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Classification of cyclic groups underlying only smooth skew morphisms

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Abstract

A skew morphism of a finite group A is a permutation φ of A fixing the identity element and for which there is an integer-valued function π on A such that $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$ for all $a, b \in A$. A skew morphism φ of A is smooth if the associated power function π is constant on the orbits of φ, that is, $\pi(\varphi(a)) \equiv \pi(a) \pmod{|\varphi|}$ for all $a \in A$. In this paper, we show that every skew morphism of a cyclic group of order n is smooth if and only if $n = 2^e n_1$, where $0 \leq e \leq 4$ and n_1 is an odd square-free number. A partial solution to a similar problem on non-cyclic abelian groups is also given.

Keywords Skew morphism · Group factorization · Solvable group

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1 Introduction

A *skew morphism* of a group A is a permutation φ of A fixing the identity element of A and for which there exists a function $\pi : A \to \mathbb{Z}$ such that

$$\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b) \text{ for all } a, b \in A.$$

The function π is referred to as the *power function* associated with φ. If φ is fixed, then the values of π are uniquely determined modulo $|\varphi|$, where $|\varphi|$ denotes the order of φ, and therefore, π may also be defined as a function from A to $\mathbb{Z}_{|\varphi|}$. It is a trivial observation that if $\pi(a) = 1$ for all $a \in A$, then φ is an automorphism of A. Skew morphisms which are not automorphisms are called *proper*.

The concept of a skew morphism was introduced by Jajcay and Širáň [17] to characterize regular Cayley maps. Without going into the details, for a Cayley map $M = \mathrm{CM}(A, X, P)$ of a group A, where X is an inverse-closed generating subset of A and P is a cyclic permutation of X, it was shown that M is regular if and only if P extends to a skew morphism of A [17, Theorem 1].

Skew morphisms are closely related to complementary factorizations of finite groups with a cyclic factor [5]. By a *complementary factorization* of a group G, we mean that G can be written as a product AB, where A and B are subgroups of G and $A \cap B = 1$. Suppose that φ is a skew morphism of A. For $a \in A$, let L_a denote the permutation of A acting as $L_a(x) = ax$ for all $x \in A$, and set $L_A = \{L_a : a \in A\}$. It is not difficult to show that the permutation group $\langle L_A, \varphi\rangle$ of A admits the complementary factorization $L_A\langle\varphi\rangle$. This group is called the *skew product group* induced by A and φ [5].

Conversely, suppose that G is any group admitting a complementary factorization $G = AY$, where Y is a cyclic subgroup and it is core-free in G. Fix a generator y of Y. Then for every $a \in A$, there is a unique element $b \in A$ and a unique number $j \in \mathbb{Z}_{|y|}$ such that $ya = by^j$. Define the mappings $\varphi : A \to A$ and $\pi : A \to \mathbb{Z}_{|y|}$ by

$$\varphi(a) = b \text{ and } \pi(a) = j \overset{\text{def}}{\iff} ya = by^j \text{ for all } a \in A. \tag{1}$$

Then, φ and π are well defined, φ is a skew morphism of A, and π is the power function associated with φ [5, Proposition 3.1(a)].

Regular Cayley maps, skew morphisms and skew product groups for a given infinite family of groups have been intensively investigated. Generally speaking, this seems a challenging problem, because even for the cyclic groups a complete classification of the skew morphisms is not at hand, apart from the celebrated classification of regular Cayley maps [7], and a partial classification of the skew morphisms and the skew product groups [1, 3, 8, 14, 19, 20]. For the elementary abelian p-groups, a characterization of the skew product groups and a complete classification of the regular Cayley maps can be found in [9, 10]. For the dihedral groups, the regular Cayley maps have been classified [18] and a characterization of the skew product groups was given recently by the authors [13]. For the non-abelian simple groups (or more explicitly, the monolithic groups), and non-abelian characteristically simple groups, the skew morphisms and skew product groups have been classified, in contrast to the fact that not much is known about the regular Cayley maps [2, 4].

In this paper, we shall continue the investigation of skew morphisms of cyclic groups. A skew morphism φ of a group A is called *smooth* if the associated power function π satisfies the following condition:

$$\pi\,(\varphi(a)) \equiv \pi(a) \pmod{|\varphi|} \text{ for all } a \in A. \tag{2}$$

It is clear that every automorphism is smooth; however, the converse is not true in general. Smooth skew morphisms were defined by Hu [12] and independently by Bachratý and Jajcay [3] under the name of *coset-preserving* skew morphisms. The smooth skew morphisms of cyclic groups and dihedral groups have been classified by Bachratý and Jajcay [3] (see also [15, Theorem 16] for an alternative proof) and Wang et al. [22], respectively.

