Journal of Algebraic Combinatorics (2023) 58:993–1017
https://doi.org/10.1007/s10801-023-01259-x

Characterisation of all integral circulant graphs with multiplicative divisor sets and few eigenvalues

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Received: 10 June 2022 / Accepted: 13 August 2023 / Published online: 22 September 2023

Abstract
We present a method which in principal allows to characterise all integral circulant graphs with multiplicative divisor set having a spectrum, i.e. the set of distinct eigenvalues, of any given size. We shall exemplify the method for spectra of up to four eigenvalues, also reproving some known results for three eigenvalues along the way. In particular we show that given any integral circulant graph of arbitrary order n with multiplicative divisor set and precisely four distinct eigenvalues, n necessarily is either a prime power or the product of two prime powers with explicitly given simply structured divisor set and set of eigenvalues in both cases.

Keywords Integral graphs · Regular graphs · Strongly regular graphs · Circulant graphs · Eigenvalues · Spectrum

Mathematics Subject Classification 05C50

1 Introduction

The adjacency matrix of an undirected graph G on n vertices is a real symmetric $(n \times n)$-matrix and thus has n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, say, which are not necessarily distinct. In other words, G has real spectrum $\mathrm{Spec}(G) := \{\lambda_i : 1 \leq i \leq n\}$. Graphs with a small number of distinct eigenvalues, i.e. $|\mathrm{Spec}(G)|$ is small, have attracted quite a bit of attention over the past decades. "*Graphs with few distinct eigenvalues tend to*

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have some kind of regularity”, Brouwer and Haemers formulated in [7, Chapter 15] (also explaining the following statements). A graph with only one eigenvalue has no edges, and a graph with only two distinct eigenvalues is complete or, in general, a union of complete subgraphs. Generally, graphs with three eigenvalues need not be regular any more. However, for a connected regular graph we have the following classic result by Shrikhande and Bhagwandas:

Theorem 1.1 [24] *A regular connected graph has precisely three distinct eigenvalues if and only if it is strongly regular.*

To this end, a graph G is called *strongly regular* with parameters (n, k, a, c) if G is k-regular with n vertices, neither complete nor empty, and every pair of adjacent (resp. non-adjacent) vertices has exactly a (resp. c) common neighbours. This notion was originally introduced by Bose [4] in 1963. Note that unconnected strongly regular graphs are characterised by $c = 0$ (equivalently $a = k - 1$) and are isomorphic to m copies of K_{k+1} for some $m > 1$ (cf. [12, Lemma 10.1.1]), resulting in m times eigenvalue k and mk times eigenvalue -1. Hence the structure of such graphs is fully understood.

Strongly regular graphs have been characterised in various ways. For example, in [18] the authors give a characterisation of strongly regular graphs by virtue of Euclidean representations of a graph. In contrast, the authors of [10] present an upper bound for the largest eigenvalue of the so-called signless Laplacian matrix of a given graph G. That bound is attained if and only if G is strongly regular, hence giving rise to a characterisation of strongly regular graphs. Graphs with few eigenvalues have also been considered with respect to eigenvalues associated with still other matrices, e.g. the normalized Laplacian matrix [5, 14, 28].

Van Dam and Omidi [29] generalize strongly regular graphs by introducing the notion of strongly walk-regular graphs. They characterise strongly walk-regular graphs by showing that this class consists of the following subclasses: empty graphs, complete graphs, strongly regular graphs, disjoint unions of complete bipartite graphs of the same size and isolated vertices, regular graphs with four eigenvalues.

Turning to various classes of connected regular graphs, it is of interest how to achieve a structural characterisation of the strongly regulars members of such a class (as opposed to the obvious algebraic characterisation by means of their spectrum). Considering the class of circulant graphs (i.e. graphs whose adjacency matrix is a circulant matrix), it has been shown in [6] that Paley graphs are the only non-trivial circulant strongly regular graphs—the “trivial” case being a complete multipartite graph or its complement. Let us now further restrict the graph class to those members whose eigenvalues are integers only. This yields the class of integral circulant graphs, cf. Sect. 2 for more details. As the authors of [27] point out, the Paley graphs in question must have p vertices, where p is a prime congruent 1 modulo 4, hence they are not integral. Thus, a characterisation of strongly regular integral circulant graphs can be achieved (see also [2, Theorem 15]).

The question whether connected graphs with four distinct eigenvalues exhibit any noticeable regularity features or how to characterise them is an active field of research. Some references can be found in [7, Chapter 15]. Most notably, authors restrict themselves to particular graphs only. For example, the authors of [30] identify all regular connected graphs with four distinct eigenvalues having at most 30 vertices. In [13] it is proven that connected graphs with four distinct eigenvalues but at least three eigenvalues of multiplicity one cannot exist. Moreover, all connected regular graphs with four distinct eigenvalues and second least eigenvalue greater than or equal to 1 are determined. The paper [11] contains characterisations of grid graphs as co-edge-regular graphs with four distinct eigenvalues.

When it comes to integral circulant graphs with four distinct eigenvalues, results are still scarce. The authors of [26] identify several classes of connected integral circulant graphs with four distinct eigenvalues. It is our purpose to address the "small" spectrum problem for integral circulant graphs which have a *multiplicative divisor set* (see Sect. 2 and the introduction to Sect. 4 for definitions). We shall establish a method which in principal allows to identify all such graphs having a spectrum of any given size. For this purpose, we introduce the combinatorial concept of the *gap number* in Sect. 3 and show in Theorem 3.1 that the number of distinct eigenvalues of an integral circulant graph of prime power order is roughly twice the corresponding gap number. The multiplicativity property of the divisor sets enables us to decipher the structure of these sets—i.e. their so-called *factorisation pattern* (cf. Section 4)—explicitly for any integral circulant graph of arbitrary order n as long as it has a small spectrum (Theorems 5.1 and 7.1 for three or four distinct eigenvalues, respectively).

In particular, we shall exemplify our method for spectra of up to four eigenvalues, also reproving some known results for three eigenvalues along the way (see Sects. 5 and 6). Incidentally this will reveal the combinatorial difficulties one has to face in case of larger spectra. Nevertheless we completely unravel the mystery for integral circulant graphs of order n with multiplicative divisor set and precisely four distinct eigenvalues by showing that for any such graph n necessarily is either a prime power or the product of two prime powers with explicitly given simply structured divisor set and set of eigenvalues in both cases (Corollary 7.1).

We hope that the new approach inspires further research in this field and formulate a couple of open problems in Sect. 8.

2 Basics about integral circulant graphs

Given a finite group Γ—assuming the operation to be written additively with identity 0—and a generating set $S \subseteq \Gamma$, the corresponding *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ is defined to have vertex set Γ and edge set $\{\{a, b\} \in \Gamma : a-b \in S\}$. If $S = -S := \{-s : s \in S\}$ and $0 \notin S$, then $\mathrm{Cay}(\Gamma, S)$ is undirected and loop-free (cf. [12] for basic definitions and properties).

Let $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ be the residue class group $\bmod\, n$, which we regularly identify with the set $\{0, 1, 2, \ldots, n-1\}$ of non-negative residues. *Circulant graphs* are Cayley graphs on finite cyclic groups, i.e. $\Gamma = \mathbb{Z}_n$ for some positive integer n, and we define $\mathrm{Circ}(n, S) := \mathrm{Cay}(\mathbb{Z}_n, S)$. In this situation S is the set of neighbours of 0. For $S = \mathbb{Z}_n^* = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$ being the set of all units in the residue class ring $\bmod\, n$, we obtain as a special case the so-called *unitary Cayley graph* $\mathrm{Circ}(n, \mathbb{Z}_n^*)$. Each circulant graph has a circulant adjacency matrix, whose j-th entry in the first row is 1 if $j \in S$ and equals 0 otherwise $(0 \leq j \leq n-1)$, where the first entry is assumed to be zero in order to avoid loops in the graph. Each of the eigenvalues of a circulant matrix can explicitly be evaluated as a sum of roots of unity (see [9] for the general theory of circulant matrices).

A graph is called *integral* if all of its eigenvalues, i.e. the eigenvalues of the graph's adjacency matrix, are rational integers. In this case the sums of roots of unity computing the eigenvalues turn out to be sums of Ramanujan sums (cf. (3) or [15]), which explains the integrality of the eigenvalues. Using this observation, So [25] and Klotz and Sander [15] showed that a circulant graph $\mathrm{Circ}(n, S)$ is integral if and only if we have for some non-empty set $\mathcal{D} \subseteq D(n) := \{d > 0 : d \mid n\}$ of positive divisors of n that

$$S = \bigcup_{d \in \mathcal{D}} G_n(d), \quad G_n(d) := \{1 \le a \le n-1 : \ \gcd(a, n) = d\}. \tag{1}$$

In other words, each integral circulant graph $\mathrm{Circ}(n, S)$ is characterised by n and a non-empty set $\mathcal{D} \subseteq D(n)$ in such a way that it has vertex set $\mathbb{Z}_n$ and edge set

$$\{(a, b) : \ a, b \in \mathbb{Z}_n, \ \gcd(a-b, n) \in \mathcal{D}\}. \tag{2}$$

The notation $\mathrm{ICG}(n, \mathcal{D}) := \mathrm{Circ}(n, \bigcup_{d\in\mathcal{D}} G_n(d))$ for integral circulant graphs is well established, where $\mathcal{D} \subseteq D(n)$, $\mathcal{D} \neq \emptyset$, is called a *divisor set* of n.
Two arithmetic properties of $\mathcal{D}$ reflected by graph-theoretical features of $\mathrm{ICG}(n, \mathcal{D})$ are:

1. If $n \in \mathcal{D}$, then apparently $\mathrm{ICG}(n, \mathcal{D})$ has loops, which is the reason why some authors preclude this possibility. However, we tolerate that $n \in D$, in particular for prime powers n, unless otherwise stated.
2. $\mathrm{ICG}(n, \mathcal{D})$ is connected if and only if the elements of $\mathcal{D}$ are coprime (cf. [3, Proposition 1]).

Let $\mathrm{ICG}(n, \mathcal{D})$ be an integral circulant graph with an arbitrary positive integer n and a divisor set $\mathcal{D} \subseteq D(n)$. As shown in [15, Theorem 16], the eigenvalues of (the adjacency matrix of) $\mathrm{ICG}(n, \mathcal{D})$, taking multiplicities into account, are integers given by

$$\lambda_j(n, \mathcal{D}) = \sum_{d\in\mathcal{D}} c\Big(j, \frac{n}{d}\Big), \quad c(j, n) := \sum_{\substack{\ell \bmod n \\ (\ell, n)=1}} e\Big(\frac{j\ell}{n}\Big) = \sum_{d \mid \gcd(j,n)} d \cdot \mu\Big(\frac{n}{d}\Big) \tag{3}$$

for $1 \le j \le n$. Here $c(j, n)$ denotes a Ramanujan sum (cf. [1, Chapters 8.3–8.4], [23, Chapter I.3]), $\mu(n)$ is Möbius' function and $e(x) := e^{2\pi i x}$ for real x. Furthermore $\vec{\lambda}(n, \mathcal{D}) := \big(\lambda_1(n, \mathcal{D}), \lambda_2(n, \mathcal{D}), \ldots, \lambda_n(n, \mathcal{D})\big)$ is called the *spectral vector* of $\mathrm{ICG}(n, \mathcal{D})$. Disregarding multiplicities we have $\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D})) := \{\lambda_j(n, \mathcal{D}) : 1 \le j \le n\}$.

The chararacterisation of the set of edges in (2) reveals that integral circulant graphs are clearly regular. Actually, circulant adjacency matrices represent regular graphs, because the number of entries 1 in each row is trivially the same, hence all circulant graphs are regular. Therefore $|S|$ (cf. (1)), counting the neighbours of 0, equals the degree of regularity of $\mathrm{ICG}(n, \mathcal{D})$). It is well known that the degree of regularity of a regular graph is also its largest eigenvalue, its its multiplicity being the number of components of the graph (cf. [8]). It can easily be seen by (3) and (1) that $\lambda_n(n, \mathcal{D})$ plays this role in $\mathrm{ICG}(n, \mathcal{D})$):

$$\lambda_n(n, \mathcal{D}) = \sum_{d \in \mathcal{D}} c\Big(n, \frac{n}{d}\Big) = \sum_{d \in \mathcal{D}} \sum_{\substack{\ell \bmod \frac{n}{d} \\ \gcd(\ell, \frac{n}{d})=1}} 1 = \sum_{d \in \mathcal{D}} \varphi\Big(\frac{n}{d}\Big) = \sum_{d \in \mathcal{D}} \big|G_n(d)\big| = |S|,$$

since $G_n(d) = \{1 \le \ell \cdot d < n : \ \gcd(\ell, \frac{n}{d}) = 1\}$, where $\varphi(n)$ denotes Euler's totient function.

3 Spec($\mathrm{ICG}(\mathcal{D}, p^k)$) and gaps in the exponent set

The knowledge about the spectral vectors $\vec{\lambda}(p^k, \mathcal{D})$ of integral circulant graphs $\mathrm{ICG}(p^k, \mathcal{D})$ having prime power order p^k is very satisfying (cf. [20–22]). A different task is to figure out the cardinality of the spectrum, i.e. the number of *distinct* eigenvalues. For its own sake as well as for later use we prove a formula for $|\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))|$, given any prime power order p^k and allowing for $p^k \in \mathcal{D}$. Moreover we explicitly determine all $\mathrm{ICG}(p^k, \mathcal{D})$ whose spectrum contains at most four distinct eigenvalues.

