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Derangements in wreath products of permutation groups

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Abstract
Given a finite group G acting on a set X let $\delta_k(G, X)$ denote the proportion of elements in G that have exactly k fixed points in X. Let $\mathcal{S}_n$ denote the symmetric group acting on $[n] = \{1, 2, \ldots, n\}$. For $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$, the permutational wreath product $A \wr B$ has two natural actions and we give formulas for both, $\delta_k(A \wr B, [m]\times[n])$ and $\delta_k(A \wr B, [m]^{[n]})$. We prove that for $k = 0$ the values of these proportions are dense in the intervals $[\delta_0(B, [n]), 1]$ and $[\delta_0(A, [m]), 1]$. Among further results, we provide estimates for $\delta_0(G, [m]^{[n]})$ for subgroups $G \leqslant \mathcal{S}_m \wr \mathcal{S}_n$ containing $\mathcal{A}_m^{[n]}$.

Keywords Permutation groups · Derangements · Fixed-point-free permutations · Wreath products

Mathematics Subject Classification 20B07 · 20B35 · 05A05

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1 Introduction

Let $\mathcal{S}_n$ and $\mathcal{A}_n$ denote the symmetric and alternating group acting on $[n] = \{1, 2, \ldots, n\}$. A permutation of $\mathcal{S}_n$ that fixes no element in $[n]$ is a *derangement*. The proportion of derangements in a subset $C \subseteq \mathcal{S}_n$ is denoted $\delta(C)$. Given $k \in \{0, 1, \ldots, n\}$, let $\mathrm{Fix}_k(C)$ denote the set of all permutations in C with precisely k fixed points. We write $\delta_k(C) = |\mathrm{Fix}_k(C)|/|C|$, and note that $\delta(C) = \delta_0(C)$. The set of derangements in C is denoted by $\mathrm{Der}(C)$ or $\mathrm{Fix}_0(C)$. Given subgroups $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$, the wreath product $A \wr B = A^{[n]} \rtimes B$ gives rise to two natural permutation subgroups: the *imprimitive* subgroup $A \wr_I B \leqslant \mathcal{S}_{mn}$ and the subgroup $A \wr_P B \leqslant \mathcal{S}_{m^n}$ with *power* (or *product*) action, see (**Iw**) and (**Pw**) in Sect. 2.2 for details. There is a large body of literature on derangements; we highlight the most relevant results in Sect. 2.

For $B \leqslant \mathcal{S}_n$ and $\ell \in [n]$ denote by $\left[{B \atop \ell}\right]$ the number of permutations in B with precisely ℓ cycles in their disjoint cycle decomposition; we note that $\left[{\mathcal{S}_n \atop \ell}\right]$ is the Stirling number $\left[{n \atop \ell}\right]$ of the first kind, see [12, Section 6.1] for properties of $\left[{n \atop \ell}\right]$. Our first result concerns $\delta_k(A \wr_I B)$ and $\delta_k(A \wr_P B)$.

Theorem 1.1 *If $m, n \geqslant 2$, $A \leqslant \mathcal{S}_m$, and $B \leqslant \mathcal{S}_n$, then*

$$\delta_k(A \wr_I B) = \sum_{\ell=0}^{n} \delta_\ell(B) \sum_{j_1+\cdots+j_\ell=k} \prod_{r=1}^{\ell} \delta_{j_r}(A) \text{ and so } \delta(A \wr_I B) = \sum_{\ell=0}^{n} \delta_\ell(B)\delta(A)^\ell;$$

$$\delta_k(A \wr_P B) = \frac{1}{|B|} \sum_{\ell=1}^{n} \begin{bmatrix} B \\ \ell \end{bmatrix} \sum_{j_1\cdots j_\ell=k} \prod_{r=1}^{\ell} \delta_{j_r}(A) \text{ and}$$

$$\delta(A \wr_P B) = 1 - \frac{1}{|B|} \sum_{\ell=1}^{n} \begin{bmatrix} B \\ \ell \end{bmatrix} (1 - \delta(A))^\ell.$$

The formulas for $\delta_k(A \wr_I B)$ and $\delta_k(A \wr_P B)$ are proved in Theorems 5.1 and 6.5. Setting $k = 1$ in Theorem 1.1 gives the simple formulas for $\delta_1(A \wr_I B)$ and $\delta_1(A \wr_P B)$ stated in Corollaries 5.2 and 6.6. Formulas for $k = 0$ were known [2, Theorems 4.3, 5.4(1)] and follow easily from our results.