To elaborate a subtler relationship between automorphisms and smooth skew morphisms, we need to introduce the concept of a *reciprocal pair* of skew morphisms of cyclic groups. Suppose that $(\varphi, \tilde{\varphi})$ is a pair of skew morphisms φ and $\tilde{\varphi}$ of the cyclic groups $\mathbb{Z}_m$ and $\mathbb{Z}_n$, and π and $\tilde{\pi}$ are the associated power functions of φ and $\tilde{\varphi}$, respectively. Then, $(\varphi, \tilde{\varphi})$ is called *reciprocal* if they satisfy the following conditions:

(a) $|\varphi|$ divides n and $|\tilde{\varphi}|$ divides m;
(b) for all $x \in \mathbb{Z}_m$ and $y \in \mathbb{Z}_n$,

$$\pi(x) \equiv -\tilde{\varphi}^{-x}(-1) \pmod{|\varphi|},$$
$$\tilde{\pi}(y) \equiv -\varphi^{-y}(-1) \pmod{|\tilde{\varphi}|}.$$

It is shown in [11, Theorem 3.5] that, for fixed m and n, the isomorphism classes of regular dessins with complete bipartite underlying graphs $K_{m,n}$ are in one-to-one correspondence with the reciprocal pairs $(\varphi, \tilde{\varphi})$ of skew morphisms of $\mathbb{Z}_m$ and $\mathbb{Z}_n$, respectively; moreover, if one of the skew morphisms is an automorphism, then the other is necessarily smooth [15, Lemma 12].

It was shown by Kovács and Nedela [19, Theorem 6.3] that every skew morphism of the cyclic groups $\mathbb{Z}_n$ is an automorphism if and only if $n = 4$ or $\gcd(n, \phi(n)) = 1$, where ϕ is Euler's totient function. Recently, Bachratý [1] showed that every skew morphism of $\mathbb{Z}_n$ is smooth whenever $n = pq, 4p, 8p, 16p$ or pqr, where p, q, r are distinct primes. Our first theorem is the following generalization.

Theorem 1.1 *Every skew morphism of the cyclic group $\mathbb{Z}_n$ is smooth if and only if $n = 2^e n_1$, where $0 \le e \le 4$ and n_1 is an odd square-free number.*

The proof of Theorem 1.1 relies on some preliminary results from group theory as well as the theory of skew morphisms, which will be collected in next section.

Conder et al. [5] classified the non-cyclic abelian groups with the property that all of their skew morphisms are automorphisms. It was shown that these are precisely the elementary abelian 2-groups [5, Theorem 7.5]. In this paper, we also propose the investigation of non-cyclic abelian groups with the property that all of their skew morphisms are smooth.

Problem 1 Classify the finite non-cyclic abelian groups which underly only smooth skew morphisms.

In Sect. 5, we give the following partial answer.

Theorem 1.2 *Let A be a non-cyclic abelian group of order $n = 2^f n_1$, where $f \geq 0$ and n_1 is odd. If A underlies only smooth skew morphisms, then n_1 is square-free, and the Sylow 2-subgroup of A contains no direct factors isomorphic to $\mathbb{Z}_{2^e}$ $(e \geq 5)$.*

2 Preliminaries

2.1 Group theory

All groups in this paper will be finite. We denote the identity element of a group G by 1_G and its order by $|G|$. The order of an element $g \in G$ is denoted by $|g|$. Let H be a subgroup of a group G. The *core of H in G* is the largest normal subgroup of G contained in H; in the case when this is trivial, H is called *core-free* in G. Moreover, $C_G(H)$ denotes the *centralizer* of H in G. If p is prime, then the largest normal p-subgroup of G is denoted by $O_p(G)$. The largest normal nilpotent subgroup of G is the *Fitting subgroup* of G, denoted by $F(G)$.

Proposition 2.1 ([21, 7.4.3, 7.4.7]) *Let G be a finite group, and $\wp = \{p_1, \ldots, p_k\}$ the set of all prime divisors of $|G|$. Then,*

(a) $F(G) = \prod_{i=1}^{k} O_{p_i}(G)$.
(b) *If G is a solvable group, then $C_G(F(G)) \leq F(G)$.*

A group G is called *supersolvable* if there is a sequence

$$1 = N_0 < N_1 < \cdots < N_k = G$$

of normal subgroups of G such that $|N_i/N_{i-1}|$ is a prime for every i, $1 \leq i \leq k$. The following results are well known.

Proposition 2.2 ([21, 13.2.9, 13.3.1]) *Suppose that a group G has a factorization $G = AB$.*

(a) *If both A and B are nilpotent, then G is solvable.*
(b) *If both A and B are cyclic, then G is supersolvable.*

The following fact is often referred to as the *Sylow tower property* of supersolvable groups.

Proposition 2.3 ([21, 7.2.19]) *Let G be supersolvable group, and let P_i be a Sylow p_i-subgroup of G, where $\wp = \{p_1, p_2, \ldots, p_r\}$ constitutes the set of prime divisors of $|G|$. If the prime divisors are ordered by $p_i > p_{i+1}$ for all i, then for each k, the product $\prod_{i=1}^{k} P_i$ is a normal subgroup of G.*

Let $\wp$ be a set of primes, a positive integer n is a *$\wp$-number* if every prime divisor of n belongs to $\wp$. If no prime divisors of n lie in $\wp$, then n will be called a $\wp'$-number. A positive divisor d of n is a *Hall divisor* if $\gcd(d, n/d) = 1$; that is, there exists some set $\wp$ of primes such that d is a $\wp$-number, while n/d is a $\wp'$-number. A group is a *$\wp$-group* if its order is a $\wp$-number. A subgroup H of a group G is a *Hall subgroup* if $|H|$ is a Hall divisor of $|G|$. Thus, H is a Hall subgroup of G if and only if, for some set $\wp$ of primes, H is a $\wp$-subgroup of G and $|G : H|$ is a $\wp'$-number, in which case H is also called a *Hall $\wp$-subgroup* of G. In particular, if $\wp = \{p\}$ consists of a single prime, then a Hall $\wp$-subgroup is indeed a *Sylow p-subgroup*.