Let p^k be a prime power. Given a divisor set $\mathcal{D} \subseteq D(p^k)$, i.e. $\mathcal{D} = \{p^{k_1}, \dots, p^{k_m}\}$ with $0 \le k_1 < k_2 < \cdots < k_m \le k$, say, we call $\mathcal{K}_\mathcal{D} := \{k_1, \dots, k_m\}$ its *exponent set*. Moreover, let $\widetilde{\mathcal{K}}_\mathcal{D} := \mathcal{K}_\mathcal{D} \backslash \{k\}$ and $\overline{\mathcal{K}}_\mathcal{D} := \{-1, 0, 1, \dots, k-1\} \backslash \mathcal{K}_\mathcal{D}$ in order to define the *leaping set*

$$\mathcal{L}_\mathcal{D} := \Big\{k - \ell : 0 \le \ell \le k-1, (\ell - 1, \ell) \in \big(\widetilde{\mathcal{K}}_\mathcal{D} \times \overline{\mathcal{K}}_\mathcal{D}\big) \cup \big(\overline{\mathcal{K}}_\mathcal{D} \times \widetilde{\mathcal{K}}_\mathcal{D}\big)\Big\} \cup \{0\} \tag{4}$$

of $\mathcal{D}$. The results of the following proposition, all of them well known, will be used to prove Theorem 3.1 below.

Proposition 3.1 *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$ a divisor set with exponent set $\mathcal{K}_\mathcal{D}$ and leaping set $\mathcal{L}_\mathcal{D}$.*

(i) (= [22, Corollary 2.4(i)]) *The (dominating) eigenvalue $\lambda_1(p^k, \mathcal{D})$ of* $\mathrm{ICG}(p^k, \mathcal{D})$ *satisfies*

$$\lambda_1(p^k, \mathcal{D}) = \begin{cases} +1 & \text{if } \ k \in \mathcal{K}_\mathcal{D} \ \text{ and } \ k-1 \notin \mathcal{K}_\mathcal{D}, \\ -1 & \text{if } \ k \notin \mathcal{K}_\mathcal{D} \ \text{ and } \ k-1 \in \mathcal{K}_\mathcal{D}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) (= [20, Proposition 3.2]) *For* $0 < \ell \leq k$ *we have*

$$\lambda_{p^\ell}(p^k, \mathcal{D}) - \lambda_{p^{\ell-1}}(p^k, \mathcal{D}) = \begin{cases} p^\ell & \text{if } k-\ell \in \mathcal{K}_\mathcal{D} \text{ and } k-\ell-1 \notin \mathcal{K}_\mathcal{D}, \\ -p^\ell & \text{if } k-\ell \notin \mathcal{K}_\mathcal{D} \text{ and } k-\ell-1 \in \mathcal{K}_\mathcal{D}, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) (= [21, Lemma 3.2]) *Let* $\mathcal{L}_\mathcal{D} = \{\ell_0, \ell_1, \ell_2, \ldots, \ell_m\}$, *where* $0 = \ell_0 < \ell_1 < \cdots < \ell_m$, *say. Setting* $\ell_{m+1} := k+1$, *we have for* $0 \leq j \leq m$ *that*

$$\lambda_{p^{\ell_j}}(p^k, \mathcal{D}) = \lambda_{p^{\ell_j+1}}(p^k, \mathcal{D}) = \lambda_{p^{\ell_j+2}}(p^k, \mathcal{D}) = \cdots = \lambda_{p^{\ell_{j+1}-1}}(p^k, \mathcal{D}).$$

(iv) (= [21, Theorem 4.1]) *The eigenvalues* $\lambda_{p^\ell}(p^k, \mathcal{D}) \in \mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$ *with* $\ell \in \mathcal{L}_\mathcal{D}$ *are pairwise distinct and satisfy* $\{\lambda_{p^\ell}(p^k, \mathcal{D}) : \ell \in \mathcal{L}_\mathcal{D}\} = \mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$.

We set $\sigma(n, \mathcal{D}) := |\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D}))|$ for any positive integer n and any divisor set $\mathcal{D} \subseteq D(n)$. To label integer intervals we use the notation $[a \cdots b] := \{a, a+1, a+2, \ldots, b\}$ for any integers $a \leq b$. Given a divisor set $\mathcal{D} = \{p^{k_1}, \ldots, p^{k_m}\} \subseteq D(p^k)$ with $0 \leq k_1 < k_2 < \cdots < k_m \leq k$, we finally introduce the notion of *gaps* in the exponent set $\mathcal{K}_\mathcal{D} := \{k_1, \ldots, k_m\}$ of $\mathcal{D}$. A *gap* in $\mathcal{K}_\mathcal{D}$ is an integer interval $[a \cdots b]$ satisfying the following conditions:

$$(i)\ 1 \leq a \leq b \leq k-1, \quad (ii)\ [a \cdots b] \cap \mathcal{K}_\mathcal{D} = \emptyset,$$
$$(iii)\ a-1 \in \mathcal{K}_\mathcal{D}, \quad (iv)\ b+1 \in \mathcal{K}_\mathcal{D} \text{ or } b = k-1.$$

The *gap number* $\gamma(p^k, \mathcal{D})$ is defined as the number of gaps in $\mathcal{K}_\mathcal{D}$. By looking at exponent sets with gaps $[1 \cdots 1], [3 \cdots 3], [5 \cdots 5], \ldots$ we clearly have $0 \leq \gamma(p^k, \mathcal{D}) \leq \frac{k}{2}$.

The main result of the subsequent theorem is that $\sigma(p^k, \mathcal{D})$ roughly equals $2 \cdot \gamma(p^k, \mathcal{D})$.

Theorem 3.1 *Let* p^k *be a prime power, and let* $\mathcal{D} \subseteq D(p^k)$ *be a divisor set of* p^k.

(i) *Then* $\sigma(p^k, \mathcal{D}) = 2\big(\gamma(p^k, \mathcal{D}) + 1\big) - \chi(p^k, \mathcal{D})$, *where*
$$\chi(p^k, \mathcal{D}) := \begin{cases} 0 & \text{if } k-1 \in \mathcal{K}_\mathcal{D}, \\ 1 & \text{if } k-1 \notin \mathcal{K}_\mathcal{D}. \end{cases}$$

(ii) *Let* $[a_i \cdots b_i - 1]$, $1 \leq i \leq \gamma(p^k, \mathcal{D})$, *be the gaps in* $\mathcal{K}_\mathcal{D}$ *(including the case* $\gamma(p^k, \mathcal{D}) = 0$ *in which no gaps exist). With* $b_0 := \min \mathcal{K}_\mathcal{D}$ *we define* $G_0 := [0 \cdots b_0 - 1]$ *if* $0 \notin \mathcal{K}_\mathcal{D}$, $G_0 := \emptyset$ *otherwise, and* $G_i := [a_i \cdots b_i - 1]$, $1 \leq i \leq \gamma(p^k, \mathcal{D})$. *Then we have*

$$\mathcal{K}_\mathcal{D} = \begin{cases} [0 \cdots k] \setminus \bigcup_{i=0}^{\gamma(p^k, \mathcal{D})} G_i & \text{if } k \in \mathcal{K}_\mathcal{D}, \\ [0 \cdots k-1] \setminus \bigcup_{i=0}^{\gamma(p^k, \mathcal{D})} G_i & \text{if } k \notin \mathcal{K}_\mathcal{D}. \end{cases} \tag{5}$$

Setting $\gamma^*(p^k, \mathcal{D}) := \gamma(p^k, \mathcal{D}) - \chi(p^k, \mathcal{D})$ *and* $a_{\gamma(p^k,\mathcal{D})+1} := k$*, the leaping set has the following characterisation in terms of the gaps:*

$$\mathcal{L}_{\mathcal{D}} = \{k - a_i : 1 \le i \le \gamma(p^k, \mathcal{D}) + 1\} \cup \{k - b_i : 0 \le i \le \gamma^*(p^k, \mathcal{D})\}. \quad (6)$$

The spectrum of $\mathrm{ICG}(p^k, \mathcal{D})$ *is given by the* $(\gamma^*(p^k, \mathcal{D}) + 1) + (\gamma(p^k, \mathcal{D}) + 1) = \sigma(p^k, \mathcal{D})$ *pairwise distinct integers*

$$\begin{aligned} &\left(\sum_{i=j}^{\gamma^*(p^k,\mathcal{D})} p^{k-b_i} - \sum_{i=j+1}^{\gamma(p^k,\mathcal{D})} p^{k-a_i} + \lambda_1(p^k, \mathcal{D}) \right) \quad (0 \le j \le \gamma^*(p^k, \mathcal{D})), \\ -&\left(\sum_{i=j}^{\gamma(p^k,\mathcal{D})} p^{k-a_i} - \sum_{i=j}^{\gamma^*(p^k,\mathcal{D})} p^{k-b_i} - \lambda_1(p^k, \mathcal{D}) \right) \quad (1 \le j \le \gamma(p^k, \mathcal{D}) + 1), \end{aligned} \quad (7)$$

where $\lambda_1(p^k, \mathcal{D})$ *was determined in Proposition* 3.1(i). *Notice that sums with an empty range of summation are to be understood as* 0 *and sets with unaccomplishable conditions equal* $\emptyset$*.*

Proof The characterisation (5) of the exponent set in terms of the gaps is a straightforward consequence of the definition of the gaps, depending only on whether $k \in \mathcal{K}_{\mathcal{D}}$ or $k \notin \mathcal{K}_{\mathcal{D}}$. Observe that the definition of G_0 takes care of a possible initial "gap" in $\mathcal{K}_{\mathcal{D}}$, i.e. if $0 \notin \mathcal{K}_{\mathcal{D}}$.

According to Proposition 3.1(iv) the distinct eigenvalues in $\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$ are characterised by the pairs $(\ell - 1, \ell) \in \left(\widetilde{\mathcal{K}}_{\mathcal{D}} \times \overline{\mathcal{K}}_{\mathcal{D}}\right) \cup \left(\overline{\mathcal{K}}_{\mathcal{D}} \times \widetilde{\mathcal{K}}_{\mathcal{D}}\right)$ in the range $0 \le \ell \le k - 1$. In our setting, these are

- (B_0) the pair $(b_0 - 1, b_0)$ (even for $b_0 = 0$, i.e. $0 \in \widetilde{\mathcal{K}}_{\mathcal{D}}$, since $-1 \in \overline{\mathcal{K}}_{\mathcal{D}}$ by definition),
- (A) all pairs $(a_i - 1, a_i)$ for $1 \le i \le \gamma(p^k, \mathcal{D})$,
- (B) all pairs $(b_i - 1, b_i)$ for $1 \le i \le \gamma^*(p^k, \mathcal{D}) = \gamma(p^k, \mathcal{D}) - \chi(p^k, \mathcal{D})$. The latter reduction of the range of eligible pairs results from the fact that the pair $(b_{\gamma(p^k,\mathcal{D})} - 1, b_{\gamma(p^k,\mathcal{D})})$ is out of range in $\mathcal{L}_{\mathcal{D}}$ if $b_{\gamma(p^k,\mathcal{D})} = k$ (cf. (4)), i.e. if $k - 1 \notin \mathcal{K}_{\mathcal{D}}$.

Taking account of all suitable pairs and—according to definition (4) of $\mathcal{L}_{\mathcal{D}}$—subjoining $0 = k - a_{\gamma(p^k,\mathcal{D})+1}$, we obtain the leaping set

$$\begin{aligned} \mathcal{L}_{\mathcal{D}} = \{k - b_0\} &\cup \{k - a_i : 1 \le i \le \gamma(p^k, \mathcal{D})\} \cup \{k - b_i : 1 \le i \le \gamma^*(p^k, \mathcal{D})\} \\ &\cup \{0\}, \end{aligned}$$

which confirms (6). Consequently Proposition 3.1(iv) implies

$$\begin{aligned} \sigma(p^k, \mathcal{D}) = |\mathcal{L}_{\mathcal{D}}| &= 1 + \gamma(p^k, \mathcal{D}) + \gamma^*(p^k, \mathcal{D}) + 1 \\ &= 2 \cdot \gamma(p^k, \mathcal{D}) - \chi(p^k, \mathcal{D}) + 2, \end{aligned}$$

which verifies (i).

Up to this point (i), (5) and (6) have been confirmed, and it remains to show (7). Let us start with the special cases $\sigma(p^k, \mathcal{D}) = |\mathcal{L}_{\mathcal{D}}| \in \{1, 2\}$, i.e. the situation where $\mathcal{K}_{\mathcal{D}}$ has no gap according to (i).

For $\sigma(p^k, \mathcal{D}) = 1$ formula (i) yields $\gamma(p^k, \mathcal{D}) = 0$ and $\chi(p^k, \mathcal{D}) = 1$, i.e. $k - 1 \notin \mathcal{K}_{\mathcal{D}}$, which necessarily implies $0, 1, \ldots, k-2 \notin \mathcal{K}_{\mathcal{D}}$, because otherwise $\mathcal{K}_{\mathcal{D}}$ would contain a gap. Therefore, $G_0 = [0 \cdots b_0{-}1] = [0 \cdots k{-}1]$, hence $\mathcal{D} = \{p^k\}$. Finally, Proposition 3.1(iv) implies that $\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D})) = \{\lambda_{p^0}(p^k, \mathcal{D})\} = \{\lambda_1(p^k, \mathcal{D})\}$, confirming (7) for $\sigma(p^k, \mathcal{D}) = 1$.

