k, x^\Theta \partial_u \in \mathcal{B}$ do not commute, then*

$$\mathrm{lev}_i([x^\Lambda \partial_k, x^\Theta \partial_u]) = \mathrm{lev}_i(x^\Lambda \partial_k) + \mathrm{lev}_i(x^\Theta \partial_u) - h_i(n-1) \tag{2}$$

and

$$\mathrm{wd}([x^\Lambda \partial_k, x^\Theta \partial_u]) = \mathrm{wd}(x^\Lambda \partial_k) + \mathrm{wd}(x^\Theta \partial_u) - (n-1). \tag{3}$$

Moreover, if $\mathrm{lev}_i(x^\Lambda \partial_k) \le i$ *and* $\mathrm{lev}_j(x^\Theta \partial_u) \le j$ *for some* $i, j \ge -1$, *then*

$$\mathrm{wd}([x^\Lambda \partial_k, x^\Theta \partial_u]) \le \min(\mathrm{wd}(x^\Lambda \partial_k), \mathrm{wd}(x^\Theta \partial_u)), \tag{4}$$

with equality if and only if one of $x^\Lambda \partial_k$ or $x^\Theta \partial_u$ is ∂_1.

Proof Let $x^\Gamma \partial_u = [x^\Lambda \partial_k, x^\Theta \partial_u]$. We have

$$\begin{aligned} \mathrm{wd}(x^\Gamma \partial_u) &= \mathrm{wt}(\Lambda) + \mathrm{wt}(\Theta) - k - \deg(x^\Lambda) - \deg(x^\Theta) + 1 + n - u \\ &= \mathrm{wd}(x^\Lambda \partial_k) + \mathrm{wd}(x^\Theta \partial_u) - (n-1). \end{aligned}$$

As a consequence,

$$\begin{aligned} \mathrm{lev}_i(x^\Gamma \partial_u) =& h_i \,\mathrm{wd}(x^\Gamma \partial_u) + \deg(x^\Gamma) - 1 \\ =& h_i\big(\mathrm{wd}(x^\Lambda \partial_k) + \mathrm{wd}(x^\Theta \partial_u) - (n-1)\big) + \deg(x^\Lambda) + \deg(x^\Theta) - 2 \\ =& \,\mathrm{lev}_i(x^\Lambda \partial_u) + \mathrm{lev}_i(x^\Theta \partial_k) - (n-1)h_i. \end{aligned}$$

The last part of the claim is a straightforward consequence of Lemma 2.2. □

Remark 1 A direct check shows that

$$\mathrm{lev}_j(x^\Lambda \partial_k) = \mathrm{lev}_i(x^\Lambda \partial_k) + (h_j - h_i)\,\mathrm{wd}(x^\Lambda \partial_k).$$

Lemma 2.4 *If* $\mathrm{lev}_i(x^\Lambda \partial_k) \le i$, *then there exists* $j \le i$ *such that* $\mathrm{lev}_j(x^\Lambda \partial_k) = j$.

Proof We argue by induction on i. If $i = -1$, then $h_{-1} = 0$ and hence

$$\mathrm{lev}_{-1}(x^\Lambda \partial_k) = \deg(x^\Lambda) - 1 \le -1$$

so $\deg(x^\Lambda) = 0$ and $\mathrm{lev}_{-1}(x^\Lambda \partial_k) = -1$.

Let now $i \geq 0$. We can assume that $\ell := \operatorname{lev}_i(x^\Lambda \partial_k) < i$. Note that

$$\operatorname{lev}_\ell(x^\Lambda \partial_k) = \operatorname{lev}_i(x^\Lambda \partial_k) + (h_\ell - h_i)\operatorname{wd}(x^\Lambda \partial_k) \leq \operatorname{lev}_i(x^\Lambda \partial_k) = \ell.$$

If $\operatorname{lev}_\ell(x^\Lambda \partial_k) = \ell$, then we have proved the claim, otherwise $\operatorname{lev}_\ell(x^\Lambda \partial_k) < \operatorname{lev}_i(x^\Lambda \partial_k) = \ell < i$ and the induction hypothesis is satisfied. Therefore, $\operatorname{lev}_j(x^\Lambda \partial_k) = j$ for some $j \leq \ell < i$. □

Lemma 2.5 *If* $\operatorname{lev}_i(x^\Lambda \partial_k) \leq i$, *then* $\operatorname{lev}_{i+(n-1)}(x^\Lambda \partial_k) \leq i + (n-1)$. *In particular,* $\operatorname{lev}_i(x^\Lambda \partial_k) > i$ *for all* $i \leq h(n-1)$ *if and only if* $\operatorname{lev}_i(x^\Lambda \partial_k) > i$ *for all* i *such that* $(h-1)(n-1)+1 \leq i \leq h(n-1)$.