Proposition 2.4 ([16, Kapitel VI, Satz 1.8]) *Let G be a solvable group, and let $\wp$ be a set consisting of some prime divisors of $|G|$. Then, the following hold:*

(a) *G contains a Hall $\wp$-subgroup.*
(b) *All Hall $\wp$-subgroups are conjugate in G.*
(c) *Every $\wp$-subgroup of G is contained in some Hall $\wp$-subgroup of G.*

If $K \leq \mathrm{Aut}(G)$ and $H \leq G$, then H is said to be *K-invariant* if $\sigma(H) = H$ for every $\sigma \in K$. We shall need the following simple lemma.

Lemma 2.5 *Let $N \cong \mathbb{Z}_p^2$ for a prime p and let $K \leq \mathrm{Aut}(N)$ such that $|K|$ is coprime to p and N contains a K-invariant subgroup of order p. Then, K is isomorphic to a subgroup of $\mathbb{Z}_{p-1}^2$.*

Proof Since $|K|$ is coprime to p, by Maschke theorem (see [21, 12.1.2]) we have $N = N_1 \times N_2$, where N_1 and N_2 are K-invariant subgroups of N such that $N_1 \cong N_2 \cong \mathbb{Z}_p$. Thus, $K \leq \mathrm{Aut}(N_1) \times \mathrm{Aut}(N_2) \cong \mathbb{Z}_{p-1}^2$. □

Suppose that G acts on a finite set Ω. For an element $x \in \Omega$, we denote by G_x the *stabilizer* of x in G, by $\mathrm{Orb}_G(x)$ the *G-orbit* containing x, and by $\mathrm{Orb}_G(\Omega)$ the set of all G-orbits. We denote by $\mathrm{Sym}(\Omega)$ the symmetric group consisting of all permutation of Ω, and by id_Ω the identity permutation. The following statement is known. It will be used a couple of times in the next section, and hence, we record it here.

Lemma 2.6 *Let $G \leq \mathrm{Sym}(\Omega)$ be a group containing a regular abelian subgroup A, and suppose that $N \lhd G$. Then, $\mathrm{Orb}_N(\Omega) = \mathrm{Orb}_B(\Omega)$ for some subgroup $B \leq A$.*

Proof It is well known that $\mathrm{Orb}_N(\Omega)$ is a block system for G. Let K be the kernel of the action of G on $\mathrm{Orb}_N(\Omega)$. It is straightforward to check that $\mathrm{Orb}_{K\cap A}(\Omega) = \mathrm{Orb}_N(\Omega)$. □

Let $\alpha_i \in \mathrm{Sym}(\Omega_i)$, $i = 1, 2$. The *direct product* $\alpha_1 \times \alpha_2$ is defined to be the permutation of $\Omega_1 \times \Omega_2$ acting as

$$(\alpha_1 \times \alpha_2)((x_1, x_2)) = (\alpha_1(x_1), \alpha_2(x_2)) \text{ for all } (x_1, x_2) \in \Omega_1 \times \Omega_2.$$

For a prime number p, the *affine group* $\mathrm{AGL}(1, p)$ consists of the permutations of the finite field $\mathbb{F}_p$ of the form $x \mapsto ax + b$, where $a, b \in \mathbb{F}_p$ and $a \neq 0$. The following result was known already by Galois.

Proposition 2.7 ([16, Kapitel II, 3.6 Satz]) *Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive and solvable group, and let $|\Omega| = p$ for a prime p. Then, G is isomorphic to a subgroup of* $\mathrm{AGL}(1, p)$.

2.2 Skew morphisms

For a skew morphism φ of a group A with associated power function π, it is well known that the subset

$$\mathrm{Ker}\,\varphi := \{x \in A : \pi(x) \equiv 1 \pmod{|\varphi|}\}$$

is subgroup of A, called the *kernel* of φ [17]. A skew morphism φ is *kernel-preserving* if $\varphi(\mathrm{Ker}\,\varphi) = \mathrm{Ker}\,\varphi$. Moreover, the subset

$$\mathrm{Core}\,\varphi := \bigcap_{i=1}^{|\varphi|} \varphi^i\,(\mathrm{Ker}\,\varphi)$$

is the core of A in the skew product group $G = A\langle\varphi\rangle$ [15, Proposition 6]. It is well known that φ is kernel-preserving if and only if $\mathrm{Ker}\,\varphi = \mathrm{Core}\,\varphi$. The following properties of $\mathrm{Ker}\,\varphi$ are well known.

Proposition 2.8 ([5, 6, 17]) *Let φ be a skew morphism of a finite group A with associated power function π. Then, we have the following:*

(a) *For all $a, b \in A$, $\pi(a) = \pi(b)$ if and only if $ab^{-1} \in \mathrm{Ker}\,\varphi$.*
(b) *If A is an abelian group, then φ is kernel-preserving.*
(c) *If A is non-trivial, then the kernel* $\mathrm{Ker}\,\varphi$ *is also non-trivial.*

The index $|A : \mathrm{Ker}\,\varphi|$ is called the *skew type* of φ. Note that $\mathrm{Ker}\,\varphi = A$ if and only if φ is an automorphism of A. Using this observation and Proposition 2.8(c), we have the following corollary.