In case $\sigma(p^k, \mathcal{D}) = 2$ it similarly follows from (i) that $\gamma(p^k, \mathcal{D}) = 0$ and $k - 1 \in \mathcal{K}_{\mathcal{D}}$, thus $\gamma^*(p^k, \mathcal{D}) = 0$ and $b_0 \le k - 1$. Hence (6) implies that $\mathcal{L}_{\mathcal{D}} = \{k - a_1, k - b_0\} = \{0, k - b_0\}$, thus $\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D})) = \{\lambda_1, \lambda_{p^{k-b_0}}\}$ by Proposition 3.1(iv). Since $\mathcal{K}_{\mathcal{D}}$ has no gaps, we necessarily have $\mathcal{K}_{\mathcal{D}} = [b_0 \cdots k]$ or $\mathcal{K}_{\mathcal{D}} = [b_0 \cdots k{-}1]$, in any case $b_0, b_0 + 1, \ldots, k - 1 \in \mathcal{K}_{\mathcal{D}}$, but $b_0 - 1 \notin \mathcal{K}_{\mathcal{D}}$. Therefore we obtain from Proposition 3.1(ii) that $\lambda_{p^{k-\ell}} = \lambda_{p^{k-\ell-1}}$ for $b_0{+}1 \le \ell \le k{-}1$ and $\lambda_{p^{k-b_0}} - \lambda_{p^{k-b_0-1}} = p^{k-b_0}$. This yields

$$\begin{aligned}\lambda_{p^{k-b_0}} - \lambda_1 = \lambda_{p^{k-b_0}} - \lambda_{p^0} &= \sum_{\ell=b_0}^{k-1} \left(\lambda_{p^{k-\ell}} - \lambda_{p^{k-\ell-1}}\right) \\ &= \lambda_{p^{k-b_0}} - \lambda_{p^{k-b_0-1}} = p^{k-b_0},\end{aligned}$$

which completes the verification of (7) for $\sigma(p^k, \mathcal{D}) = 2$.

Therefore, we may assume that $\sigma(p^k, \mathcal{D}) \ge 3$, whence $\gamma(p^k, \mathcal{D}) \ge 1$ according to (i), and thus $b_0 < k$. Without loss of generality we may assume that the gaps G_i in $\mathcal{K}_{\mathcal{D}}$ are indexed in such a way that $a_1 < a_2 < \cdots < a_{\gamma(p^k,\mathcal{D})}$, hence

$$\begin{aligned}0 \le b_0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{\gamma(p^k,\mathcal{D})-1} < b_{\gamma(p^k,\mathcal{D})-1} \\ < a_{\gamma(p^k,\mathcal{D})} \Big[< b_{\gamma(p^k,\mathcal{D})} \Big] \le k,\end{aligned}$$

where $b_{\gamma(p^k,\mathcal{D})}$ occurs if and only if $k - 1 \in \mathcal{K}_{\mathcal{D}}$ (cf. (B) above). By (6) and Proposition 3.1(iv), and abbreviating $\lambda_{p^j} := \lambda_{p^j}(p^k, \mathcal{D})$, this implies that

$$\begin{aligned}\lambda_{p^{k-a_{\gamma(p^k,\mathcal{D})+1}}}, \Big[\lambda_{p^{k-b_{\gamma(p^k,\mathcal{D})}}}, \Big] \lambda_{p^{k-a_{\gamma(p^k,\mathcal{D})}}}, \lambda_{p^{k-b_{\gamma(p^k,\mathcal{D})-1}}}, \lambda_{p^{k-a_{\gamma(p^k,\mathcal{D})-1}}}, \ldots \\ \ldots, \lambda_{p^{k-b_2}}, \lambda_{p^{k-a_2}}, \lambda_{p^{k-b_1}}, \lambda_{p^{k-a_1}}, \lambda_{p^{k-b_0}}\end{aligned} \tag{8}$$

is a complete collection of the $2\gamma(p^k, \mathcal{D}) + 2 - \chi(p^k, \mathcal{D})$ distinct eigenvalues of $\mathrm{ICG}(p^k, \mathcal{D})$, arranged according to increasing indices, i.e. increasing exponents of p.

Let $\lambda_{p^{k-\ell'}}, \lambda_{p^{k-\ell}}$ with $\ell < \ell'$ be any pair of successive eigenvalues in (8). According to Proposition 3.1(iii) we have

$$\lambda_{p^{k-\ell'}} = \lambda_{p^{k-\ell'+1}} = \lambda_{p^{k-\ell'+2}} = \cdots = \lambda_{p^{k-\ell-1}}.$$

Therefore Proposition 3.1(ii) implies

$$\lambda_{p^{k-\ell}} - \lambda_{p^{k-\ell'}} = \sum_{j=\ell}^{\ell'-1} \left(\lambda_{p^{k-j}} - \lambda_{p^{k-j-1}}\right) = \lambda_{p^{k-\ell}} - \lambda_{p^{k-\ell-1}}$$
$$= \begin{cases} p^{k-\ell} & \text{if} \quad \ell \in \mathcal{K}_{\mathcal{D}} \quad \text{and} \quad \ell - 1 \notin \mathcal{K}_{\mathcal{D}}, \\ -p^{k-\ell} & \text{if} \quad \ell \notin \mathcal{K}_{\mathcal{D}} \quad \text{and} \quad \ell - 1 \in \mathcal{K}_{\mathcal{D}}. \end{cases}$$

Since $a_i \notin \mathcal{K}_{\mathcal{D}}$ and $a_i - 1 \in \mathcal{K}_{\mathcal{D}}$ for $1 \leq i \leq \gamma(p^k, \mathcal{D})$, while $b_i \in \mathcal{K}_{\mathcal{D}}$ and $b_i - 1 \notin \mathcal{K}_{\mathcal{D}}$ for $0 \leq i \leq \gamma(p^k, \mathcal{D}) - 1$, we thus obtain

$$\lambda_{p^{k-\ell}} - \lambda_{p^{k-\ell'}} = \begin{cases} -p^{k-a_i} & \text{for} \quad \ell = a_i, \quad \ell' = b_i (1 \leq i \leq \gamma(p^k, \mathcal{D})) \\ & \text{and for} \quad \ell = a_{\gamma(p^k,\mathcal{D})}, \quad \ell' = a_{\gamma(p^k,\mathcal{D})+1}, \\ p^{k-b_i} & \text{for} \quad \ell = b_i, \quad \ell' = a_{i+1} (0 \leq i \leq \gamma(p^k, \mathcal{D}) - 1). \end{cases}$$

By using these identities inductively, collection (8) can be written as

$$\begin{aligned} &\lambda_1, -p^{k-a_{\gamma(p^k,\mathcal{D})}} + \lambda_1, p^{k-b_{\gamma(p^k,\mathcal{D})-1}} - p^{k-a_{\gamma(p^k,\mathcal{D})}} + \lambda_1, \\ &- p^{k-a_{\gamma(p^k,\mathcal{D})-1}} + p^{k-b_{\gamma(p^k,\mathcal{D})-1}} - p^{k-a_{\gamma(p^k,\mathcal{D})}} + \lambda_1, \ldots \\ &\quad \ldots, p^{k-b_0} - p^{k-a_1} \pm \ldots + p^{k-b_{\gamma(p^k,\mathcal{D})-1}} - p^{k-a_{\gamma(p^k,\mathcal{D})}} + \lambda_1 \end{aligned}$$

in case $k - 1 \notin \mathcal{K}_{\mathcal{D}}$, and

$$\begin{aligned} &\lambda_1, p^{k-b_{\gamma(p^k,\mathcal{D})}} + \lambda_1, -p^{k-a_{\gamma(p^k,\mathcal{D})}} + p^{k-b_{\gamma(p^k,\mathcal{D})}} + \lambda_1, \\ &p^{k-b_{\gamma(p^k,\mathcal{D})-1}} - p^{k-a_{\gamma(p^k,\mathcal{D})}} + p^{k-b_{\gamma(p^k,\mathcal{D})}} + \lambda_1, \ldots \\ &\quad \ldots, p^{k-b_0} - p^{k-a_1} \pm \ldots + p^{k-b_{\gamma(p^k,\mathcal{D})-1}} - p^{k-a_{\gamma(p^k,\mathcal{D})}} + p^{k-b_{\gamma(p^k,\mathcal{D})}} + \lambda_1 \end{aligned}$$

in case $k - 1 \in \mathcal{K}_{\mathcal{D}}$. This, finally, proves (7) and completes the proof of the theorem. $\square$

The following result exemplifies the use of Theorem 3.1 in order to determine all ICG$(p^k, \mathcal{D})$ satisfying $\sigma(p^k, \mathcal{D}) = s$ for any given s.

Corollary 3.1 *The values of $\gamma(p^k, \mathcal{D})$ and the leaping set $\mathcal{L}_{\mathcal{D}}$, the exponent set $\mathcal{K}_{\mathcal{D}}$ and the spectrum* Spec(ICG$(p^k, \mathcal{D})$) *corresponding to integral circulant graphs* ICG$(p^k, \mathcal{D})$ *with $1 \leq \sigma(p^k, \mathcal{D}) \leq 4$ are given in Table 1.*

Proof All entries for $\sigma(p^k, \mathcal{D}) = 1$ and $\sigma(p^k, \mathcal{D}) = 2$ in Table 1 have explicitly been discussed in the proof of (7) of Theorem 3.1, and the required values of $\lambda_1(p^k, \mathcal{D})$ can be found in Proposition 3.1(i). It remains to check the cases $\sigma(p^k, \mathcal{D}) \geq 3$.

First of all we observe that $k \in \mathcal{K}_{\mathcal{D}}$ versus $k \notin \mathcal{K}_{\mathcal{D}}$ has an effect on the value of $\lambda_1(p^k, \mathcal{D})$ (cf. Proposition 3.1(i)) and consequently on all eigenvalues (see (7)), but does not influence the leaping set $\mathcal{L}_{\mathcal{D}}$ (cf. its definition in (4)) nor the set of

Table 1 All integral circulant graphs $\mathrm{ICG}(p^k, \mathcal{D})$ with at most four distinct eigenvalues

$\sigma(p^k, \mathcal{D})$	$\gamma(p^k, \mathcal{D})$	$k-1 \in \mathcal{K}_{\mathcal{D}}$?	$\mathcal{L}_{\mathcal{D}}$	Type	$\mathcal{K}_{\mathcal{D}}$	$\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$
1	0	$\notin$	$\{0\}$	A	$\{k\}$	$\{1\}$
2	0	$\in$	$\{0, k-u\}$,	B1	$[u \cdots k-1]$	$\{-1, p^{k-u}-1\}$
			$0 \leq u < k$	B2	$[u \cdots k]$	$\{0, p^{k-u}\}$
3	1	$\notin$	$\{0, k-v, k-u\}$,	C1	$[u \cdots v-1]$	$\{0, -p^{k-v}, p^{k-u}-p^{k-v}\}$
			$0 \leq u < v < k$	C2	$[u \cdots v-1] \cup \{k\}$	$\{1, -p^{k-v}+1, p^{k-u}-p^{k-v}+1\}$
4	1	$\in$	$\{0, k-w, k-v, k-u\}$,	D1	$[u \cdots v-1] \cup [w \cdots k-1]$	$\{-1, p^{k-w}-1, -p^{k-v}+p^{k-w}-1,$
			$0 \leq u < v < w < k$			$p^{k-u}-p^{k-v}+p^{k-w}-1\}$
				D2	$[u \cdots v-1] \cup [w \cdots k]$	$\{0, p^{k-w}, -p^{k-v}+p^{k-w},$
						$p^{k-u}-p^{k-v}+p^{k-w}\}$

gaps. In other words, $\sigma(p^k, \mathcal{D}) = \sigma(p^k, \mathcal{D} \cup \{p^k\})$ and $\sigma(p^k, \mathcal{D}) = \sigma(p^k, \mathcal{D}\backslash\{p^k\})$, respectively.

For $\sigma(p^k, \mathcal{D}) = 3$, we obtain $\gamma(p^k, \mathcal{D}) = 1$ and $\chi(p^k, \mathcal{D}) = 1$ by Theorem 3.1(i), hence $k-1 \notin \mathcal{K}_{\mathcal{D}}$. Since $k-1 \notin \mathcal{K}_{\mathcal{D}}$ and $\mathcal{K}_{\mathcal{D}}$ has precisely one gap, we necessarily have $\mathcal{K}_{\mathcal{D}} = [u \cdots v{-}1]$ or $\mathcal{K}_{\mathcal{D}} = [u \cdots v{-}1] \cup \{k\}$ for some integers $0 \le u < v < k$ by (5). The corresponding sets $\mathcal{L}_{\mathcal{D}}$ and $\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$, as given in Table 1, follow from (6) and (7), respectively.

For $\sigma(p^k, \mathcal{D}) = 4$, Theorem 3.1(i) implies $\gamma(p^k, \mathcal{D}) = 1$ and $\chi(p^k, \mathcal{D}) = 0$, hence $k-1 \in \mathcal{K}_{\mathcal{D}}$. According to (5) these facts require $\mathcal{K}_{\mathcal{D}} = [u \cdots v{-}1] \cup [w \cdots k{-}1]$ or $\mathcal{K}_{\mathcal{D}} = [u \cdots v{-}1] \cup [w \cdots k]$ for some integers $0 \le u < v < w < k$. Then $\mathcal{L}_{\mathcal{D}}$ and $\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$ can again be determined by (6) and (7), respectively. □

4 ICG($n, \mathcal{D}$) with multiplicative $\mathcal{D}$ and its spectrum

All integral circulant graphs considered in the preceding section had prime power order. Contrary to the spectral vectors of such graphs, $\vec{\lambda}(n, \mathcal{D})$ for arbitrary composite n is not well understood. Therefore the idea to consider integral circulant graphs $\mathrm{ICG}(n, \mathcal{D})$ with arbitrary n, but *multiplicative divisor sets* $\mathcal{D}$ was introduced and applied by Le and the first author in [16, 17], and again used by the first author in [19]. For a positive integer d and a number p in the set $\mathbb{P}$ of all primes, we denote by $e_p(d)$ the order of p in d. A non-empty finite set $\mathcal{D}$ of positive integers—as for instance a divisor set—is called *multiplicative* if $\mathcal{D} = \prod_{p \in \mathbb{P}} \mathcal{D}_p$, where $\mathcal{D}_p := \{p^{e_p(d)} : d \in \mathcal{D}\}$ for each prime p, and the product of sets $D_1, \ldots, D_t$ of positive integers is defined as $\prod_{i=1}^t D_i := \{d_1 \cdot \ldots \cdot d_t : d_i \in D_i\}$.

The following formula to determine $\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D}))$ for multiplicative divisor sets $\mathcal{D}$ is a consequence of Proposition 3.1(iv).