Proof By Lemma 2.2, we have that $\operatorname{wd}(x^\Lambda \partial_k) \leq n-1$. Thus, $h_{i+(n-1)} = h_i + 1$ and

$$\operatorname{lev}_{i+(n-1)}(x^\Lambda \partial_k) = \operatorname{wd}(x^\Lambda \partial_k) + \operatorname{lev}_i(x^\Lambda \partial_k) \leq (n-1) + i.$$

□

2.2 Introducing the chain

We are now ready to introduce a chain of subsets of $\mathcal{B}$. We will prove later that they span, as free modules, the idealizers of Eq. (1).

Definition 2.6 For $i \geq -1$ let

$$\mathcal{N}_i := \left\{x^\Lambda \partial_k \mid \operatorname{lev}_j(x^\Lambda \partial_k) \leq j \text{ for some } j \leq i\right\}.$$

Moreover, for $i \geq 0$ let

$$\mathcal{L}_i := \mathcal{N}_i \setminus \mathcal{N}_{i-1}.$$

Remark 2 Note that $\mathcal{N}_{-1} = \{\partial_1, \dots, \partial_n\}$, that $\mathfrak{T}$ is the free $\mathbb{Z}$-module spanned by $\mathcal{N}_{-1}$, and that the subsets $\{\mathcal{N}_i\}_{i \geq -1}$ constitute an ascending chain of $\mathcal{B}$. Note also that an easy application of Lemma 2.4 gives

$$\mathcal{N}_i = \left\{x^\Lambda \partial_k \mid \operatorname{lev}_j(x^\Lambda \partial_k) = j \text{ for some } j \leq i\right\}.$$

Furthermore, $x^\Lambda \partial_k \in \mathcal{L}_i$ if and only if i is minimum such that $\operatorname{lev}_i(x^\Lambda \partial_k) = i$, i.e.,

$$\mathcal{L}_i = \left\{x^\Lambda \partial_k \in \mathcal{B} \,\middle|\, i = \min_j \left\{j = \operatorname{lev}_j(x^\Lambda \partial_k)\right\}\right\}.$$

Proposition 2.7 *Let* $i \geq 0$. *If* $x^\Lambda \partial_k \in \mathcal{L}_i$, *then* $r_i > \operatorname{wd}(x^\Lambda \partial_k)$. *In particular if* $k < n - r_i + 1$, *then* $\mathcal{L}_i \cap \mathcal{B}_k = \emptyset$.

Proof Assume by contradiction that $x^{\Lambda}\partial_k \in \mathcal{L}_i$ and $r_i \le \mathrm{wd}(x^{\Lambda}\partial_k)$. We have that i is minimum such that $i = \mathrm{lev}_i(x^{\Lambda}\partial_k)$. Thus, $j := i - \mathrm{wd}(x^{\Lambda}\partial_k) > i - (n-1)$ is such that

$$(h_i - 2)(n-1) < j = (h_i - 1)(n-1) + r_i - \mathrm{wd}(x^{\Lambda}\partial_k) \le (h_i - 1)(n-1).$$

Hence $h_j = (h_i - 1)$ and, by Remark 1,

$$j = i - \mathrm{wd}(x^{\Lambda}\partial_k) = \mathrm{lev}_i(x^{\Lambda}\partial_k) - \mathrm{wd}(x^{\Lambda}\partial_k) = \mathrm{lev}_j(x^{\Lambda}\partial_k),$$

where $j < i$, a contradiction.

If $k < n - r_i + 1$, then $r_i \le n - k \le \mathrm{wd}(x^{\Lambda}\partial_k)$. From above, we have $\mathcal{L}_i \cap \mathcal{B}_k = \emptyset$. □

The following corollary is straightforward.

Corollary 2.8 *If* $r_t = 1$ *then* $\mathcal{L}_t = \left\{x_1^{t+1}\partial_n\right\}$.

2.3 Periodicity

We can now define a concept of *periodicity* for the sequence $\{\mathcal{L}_i\}_{i\ge 0}$. The following definition is crucial.

Definition 2.9 The *period function* $\nu\colon \mathcal{B} \to \mathcal{B}$ is defined by

$$\nu(x^{\Lambda}\partial_k) := x_1^{n-1-\mathrm{wd}(x^{\Lambda}\partial_k)} x^{\Lambda}\partial_k.$$