Corollary 2.9 *[5] If p is a prime number, then all skew morphisms of $\mathbb{Z}_p$ are automorphisms.*

The direct product of skew morphisms of groups A and B, respectively, may not be a skew morphism of the group $A \times B$. The following criterion will be useful.

Proposition 2.10 ([23]) *Let $G = A \times B$, and let φ and ψ be skew morphisms of A and B with associated power functions π_φ and π_ψ, respectively.*

(a) *The direct product $\varphi \times \psi$ is a skew morphism of G if and only if*

$$\pi_\varphi(a) \equiv \pi_\psi(b) \equiv 1 \pmod{d} \text{ for all } a \in A \text{ and } b \in B,$$

where $d = \gcd(|\varphi|, |\psi|)$.

(b) *Suppose that $\varphi \times \psi$ is a skew morphism of G with associated power function π. Then for all $a \in A$ and $b \in B$,*

$$\pi((a,b)) \equiv \pi_\varphi(a) \pmod{|\varphi|} \quad \textit{and} \quad \pi((a,b)) \equiv \pi_\psi(b) \pmod{|\psi|}.$$

Finally, if φ is a skew morphism of a group A and θ is an automorphism of A, then $\theta\varphi\theta^{-1}$ is also a skew morphism of A (see [22, Lemma 2.2]), so the automorphism group $\mathrm{Aut}(A)$ acts on the set of skew morphisms of A by conjugation. Two skew morphisms φ and ψ of A are *equivalent* if they belong to the same orbit of this action.

3 Construction

In this section, for certain n, we construct non-smooth skew morphisms of the cyclic group $\mathbb{Z}_n$. The following lemma is very useful.

Lemma 3.1 *Let $G = A \times B$, and let φ and ψ be skew morphisms of A and B, respectively. If $\varphi \times \psi$ is a skew morphism of $A \times B$, then $\varphi \times \psi$ is smooth if and only if both φ and ψ are smooth.*

Proof Let π, π_φ and π_ψ be the power functions associated with $\varphi \times \psi, \varphi$ and ψ, respectively. Assume that $\varphi \times \psi$ is smooth. For any $a \in A$, by Proposition 2.10(b) we have

$$\pi\left((\varphi \times \psi)(a, 1_B)\right) = \pi\left(a, 1_B\right) \equiv \pi_\varphi(a) \pmod{|\varphi|}$$

and

$$\pi\left((\varphi \times \psi)(a, 1_B)\right) = \pi\left(\varphi(a), 1_B\right) \equiv \pi_\varphi(\varphi(a)) \pmod{|\varphi|}.$$

This implies that $\pi_\varphi\left(\varphi(a)\right) \equiv \pi_\varphi(a) \pmod{|\varphi|}$, and hence, φ is smooth. Similarly, ψ is also smooth.

Conversely, suppose that both φ and ψ are smooth. Then, for any $(a,b) \in A \times B$, also by Proposition 2.10(b), we have

$$\pi((\varphi \times \psi)(a,b)) = \pi(\varphi(a), \psi(b)) \equiv \pi_\varphi(\varphi(a)) \equiv \pi_\varphi(a) \equiv \pi(a,b) \pmod{|\varphi|}$$

and

$$\pi((\varphi \times \psi)(a,b)) = \pi(\varphi(a), \psi(b)) \equiv \pi_\psi(\psi(b)) \equiv \pi_\psi(b) \equiv \pi(a,b) \pmod{|\psi|}.$$

This implies that $\pi((\varphi \times \psi)(a,b)) \equiv \pi(a,b) \pmod{|\varphi \times \psi|}$, and hence, $\varphi \times \psi$ is smooth. □

For positive integers s and t, we define $\tau(s,t) := \sum_{i=1}^{t} s^{i-1}$. For integers n and r with $n \geq 1$, we write $r \in \mathbb{Z}_n^*$ if $\gcd(r,n) = 1$. In this case, the *multiplicative order* of r in $\mathbb{Z}_n$ is defined to be the smallest positive integer l for which $r^l \equiv 1 \pmod{n}$.

Proposition 3.2 ([3, 15]) *For $n > 1$, the proper smooth skew morphisms of $\mathbb{Z}_n$ and the associated power functions are given by the formula*

$$\varphi(x) \equiv x + rk\frac{\tau(s,t)^x - 1}{\tau(s,t) - 1} \pmod{n} \text{ and } \pi(x) \equiv t^x \pmod{m}, \tag{3}$$

where $k > 1$ is a proper divisor of n, and $r \in \mathbb{Z}_{n/k}$, $s \in \mathbb{Z}^*_{n/k}$ and $t \in \mathbb{Z}^*_m$ are positive integers satisfying the following conditions:

(a) *m is the smallest positive integer such that $r\sum_{i=1}^m s^{i-1} \equiv 0 \pmod{n/k}$.*
(b) *t has multiplicative order k in $\mathbb{Z}_m$.*
(c) $s - 1 \equiv r\big((\sum_{i=1}^t s^{i-1})^k - 1\big)/(\sum_{i=1}^t s^{i-1} - 1) \pmod{n/k}$.
(d) $s^{t-1} \equiv 1 \pmod{n/k}$.

Moreover, k is equal to the skew type of φ, and m is equal to the order of φ.