Proposition 4.1 (= [21, Theorem 5.1(i)]) *Let $n > 1$ be an integer with prime factorisation $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$. For $1 \le i \le r$ let $\mathcal{D}_i \subseteq D(p_i^{k_i})$ be a divisor set with leaping set $\mathcal{L}_{\mathcal{D}_i}$, and define the multiplicative divisor set $\mathcal{D} := \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r \subseteq D(n)$. Then*

$$\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D})) = \prod_{i=1}^{r} \mathrm{Spec}\left(\mathrm{ICG}(p_i^{k_i}, \mathcal{D}_i)\right) = \prod_{i=1}^{r} \left\{ \lambda_{p_i^{\ell_i}} \left(p_i^{k_i}, \mathcal{D}_i\right) : \ell_i \in \mathcal{L}_{\mathcal{D}_i} \right\}. \tag{9}$$

We continue with a useful observation, namely a kind of monotonicity property of the function $\sigma(n, \mathcal{D})$ with respect to the number $\omega(n)$ of distinct prime factors of n.

Lemma 4.1 *Let $n > 1$ have prime factorisation $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$, and let $\mathcal{D}_i \subseteq D(p_i^{k_i})$, $1 \le i \le r$, and $\mathcal{D} := \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r \subseteq D(n)$ be divisor sets.*

(i) *If $\emptyset \ne \{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, r\}$, then $\sigma(n', \mathcal{D}') \le \sigma(n, \mathcal{D})$ for $n' := p_{i_1}^{k_{i_1}} \cdot \ldots \cdot p_{i_m}^{k_{i_m}}$ and $\mathcal{D}' := \mathcal{D}_{i_1} \cdot \ldots \cdot \mathcal{D}_{i_m} \subseteq D(n')$.*

(ii) *If $\sigma(n, \mathcal{D}) \le s$ for some positive integer s, then $\sigma(p_i^{k_i}, \mathcal{D}_i) \le s$ for $1 \le i \le r$.*

Proof The assertion in (ii) is an immediate corollary to (i). In order to verify (i), let $\lambda_1, \ldots, \lambda_{\sigma'}$ be the $\sigma' := \sigma(n', \mathcal{D}')$ distinct eigenvalues of $\mathrm{ICG}(n', \mathcal{D}')$. If $\sigma(p_j^{k_j}, \mathcal{D}_j) = 1$ for some $j \in \{1, 2, \ldots, r\} \backslash \{i_1, i_2, \ldots, i_m\}$, the single eigenvalue of $\mathrm{ICG}(p_j^{k_j}, \mathcal{D}_j)$ is 1 by Corollary 3.1, and if $\sigma(p_j^{k_j}, \mathcal{D}_j) \geq 2$, there is a nonzero eigenvalue anyway. Thus $\mathrm{ICG}(p_j^{k_j}, \mathcal{D}_j)$ always has an eigenvalue $\lambda^* \neq 0$. Consequently, $\lambda_1 \cdot \lambda^*, \ldots, \lambda_{\sigma'} \cdot \lambda^*$ apparently are pairwise distinct and all lie in $\mathrm{Spec}(\mathrm{ICG}(n' \cdot p_j^{k_j}, \mathcal{D}' \cdot \mathcal{D}_j))$ by Proposition 4.1. This shows that $\sigma(n', \mathcal{D}') \leq \sigma(n' \cdot p_j^{k_j}, \mathcal{D}' \cdot \mathcal{D}_j)$, and iterating the argument proves (i). $\square$

We shall say that a divisor set $\mathcal{D} \subseteq D(p^k)$ for a prime power p^k *is of type* $\mathbf{X} \in \{\mathbf{A}, \mathbf{B1}, \mathbf{B2}, \mathbf{C1}, \mathbf{C2}, \mathbf{D1}, \mathbf{D2}\}$ if it has the corresponding form in Table 1 of Corollary 3.1 (with admissible integers $0 \leq u < k$ if $\mathbf{X} \in \{\mathbf{B1}, \mathbf{B2}\}$, or $0 \leq u < v < k$ if $\mathbf{X} \in \{\mathbf{C1}, \mathbf{C2}\}$, or $0 \leq u < v < w < k$ if $\mathbf{X} \in \{\mathbf{D1}, \mathbf{D2}\}$, respectively). Given prime powers $p_i^{k_i}$ with distinct p_i and $\mathbf{X}_i \in \{\mathbf{A}, \mathbf{B1}, \mathbf{B2}, \mathbf{C1}, \mathbf{C2}, \mathbf{D1}, \mathbf{D2}\}$ for $1 \leq i \leq r$, we call $\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r$ the *factorisation pattern* of the—apparently multiplicative—divisor set $\mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r$ if $\mathcal{D}_i \subseteq D(p_i^{k_i})$ is of type $\mathbf{X}_i$ for $1 \leq i \leq r$. Moreover, we define

$$\begin{aligned}
\underline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r} := \min \Big\{ & \sigma\left(p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}, \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r\right) : \\
& \mathcal{D}_i \subseteq D\left(p_i^{k_i}\right) \text{ is of type } \mathbf{X}_i \, (1 \leq i \leq r) \Big\}, \\
\overline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r} := \max \Big\{ & \sigma\left(p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}, \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r\right) : \\
& \mathcal{D}_i \subseteq D\left(p_i^{k_i}\right) \text{ is of type } \mathbf{X}_i \, (1 \leq i \leq r) \Big\}.
\end{aligned}$$

Only if $\underline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r} = \overline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}$, we shall write $\sigma_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}$ for the coinciding values of minimum and maximum. This is the case when $\sigma(p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}, \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r)$ has the same value for all $\mathcal{D}_i \subseteq D(p_i^{k_i})$ of type $\mathbf{X}_i$ $(1 \leq i \leq r)$, i.e. it depends only on the types $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r$, but does not depend on the special values of p_i or k_i or any of the other parameters involved. Trivially, the order of the $\mathbf{X}_i$ in a factorisation pattern $\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r$ influences neither $\underline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}$ nor $\overline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}$ nor $\sigma_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}$.

The following proposition analyses the sizes of spectra of integral circulant graphs whose multiplicative divisor sets have factorisation patterns composed of divisor sets of the types occurring in Table 1.

Proposition 4.2 *Given any positive integer r, let $p_1, p_2, \ldots, p_r$ be distinct primes and let $k_1, k_2, \ldots, k_r$ be positive integers. For each $i \in [1 \cdots r]$ and $\mathbf{X}_i \in \{\mathbf{A}, \mathbf{B1}, \mathbf{B2}, \mathbf{C1}, \mathbf{C2}, \mathbf{D1}, \mathbf{D2}\}$ let $\mathcal{D}_i \subseteq D(p_i^{k_i})$ be a divisor set of type $\mathbf{X}_i$. For any positive integer m and $\mathbf{X} \in \{\mathbf{A}, \mathbf{B1}, \mathbf{B2}\}$ let $\mathbf{X}^m := \underbrace{\mathbf{X} \cdot \mathbf{X} \cdot \ldots \cdot \mathbf{X}}_{m-fold}$. Then we have*

(i) *$\sigma_{\mathbf{A}^m} = 1$ and $\underline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r \cdot \mathbf{A}^m} = \underline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}$; in particular $\sigma_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r \cdot \mathbf{A}^m} = \sigma_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}$ in case $\underline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r} = \overline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}$;*

(ii) *$\sigma_{\mathbf{B1}^2} = 4$, $\underline{\sigma}_{\mathbf{B1}^3} \geq 5$, $\underline{\sigma}_{\mathbf{B1}^2 \cdot \mathbf{B2}} \geq 5$, $\sigma_{\mathbf{B2}^m} = 2$;*

Table 2 $\underline{\sigma}_{\mathbf{X}\cdot\mathbf{Y}}$ or $\sigma_{\mathbf{X}\cdot\mathbf{Y}}$, respectively, for all $\mathbf{X}, \mathbf{Y} \in \{\mathbf{B1}, \mathbf{B2}, \mathbf{C1}, \mathbf{C2}, \mathbf{D1}, \mathbf{D2}\}$

	B1	B2	C1	C2	D1	D2
B1	$\sigma_{\mathbf{B1}^2} = 4$	$\sigma_{\mathbf{B1}\cdot\mathbf{B2}} = 3$	$\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{C1}} \geq 5$	$\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{C2}} \geq 5$	$\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{D1}} \geq 5$	$\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{D2}} \geq 5$
B2	$\sigma_{\mathbf{B2}\cdot\mathbf{B1}} = 3$	$\sigma_{\mathbf{B2}^2} = 2$	$\sigma_{\mathbf{B2}\cdot\mathbf{C1}} = 3$	$\sigma_{\mathbf{B2}\cdot\mathbf{C2}} = 4$	$\underline{\sigma}_{\mathbf{B2}\cdot\mathbf{D1}} \geq 5$	$\sigma_{\mathbf{B2}\cdot\mathbf{D2}} = 4$
C1	$\underline{\sigma}_{\mathbf{C1}\cdot\mathbf{B1}} \geq 5$	$\sigma_{\mathbf{C1}\cdot\mathbf{B2}} = 3$	$\underline{\sigma}_{\mathbf{C1}^2} \geq 5$	$\underline{\sigma}_{\mathbf{C1}\cdot\mathbf{C2}} \geq 5$	$\underline{\sigma}_{\mathbf{C1}\cdot\mathbf{D1}} \geq 5$	$\underline{\sigma}_{\mathbf{C1}\cdot\mathbf{D2}} \geq 5$
C2	$\underline{\sigma}_{\mathbf{C2}\cdot\mathbf{B1}} \geq 5$	$\sigma_{\mathbf{C2}\cdot\mathbf{B2}} = 4$	$\underline{\sigma}_{\mathbf{C2}\cdot\mathbf{C1}} \geq 5$	$\underline{\sigma}_{\mathbf{C2}^2} \geq 5$	$\underline{\sigma}_{\mathbf{C2}\cdot\mathbf{D1}} \geq 5$	$\underline{\sigma}_{\mathbf{C2}\cdot\mathbf{D2}} \geq 5$
D1	$\underline{\sigma}_{\mathbf{D1}\cdot\mathbf{B1}} \geq 5$	$\underline{\sigma}_{\mathbf{D1}\cdot\mathbf{B2}} \geq 5$	$\underline{\sigma}_{\mathbf{D1}\cdot\mathbf{C1}} \geq 5$	$\underline{\sigma}_{\mathbf{D1}\cdot\mathbf{C2}} \geq 5$	$\underline{\sigma}_{\mathbf{D1}^2} \geq 5$	$\underline{\sigma}_{\mathbf{D1}\cdot\mathbf{D2}} \geq 5$
D2	$\underline{\sigma}_{\mathbf{D2}\cdot\mathbf{B1}} \geq 5$	$\sigma_{\mathbf{D2}\cdot\mathbf{B2}} = 4$	$\underline{\sigma}_{\mathbf{D2}\cdot\mathbf{C1}} \geq 5$	$\underline{\sigma}_{\mathbf{D2}\cdot\mathbf{C2}} \geq 5$	$\underline{\sigma}_{\mathbf{D2}\cdot\mathbf{D1}} \geq 5$	$\underline{\sigma}_{\mathbf{D2}^2} \geq 5$

(iii) $\underline{\sigma}_{\mathbf{X}\cdot\mathbf{Y}}$ *or, if defined,* $\sigma_{\mathbf{X}\cdot\mathbf{Y}}$*, respectively, for all* $\mathbf{X}, \mathbf{Y} \in \{\mathbf{B1}, \mathbf{B2}, \mathbf{C1}, \mathbf{C2}, \mathbf{D1}, \mathbf{D2}\}$ *as given in Table 2;*

(iv) $\sigma_{\mathbf{B1}\cdot\mathbf{B2}^m} = \sigma_{\mathbf{B2}^m\cdot\mathbf{C1}} = 3$, $\sigma_{\mathbf{B2}^m\cdot\mathbf{C2}} = \sigma_{\mathbf{B2}^m\cdot\mathbf{D2}} = 4$.

Proof For given divisor sets $\mathcal{D}_i \subseteq D(p_i^{k_i})$ of type $\mathbf{X}_i$, $1 \leq i \leq r$, $\mathcal{D} := \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r$ is a multiplicative divisor set of $n := p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$. Therefore

$$\operatorname{Spec}(\operatorname{ICG}(n, \mathcal{D})) = \prod_{i=1}^{r} \operatorname{Spec}\left(\operatorname{ICG}(p_i^{k_i}, \mathcal{D}_i)\right) \tag{10}$$

by Proposition 4.1. In particular the order of the $\mathcal{D}_i$ does not influence the sets $\mathcal{D}$ or $\operatorname{Spec}(\operatorname{ICG}(n, \mathcal{D}))$. Consequently, $\underline{\sigma}_{\mathbf{X}_1\cdot\mathbf{X}_2\cdot\ldots\cdot\mathbf{X}_r}$ and, if defined, $\sigma_{\mathbf{X}_1\cdot\mathbf{X}_2\cdot\ldots\cdot\mathbf{X}_r}$ are unaffected by the order of the $\mathbf{X}_i$.

(i) Let $\mathcal{D}_i \subseteq D(p_i^{k_i})$, $r+1 \leq i \leq r+m$, be m further divisor sets all of type $\mathbf{A}$, i.e. $\operatorname{Spec}(\operatorname{ICG}(p_i^{k_i}, \mathcal{D}_i)) = \{1\}$ for $r+1 \leq i \leq r+m$ according to Corollary 3.1. It follows by (10) that $\operatorname{Spec}(\operatorname{ICG}(p_{r+1}^{k_{r+1}} \cdot \ldots \cdot p_{r+m}^{k_{r+m}}, \mathcal{D}_{r+1} \cdot \ldots \cdot \mathcal{D}_{r+m})) = \{1\}$, hence $\sigma_{\mathbf{A}^m} = 1$, independently of the p_i and the k_i $(r+1 \leq i \leq r+m)$.