Remark 3 Note that if $x^{\Lambda}\partial_k \in \mathcal{L}_i$, then

$$\begin{aligned}
\mathrm{lev}_{i+n-1}(\nu(x^{\Lambda}\partial_k)) &= \mathrm{lev}_i(\nu(x^{\Lambda}\partial_k)) + \mathrm{wd}(\nu(x^{\Lambda}\partial_k)) \\
&= \mathrm{lev}_i(\nu(x^{\Lambda}\partial_k)) + \mathrm{wd}(x^{\Lambda}\partial_k) \\
&= \mathrm{lev}_i(x^{\Lambda}\partial_k) + \big(n - 1 - \mathrm{wd}(x^{\Lambda}\partial_k)\big) + \mathrm{wd}(x^{\Lambda}\partial_k) \\
&= \mathrm{lev}_i(x^{\Lambda}\partial_k) + n - 1 = i + n - 1,
\end{aligned}$$

so $\nu(x^{\Lambda}\partial_k) \in \mathcal{L}_{i+n-1}$, as $t = i+n-1$ is trivially minimum such that $\mathrm{lev}_t(\nu(x^{\Lambda}\partial_k)) = t$. Moreover, since the period map is injective, we have that $|\mathcal{L}_i| \le |\mathcal{L}_{i+n-1}|$.

Remark 4 Let $i \ge 1$ and $k > u \ge 1$. Suppose that $x^{\Lambda}\partial_k \in \mathcal{L}_i$ and $x^{\Theta}\partial_u \in \mathcal{L}_{h_i(n-1)}$. Equation (2) shows that $\lambda_u x^{\Gamma}\partial_k := [x^{\Lambda}\partial_k, x^{\Theta}\partial_u] \in \mathbb{Z}\mathcal{L}_i$, as $\mathrm{lev}_i(x^{\Gamma}\partial_k) = \mathrm{lev}_i(x^{\Lambda}\partial_k)$, whereas Eq. (3) shows that $\mathrm{wd}(x^{\Gamma}\partial_k) < \mathrm{wd}(x^{\Lambda}\partial_k)$. Note also that the element $b_u = x_1^{h_i} x_{u-1}\partial_u \in \mathcal{L}_{h_i(n-1)}$. Hence, given $x^{\Lambda}\partial_k \in \mathcal{L}_i$, there exists a sequence $u_1, u_2, \ldots, u_s$ such that

$$[x^{\Lambda}\partial_k, b_{u_1}, b_{u_2}, \ldots, b_{u_s}] = \lambda x_1^{i-h_i(n-k)+1}\partial_k \in \mathbb{Z}\mathcal{L}_i$$

for a suitable non-zero $\lambda \in \mathbb{Z}$. Equation (3) shows that $\mathrm{wd}([x^\Lambda \partial_k, b_u]) = \mathrm{wd}(x^\Lambda \partial_k) - 1$. Hence, from Lemma 2.2, we have $s \le k-2$. Moreover, if $\lambda_u \neq 0$, then

$$[x^\Lambda \partial_k, \underbrace{b_u, \dots, b_u}_{\lambda_u \text{ times}}, \underbrace{b_{u-1}, \dots, b_{u-1}}_{\lambda_u \text{ times}}, \dots, \underbrace{b_2, \dots, b_2}_{\lambda_u \text{ times}}] = \lambda x_u^{-\lambda_u} x_1^{\lambda_u(u+(h_i-1)(u-1))} x^\Lambda \partial_k$$

for some nonzero $\lambda \in \mathbb{Z}$, so that $\sum \lambda_u(u-1) = s \le k-2$, a condition which is trivially equivalent to $\mathrm{wd}(x^\Lambda \partial_k) \le n-2$, as already seen in Lemma 2.2.

The previous remark together with Eq. (3) and Lemma 2.2 yields the following result.

Proposition 2.10 *If $\mathcal{L}_i \cap \mathcal{B}_k \neq \emptyset$, then there exists at most one element in $\mathcal{L}_i \cap \mathcal{B}_k$ of the form $x_1^t \partial_k$, in which case it is the unique element having minimum weight degree in $\mathcal{L}_i \cap \mathcal{B}_k$. The exponent $t = i - h_i(n-k)+1$ is determined by the condition $x_1^t \partial_k \in \mathcal{L}_i$.*

Now we can prove one of the main contributions of this work where we give a precise characterization of the elements of $\mathcal{L}_i$.

Theorem 2.11 *Let $i \ge 0$. A basis element $x^\Lambda \partial_k$ belongs to $\mathcal{L}_i \cap \mathcal{B}_k$ if and only if the following conditions are satisfied:*

(a) $n-k \le \mathrm{wd}(x^\Lambda \partial_k) < r_i$,
(b) $i = \mathrm{lev}_i(x^\Lambda \partial_k)$.