We remark that even though all smooth skew morphisms of $\mathbb{Z}_n$ are determined, it is not clear from Proposition 3.2 that, for which n, the group $\mathbb{Z}_n$ underlies a non-smooth skew morphism. We also observe in (3) that the proper smooth skew morphisms of $\mathbb{Z}_n$ are all defined by exponential functions on $\mathbb{Z}_n$. By contrast, it is well known that every automorphism of $\mathbb{Z}_n$ is a linear function of the form $x \mapsto rx$, where $r \in \mathbb{Z}^*_n$. Surprisingly enough, it was shown in [14] that the proper skew morphisms of $\mathbb{Z}_n$ which are square roots of automorphisms are quadratic polynomials over the ring $\mathbb{Z}_n$.

Proposition 3.3 *[14] Every proper skew morphism φ of $\mathbb{Z}_n$ such that φ^2 is an automorphism of $\mathbb{Z}_n$ is equivalent to a skew morphism of the form*

$$\varphi(x) \equiv sx - \frac{x(x-1)n}{2k} \pmod{n},$$

where k and s are positive integers satisfying the following conditions:

(a) *k^2 divides n and $s \in \mathbb{Z}^*_n$ if k is odd, and $2k^2$ divides n and $s \in \mathbb{Z}^*_{n/2}$ if k is even.*
(b) *$s \equiv -1 \pmod{k}$, s has multiplicative order 2ℓ in $\mathbb{Z}_{n/k}$, and $\gcd(w, k) = 1$, where*

$$w \equiv \frac{k}{n}(s^{2\ell} - 1) - \frac{s(s-1)}{2}\ell \pmod{k}.$$

The skew type of φ is equal to k, and the order of φ is equal to $m := 2k\ell$, and finally, the power function of φ is given by

$$\pi(x) \equiv 1 + 2xw'\ell \pmod{m},$$

where w' is determined by the congruence $w'w \equiv 1 \pmod{k}$.

We are ready to show that $\mathbb{Z}_n$ underlies a non-smooth skew morphism for certain n.

Lemma 3.4 (a) *For any $e \geq 2$, the cyclic group $\mathbb{Z}_{p^e}$ underlies a non-smooth skew morphism, where p is an odd prime.*

(b) *For any $e \geq 5$, the cyclic group $\mathbb{Z}_{2^e}$ underlies a non-smooth skew morphism.*

Proof (a) In Proposition 3.3, take $n := p^e$, where $e \geq 2$, and set $k := p$, $s := -1$. Then, $\ell = 1$, $w = w' = -1$ and $m = 2p$. It is easy to verify that the stated conditions are satisfied, so we obtain a skew morphism of $\mathbb{Z}_{p^e}$ and the associated power function given by

$$\varphi(x) \equiv -x - p^{e-1}\frac{x(x-1)}{2} \pmod{p^e} \quad \text{and} \quad \pi(x) \equiv 1 - 2x \pmod{2p}.$$

Since $\pi(\varphi(1)) \equiv \pi(-1) \equiv 3 \pmod{2p}$ and $\pi(1) \equiv -1 \pmod{2p}$, we have $\pi(\varphi(1)) \not\equiv \pi(1) \pmod{2p}$, so φ is not smooth.

(b) In Proposition 3.3, let $n := 2^e$ where $e \geq 5$, and take $k := 4$ and $s := -1$. As before we have $\ell = 1$, $w = w' = -1$ and $m = 8$. It is easy to verify that the stated conditions are satisfied, so we obtain a skew morphism of $\mathbb{Z}_{2^e}$ and the associated power function given by

$$\varphi(x) = -x - 2^{e-3}x(x-1) \pmod{2^e} \quad \text{and} \quad \pi(x) \equiv 1 - 2x \pmod{8}.$$

Since $\pi(\varphi(1)) \equiv \pi(-1) \equiv 3 \pmod{8}$ and $\pi(1) \equiv -1 \pmod{8}$, $\pi(1) \not\equiv \pi(\varphi(1)) \pmod{8}$, φ is also not smooth. □

Lemma 3.5 *Let n be a positive integer with a decomposition $n = 2^e n_1$, where n_1 is odd. If $e \geq 5$ or n_1 is not square-free, then $\mathbb{Z}_n$ underlies a non-smooth skew morphism.*

Proof Note that $\mathbb{Z}_n \cong \mathbb{Z}_{2^e} \times \mathbb{Z}_{n_1}$. If $e \geq 5$, then by Lemma 3.4(b), the cyclic group $\mathbb{Z}_{2^e}$ has a non-smooth skew morphism α, so by Lemma 3.1, $\varphi := \alpha \times \mathrm{id}_{n_1}$ is a non-smooth skew morphism of $\mathbb{Z}_n$, where id_{n_1} denotes the identity permutation on $\mathbb{Z}_{n_1}$. If n_1 is not square-free, then there is an odd prime p such that $p^e \mid n_1$ where $e \geq 2$, using Lemma 3.4(a) and similar techniques we can construct a non-smooth skew morphism of $\mathbb{Z}_n$, as required. □

4 Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1. Before doing this, we make the following convention. Suppose that $G = AY$ is a complementary factorization, where the subgroup Y is cyclic and core-free in G. Let y be a fixed generator of Y, then a skew morphism φ of A together with the associated power function π are defined according to (1). The skew morphism φ will be referred to as the *skew morphism of A induced by y*. Note that the action of G on the set $\Omega := \{gY \mid g \in G\}$ of left cosets of Y in G is faithful; in particular, the subgroup A acts regularly on Ω and Y is a point stabilizer. Therefore, we may assume that $G \leq \mathrm{Sym}(\Omega)$ with $G = AY$, where A is a regular subgroup of G and $Y = G_x$ is cyclic for some $x \in \Omega$. This assumption will be used throughout this section and for the sake of convenience we record it here.