Moreover, (10) and $\sigma_{\mathbf{A}^m} = 1$ imply that

$$\begin{aligned}
&\underline{\sigma}_{\mathbf{X}_1\cdot\ldots\cdot\mathbf{X}_r\cdot\mathbf{A}^m} \\
&\quad = \min\left\{\sigma\left(p_1^{k_1}\cdot\ldots\cdot p_{r+m}^{k_{r+m}}, \mathcal{D}_1\cdot\ldots\cdot\mathcal{D}_{r+m}\right) : \mathcal{D}_i \subseteq D\left(p_i^{k_i}\right) \text{ of type } \mathbf{X}_i\,(1\leq i\leq r+m)\right\}, \\
&\quad = \min\left\{\left|\operatorname{Spec}\left(\operatorname{ICG}(\prod_{i=1}^{r+m} p_i^{k_i}, \prod_{i=1}^{r+m} \mathcal{D}_i)\right)\right| : \mathcal{D}_i \subseteq D(p_i^{k_i}) \text{ of type } \mathbf{X}_i\,(1\leq i\leq r+m)\right\} \\
&\quad = \min\left\{\left|\operatorname{Spec}\left(\operatorname{ICG}(\prod_{i=1}^{r} p_i^{k_i}, \prod_{i=1}^{r} \mathcal{D}_i)\right)\right| \cdot \left|\operatorname{Spec}\left(\operatorname{ICG}(\prod_{i=r+1}^{r+m} p_i^{k_i}, \prod_{i=r+1}^{r+m} \mathcal{D}_i)\right)\right|\right. \\
&\qquad \left. : \mathcal{D}_i \text{ of type } \mathbf{X}_i\right\} \\
&\quad = \min\left\{\left|\operatorname{Spec}\left(\operatorname{ICG}(\prod_{i=1}^{r} p_i^{k_i}, \prod_{i=1}^{r} \mathcal{D}_i)\right)\right| \cdot \sigma_{\mathbf{A}^m} : \mathcal{D}_i \subseteq D(p_i^{k_i}) \text{ of type } \mathbf{X}_i\ (1\leq i\leq r)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \min\left\{ \left| \mathrm{Spec}\left(\mathrm{ICG}(\prod_{i=1}^{r} p_i^{k_i}, \prod_{i=1}^{r} \mathcal{D}_i) \right) \right| : \mathcal{D}_i \subseteq D(p_i^{k_i}) \text{ of type } \mathbf{X}_i \, (1 \le i \le r) \right\} \\
&= \underline{\sigma}_{\mathbf{X}_1 \cdot \mathbf{X}_2 \cdot \ldots \cdot \mathbf{X}_r}.
\end{aligned}
$$

If $\underline{\sigma}_{\mathbf{X}_1 \cdot \ldots \cdot \mathbf{X}_r} = \overline{\sigma}_{\mathbf{X}_1 \cdot \ldots \cdot \mathbf{X}_r}$ the corresponding formula for $\sigma_{\mathbf{X}_1 \cdot \ldots \cdot \mathbf{X}_r \cdot \mathbf{A}^m}$ follows immediately.

(ii) Let $\mathcal{D}_i \subseteq D(p_i^{k_i})$, $1 \le i \le 3$, be three divisor sets of type **B1**, i.e. $\mathrm{Spec}(\mathrm{ICG}(p_i^{k_i}, \mathcal{D}_i)) = \{-1, p_i^{k_i - u_i} - 1\}$ for suitable $0 \le u_i < k_i$ by Corollary 3.1. By (10) it follows that

$$
\begin{aligned}
&\mathrm{Spec}\left(\mathrm{ICG}(p_1^{k_1} \cdot p_2^{k_2}, \mathcal{D}_1 \cdot \mathcal{D}_2) \right) \\
&\quad = \left\{ 1, -\left(p_1^{k_1 - u_1} - 1 \right), -\left(p_2^{k_2 - u_2} - 1 \right), \left(p_1^{k_1 - u_1} - 1 \right) \left(p_2^{k_2 - u_2} - 1 \right) \right\}, \qquad (11)
\end{aligned}
$$

and this set always contains two distinct positive and two distinct negative entries by the fact that p_1 and p_2 are distinct primes and $k_1 > u_1, k_2 > u_2$. Therefore, $\sigma_{\mathbf{B1}^2} = 4$. Furthermore, by (10)

$$
\begin{aligned}
&\mathrm{Spec}\left(\mathrm{ICG}(p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3}, \mathcal{D}_1 \cdot \mathcal{D}_2 \cdot \mathcal{D}_3) \right) \\
&\quad = \mathrm{Spec}\left(\mathrm{ICG}(p_1^{k_1} \cdot p_2^{k_2}, \mathcal{D}_1 \cdot \mathcal{D}_2) \right) \cdot \mathrm{Spec}\left(\mathrm{ICG}(p_3^{k_3}, \mathcal{D}_3) \right) \\
&\quad = \left\{ 1, -\left(p_1^{k_1 - u_1} - 1 \right), -\left(p_2^{k_2 - u_2} - 1 \right), \left(p_1^{k_1 - u_1} - 1 \right) \left(p_2^{k_2 - u_2} - 1 \right) \right\} \\
&\quad \cdot \left\{ -1, p_3^{k_3 - u_3} - 1 \right\}.
\end{aligned}
$$

This set apparently contains the four distinct elements of $(-1) \cdot \mathrm{Spec}(\mathrm{ICG}(p_1^{k_1} \cdot p_2^{k_2}, \mathcal{D}_1 \cdot \mathcal{D}_2))$ (cf. (11)) and additionally the integer $\prod_{i=1}^{3} (p_i^{k_i - u_i} - 1)$, being bigger than any of the former four integers. This shows that $\underline{\sigma}_{\mathbf{B1}^3} \ge 5$.

Assume alternatively that $\mathcal{D}_3 \subseteq D(p_3^{k_3})$ is a divisor sets of type **B2**, i.e. $\mathrm{Spec}(\mathrm{ICG}(p_3^{k_3}, \mathcal{D}_3)) = \{0, p_3^{k_3 - u_3}\}$ for suitable $0 \le u_3 < k_3$ by Corollary 3.1. Using (11) we obtain

$$
\begin{aligned}
&\mathrm{Spec}\left(\mathrm{ICG}(p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3}, \mathcal{D}_1 \cdot \mathcal{D}_2 \cdot \mathcal{D}_3) \right) \\
&\quad = \mathrm{Spec}\left(\mathrm{ICG}(p_1^{k_1} \cdot p_2^{k_2}, \mathcal{D}_1 \cdot \mathcal{D}_2) \right) \cdot \mathrm{Spec}\left(\mathrm{ICG}(p_3^{k_3}, \mathcal{D}_3) \right) \\
&\quad = \left\{ 1, -\left(p_1^{k_1 - u_1} - 1 \right), -\left(p_2^{k_2 - u_2} - 1 \right), \left(p_1^{k_1 - u_1} - 1 \right) \left(p_2^{k_2 - u_2} - 1 \right) \right\} \\
&\quad \cdot \left\{ 0, p_3^{k_3 - u_3} \right\}
\end{aligned}
$$

containing the four distinct nonzero elements of $p_3^{k_3 - u_3} \cdot \mathrm{Spec}(\mathrm{ICG}(p_1^{k_1} \cdot p_2^{k_2}, \mathcal{D}_1 \cdot \mathcal{D}_2))$ and zero as well. This implies that $\underline{\sigma}_{\mathbf{B1}^2 \cdot \mathbf{B2}} \ge 5$.

Finally it is an easy consequence of Corollary 3.1 and (10) that $\mathrm{Spec}(\mathrm{ICG}(p_i^{k_i}, \mathcal{D}_i)) = \{0, \prod_{i=1}^m p_i^{k_i-u_i}\}$ for divisor sets $\mathcal{D}_i \subseteq D(p_i^{k_i})$, $1 \le i \le m$, which are all of type **B2**. This confirms that $\sigma_{\mathbf{B2}^m} = 2$.

(iii) Let us show in an exemplary way that $\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{C2}} \ge 5$. To this end, let $\mathcal{D}_1 \subseteq D(p_1^{k_1})$ be of type **B1**, i.e. $\mathrm{Spec}(\mathrm{ICG}(p_1^{k_1}, \mathcal{D}_1)) = \{-1, p_1^{k_1-u_1}-1\}$ for suitable $0 \le u_1 < k_1$, and let $\mathcal{D}_2 \subseteq D(p_2^{k_2})$ be of type **C2**, i.e. $\mathrm{Spec}(\mathrm{ICG}(p_2^{k_2}, \mathcal{D}_2)) = \{1, -(p_2^{k_2-v_2}-1), p_2^{k_2-u_2} - p_2^{k_2-v_2}+1\}$ for suitable $0 \le u_2 < v_2 < k_2$ by Corollary 3.1. Using (10) we conclude that

$$\begin{aligned}
&\mathrm{Spec}\left(\mathrm{ICG}(p_1^{k_1}\cdot p_2^{k_2}, \mathcal{D}_1\cdot\mathcal{D}_2)\right)\\
&\quad= \mathrm{Spec}\left(\mathrm{ICG}(p_1^{k_1}, \mathcal{D}_1)\right)\cdot \mathrm{Spec}\left(\mathrm{ICG}(p_2^{k_2}, \mathcal{D}_2)\right)\\
&\quad= \left\{-1, p_1^{k_1-u_1}-1\right\}\left\{1, -\left(p_2^{k_2-v_2}-1\right), p_2^{k_2-u_2} - p_2^{k_2-v_2}+1\right\}\\
&\quad= \left\{-1, -\left(p_2^{k_2-u_2} - p_2^{k_2-v_2}+1\right), -\left(p_1^{k_1-u_1}-1\right)\left(p_2^{k_2-v_2}-1\right)\right\}\\
&\qquad\cup \left\{p_1^{k_1-u_1}-1, p_2^{k_2-v_2}-1, \left(p_1^{k_1-u_1}-1\right)\left(p_2^{k_2-u_2} - p_2^{k_2-v_2}+1\right)\right\}\\
&\quad=: S_1 \cup S_2,
\end{aligned}$$

say. The second and third element of S_1 are easily seen to be different from -1. Moreover, $p_2^{k_2-u_2} - p_2^{k_2-v_2}+1 \ne \left(p_1^{k_1-u_1}-1\right)\left(p_2^{k_2-v_2}-1\right)$, because otherwise $p_2^{k_2-u_2} = p_1^{k_1-u_1}\left(p_2^{k_2-v_2}-1\right)$ would follow, which is impossible since p_1 and p_2 are different primes. Hence $|S_1| = 3$. By the same argument we also conclude that the two positive elements $p_1^{k_1-u_1}-1, p_2^{k_2-v_2}-1 \in S_2$ are distinct. Altogether $\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{C2}} \ge 5$ has been verified.

By careful reasoning along the line of the argument used above one can easily fill Table 2 with the alleged entries.

(iv) By the observation already used in (ii) that $\mathrm{Spec}(\mathrm{ICG}(p_i^{k_i}, \mathcal{D}_i)) = \{0, \prod_{i=1}^r p_i^{k_i-u_i}\}$ for divisor sets $\mathcal{D}_i \subseteq D(p_i^{k_i})$, $1 \le i \le r$, of type **B1**, it is easy to check the four formulae in (iv). $\square$

5 ICG$(n, \mathcal{D})$ with multiplicative $\mathcal{D}$ and $\sigma(n, \mathcal{D}) \le 3$

Applying the results quoted in the introduction to integral circulant graphs $\mathrm{ICG}(n, \mathcal{D})$ reveals that the cases $\sigma(n, \mathcal{D}) = 1$ and $\sigma(n, \mathcal{D}) = 2$ are trivial, and that in case of connectivity $\sigma(n, \mathcal{D}) = 3$ characterises strongly regular graphs. Although the following result is essentially known (cf. commentary preceding Corollary 5.1) we use it to illustrate our method. We agree to define empty products of integers as 1 and empty products of sets to be $\{1\}$.

Theorem 5.1 *Let $n > 1$ be an integer with $r := \omega(n)$ distinct prime factors and let $\mathcal{D} \subseteq D(n)$ be a multiplicative divisor set. Then $\sigma(n, \mathcal{D}) = 3$ if and only if $\mathcal{D}$ has*

Table 3 All factorisation patterns producing $\sigma(n, \mathcal{D}) = 3$

Fact. pattern	$\boldsymbol{\mathcal{D}}$ Spec(ICG($n, \mathcal{D}$)) (range of parameters)
$\mathbf{A}^{r-1}{\cdot}\mathbf{C2}$ $(r \geq 1)$	$\prod_{i=1}^{r-1} p_i^{k_i} \cdot \left\{p_r^{i_r} \,:\, i_r \in [u_r \cdots v_r{-}1] \cup \{k_r\}\right\}$ $\left\{1, -p_r^{k_r-v_r}{+}1, p_r^{k_r-u_r}{-}p_r^{k_r-v_r}{+}1\right\}$ $(0 \leq u_r < v_r < k_r)$
$\mathbf{A}^{r-m-1}{\cdot}\mathbf{B2}^m{\cdot}\mathbf{C1}$ $(0 \leq m < r)$	$\prod_{i=1}^{r-m-1} p_i^{k_i} \cdot \left\{p_{r-m}^{i_{r-m}} \cdot \ldots \cdot p_r^{i_r} \,:\, i_j{\in}[u_j \cdots k_j]\ (r{-}m{\leq}j{<}r),\ i_r{\in}[u_r \cdots v_r{-}1]\right\}$ $\prod_{i=r-m}^{r-1} p_i^{k_i-u_i} \cdot \left\{0, -p_r^{k_r-v_r}, p_r^{k_r-u_r} - p_r^{k_r-v_r}\right\}$ $(0 \leq u_j < k_j$ for $r-m \leq j < r,\ 0 \leq u_r < v_r < k_r)$
$\mathbf{A}^{r-m-1}{\cdot}\mathbf{B1}{\cdot}\mathbf{B2}^m$ $(0 < m < r)$	$\prod_{i=1}^{r-m-1} p_i^{k_i} \cdot \left\{p_{r-m}^{i_{r-m}} \cdot \ldots \cdot p_r^{i_r} \,:\, i_j{\in}[u_j \cdots k_j]\ (r{-}m{\leq}j{<}r),\ i_r{\in}[u_r \cdots k_r{-}1]\right\}$ $\prod_{i=r-m}^{r-1} p_i^{k_i-u_i} \cdot \left\{0, -1, p_r^{k_r-u_r}{-}1\right\}$ $(0 \leq u_j < k_j$ for $r-m \leq j \leq r)$

factorisation pattern $\mathbf{A}^{r-1}{\cdot}\mathbf{C2}$*,* $\mathbf{A}^{r-m-1}{\cdot}\mathbf{B2}^m{\cdot}\mathbf{C1}$ $(0 \leq m < r)$ *or* $\mathbf{A}^{r-m-1}{\cdot}\mathbf{B1}{\cdot}\mathbf{B2}^m$ $(0 < m < r)$.