Proof Suppose first that (a) and (b) are satisfied. We have $\mathrm{lev}_i(x^\Lambda \partial_k) = i$, and so $x^\Lambda \partial_k \in \mathcal{L}_j \cap \mathcal{B}_k$, for some $j \le i$. By contradiction, assume that $j = \mathrm{lev}_j(x^\Lambda \partial_k)$ and that $i \neq j$. We have $i > j = (h_j-1)(n-1) + r_j = \mathrm{lev}_j(x^\Lambda \partial_k) = h_j \,\mathrm{wd}(x^\Lambda \partial_k) + \deg(x^\Lambda) - 1$. Note that at least one of the two conditions $h_i > h_j$ or $r_i > r_j$ must be satisfied, so we have

$$\begin{aligned} r_i &= i - (h_i-1)(n-1) \\ &= \mathrm{lev}_i(x^\Lambda \partial_k) - (h_i-1)(n-1) \\ &= h_i \,\mathrm{wd}(x^\Lambda \partial_k) + \deg(x^\Lambda) - 1 - (h_i-1)(n-1) \\ &= (h_i - h_j)(\mathrm{wd}(x^\Lambda \partial_k) - n + 1) + \mathrm{lev}_j(x^\Lambda \partial_k) - (h_j-1)(n-1) \\ &= (h_i - h_j)(\mathrm{wd}(x^\Lambda \partial_k) - n + 1) + j - (h_j-1)(n-1) \\ &= (h_i - h_j)(\mathrm{wd}(x^\Lambda \partial_k) - n + 1) + r_j \\ &\le \begin{cases} \mathrm{wd}(x^\Lambda \partial_k) + (r_j - n + 1) \le \mathrm{wd}(x^\Lambda \partial_k) < r_i & \text{if } h_i > h_j \\ r_j < r_i & \text{if } h_i = h_j, \end{cases} \end{aligned}$$

which is, in both the cases, a contradiction. Hence $j = i$.

Conversely, suppose that $x^\Lambda \partial_k \in \mathcal{L}_i \cap \mathcal{B}_k$, so $i = \mathrm{lev}_i(x^\Lambda \partial_k)$. By Proposition 2.7 and Lemma 2.2, we obtain that $n-k \le \mathrm{wd}(x^\Lambda \partial_k) < r_i$.

∎

2.4 Connection to the sequence of partitions

Let us give a description of the behavior of the chain $\{\mathcal{N}_i\}_i$ in terms of integer partitions. Let $\{a_n\}_{n=0}^{\infty}$ be the sequence whose term a_n is equal to the number of partitions of n. Let also $b_n = \sum_{i=0}^{n} a_i$ be the partial sum of $\{a_i\}$ and $c_n = \sum_{i=0}^{n} b_n$ be the partial sum of $\{b_i\}$, or the *second partial sum* of $\{a_i\}$. The first values of the sequences are displayed in Table 1, which also contains the corresponding OEIS references [5].

We are now ready to show that, after a threshold value with depends quadratically on n, the sequence $\{|\mathcal{L}_i|\}$ is periodic and how it depends on $\{c_i\}$. Here the value r_i is as in Definition 2.1.

Corollary 2.12 *If* $i > (n-4)(n-1)$ *and* $1 \le k \le n$*, then*

$$|\mathcal{L}_i \cap \mathcal{B}_k| = b_{r_i+k-n-1}$$
$$\textit{and}\ |\mathcal{L}_i| = c_{r_i-1}.$$

Proof Let $x^{\Lambda}\partial_k \in \mathcal{L}_i \cap \mathcal{B}_k$. By Theorem 2.11, $x^{\Lambda}\partial_k$ satisfies

(a) $n-k \le \mathrm{wd}(x^{\Lambda}\partial_k) < r_i$,
(b) $i = \mathrm{lev}_i(x^{\Lambda}\partial_k)$.

Letting $\theta_u = \lambda_{u+1}$, the condition (a) can be rewritten as $0 \le \sum_{u\ge 1} u\theta_u < r_i + k - n$, where θ_u is a non-negative integer. There are exactly $b_{r_i+k-n-1}$ sequences $\Theta = \{\theta_u\}_{u=1}^{\infty}$, including the trivial one, satisfying this condition and determining the values of λ_u for $u \ge 2$. The value of λ_1 is uniquely determined by the condition (b). Indeed, $\mathrm{wd}(x^{\Lambda}\partial_k)$ does not depend on λ_1 and by (b) we have $i = h_i\,\mathrm{wd}(x^{\Lambda}\partial_k) + \lambda_1 + \sum_{u\ge 2}\lambda_u - 1$. Moreover, by the hypotheses we have $h_i \ge n-3$. Thus,

$$\begin{aligned}
\lambda_1 &= i - h_i\,\mathrm{wd}(x^{\Lambda}\partial_k) - \sum_{u\ge 2}\lambda_u + 1 \\
&= (h_i-1)(n-1) + r_i - h_i\,\mathrm{wd}(x^{\Lambda}\partial_k) - \sum_{u\ge 2}\lambda_u + 1 \\
&= h_i\left(n-1-\mathrm{wd}(x^{\Lambda}\partial_k)\right) - n + 1 + r_i - \sum_{u\ge 2}\lambda_u + 1 \\
&\ge h_i\,(n-1-r_i+1) - n + 1 + r_i - r_i - k + n + 2 \\
&= h_i\,(n-r_i) + 3 - k \ge h_i + 3 - k \ge n - k \ge 0
\end{aligned}$$

is uniquely determined and non-negative.