We now summarise our density results; as usual, for $a \leqslant b$ we write $[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$.

Theorem 1.2 *For fixed $B \leqslant \mathcal{S}_n$, the set $\{\delta(A \wr_I B) \mid A \leqslant \mathcal{S}_m$ primitive, $m \in \mathbb{N}\}$ is dense in $[\delta(B), 1]$. For fixed $A \leqslant \mathcal{S}_m$, the set $\{\delta(A \wr_I B) \mid B \leqslant \mathcal{S}_n$ imprimitive, $n \in \mathbb{N}\}$ is dense in $[\delta(A), 1]$.*

Theorem 1.3 *For fixed $B \leqslant \mathcal{S}_n$, the set $\{\delta(A \wr_P B) \mid A \leqslant \mathcal{S}_m$ primitive, $m \in \mathbb{N}\}$ is dense in $[0, 1]$. For fixed $A \leqslant \mathcal{S}_m$, the set $\{\delta(A \wr_P B) \mid B \leqslant \mathcal{S}_n$ regular, $n \in \mathbb{N}\}$ is dense in $[\delta(A), 1]$.*

Theorem 1.4 *Fix $n \geqslant 2$ and $B \leqslant \mathcal{S}_n$. Let $(C_m)_{m \geqslant 1}$ and $(A_m)_{m \geqslant 1}$ be sequences of subgroups such that $C_m \trianglelefteq A_m \leqslant \mathcal{S}_m$. Let $(G_m)_{m \geqslant 1}$ be a sequence of subgroups* *such that* $G_m \leqslant \mathcal{S}_{m^n}$ *satisfies* $C_m^{[n]} \trianglelefteq G_m \leqslant A_m \wr_P B$*, where* $C_m^{[n]} \leqslant A_m^{[n]}$ *is the subgroup of all functions* $[n] \to C_m$*, and* $B = \pi_n(G)$ *is the image of the projection map* $\pi_n \colon \mathcal{S}_m \wr_P \mathcal{S}_n \to \mathcal{S}_n$*. Suppose there exists* δ_0 *such that* $\lim_{m\to\infty} \delta(C_m a_m) = \delta_0$ *for each sequence* $(a_m)_{m\geqslant 1}$ *of elements* $a_m \in A_m$*. Then*

$$\lim_{m\to\infty} \delta(G_m) = 1 - \frac{1}{|B|} \sum_{\ell=1}^{n} \begin{bmatrix} B \\ \ell \end{bmatrix} (1 - \delta_0)^{\ell}.$$

Theorem 1.2 is proved in Sect. 5. Most of the paper deals with power action in Sect. 6 where Theorems 1.3 and 1.4 are proved. Theorem 6.5(b) is a generalisation of [2, Theorem 5.4(2)] (see also Theorem 2.1(b) below). Derangements of 'large' primitive subgroups $G \leqslant \mathcal{S}_m \wr_P \mathcal{S}_n$, i.e. those satisfying $\mathcal{A}_m^{[n]} \trianglelefteq G \leqslant \mathcal{S}_m \wr_P \mathcal{S}_n$, are considered in Corollary 6.10.

Further results of this paper include the determination of $\delta_k(G)$ for sharply t-transitive subgroups $G \leqslant \mathcal{S}_n$, see Sect. 3, and the determination of $\delta_k(C)$ for direct products $C_1 \times \cdots \times C_r$ with intransitive or product actions, see Theorems 4.1 and 4.3.