Given the prime factorisation $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ *(indexed in suitable order), Table 3 provides the corresponding parameterised forms of* $\mathcal{D}$ *and the associated spectra* Spec(ICG($n, \mathcal{D}$)).

Proof The formulae for the divisor sets $\mathcal{D}$ and the spectra Spec(ICG($n, \mathcal{D}$)) corresponding to the factorisation patterns in the four cases (cf. Table 3) are immediately obtained by use of Table 1 in Corollary 3.1. Therefore it suffices to prove that $\sigma(n, \mathcal{D}) = 3$ if and only if the multiplicative divisor set $\mathcal{D}$ has one of the three factorisation patterns in Table 3.

By Lemma 4.1(ii), $\sigma(n, \mathcal{D}) \leq 3$ for $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ and multiplicative $\mathcal{D} = \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r \subseteq D(n)$ with $\mathcal{D}_i \subseteq D(p_i^{k_i})$, $1 \leq i \leq r$, implies $\sigma(p_i^{k_i}, \mathcal{D}_i) \leq 3$ for $1 \leq i \leq r$, and consequently the divisor sets $\mathcal{D}_i$ are of type **A**, **B1**, **B2**, **C1**, **C2** by Corollary 3.1.

Let us denote by $\alpha(n, \mathcal{D})$, $\beta_1(n, \mathcal{D})$, $\beta_2(n, \mathcal{D})$, $\gamma_1(n, \mathcal{D})$ and $\gamma_2(n, \mathcal{D})$ the numbers of $\mathcal{D}_i$ of types **A**, **B1**, **B2**, **C1** or **C2** in $\mathcal{D} = \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r$, respectively. Clearly $\alpha(n, \mathcal{D}) + \beta_1(n, \mathcal{D}) + \beta_2(n, \mathcal{D}) + \gamma_1(n, \mathcal{D}) + \gamma_2(n, \mathcal{D}) = r$.

Since $\sigma_{\mathbf{B1}^2} = 4$ by Proposition 4.2(ii), we have $\beta_1(n, \mathcal{D}) \in \{0, 1\}$ by use of Lemma 4.1(i). Moreover $\underline{\sigma}_{\mathbf{C1}^2} \geq 5$, $\underline{\sigma}_{\mathbf{C2}^2} \geq 5$ and $\underline{\sigma}_{\mathbf{C1}\cdot\mathbf{C2}} \geq 5$ by Proposition 4.2(iii), hence it follows that $\gamma_1(n, \mathcal{D}), \gamma_2(n, \mathcal{D}) \in \{0, 1\}$, but $\gamma_1(n, \mathcal{D}) + \gamma_2(n, \mathcal{D}) \leq 1$. As for the proof of these statements Lemma 4.1(i) is repeatedly applied in what follows.

Case 1: $\beta_1(n, \mathcal{D}) = 0$.

Case 1.1: $\beta_2(n, \mathcal{D}) = 0$.
Proposition 4.2(i) and Corollary 3.1 imply that $\sigma_{\mathbf{A}^m\cdot\mathbf{C1}} = \sigma_{\mathbf{C1}} = 3$ and $\sigma_{\mathbf{A}^m\cdot\mathbf{C2}} = \sigma_{\mathbf{C2}} = 3$ for any $m \geq 0$, and $\mathbf{A}^m \cdot \mathbf{C1}$ and $\mathbf{A}^m \cdot \mathbf{C2}$ are the only possible factorisation patterns in this subcase, i.e. the second pattern with $m = 0$ or the first pattern in Table 3.

Case 1.2: $\beta_2(n,\mathcal{D}) \geq 1$.
By Proposition 4.2(iii) we have $\sigma_{\mathbf{B2}\cdot\mathbf{C2}} = 4$, hence $\gamma_2(n,\mathcal{D}) = 0$. Moreover Proposition 4.2(iv) shows that $\sigma_{\mathbf{B2}^m\cdot\mathbf{C1}} = 3$ for any $m \geq 1$, and by Proposition 4.2(i) $\sigma_{\mathbf{A}^{m_1}\cdot\mathbf{B2}^{m_2}\cdot\mathbf{C1}} = 3$, which yields the only possible factorisation pattern $\mathbf{A}^{m_1}\cdot\mathbf{B2}^{m_2}\cdot\mathbf{C1}$, i.e. the second pattern with $m \geq 1$ in Table 3.

Case 2: $\beta_1(n,\mathcal{D}) = 1$.
By Proposition 4.2(iii) we have $\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{C1}} \geq 5$ and $\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{C2}} \geq 5$, hence $\gamma_1(n,\mathcal{D}) = \gamma_2(n,\mathcal{D}) = 0$. According to Proposition 4.2(iv) and Proposition 4.2(i) we have $\sigma_{\mathbf{A}^{m_1}\cdot\mathbf{B1}\cdot\mathbf{B2}^{m_2}} = 3$, which yields the only possible pattern $\mathbf{A}^{m_1}\cdot\mathbf{B1}\cdot\mathbf{B2}^{m_2}$, i.e. the third pattern in Table 3. □

As stated in Sect. 2, $n \in \mathcal{D}$ produces loops in $\mathrm{ICG}(n,\mathcal{D})$. Moreover, $1 \in \mathcal{D}$ guarantees that $\mathrm{ICG}(n,\mathcal{D})$ is connected (see also Sect. 2). These are the graph-theoretical reasons why we require $1 \in \mathcal{D}$, but $n \notin \mathcal{D}$ in the following straightforward consequence of Theorem 5.1 (cf. [27, Theorems 4.1 and 4.2]). Recall the notation $e_p(n)$ (≥ 0) for the order of a prime p in the prime factorisation of $n > 1$.

Corollary 5.1 *Let $n > 1$ be an integer, and let $\mathcal{D} \subseteq D(n)$ be a multiplicative divisor set with $1 \in \mathcal{D}$, $n \notin \mathcal{D}$. Then $\sigma(n,\mathcal{D}) = 3$ if and only if $\mathcal{D} = \{d \in D(n) : \; p^t \nmid d\} = D\big(\frac{n}{p^{e_p(n)-t+1}}\big)$ for some prime $p \mid n$, where $1 \leq t \leq e_p(n)$ and $\omega(n) \geq 2$ in case $t = e_p(n)$. The corresponding spectrum is* $\mathrm{Spec}(\mathrm{ICG}(n,\mathcal{D})) = \big\{0, -\frac{n}{p^t}, n - \frac{n}{p^t}\big\}$.

Proof Let $n = p_1^{k_1}\cdot\ldots\cdot p_r^{k_r}$ with $r = \omega(n)$ be the prime factorisation of n. Since $\mathcal{D}$ is multiplicative, there are $\mathcal{D}_i \in D(p_i^{k_i})$ such that $\mathcal{D} = \prod_{i=1}^r \mathcal{D}_i$. The condition $1 \in \mathcal{D}$ implies $1 \in \mathcal{D}_i$ for $1 \leq i \leq r$. Now our assertion is a special case of Theorem 5.1. Since 1 lies in every $\mathcal{D}_i$, we cannot have any set $\mathcal{D}_i = \{p_i^{k_i}\}$ of type $\mathbf{A}$, i.e. $\alpha(n,\mathcal{D}) = 0$.

For the first pattern in Table 3 of Theorem 5.1, $\alpha(n,\mathcal{D}) = 0$ implies $\omega(n) = r = 1$, i.e. $n = p^k$ is a prime power and $p^k \in \mathcal{D}$, contradicting our assumptions.

We are left with the second or third pattern in Table 3, where $\alpha(n,\mathcal{D}) = 0$ yields $m = r-1$, therefore $\mathbf{B2}^{r-1}\cdot\mathbf{C1}$ for some $r \geq 1$ or $\mathbf{B1}\cdot\mathbf{B2}^{r-1}$ for some $r \geq 2$. Moreover, in both cases $u_i = 0$ for $1 \leq i \leq r$, because 1 lies in every $\mathcal{D}_i$. Consequently, the corresponding divisor sets look like $\mathcal{D} = \prod_{i=1}^{r-1} D\big(p_i^{k_i}\big)\cdot D\big(p_r^{v_r-1}\big)$ with $r \geq 1$ and $1 \leq v_r \leq k_r - 1$ or like $\mathcal{D} = \prod_{i=1}^{r-1} D\big(p_i^{k_i}\big)\cdot D\big(p_r^{k_r-1}\big)$ with $r \geq 2$. For $p := p_r$ and setting $t := v_r$ in the former case and $t := k_r = e_p(n)$ in the latter case, we combine the two cases to obtain $\mathcal{D} = \{d \in D(n) : \; p^t \nmid d\}$, where $1 \leq t \leq e_p(n)$ and $r = \omega(n) \geq 2$ in case $t = e_p(n)$.

It remains to determine the corresponding spectra. Observe that $\mathrm{Spec}(\mathrm{ICG}(n,\mathcal{D}))$ for the third pattern in Table 3 equals the extension of the spectrum for the second pattern to the case $v_r = k_r$. Since $u_i = 0$ for $1 \leq i \leq r$, combining the cases yields

$$\begin{aligned}\mathrm{Spec}(\mathrm{ICG}(n,\mathcal{D})) &= \prod_{i=1}^{r-1} p_i^{k_i}\cdot\left\{0,-p_r^{k_r-v_r},\, p_r^{k_r}-p_r^{k_r-v_r}\right\} = \left\{0,-\frac{n}{p_r^{v_r}},\, n-\frac{n}{p_r^{v_r}}\right\}\\ &= \left\{0,-\frac{n}{p^t},\, n-\frac{n}{p^t}\right\}.\end{aligned}$$

□

6 Strongly regular ICG$(n, \mathcal{D})$ with multiplicative $\mathcal{D}$

The goal of this section is to rephrase the results from the previous section such that their connection to previously known results about strong regularity of integral circulant graphs from other sources (e.g. [27, Theorems 4.1 and 4.2]) becomes more transparent. To this end, Corollary 5.1 and Theorem 1.1 immediately imply the following characterisation, which has also been obtained in [2, Theorem 15] by use of different tools.

Theorem 6.1 *Let $n > 1$ be an integer, and let* ICG$(n, \mathcal{D})$ *be a connected (loopless) integral circulant graph with multiplicative $\mathcal{D}$. Then* ICG$(n, \mathcal{D})$ *is strongly regular if and only if $\mathcal{D} = \{d \in D(n) :\ p^t \nmid d\}$ for some prime $p \mid n$, where $1 \le t \le e_p(n)$ and $\omega(n) \ge 2$ in case $t = e_p(n)$.*

Proof Since ICG$(n, \mathcal{D})$ is assumed to be connected, the elements of $\mathcal{D}$ are coprime according to property 2. in Sect. 2. Moreover $\mathcal{D}$ is multiplicative, i.e. $\mathcal{D} = \prod_{i=1}^{r} \mathcal{D}_i$ with $\mathcal{D}_i \in D(p_i^{k_i})$ for $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$, say. Consequently $1 \in \mathcal{D}_i$ for all i, because otherwise some p_i would divide all elements of $\mathcal{D}$. Thus $1 \in \mathcal{D}$, and $n \notin \mathcal{D}$ since ICG$(n, \mathcal{D})$ is loopless (cf. property 1 in Sect. 2). Now all assertions follow from Corollary 5.1 by use of Theorem 1.1. □

By definition (cf. (1)) we have ICG$(n, \mathcal{D}) = $ Circ(n, S) for $S = \bigcup_{d\in\mathcal{D}} G_n(d)$ and some divisor set $\mathcal{D}$ of n, where $G_n(d) = \{1 \le a \le n-1 :\ \gcd(a, n) = d\}$. We prepare the translation of Theorem 6.1 into the terminology of Circ(n, S) by

Lemma 6.1 *Let $n > 1$ have prime factorisation $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$, and let $\mathcal{D}_i = \{1, p_i, p_i^2, \ldots, p_i^{s_i}\}$ for some $0 \le s_i \le k_i$ $(1 \le i \le r)$. Then* ICG$(n, \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r) =$ Circ(n, S) *satisfies*

$$S = \{1, 2, 3, \ldots, n-1\} \setminus \bigcup_{\substack{i=1\\ s_i<k_i}}^{r} M_n\left(p_i^{s_i+1}\right),$$

where $M_n(d) := \{1 \le j \cdot d < n :\ j = 1, 2, 3, \ldots\} = \{d, 2d, 3d, \ldots, n-d\}$ for any $d \mid n$. In case $s_i = k_i$ $(1 \le i \le r-1)$ in particular we have $S = \{1, 2, 3, \ldots, n-1\} \setminus M_n(p_r^{s_r+1})$.