Finally, the equality $|\mathcal{L}_i| = c_{r_i-1}$ is obtained computing $\sum_{k=2}^{n} |\mathcal{L}_i \cap \mathcal{B}_k|$. □

A straightforward consequence is the following result, which proves that the sequence $\{|\mathcal{L}_i|\}_{i\ge 0}$ is ultimately periodic, i.e., there exist integers k and j such that $|\mathcal{L}_i| = |\mathcal{L}_{i+k}|$ for all $i \ge j$.

Corollary 2.13 *If* $i > (n-4)(n-1)$*, then the period function* ν *is a bijection from* $\mathcal{L}_i$ *to* $\mathcal{L}_{i+n-1}$.

Table 1 First values of the sequences $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	OEIS
a_i	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	A000041
$b_i - 1$	0	1	3	6	11	18	29	44	66	96	138	194	271	372	507	A026905
$c_i - i - 1$	0	1	4	10	21	39	68	112	178	274	412	606	877	1249	1756	A085360

We conclude this section with an explicit example where we highlight the periodic structure of the sequence $\{\mathcal{L}_i\}$.

Example 2.14 Let $n = 5$ and $i \geq 5$. We list the non-empty sets $\mathcal{L}_i \cap \mathcal{B}_k$.

$r_i = 1$

$$\mathcal{L}_i \cap \mathcal{B}_5 = \left\{ x_1^{i+1} \partial_5 \right\};$$

$r_i = 2$

$$\begin{aligned} \mathcal{L}_i \cap \mathcal{B}_5 &= \left\{ x_1^{i+1} \partial_5,\ x_1^{i-h_i} x_2 \partial_5 \right\} \\ \mathcal{L}_i \cap \mathcal{B}_4 &= \left\{ x_1^{i+1-h_i} \partial_4 \right\}; \end{aligned}$$

$r_i = 3$

$$\begin{aligned} \mathcal{L}_i \cap \mathcal{B}_5 &= \left\{ x_1^{i+1} \partial_5,\ x_1^{i-h_i} x_2 \partial_5,\ x_1^{i+1-2h_i} x_2^2 \partial_5,\ x_1^{i-2h_i} x_3 \partial_5 \right\} \\ \mathcal{L}_i \cap \mathcal{B}_4 &= \left\{ x_1^{i+1-h_i} \partial_4,\ x_1^{i-2h_i} x_2 \partial_4 \right\} \\ \mathcal{L}_i \cap \mathcal{B}_3 &= \left\{ x_1^{i+1-2h_i} \partial_3 \right\}; \end{aligned}$$

$r_i = 4$

$$\begin{aligned} \mathcal{L}_i \cap \mathcal{B}_5 &= \left\{ x_1^{i+1} \partial_5,\ x_1^{i-h_i} x_2 \partial_5,\ x_1^{i+1-2h_i} x_2^2 \partial_5,\ x_1^{i-2h_i} x_3 \partial_5, \right. \\ &\qquad \left. x_1^{i-2-3h_i} x_2^3 \partial_5,\ x_1^{i-1-3h_i} x_2 x_3 \partial_5,\ x_1^{i-3h_i} x_4 \partial_5 \right\} \\ \mathcal{L}_i \cap \mathcal{B}_4 &= \left\{ x_1^{i+1-h_1} \partial_4,\ x_1^{i-2h_i} x_2 \partial_4,\ x_1^{i+1-3h_i} x_2^2 \partial_4,\ x_1^{i-3h_i} x_3 \partial_4 \right\} \\ \mathcal{L}_i \cap \mathcal{B}_3 &= \left\{ x_1^{i+1-2h_i} \partial_3,\ x_1^{i+1-3h_i} x_2 \partial_3 \right\} \\ \mathcal{L}_i \cap \mathcal{B}_2 &= \left\{ x_1^{i+1-3h_i} \partial_2 \right\}. \end{aligned}$$

3 The idealizer chain over the ring of integers

We conclude the paper by proving that the idealizer chain is generated by the subsets $\mathcal{N}_i$ of Definition 2.6. We start by describing the commutator structure of the chain $\{\mathcal{N}_i\}_{i \geq -1}$.

Lemma 3.1 *If $i < j$, then $[\mathcal{N}_i, \mathcal{N}_j] \subseteq \mathbb{Z}\mathcal{N}_{j-1}$.*

Proof Let $x^\Lambda \partial_k \in \mathcal{N}_i$ and $x^\Theta \partial_u \in \mathcal{N}_j$ be such that $[x^\Lambda \partial_k, x^\Theta \partial_u] \neq 0$. By definition of $\mathcal{N}_j$ there exist $\ell \leq j$ and $m \leq i$ such that $\mathrm{lev}_\ell(x^\Theta \partial_u) \leq \ell$ and $\mathrm{lev}_m(x^\Lambda \partial_k) \leq m$.