1.1 Motivation

The motivation that led us to Theorem 1.4 involved the study of primitive permutation groups that are 'large', or are 'diagonal' [16]. We explain the former. A *base* for a permutation group $G \leqslant \mathcal{S}_n$ is a set $\{x_1, \ldots, x_m\}$ of points of $[n]$ such that the elementwise stabiliser $G_{(x_1,\ldots,x_m)}$ is trivial. The minimal size of a base for $G \leqslant \mathcal{S}_n$ is denoted $b(G, [n])$, or $b(G)$. It is clear that $|G| \leqslant \prod_{i=0}^{b(G)-1}(n-i)$. In fact, the size of $b(G)$ is a good proxy for the size of G for reasons that are related to Pyber's (now solved) base size conjecture, see [9]. When considering permutation groups computationally or theoretically, groups with small base size (and hence small order) are treated very differently. For example, Seress describes very different algorithms for small and large base groups in [23], and the 'large' primitive groups with product action are considered separately by Maróti [19]. In particular, primitive groups are frequently divided into two categories ('small' and 'large'), see [19, Theorem 1.1]. The *'small'* primitive subgroups $G \leqslant \mathcal{S}_N$ satisfy $|G| \leqslant N^{1+\lfloor \log_2(N) \rfloor}$ or are one of four simple groups and the *'large'* primitive groups satisfy $\mathcal{A}_m^{[n]} \trianglelefteq G \leqslant \mathcal{S}_m \wr \mathcal{S}_n$ where $\mathcal{S}_m$ acts on k-subsets of $[m]$ and $N = \binom{m}{k}^n$. 'Large' primitive groups arise when considering orders [19, Theorem 1.1], base sizes [15, Theorem], and minimal degrees [17, Theorem 2] of primitive groups. We note that the definition of 'large' groups does not depend on the Classification of Finite Simple Groups, but the fact that they are almost always larger than 'small' groups does.

The 'large' primitive permutation groups are important in many computational and theoretical contexts. Given such a group, we wondered whether knowing only $\delta(G)$ was enough to determine the projection $\pi(G) \leqslant \mathcal{S}_n$ in Theorem 1.4. In the light of Theorem 1.3 this may seem like an impossible hope. However, if $G = A \wr_P B$ is the full wreath product, then $\delta(G) = 1 - \mathcal{C}_B(1 - \delta(A))$ holds by [2, Theorem 5.4(2)]. If $A = \mathcal{S}_m$, then $\delta(G)$ strongly influences the polynomial $\mathcal{C}_B(x)$ when n is small, and we can sometimes recover the group B. (For example, if $n < 6$, then the polynomial $\mathcal{C}_B(x)$ determines the group B, and if B is primitive, then $\mathcal{C}_B(x)$ determines B if $n < 64$.) By taking $C = \mathcal{A}_m$, Theorem 1.4 gives hope for recovering $\pi(G)$ from $\delta(G)$ even when G is *not* a wreath product. In order to estimate $\delta(G)$ when $\mathcal{A}_m^{[n]} \trianglelefteq G \leqslant \mathcal{S}_m \wr_P \mathcal{S}_n$, we consider proportions of elements in *subsets* of G.

Our interest was piqued by a recent announcement [21, Theorem 1.1] that if $G \leqslant \mathcal{S}_n$ is a subgroup with $\delta(Gc) = \delta(\mathcal{S}_n)$ for some $c \in \mathcal{S}_n$, then $Gc = \mathcal{S}_n$ and hence $G = \mathcal{S}_n$; see [21, Remark 1.2] for a comment on subsets $C \leqslant \mathcal{S}_n$ with $\delta(C) = \delta(\mathcal{S}_n)$. On a computational note, we mention that Arvind [1] presents an algorithm which takes as input $k \in [n]$ and a subgroup $\langle S\rangle \leqslant \mathcal{S}_n$ and outputs a permutation in $\langle S\rangle$ that moves at least k points: to find a derangement set $k = n$.

2 Background

If $G \leqslant \mathcal{S}_n$ is a transitive subgroup and $n \geqslant 2$, then the Orbit-Counting Theorem implies that $\mathrm{Der}(G)$ is non-empty; this result is due to Jordan, see [24, Theorem 4]. Given a transitive permutation group $G \leqslant \mathcal{S}_n$, its *rank* r is the number of orbits of a point-stabiliser on $[n]$. For such a G, the following bounds hold:

$$\frac{r-1}{n} \leqslant \delta(G) \leqslant 1 - \frac{1}{r}. \tag{1}$$

The lower bound was proved by Cameron and Cohen [4], and the upper bound by Diaconis, Fulman, and Guralnick [7, Theorem 3.1], see also Guralnick, Isaacs, and Spiga [13] for a short proof.