Proof According to [25, Theorem 7.1] we know that

$$
\begin{aligned}
S &= \bigcup_{d \in \mathcal{D}} G_n(d) = \{1 \le a \le n-1 : \ \gcd(a, n) = d \text{ for some } d \in \mathcal{D}\} \\
&= \{1 \le a \le n-1 : \ \gcd(a, p_i^{k_i}) \le p_i^{s_i} \ (1 \le i \le r)\} \\
&= \{1 \le a \le n-1 : \ \gcd(a, p_i^{k_i}) \le p_i^{s_i} \text{ for all } 1 \le i \le r \text{ satisfying } s_i < k_i\} \\
&= \{1, 2, 3, \dots, n-1\} \backslash \{1 \le a \le n-1 : \ \gcd(a, p_i^{k_i}) > p_i^{s_i} \text{ for some } 1 \\
&\le i \le r \text{ satisfying } s_i < k_i\} \\
&= \{1, 2, 3, \dots, n-1\} \backslash \{1 \le a \le n-1 : \ p_i^{s_i+1} \mid a \text{ for some } 1 \\
&\le i \le r \text{ satisfying } s_i < k_i\} \\
&= \{1, 2, 3, \dots, n-1\} \backslash \bigcup_{\substack{i=1 \\ s_i < k_i}}^{r} \left\{p_i^{s_i+1} j : \ j = 1, 2, 3, \dots\right\}.
\end{aligned}
$$

□

Theorem 6.1 can immediately be translated into

Theorem 6.2 *Let $n > 1$ be an integer and let $\mathrm{Circ}(n, S)$ be a loop-free connected integral circulant graph, i.e. $S = \bigcup_{d \in \mathcal{D}} G_n(d)$ for some $\mathcal{D} \subseteq D(n)$ satisfying $1 \in \mathcal{D}$ and $n \notin \mathcal{D}$. Furthermore assume that $\mathcal{D}$ is multiplicative. Then $\mathrm{Circ}(n, S)$ is strongly regular if and only if $S = \{1, 2, 3, \dots, n-1\} \backslash M_n(p^t)$ for some prime power divisor p^t of n, where $1 \le t \le e_p(n)$ and $\omega(n) \ge 2$ in case $t = e_p(n)$.*

In case of strong regularity $\mathrm{Circ}(n, S)$ has parameters $\left(n, n - \frac{n}{p^t}, n - \frac{2n}{p^t}, n - \frac{n}{p^t}\right)$.

Proof By the conditions imposed on $\mathrm{Circ}(n, S)$ we have $\mathrm{Circ}(n, S) = \mathrm{ICG}(n, \mathcal{D})$ for some multiplicative $\mathcal{D} \subseteq D(n)$ with $1 \in \mathcal{D}$, but $n \notin \mathcal{D}$.

First assuming that $\mathrm{Circ}(n, S)$ is strongly regular, Theorem 6.1 implies that $\mathcal{D} = \{d \in D(n) : \ p^t \nmid d\}$ for some prime $p \mid n$, where $1 \le t \le e_p(n)$ and $\omega(n) \ge 2$ in case $t = e_p(n)$. Let $n = p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$ be the prime factorisation of n. Since $\mathcal{D}$ is multiplicative, we have $\mathcal{D} = \mathcal{D}_1 \cdot \dots \cdot \mathcal{D}_r$, say, with $\mathcal{D}_i \subseteq D(p_i^{k_i})$ for $1 \le i \le r$. It follows from $\mathcal{D} = \{d \in D(n) : \ p^t \nmid d\}$ that $\mathcal{D}_i = D(p_i^{k_i})$ for $p_i \ne p$ and $\mathcal{D}_i = D(p^{t-1})$ for $p_i = p$. Setting $s_i = k_i$ for $p_i \ne p$ and $s_i = t-1$ for $p_i = p$, Lemma 6.1 implies $S = \{1, 2, 3, \dots, n-1\} \backslash M_n(p^t)$.

Now suppose conversely that $S = \{1, 2, 3, \dots, n-1\} \backslash M_n(p^t)$ as specified above. For composite n, i.e. in case $\omega(n) \ge 2$, the strong regularity of $\mathrm{Circ}(n, S)$ as well as its parameters have been verified in [27, Theorem 4.1]. It remains to consider prime powers $n = p^k$, say. Clearly, the degree ρ of regularity of $\mathrm{Circ}(n, S)$ is $\rho = |S| = n - \frac{n}{p^t}$. Let λ denote the number of common neighbours of every two adjacent vertices of $\mathrm{Circ}(n, S)$, and let μ be the number of common neighbours of every two non-adjacent vertices. Since $1 \in \mathcal{D}$, the vertices 0 and 1 are adjacent in $\mathrm{Circ}(n, S)$, and clearly their common neighbours are all $1, 2, \dots, n-1$ except for the vertices of type

$j \cdot p^t$ and those of type $j \cdot p^t + 1$. Hence $\lambda = n - 2 \cdot \frac{n}{p^t}$. Similarly 0 and p^t are non-adjacent vertices and have as common neighbours all vertices $1, 2, \ldots, n-1$ except for the vertices of type $j \cdot p^t$, thus $\mu = n - \frac{n}{p^t}$. Altogether, we have the parameters $(n, \rho, \lambda, \mu) = \left(n, n - \frac{n}{p^t}, n - \frac{2n}{p^t}, n - \frac{n}{p^t}\right)$ as required. □

Remark 6.1 A connected circulant graph $\mathrm{Circ}(n, S)$ with exactly three integral eigenvalues satisfies $S = \{1, 2, 3, \ldots, n-1\} \backslash M_n(d)$ for some proper divisor $d \in D(n) \backslash \{1, n\}$ of n (cf. [27, Theorem 4.2]). Our Theorem 6.2 verifies this result for the subclass of integral circulant graphs with multiplicative divisor sets and shows, however, that admissable divisors d in $M_n(d)$ for our subclass necessarily have to be prime power divisors p^t of n (see Corollary 6.2). Lemma 6.1 reveals that this special arithmetic feature of possible "exceptional" d, namely that d cannot be a composite divisor, is a consequence of two facts, which are (i) $1 \in \mathcal{D}$, in other words enforced by the connectedness of $\mathrm{Circ}(n, S)$, and (ii) $s_i = k_i$ for all but one index i.

7 $\mathrm{ICG}(n, \mathcal{D})$ with multiplicative $\mathcal{D}$ and $\sigma(n, \mathcal{D}) = 4$

As already mentioned in the introduction, achieving a characterisation of integral circulant graphs with exactly four distinct eigenvalues is currently an open and elusive research goal. Hence it is useful to focus on subclasses first. The following theorem provides a complete list of factorisation patterns of multiplicative divisor sets $\mathcal{D}$ with $\sigma(n, \mathcal{D}) = 4$. Observe that the primes dividing n are completely irrelevant for our result. Even the arithmetic nature of n, e.g. the number $\omega(n)$ of prime factors or the order of primes in n is negligible. All that is required for some factorisation patterns to occur is $\omega(n) \geq 2$ and that some prime factor of n has a certain minimal order of 2 or 3 in n.

Theorem 7.1 *Let $n > 1$ be an integer with $r := \omega(n)$ distinct prime factors and let $\mathcal{D} \subseteq D(n)$ be a multiplicative divisor set. Then $\sigma(n, \mathcal{D}) = 4$ if and only if $\mathcal{D}$ has factorisation pattern $\mathbf{A}^{r-1} \cdot \mathbf{D1}$, $\mathbf{A}^{r-2} \cdot \mathbf{B1}^2$ $(r \geq 2)$, $\mathbf{A}^{r-m-1} \cdot \mathbf{B2}^m \cdot \mathbf{D2}$ $(0 \leq m < r)$ or $\mathbf{A}^{r-m-1} \cdot \mathbf{B2}^m \cdot \mathbf{C2}$ $(0 < m < r)$. Given the prime factorisation $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ (indexed in suitable order), Table 4 provides the corresponding parameterised forms of $\mathcal{D}$ and the associated spectra* $\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D}))$.

Proof The formulae for the divisor sets $\mathcal{D}$ and the spectra $\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D}))$ corresponding to the factorisation patterns in the four cases (cf. Table 4) are immediately obtained by use of Table 1 in Corollary 3.1. Therefore it suffices to prove that $\sigma(n, \mathcal{D}) = 4$ if and only if the multiplicative divisor set $\mathcal{D}$ has one of the four factorisation patterns in Table 4.

By Lemma 4.1(ii), $\sigma(n, \mathcal{D}) \leq 4$ for $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ and multiplicative $\mathcal{D} = \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r \subseteq D(n)$ with $\mathcal{D}_i \subseteq D(p_i^{k_i})$, $1 \leq i \leq r$, implies $\sigma(p_i^{k_i}, \mathcal{D}_i) \leq 4$ for $1 \leq i \leq r$, and consequently the divisor sets $\mathcal{D}_i$ are of type **A**, **B1**, **B2**, **C1**, **C2**, **D1** or **D2** by Corollary 3.1.

Let us denote by $\alpha(n, \mathcal{D})$, $\beta_1(n, \mathcal{D})$, $\beta_2(n, \mathcal{D})$, $\gamma_1(n, \mathcal{D})$, $\gamma_2(n, \mathcal{D})$, $\delta_1(n, \mathcal{D})$ and $\delta_2(n, \mathcal{D})$ the numbers of $\mathcal{D}_i$ of types **A**, **B1**, **B2**, **C1**, **C2**, **D1** or **D2** in $\mathcal{D} = \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r$,

Table 4 All factorisation patterns producing $\sigma(n, \mathcal{D}) = 4$

Fact. pattern	$\mathcal{D}$ Spec(ICG$(n, \mathcal{D})$) (range of parameters)
$\mathbf{A}^{r-1} \cdot \mathbf{D1}$ $(r \geq 1)$	$\prod_{i=1}^{r-1} p_i^{k_i} \cdot \left\{ p_r^{i_r} : i_r \in [u_r \cdots v_r - 1] \cup [w_r \cdots k_r - 1] \right\}$ $\left\{ -1, p_r^{k_r - w_r} - 1, -p_r^{k_r - v_r} + p_r^{k_r - w_r} - 1, p_r^{k_r - u_r} - p_r^{k_r - v_r} + p_r^{k_r - w_r} - 1 \right\}$ $(0 \leq u_r < v_r < w_r < k_r)$
$\mathbf{A}^{r-2} \cdot \mathbf{B1}^2$ $(r \geq 2)$	$\prod_{i=1}^{r-2} p_i^{k_i} \cdot \left\{ p_{r-1}^{i_{r-1}} \cdot p_r^{i_r} : i_{r-1} \in [u_{r-1} \cdots k_{r-1} - 1], \ i_r \in [u_r \cdots k_r - 1] \right\}$ $\left\{ 1, -(p_{r-1}^{k_{r-1} - u_{r-1}} - 1), -(p_r^{k_r - u_r} - 1), (p_{r-1}^{k_{r-1} - u_{r-1}} - 1)(p_r^{k_r - u_r} - 1) \right\}$ $(0 \leq u_{r-1} < k_{r-1}, \ 0 \leq u_r < k_r)$
$\mathbf{A}^{r-m-1} \cdot \mathbf{B2}^m \cdot \mathbf{D2}$ $(0 \leq m < r)$	$\prod_{i=1}^{r-m-1} p_i^{k_i} \cdot \left\{ p_{r-m}^{i_{r-m}} \cdot \ldots \cdot p_r^{i_r} : i_j \in [u_j \cdots k_j] \ (r-m \leq j < r), \ i_r \in [u_r \cdots v_r - 1] \cup [w_r \cdots k_r] \right\}$ $\prod_{i=r-m}^{r-1} p_i^{k_i - u_i} \cdot \left\{ 0, p_r^{k_r - w_r}, -p_r^{k_r - v_r} + p_r^{k_r - w_r}, p_r^{k_r - u_r} - p_r^{k_r - v_r} + p_r^{k_r - w_r} \right\}$ $(0 \leq u_j < k_j$ for $r - m \leq j < r, \ 0 \leq u_r < v_r < w_r < k_r)$
$\mathbf{A}^{r-m-1} \cdot \mathbf{B2}^m \cdot \mathbf{C2}$ $(0 < m < r)$	$\prod_{i=1}^{r-m-1} p_i^{k_i} \cdot \left\{ p_{r-m}^{i_{r-m}} \cdot \ldots \cdot p_r^{i_r} : i_j \in [u_j \cdots k_j] \ (r-m \leq j < r), \ i_r \in [u_r \cdots v_r - 1] \cup \{k_r\} \right\}$ $\prod_{i=r-m}^{r-1} p_i^{k_i - u_i} \cdot \left\{ 0, 1, -p_r^{k_r - v_r} + 1, p_r^{k_r - u_r} - p_r^{k_r - v_r} + 1 \right\}$ $(0 \leq u_j < k_j$ for $r - m \leq j < r, \ 0 \leq u_r < v_r < k_r)$

respectively. Clearly $\alpha(n,\mathcal{D})+\beta_1(n,\mathcal{D})+\beta_2(n,\mathcal{D})+\gamma_1(n,\mathcal{D})+\gamma_2(n,\mathcal{D})+\delta_1(n,\mathcal{D})+\delta_2(n,\mathcal{D})=r$.

Since $\underline{\sigma}_{\mathbf{B1}^3}\geq 5$ by Proposition 4.2(ii), we have $\beta_1(n,\mathcal{D})\in\{0,1,2\}$. Moreover $\underline{\sigma}_{\mathbf{X}\cdot\mathbf{Y}}\geq 5$ for $\mathbf{X},\mathbf{Y}\in\{\mathbf{C1},\mathbf{C2},\mathbf{D1},\mathbf{D2}\}$ by Proposition 4.2(iii), hence it follows that $\gamma_1(n,\mathcal{D}),\gamma_2(n,\mathcal{D}),\delta_1(n,\mathcal{D}),\delta_2(n,\mathcal{D})\in\{0,1\}$, but $\gamma_1(n,\mathcal{D})+\gamma_2(n,\mathcal{D})+\delta_1(n,\mathcal{D})+\delta_2(n,\mathcal{D})\leq 1$.