We may assume $m \leq \ell$, otherwise we interchange the roles of i and j in the statement and we argue by induction.

If $x^\Lambda \partial_k = \partial_1$, then

$$\mathrm{lev}_{\ell-1}([x^\Lambda \partial_k, x^\Theta \partial_u]) \leq \mathrm{lev}_\ell([x^\Lambda \partial_k, x^\Theta \partial_u]) = \mathrm{lev}_\ell(x^\Theta \partial_u) - 1 \leq \ell - 1.$$

Thus, $[x^\Lambda \partial_k, x^\Theta \partial_u] \in \mathcal{N}_{j-1}$.

Assume now that $x^\Lambda \partial_k \neq \partial_1$.

By Proposition 2.3, $\mathrm{wd}(x^\Lambda \partial_k) < n-1$. If $h_m = h_\ell$, then

$$\begin{aligned}\mathrm{lev}_\ell([x^\Lambda \partial_k, x^\Theta \partial_u]) &= \mathrm{lev}_\ell(x^\Lambda \partial_k) + \mathrm{lev}_\ell(x^\Theta \partial_u) - h_\ell(n-1)\\ &= \mathrm{lev}_m(x^\Lambda \partial_k) + \mathrm{lev}_\ell(x^\Theta \partial_u) - h_\ell(n-1)\\ &\leq \mathrm{lev}_m(x^\Lambda \partial_k) + \ell - h_\ell(n-1) = \mathrm{lev}_m(x^\Lambda \partial_k) + r_\ell - (n-1) \leq m.\end{aligned}$$

If $m = \ell$, then $\ell \leq j-1$ and so $[x^\Lambda \partial_k, x^\Theta \partial_u] \in \mathcal{N}_{j-1}$. Otherwise $m < \ell \leq j$ and

$$\mathrm{lev}_m([x^\Lambda \partial_k, x^\Theta \partial_u]) \leq \mathrm{lev}_\ell([x^\Lambda \partial_k, x^\Theta \partial_u]) \;\leq m < j,$$

which implies $[x^\Lambda \partial_k, x^\Theta \partial_u] \in \mathcal{N}_{j-1}$.

The remaining possibility is that $h_m < h_\ell$. In this case $m < \ell$ and $\mathrm{wd}(x^\Lambda \partial_k)(h_\ell - h_m) \leq (n-1)(h_\ell - h_m) - 1$, thus

$$\begin{aligned}\mathrm{lev}_\ell([x^\Lambda \partial_k, x^\Theta \partial_u]) &= \mathrm{lev}_\ell(x^\Lambda \partial_k) + \mathrm{lev}_\ell(x^\Theta \partial_u) - h_\ell(n-1)\\ &= \mathrm{lev}_m(x^\Lambda \partial_k) + \mathrm{lev}_\ell(x^\Theta \partial_u) - h_\ell(n-1) + \mathrm{wd}(x^\Lambda \partial_k)(h_\ell - h_m)\\ &\leq \mathrm{lev}_m(x^\Lambda \partial_k) + \mathrm{lev}_\ell(x^\Theta \partial_u) - h_\ell(n-1) + (n-1)(h_\ell - h_m) - 1\\ &\leq m - h_m(n-1) + \ell - 1 = r_m - (n-1) + \ell - 1 \leq \ell - 1.\end{aligned}$$

Hence, $\mathrm{lev}_{\ell-1}([x^\Lambda \partial_k, x^\Theta \partial_u]) \leq \mathrm{lev}_\ell([x^\Lambda \partial_k, x^\Theta \partial_u]) \leq \ell - 1$, therefore $[x^\Lambda \partial_k, x^\Theta \partial_u] \in \mathcal{N}_{j-1}$.

□

As straightforward consequence is the following corollary.

Corollary 3.2 $\mathcal{N}_j \subseteq N_{\mathcal{B}}(\mathcal{N}_{j-1})$.

We prove now the opposite inclusion.

Proposition 3.3 $\mathcal{N}_j = N_{\mathcal{B}}(\mathcal{N}_{j-1})$.