Assumption 4.1 Let $G \leq \mathrm{Sym}(\Omega)$ be a group such that $G = AY$, where A is abelian and regular on Ω, $Y = G_x = \langle y \rangle$ for a fixed element $x \in \Omega$, and let φ be the skew morphism induced by y.

The proof of Theorem 1.1 will be given after four preparatory lemmas.

Lemma 4.2 *With the notations in Assumption 4.1, suppose that $O_p(G) \leq A$ for every prime divisor p of $|G|$. Then, $A \lhd G$.*

Proof Denote by $\wp = \{p_1, p_2, \ldots, p_k\}$ the set of prime divisors of $|G|$, then by Proposition 2.1(a) $F(G) = \prod_{i=1}^{k} O_{p_i}(G) \leq A$. On the other hand, by Proposition 2.2(a) G is solvable, and by Proposition 2.1(b) we have $A \leq C_G(F(G)) \leq F(G)$. Therefore, $A = F(G)$, in particular, $A \lhd G$. □

For a prime divisor p of $|A|$, we denote by A_p the (unique) Sylow p-subgroup of the abelian group A and by $A_{p'}$ the (unique) Hall p'-subgroup of A.

Lemma 4.3 *With the notations in Assumption 4.1, suppose that $O_p(G) \neq 1$ for some prime divisor p of $|A|$ and $|A_p| = p$. Then, $O_p(G) = A_p$ or $A_p \times Y^*$, where $Y^* \leq Y$ and $|Y^*| = p$.*

Proof Write $N := O_p(G)$. By Lemma 2.6, we know that $\mathrm{Orb}_N(\Omega) = \mathrm{Orb}_B(\Omega)$ for some $B \leq A$. Since the size of each N-orbit is a power of p, we have $B = A_p$. Thus, $\mathrm{Orb}_N(\Omega)$ is a block system for G in which each block has size p. Let K be the kernel of the action of G on $\mathrm{Orb}_N(\Omega)$. By Proposition 2.2(a) G is solvable, so K is solvable as well. It follows from Proposition 2.7 that K is isomorphic to a subgroup of the direct product $\mathrm{AGL}(1, p) \times \cdots \times \mathrm{AGL}(1, p)$ with $|A|/p$ factors. It is clear that the Sylow p-subgroup P of K is normal in K and it is elementary abelian. Thus, P char $K \lhd G$ and hence $P \lhd G$, and we obtain $P \leq O_p(G)$. On the other hand, since $O_p(G) = N \leq K$, we also have $O_p(G) \leq P$. Therefore, $O_p(G) = P$, and we conclude that $O_p(G) = A_p$ or $A_p \times Y^*$, as required. □

Lemma 4.4 *With the notations in Assumption 4.1, suppose that N is a normal subgroup of G. For $g \in G$, denote by $\bar{g}$ the image of g under its action on $\mathrm{Orb}_N(\Omega)$ and for $H \leq G$, set $\bar{H} = \{\bar{h} : h \in H\}$.*

(a) *$\bar{G} = \bar{A}\bar{Y}$, $\bar{A} \cap \bar{Y} = 1$ and $\bar{Y}$ is core-free in $\bar{G}$.*
(b) *Let $\bar{\varphi}$ be the skew morphism of $\bar{A}$ induced by $\bar{y}$. Then,*

$$\bar{\varphi}(\bar{a}) = \overline{(\varphi(a))} \text{ for all } a \in A.$$

(c) *Let $\bar{\pi}$ be a power function associated with $\bar{\varphi}$. Then*

$$\bar{\pi}(\bar{a}) \equiv \pi(a) \pmod{|\bar{\varphi}|} \text{ for all } a \in A.$$

Proof The mapping $g \mapsto \bar{g}$ is an epimorphism from G onto $\bar{G}$; therefore, $\bar{G} = \bar{A}\bar{Y}$. Since $\bar{A}$ is abelian and transitive on $\mathrm{Orb}_N(\Omega)$, it is regular. It is evident that $\bar{Y}$ is contained in the stabilizer of the N-orbit containing the identity element 1_A. Denoting this stabilizer by Z, we have $\bar{A}\bar{Y} = \bar{G} = \bar{A}Z$. Since $\bar{A}$ is regular, we have $\bar{A} \cap \bar{Y} = \bar{1} = \bar{A} \cap Z$, so $Z = \bar{Y}$. This proves (a).

Fix any $a \in A$, we have $ya = \varphi(a)y^{\pi(a)}$, so $\bar{y}\bar{a} = \overline{\varphi(a)}(\bar{y})^{\pi(a)}$. Then, (1) can be applied to the factorization $\bar{A}\bar{Y}$ and generator $\bar{y} \in \bar{Y}$, giving rise to the equality $\bar{\varphi}(\bar{a}) = \overline{\varphi(a)}$ and the congruence $\bar{\pi}(\bar{a}) \equiv \pi(a) \pmod{|\bar{\varphi}|}$, and this proves (b) and (c). □

In what follows, the skew morphism $\bar{\varphi}$ defined as in the lemma above will be called *the quotient of* φ *determined by* N, and denoted by φ_N.