Case 1: $\beta_1(n,\mathcal{D})=0$.

Case 1.1: $\beta_2(n,\mathcal{D})=0$.
Proposition 4.2(i) and Corollary 3.1 imply that $\sigma_{\mathbf{A}^m\cdot\mathbf{D1}}=\sigma_{\mathbf{D1}}=4$ and $\sigma_{\mathbf{A}^m\cdot\mathbf{D2}}=\sigma_{\mathbf{D2}}=4$ for any $m\geq 0$, and $\mathbf{A}^m\cdot\mathbf{D1}$ and $\mathbf{A}^m\cdot\mathbf{D2}$ are the only possible factorisation patterns in this subcase, i.e. the first pattern or the third pattern with $m=0$ in Table 4.

Case 1.2: $\beta_2(n,\mathcal{D})\geq 1$.
By Proposition 4.2(iv) we have $\sigma_{\mathbf{B2}^m\cdot\mathbf{C1}}=3$ and $\underline{\sigma}_{\mathbf{B2}^m\cdot\mathbf{D1}}\geq 5$, hence Proposition 4.2(i) implies $\gamma_1(n,\mathcal{D})=\delta_1(n,\mathcal{D})=0$. Moreover Proposition 4.2(iv) shows that $\sigma_{\mathbf{B2}^m\cdot\mathbf{C2}}=\sigma_{\mathbf{B2}^m\cdot\mathbf{D2}}=4$ for any $m\geq 1$, thus $\sigma_{\mathbf{A}^{m_1}\cdot\mathbf{B2}^{m_2}\cdot\mathbf{C2}}=\sigma_{\mathbf{A}^{m_1}\cdot\mathbf{B2}^{m_2}\cdot\mathbf{D2}}=4$ according to Proposition 4.2(i). This yields the only possible two factorisation patterns $\mathbf{A}^{m_1}\cdot\mathbf{B2}^{m_2}\cdot\mathbf{C2}$ and $\mathbf{A}^{m_1}\cdot\mathbf{B2}^{m_2}\cdot\mathbf{D2}$, i.e. we have the fourth patterns or the third patterns with $m\geq 1$ in Table 4.

Case 2: $\beta_1(n,\mathcal{D})\in\{1,2\}$.
By Proposition 4.2(iii) we have $\underline{\sigma}_{\mathbf{B1}\cdot\mathbf{X}}\geq 5$ for $\mathbf{X}\in\{\mathbf{C1},\mathbf{C2},\mathbf{D1},\mathbf{D2}\}$, hence $\gamma_1(n,\mathcal{D})=\gamma_2(n,\mathcal{D})=\delta_1(n,\mathcal{D})=\delta_2(n,\mathcal{D})=0$. However, $\sigma_{\mathbf{A}^{m_1}\cdot\mathbf{B1}\cdot\mathbf{B2}^{m_2}}=3$ according to Propositions 4.2(iv) and 4.2(i). Therefore, $\beta_1(n,\mathcal{D})=2$. Since Proposition 4.2(ii) shows that $\underline{\sigma}_{\mathbf{B1}^2\cdot\mathbf{B2}}\geq 5$, we also have $\beta_2(n,\mathcal{D})=0$. By Proposition 4.2(ii) and Proposition 4.2(i), the only possible factorisation pattern is $\mathbf{A}^m\cdot\mathbf{B1}^2$, i.e. the second pattern in Table 4. $\square$

In analogy to the characterisation of strongly regular integral circulant graphs—i.e. those connected regular graphs with exactly three distinct eigenvalues—in Corollary 5.1 and Sect. 6 we now consider integral circulant graphs with precisely four distinct eigenvalues, again imposing the graph-theoretical conditions that $\mathrm{ICG}(n,\mathcal{D})$ is connected, i.e. $1\in\mathcal{D}$, and loopfree, i.e. $n\notin\mathcal{D}$. These "natural" graph-theoretical conditions imply the strong arithmetic restriction that n may not have more than two distinct prime factors.

Corollary 7.1 *Let $n>1$ be an integer, and let $\mathcal{D}\subseteq D(n)$ be a multiplicative divisor set with $1\in\mathcal{D}$, $n\notin\mathcal{D}$. Then $\sigma(n,\mathcal{D})=4$ if and only if $\omega(n)\leq 2$ with*

$$\begin{aligned}\mathcal{D}&=D\big(p^{k-1}\big)\backslash\big\{p^v,\ldots,p^{w-1}\big\}\\ \mathrm{Spec}(\mathrm{ICG}(n,\mathcal{D}))&=\big\{-1,\,p^{k-w}-1,\,-p^{k-v}+p^{k-w}-1,\,p^k-p^{k-v}+p^{k-w}-1\big\}\end{aligned}\tag{12}$$

for $n=p^k$ and some $0<v<w<k$ in case $\omega(n)=1$, or

$$\begin{aligned}\mathcal{D}&=D\big(p_1^{k_1-1}\cdot p_2^{k_2-1}\big),\ \mathrm{Spec}(\mathrm{ICG}(n,\mathcal{D}))\\ &=\Big\{1,-\Big(p_1^{k_1}-1\Big),-\Big(p_2^{k_2}-1\Big),\Big(p_1^{k_1}-1\Big)\Big(p_2^{k_2}-1\Big)\Big\}\end{aligned}\tag{13}$$

for $n = p_1^{k_1} \cdot p_2^{k_2}$ *in case* $\omega(n) = 2$.

Proof Let $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ with $r = \omega(n)$ be the prime factorisation of n. Since $\mathcal{D}$ is multiplicative, there are $\mathcal{D}_i \in D(p_i^{k_i})$ such that $\mathcal{D} = \prod_{i=1}^{r} \mathcal{D}_i$. The condition $1 \in \mathcal{D}$ implies $1 \in \mathcal{D}_i$ for $1 \le i \le r$. Now our assertion is a special case of Theorem 7.1. Since 1 lies in every $\mathcal{D}_i$, we cannot have any set $\mathcal{D}_i = \{p_i^{k_i}\}$ of type **A**, i.e. $\alpha(n, \mathcal{D}) = 0$.

In case (i) of Theorem 7.1, $\alpha(n, \mathcal{D}) = 0$ implies $\omega(n) = r = 1$, i.e. $n = p^k$ is a prime power, and by the fact that $1 \in \mathcal{D}$ we obtain $\mathcal{D} = \{p^i \;:\; i \in [0 \cdots v{-}1] \cup [w \cdots k{-}1]\}$ and $\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D})) = \{-1, p^{k-w}{-}1, -p^{k-v}{+}p^{k-w}{-}1, p^k{-}p^{k-v} +p^{k-w}{-}1\}$, which prove (12).

In case (ii) of Theorem 7.1, $\alpha(n, \mathcal{D}) = 0$ implies $\omega(n) = r = 2$, i.e. $n = p_1^{k_1} \cdot p_2^{k_2}$, and $1 \in \mathcal{D}$ yields $\mathcal{D} = \{p_1^{i_1} \cdot p_2^{i_2} \;:\; i_1 \in [0 \cdots k_1{-}1],\; i_2 \in [0 \cdots k_2{-}1]\}$ and $\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D})) = \{1, -(p_1^{k_1}{-}1), -(p_2^{k_2}{-}1), (p_1^{k_1}{-}1)(p_2^{k_2}{-}1)\}$ for $n = p_1^{k_1} \cdot p_2^{k_2}$ in case $\omega(n) = 2$, thus (13) follows.

We are left with cases (iii) and (iv) of Theorem 7.1, where $\alpha(n, \mathcal{D}) = 0$ yields $m = r - 1$, therefore $\pi_4(n, \mathcal{D}) = (0, 0, r-1, 0, 0, 0, 1)$ for some $r \ge 1$ or $\pi_4(n, \mathcal{D}) = (0, 0, r-1, 0, 1, 0, 0)$ for some $r \ge 2$. Moreover, in both cases $u_i = 0$ for $1 \le i \le r$, because 1 lies in every $\mathcal{D}_i$. Consequently, the corresponding divisor sets are of shape $\mathcal{D} = \prod_{i=1}^{r-1} D(p_i^{k_i}) \cdot \{p_r^{i_r} \;:\; i_r \in [0 \cdots v_r{-}1] \cup [w_r \cdots k_r]\}$ with $r \ge 1$ and $1 \le v_r < w_r \le k_r - 1$ in case (iii) of Theorem 7.1, and of shape $\mathcal{D} = \prod_{i=1}^{r-1} D(p_i^{k_i}) \cdot \{p_r^{i_r} \;:\; i_r \in [0 \cdots v_r{-}1] \cup \{k_r\}\}$ with $r \ge 2$ in case (iv). We observe that $n \in \mathcal{D}$ in both of these cases, which contradicts our assumptions and completes the proof. □

8 Some open problems

Let us conclude with a couple of open questions.

Problem 1. Given an integer $n > 1$, characterise all multiplicative divisor sets $\mathcal{D} \subseteq D(n)$ with $\sigma(n, \mathcal{D}) = 5$ (by their factorisation patterns).

Probably more difficult to solve might be

Problem 2. Given an integer $n > 1$, characterise all divisor sets $\mathcal{D} \subseteq D(n)$ with $\sigma(n, \mathcal{D}) = 4$.

Funding Open Access funding enabled and organized by Projekt DEAL. The authors did not receive support from any organization for the submitted work.

Data availability All data generated or analysed during this study are included in this published article.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

References

1. Apostol, T.M.: Introduction to Analytic Number Theory. Springer, Berlin (1976)
2. Bašić, M.: Characterization of strongly regular integral circulant graphs by spectral approach. Appl. Anal. Discrete Math. **16**, 288–306 (2022)
3. Boesch, F., Tindell, R.: Circulants and their connectivities. J. Graph Theory **8**, 487–499 (1984)
4. Bose, R.C.: Strongly regular graphs, partial geometries and partially balanced designs. Pacific J. Math. **13**, 389–419 (1963)
5. Braga, R.O., Del-Vecchio, R.R., Rodrigues, V.M., Trevisan, V.: Trees with 4 or 5 distinct normalized Laplacian eigenvalues. Linear Algebra Appl. **471**, 615–635 (2015)
6. Bridges, W.G., Mena, R.A.: Rational circulants with rational spectra and cyclic strongly regular graphs. Ars Combin. **8**, 143–161 (1979)
7. Brouwer, A.E., Haemers, W.H.: Spectra of Graphs. Springer, Berlin (2012)
8. Cvetković, D.M., Doob, M., Sachs, H.: Spectra of Graphs: Theory and Applications, 3rd edn. Wiley, Hoboken (1998)
9. Davis, P.J.: Circulant Matrices. Wiley, New York (1979)
10. Fan, F., Weng, C.: A characterization of strongly regular graphs in terms of the largest signless Laplacian eigenvalues. Linear Algebra Appl. **506**, 1–5 (2016)
11. Gebremichel, B., Cao, M.-Y., Koolen, J.H.: Two characterizations of the grid graphs. Discrete Math. **344**, 11p (2021)
12. Godsil, C.D., Royle, G.: Algebraic Graph Theory. Springer, New York (2001)
13. Huang, X., Huang, Q.: On regular graphs with four distinct eigenvalues. Linear Algebra Appl. **512**, 219–233 (2017)
14. Huang, X., Huang, Q.: On graphs with three or four distinct normalized Laplacian eigenvalues. Algebra Colloq. **26**, 65–82 (2019)
15. Klotz, W., Sander, T.: Some properties of unitary Cayley graphs. Electron. J. Comb. **14** (2007), (Research Paper #R45, electronic)
16. Le, T.A., Sander, J.W.: Convolutions of Ramanujan sums and integral circulant graphs. Int. J. Number Theory **8**, 1777–1788 (2012)
17. Le, T.A., Sander, J.W.: Extremal energies of integral circulant graphs via multiplicativity. Linear Algebra Appl. **437**, 1408–1421 (2012)
18. Nozaki, H., Shinohara, M.: A geometrical characterization of strongly regular graphs. Linear Algebra Appl. **437**, 2587–2600 (2012)
19. Sander, J.W.: Integral circulant Ramanujan graphs via multiplicativity and ultrafriable integers. Linear Algebra Appl. **477**, 21–41 (2015)
20. Sander, J.W.: On the kernel of integral circulant graphs. Linear Algebra Appl. **549**, 79–85 (2018)
21. Sander, J.W.: Structural propertiers and formulae of the spectra of integral circulant graphs. Acta Arith. **184**, 297–315 (2018)
22. Sander, J.W., Sander, T.: On So's conjecture for integral circulant graphs. Appl. Anal. Discrete Math. **9**, 59–72 (2015)
23. Schwarz, W., Spilker, J.: Arithmetical functions. In: London Mathematical Society Lecture Note in Mathematics, vol. **184**. Cambridge University Press (1994)
24. Shrikhande, S.S.: Bhagwandas: duals of incomplete block designs. J. Indian Statist. Assoc. **3**, 30–37 (1965)
25. So, W.: Integral circulant graphs. Discrete Math. **306**, 153–158 (2005)
26. Tamizh Chelvam, T., Raja, S.: Integral circulant graphs with four distinct eigenvalues. Discrete Math. Algorithms Appl. **10**, 1850057 (2018)
27. Tamizh Chelvam, T., Raja, S., Gutman, I.: Strongly regular integral circulant graphs and their energies. Bull. Int. Math. Virtual Inst. **2**, 9–16 (2012)

28. van Dam, E.R., Omidi, G.R.: Graphs whose normalized Laplacian has three eigenvalues. Linear Algebra Appl. **435**, 2560–2569 (2011)
29. van Dam, E.R., Omidi, G.R.: Strongly walk-regular graphs. J. Combin. Theory Ser. A **120**, 803–810 (2013)
30. van Dam, E.R., Spence, E.: Small regular graphs with four eigenvalues. Discrete Math. **189**, 233–257 (1998)