Proof By the previous corollary it suffices to show that $\mathcal{N}_j \supseteq N_{\mathcal{B}}(\mathcal{N}_{j-1})$. Looking for a contradiction, we assume that $x^\Lambda \partial_k \in N_{\mathcal{B}}(\mathcal{N}_{j-1})$ is such that $\mathrm{lev}_i(x^\Lambda \partial_k) > i$ for all $i \leq j$. Set

$$x^\Theta \partial_u =: x_1^{(h_j-1)(u-\ell)} x_\ell \partial_u,$$

where $1 \leq \ell < u$, and $s := (h_j - 1)(n - 1) \leq j - 1$. We have $h_s = h_j - 1$ and

$$\operatorname{lev}_s(x^{\Theta}\partial_u) = (h_j - 1)(\ell - 1 + n - u) + (h_j - 1)(u - \ell) = (h_j - 1)(n - 1) = s.$$

Hence, both $x^{\Theta}\partial_u$ and $[x^{\Lambda}\partial_k, x^{\Theta}\partial_u]$ belong to $\mathcal{N}_{j-1}$. Assume that either $\lambda_u \neq 0$ for some $1 < u < n$ or $\ell = k$, so $[x^{\Lambda}\partial_k, x^{\Theta}\partial_u] \neq 0$. For some $i \leq j - 1$, we have $\operatorname{lev}_i([x^{\Lambda}\partial_k, x^{\Theta}\partial_u]) = i$. We start with assuming that $i > s$ and that $h_j = h_{j-1}$, which in turn implies $h_i = h_{j-1} = h_j$. We have

$$\begin{aligned}
i &= \operatorname{lev}_i([x^{\Lambda}\partial_k, x^{\Theta}\partial_u])\\
&= \operatorname{lev}_i(x^{\Lambda}\partial_k) + \operatorname{lev}_i(x^{\Theta}\partial_u) - h_i(n-1)\\
&= \operatorname{lev}_j(x^{\Lambda}\partial_k) + h_i(\ell - 1 + n - u) + (h_j - 1)(u - \ell) - h_i(n-1)\\
&= \operatorname{lev}_j(x^{\Lambda}\partial_k) + h_j(\ell - 1 + n - u) + (h_j - 1)(u - \ell) - h_j(n-1)\\
&= \operatorname{lev}_j(x^{\Lambda}\partial_k) + (\ell - u) > j + (\ell - u).
\end{aligned}$$

As a consequence, we have $j - 1 \geq i > j + (\ell - u)$. Since we may alternatively choose ℓ or u, with respect to our assumption on $x^{\Theta}\partial_u$ such that $u - \ell = 1$, we have a contradiction.

Suppose now that $i \leq s$ and that $h_j = h_{j-1}$, and so $h_i + 1 \leq h_{j-1} = h_j$. We have

$$\begin{aligned}
i &= \operatorname{lev}_i([x^{\Lambda}\partial_k, x^{\Theta}\partial_u])\\
&= \operatorname{lev}_i(x^{\Lambda}\partial_k) + \operatorname{lev}_i(x^{\Theta}\partial_u) - h_i(n-1)\\
&= \operatorname{lev}_i(x^{\Lambda}\partial_k) + \operatorname{lev}_s(x^{\Theta}\partial_u) + (h_i - h_s)\operatorname{wd}(x^{\Theta}\partial_u) - h_i(n-1)\\
&= \operatorname{lev}_i(x^{\Lambda}\partial_k) + \operatorname{lev}_s(x^{\Theta}\partial_u) + (h_i - h_s)(\ell - 1 + n - u) - h_i(n-1)\\
&= \operatorname{lev}_i(x^{\Lambda}\partial_k) + h_s(n-1) + (h_i - h_s)(\ell - 1 + n - u) - h_i(n-1)\\
&= \operatorname{lev}_i(x^{\Lambda}\partial_k) + (h_i - h_s)(\ell - u) \geq \operatorname{lev}_i(x^{\Lambda}\partial_k),
\end{aligned}$$

a contradiction.

Suppose now that $j - 1 = s$ so that $h_{j-1} = h_s = h_j - 1$. By a repeated use of Lemma 2.5, we may assume $\operatorname{lev}_t([x^{\Lambda}\partial_k, x^{\Theta}\partial_u]) \leq t$ for some t such that $s - (n-1) < t \leq s = j - 1$. Hence,

$$\begin{aligned}
t &\geq \operatorname{lev}_t([x^{\Lambda}\partial_k, x^{\Theta}\partial_u])\\
&= \operatorname{lev}_s([x^{\Lambda}\partial_k, x^{\Theta}\partial_u])\\
&= \operatorname{lev}_s(x^{\Lambda}\partial_k) + \operatorname{lev}_s(x^{\Theta}\partial_u) - h_s(n-1)\\
&= \operatorname{lev}_{j-1}(x^{\Lambda}\partial_k) + s - s = \operatorname{lev}_{j-1}(x^{\Lambda}\partial_k)
\end{aligned}$$

giving the contradiction $\operatorname{lev}_{j-1}(x^{\Lambda}\partial_k) \leq t \leq s = j - 1$.