Lemma 4.5 *With the notations in Assumption 4.1, suppose that* G *has two distinct normal subgroups* M *and* N *satisfying the following conditions:*

(a) *There are subgroups* $B, C \leq A$ *such that* $B \cap C = 1$*, and*

$$\mathrm{Orb}_M(\Omega) = \mathrm{Orb}_B(\Omega) \text{ and } \mathrm{Orb}_N(\Omega) = \mathrm{Orb}_C(\Omega).$$

(b) *Both skew morphisms* φ_M *and* φ_N *are smooth.*

Then, φ *is smooth.*

Proof For any $a \in A$, since φ_M is smooth, by Lemma 4.4(a)–(b), we have

$$\pi(\varphi(a)) \equiv \bar{\pi}(\overline{\varphi(a)}) = \bar{\pi}(\varphi_M(\bar{a})) \equiv \bar{\pi}(\bar{a}) \equiv \pi(a) \pmod{|\varphi_M|}.$$

Similarly, $\pi(\varphi(a)) \equiv \pi(a) \pmod{|\varphi_N|}$.

In what follows, we show that $|\varphi| = \mathrm{lcm}(|\varphi_M|, |\varphi_N|)$, which together with the above congruences will imply $\pi(\varphi(a)) \equiv \pi(a) \pmod{|\varphi|}$ for all $a \in A$, and hence, φ is smooth. Indeed, let K and L be the kernels of the actions of G on $\mathrm{Orb}_M(\Omega)$ and $\mathrm{Orb}_N(\Omega)$, respectively. Then,

$$|Y| = |\varphi_M||Y \cap K| = |\varphi_N||Y \cap L|. \tag{4}$$

Let $d = \gcd(|\varphi_M|, |\varphi_N|)$. Now by (4) we have

$$\frac{|\varphi_M|}{d} \cdot |Y \cap K| = \frac{|\varphi_N|}{d} \cdot |Y \cap L|. \tag{5}$$

If $\psi \in (Y \cap K) \cap (Y \cap L)$, then for any $x \in \Omega$, ψ fixes the orbits $\mathrm{Orb}_M(x)$ and $\mathrm{Orb}_N(x)$, so by (a) we have $\psi(x) \in \mathrm{Orb}_B(x) \cap \mathrm{Orb}_C(x)$. Since $B \cap C = 1$ and A is regular, we have $\mathrm{Orb}_B(x) \cap \mathrm{Orb}_C(x) = \{x\}$, and so $\psi = \mathrm{id}_\Omega$. But Y is a cyclic group, we obtain $\gcd(|Y \cap K|, |Y \cap L|) = 1$, and from (5) we deduce that $|Y \cap K|$ divides $|\varphi_N|/d$. On the other hand, $|\varphi_N|/d$ divides $|Y \cap K|$ because it is coprime to $|\varphi_M|/d$, and we find $|Y \cap K| = |\varphi_N|/d$. Thus, by (4) $|\varphi| = |Y| = \mathrm{lcm}(|\varphi_M|, |\varphi_N|)$. □

Proof of Theorem 1.1 We keep the notations set in Assumption 4.1 and, in addition, assume that $A \cong \mathbb{Z}_n$ for some $n \geq 1$. We have to show that φ is smooth if and only if $n = 2^e n_1$, where $0 \leq e \leq 4$ and n_1 is an odd square-free number. The necessity has been proved in Lemma 3.5. For the converse, we proceed by induction on n. If $n = 2^e$ $(0 \leq e \leq 4)$, then by the census in [1] we know that φ is smooth; if n is an odd prime, then φ is an automorphism of A due to Corollary 2.9; in particular, it is smooth.

From now on, we will assume that n is either an odd composite number or an even number with an odd prime divisor and also that the theorem holds for any cyclic group whose order is a proper divisor of n. By Proposition 2.2(b), G is supersolvable, and therefore, Proposition 2.3 can be applied to G. In particular, if p is the largest prime divisor of $|G|$, then the Sylow p-subgroup P of G is normal in G. If $O_q(G) \neq 1$ for another prime divisor q of $|G|$, then Lemma 4.5 can be applied to G with $M = P$ and $N = O_q(G)$. Using also the induction hypothesis, we obtain that φ is smooth.

Thus, we may assume that $F(G) = O_p(G) = P$. Notice that $p > 2$, so $|A_p| = p$. By Lemma 4.3, we have $P = A_p$ or $P = A_p \times Y^* \cong \mathbb{Z}_p^2$, where $Y^* \leq Y$ and $|Y^*| = p$. If $P = A_p$, then from Lemma 4.2 we deduce that $A \lhd G$, and so φ is an automorphism; in particular, φ is smooth. In what follows, we assume $P = A_p \times Y^*$.