We are now left with the case $x^{\Lambda}\partial_k = x_1^{\lambda_1}\partial_n$. In order to have $[x^{\Lambda}\partial_k, x^{\Theta}\partial_u] \neq 0$, $x^{\Theta}\partial_u$ must be set to ∂_1. In particular, $[x^{\Lambda}\partial_k, x^{\Theta}\partial_u] = \lambda_1 x_1^{\lambda_1 - 1}\partial_n \in \mathbb{Z}\mathcal{N}_{j-1}$, and also

$\mathrm{lev}_i(x_1^{\lambda_1-1}\partial_n) = \lambda_1 - 2$ is independent of i. Thus, for some $i \le j-1$ we have

$$\mathrm{lev}_{j-1}(x_1^{\lambda_1-1}\partial_n) = \mathrm{lev}_i(x_1^{\lambda_1-1}\partial_n) = \lambda_1 - 2 \le i \le j-1.$$

Hence, $j \ge \lambda_1 - 1 = \mathrm{lev}_j(x^\Lambda\partial_k) > j$, a contradiction. □

We are now able to prove the claimed result on the idealizer chain.

Theorem 3.4 *Let $\mathfrak{M}_i$ be the free $\mathbb{Z}$-module spanned by $\mathcal{N}_i$, for $i \ge -1$. Then $\mathfrak{M}_i$ is a homogeneous subring and*

$$\mathfrak{M}_i = N_{\mathfrak{L}(n)}(\mathfrak{M}_{i-1}) = \mathfrak{N}_i.$$

Proof The statement follows directly from Theorem 1.2 and Proposition 3.3 noting that $\mathfrak{T} = \mathbb{Z}\,\partial_1 + \cdots + \mathbb{Z}\,\partial_n = \mathfrak{N}_{-1}$ is an abelian homogeneous subring. □

Remark 5 A straightforward consequence of Lemma 2.2 is that $\mathfrak{L}(n) \ne \bigcup_{i\ge -1}\mathfrak{N}_i$ for $n \ge 3$. For example, the element $x_2^3\partial_3$, which has weight-degree n, cannot belong to any of the $\mathfrak{N}_i$s. We point out that, unlike the case of the Lie ring $\mathfrak{L}_m(n)$ $(m > 0)$, this shows that $\mathfrak{L}(n)$ is not nilpotent, beside not being finitely generated.

We can now conclude the paper with the characterization of the idealizers of Eq. (1).

Theorem 3.5 *If $i \ge 0$, then $x^\Lambda\partial_k \in \mathfrak{N}_i \setminus \mathfrak{N}_{i-1}$ if and only if i is the least non-negative integer such that $i = \mathrm{lev}_i(x^\Lambda\partial_k)$.*

Proof The proof follows by noticing that, by Remark 2, we have that $x^\Lambda\partial_k \in \mathcal{N}_i \setminus \mathcal{N}_{i-1}$ if and only if i is the least non-negative integer such that $i = \mathrm{lev}_i(x^\Lambda\partial_k)$. □

A trivial consequence of Corollary 2.12 is the following conclusive result.

Corollary 3.6 *For $i > (n-4)(n-1)$, the $\mathbb{Z}$-module $\mathfrak{N}_i/\mathfrak{N}_{i-1}$ is free of rank c_{r_i-1}.*

Acknowledgements We express our sincere gratitude to the referees for their valuable comments. Special thanks to Ruggero Marino for diligently reviewing a preliminary version of this work and identifying certain inaccuracies.

Funding Open access funding provided by Università degli Studi dell'Aquila within the CRUI-CARE Agreement.

Data availability All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The authors declare that no financial interests are directly or indirectly related to the work submitted for publication.

References

1. Aragona, R., Civino, R., Gavioli, N.: A modular idealizer chain and unrefinability of partitions with repeated parts. Israel J. Math. (2023). https://doi.org/10.1007/s11856-023-2559-8
2. Aragona, R., Civino, R., Gavioli, N., Scoppola, C.M.: Regular subgroups with large intersection. Ann. Mat. Pura Appl. (4) **198**(6), 2043–2057 (2019)
3. Aragona, R., Civino, R., Gavioli, N., Scoppola, C.M.: Rigid commutators and a normalizer chain. Monatsh. Math. **196**(3), 431–455 (2021)
4. Kassel, C.: Quantum groups, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York (1995). MR1321145
5. Sloane, N. J. A.: The On-Line Encyclopedia of Integer Sequences, Published electronically at. Accessed 21 Feb 2023. https://oeis.org
6. Petrogradsky, V.M., Razmyslov, Yu.P., Shishkin, E.O.: Wreath products and Kaluzhnin-Krasner embedding for Lie algebras. Proc. Am. Math. Soc. **135**(3), 625–636 (2007)
7. Sushchansky, V. I., Netreba, N. V.: Wreath product of Lie algebras and Lie algebras associated with Sylow p-subgroups of finite symmetric groups. Algebra Discrete Math. (1): 122–132 (2005)
8. Strade, H. : Simple Lie algebras over fields of positive characteristic. Vol. 1. De Gruyter Expositions in Mathematics, vol. 38, De Gruyter, Berlin (2017)