By Proposition 2.4, we may assume that Q is a Hall p'-subgroup of G containing $A_{p'}$. Then, $G = P \rtimes Q$ and by Proposition 2.1 $C_G(P) = C_G(F(G)) = F(G) = P$. It follows that Q is isomorphic to a subgroup of $\mathrm{Aut}(P)$. By Proposition 2.8(b)–(c), $1 \neq \operatorname{Ker}\varphi \lhd G$, and so $1 \neq \operatorname{Ker}\varphi \leq F(G) \cap A = A_p$. But $|A_p| = p$, we get $A_p = \operatorname{Ker}\varphi$ and A_p is a normal subgroup of G. Thus, A_p is Q-invariant. Since $p \nmid |Q|$, by Lemma 2.5 the subgroup Q is isomorphic to a subgroup of $\mathbb{Z}_{p-1}^2$, so it is abelian. Now, $Q \cong G/P \cong \bar{G}$, where $\bar{G}$ is the image of G induced by its action on $\mathrm{Orb}_P(\Omega)$. By Lemma 4.4 $Q \cong \bar{A}\bar{Y}$ and the latter group is the skew product group induced by $\bar{A}$ and the quotient skew morphism $\bar{\varphi}$. Since it is abelian, $\bar{Y} = 1$, so $Y \leq P$. Therefore, $|\varphi| = |Y| = p$, and hence, φ is smooth. □

5 Non-cyclic abelian groups

In this section, on the basis of the results obtained in the previous sections, we shall give a partial solution to Problem 1. The following classification of proper skew morphisms of the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$ will be useful.

Proposition 5.1 ([5, Theorem 5.10]) *Let (a, x) be a basis of the elementary abelian group $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is an odd prime. Then, every proper skew morphisms of A can be expressed as the form*

$$\varphi(a^i x^j) = a^{ri+\frac{1}{2}dj(j-1)rn}(bx^r)^j,$$

where b is an element in the kernel $\operatorname{Ker}\varphi = \langle a \rangle$ uniquely determined by the triple (d, n, r), for all $d, n \in \mathbb{Z}_p^*$ and $r \in \{2, \dots, p-1\}$. The skew morphism φ has order pk, where k is the multiplicative order of r modulo p, and its power function is given by

$$\pi(a^i x^j) \equiv 1 + jnk \pmod{pk}.$$

Lemma 5.2 *Let p be an odd prime, then every proper skew morphism of the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$ is non-smooth.*

Proof Using the notation in Proposition 5.1, we have $\pi(x) \equiv 1 + nk \pmod{pk}$ and $\pi(\varphi(x)) \equiv \pi(bx^r) \equiv \pi(x^r) \equiv 1+rnk \pmod{pk}$, and so $\pi(x) \not\equiv \pi(\varphi(x)) \pmod{pk}$. Therefore, φ is not smooth. □

Lemma 5.3 *Let $G = A \times B$ be an abelian group such that $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is an odd prime. Then, G underlies a non-smooth skew morphism.*

Proof By Lemma 5.1, the elementary abelian group $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$ has a non-smooth skew morphism α, so by Lemma 3.1 $\varphi := \alpha \times \mathrm{id}_B$ is a non-smooth skew morphism of G. □

Proof of Theorem 1.2 If the Sylow 2-group P of A contains a direct factor isomorphic to $\mathbb{Z}_{2^e}$ for some $e \geq 5$, then by Lemma 3.1 and Lemma 3.4 we can construct a non-smooth skew morphism of A.

If n_1 is not square-free, then there is an odd prime p such that $p^2 \mid n$, and so either $A \cong \mathbb{Z}_p \times \mathbb{Z}_p \times B$, or $A \cong \mathbb{Z}_{p^i} \times B$ for some $i \geq 2$ and some subgroup $B \leq A$. By Lemmas 3.4(a) and 5.3, both $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^i}$ underly a non-smooth skew morphism α. Therefore, by Lemma 3.1 the skew morphism $\varphi = \alpha \times \mathrm{id}_B$ is a non-smooth skew morphism of A. □

Let A be a non-cyclic abelian group of order $n = 2^f n_1$, where n_1 is odd. If every skew morphism of A is smooth, then by Theorem 1.2, n_1 is square-free and the Sylow 2-subgroup P of A contains no direct factors isomorphic to $\mathbb{Z}_{2^e}$ for any $e \geq 5$. Thus, $P \cong \mathbb{Z}_{2^{e_1}} \times \cdots \times \mathbb{Z}_{2^{e_r}}$ for some positive integer r, where $1 \leq e_i \leq 4$ for all $1 \leq i \leq r$. In particular, if $e_1 = e_2 = \cdots = e_r = 1$, it is known that every skew morphism of P is an automorphism (see [5, Theorem 5.8]), so it is smooth. But for other cases, little is known. For an abelian 2-group $\mathbb{Z}_{2^{e_1}} \times \cdots \times \mathbb{Z}_{2^{e_r}}$, we call the r-tuple $(e_1, \ldots, e_r)$ *admissible* if every skew morphism of the abelian group $\mathbb{Z}_{2^{e_1}} \times \cdots \times \mathbb{Z}_{2^{e_r}}$ is smooth.

Problem 2 Determine the admissible r-tuples for all $r \geq 1$.

Problem 3 Suppose that $(e_1, \ldots, e_r)$ is an admissible r-tuple, and A is an abelian group such that its Sylow 2-subgroup $P \cong \mathbb{Z}_{2^{e_1}} \times \cdots \times \mathbb{Z}_{e^r}$ and the order of its Hall $2'$-subgroup is square-free, is it true that every skew morphism of A is smooth?

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Declarations

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