Serre [24] describes some interesting consequences of the existence of derangements to number theory and topology. Fulman and Guralnick [11] show that if G is a sufficiently large finite simple group acting faithfully and transitively on $[n]$, then $\delta(G) \geqslant 0.016$. This result completes the proof of the Boston–Shalev Conjecture, which claims that there is a constant $\varepsilon > 0$ such that $\delta(G) > \varepsilon$ for any such group G.

In the course of our research, we proved three (quite natural) results that were previously proved by Boston et al. [2, Theorems 4.3, 5.4(2), 5.11].

Theorem 2.1 *(Boston et al. [2]) Let $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$. Then*

(a) $\delta(A \wr_I B) = \mathcal{P}_B(\delta(A))$ *where* $\mathcal{P}_B(x) = \sum_{\ell=0}^{n} \delta_\ell(B)x^\ell$.
(b) $\delta(A \wr_P B) = 1 - \mathcal{C}_B(1-\delta(A))$ *where* $\mathcal{C}_B(x) = \frac{1}{|B|}\sum_{\ell=1}^{n} \left[{B \atop \ell}\right] x^\ell$.
(c) *The set* $\{\delta(C) \mid C \leqslant \mathcal{S}_n \text{ primitive},\ n \in \mathbb{N}\}$ *is dense in* $[0, 1]$.

2.1 Cycle index polynomials and derangements

For $g \in \mathcal{S}_n$ and $i \in [n]$ let $c_i(g)$ denote the number of i-cycles in the disjoint cycle decomposition of g, and let $c(g)$ denote the number of cycles of g, so that $c(g) = \sum_{i=1}^{n} c_i(g)$. The well-known *cycle index polynomial* $\mathcal{Z}(G)$ of $G \leqslant \mathcal{S}_n$ is defined to be the multivariate polynomial

$$\mathcal{Z}(G) = \mathcal{Z}(G; x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x_1^{c_1(g)} \cdots x_n^{c_n(g)}.$$

Specialising the variables x_i allows one to obtain polynomials with fewer variables which are easier to calculate. For example, the probability generating functions $\mathcal{P}_G(x)$ and $\mathcal{C}_G(x)$ for the number of fixed points and the number of permutations with ℓ cycles, respectively, are

$$\mathcal{P}_G(x) = \mathcal{Z}(G; x, 1, \ldots, 1) = \frac{1}{|G|} \sum_{\ell=0}^{n} |\mathrm{Fix}_\ell(G)| x^\ell = \sum_{\ell=0}^{n} \delta_\ell(G) x^\ell \qquad \text{and}$$
$$\mathcal{C}_G(x) = \mathcal{Z}(G; x, x, \ldots, x) = \frac{1}{|G|} \sum_{g \in G} x^{c(g)} = \frac{1}{|G|} \sum_{\ell=1}^{n} \begin{bmatrix} G \\ \ell \end{bmatrix} x^\ell,$$

where, as in Sect. 1, we denote the number of $g \in G$ with $c(g) = \ell$ by

$$\begin{bmatrix} G \\ \ell \end{bmatrix} = |\{g \in G \mid c(g) = \ell\}|\,. \tag{2}$$

Clearly, $g \in \mathrm{Fix}_k(G)$ if and only if $c_1(g) = k$; in particular, $\delta(G) = \mathcal{P}_G(0)$. We note that for $k = 0$, the formulas in Theorems 1.1 and 1.4 can be expressed using the cycle index polynomial.

Formulas for the cycle index polynomials $\mathcal{Z}(A \wr_I B)$ and $\mathcal{Z}(A \wr_P B)$ date back to Pólya and to Palmer and Robinson, respectively [20, Theorems 1, 2]. Our proof of Theorem 1.1 does not use these formulas, as they are rather complicated, especially for $\mathcal{Z}(A \wr_P B)$. The GAP package WPE [22] is a useful research tool which, in particular, can compute $\mathcal{Z}(A \wr_P B)$. We note that the fixed point polynomials and cycle polynomials are also studied in [6, 14]. For example, $\mathcal{C}_{A \wr_I B}(x) = \mathcal{C}_B(\mathcal{C}_A(x))$ holds for all $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$ by [6, Proposition 8].

2.2 Group actions

We denote a group G acting on a set X by (G, X). The corresponding homomorphism $\varphi\colon G \to \mathrm{Sym}(X)$ allows us to identify $G/\ker\varphi$ with the subgroup $H = \varphi(G) \leqslant \mathcal{S}_n$ where $|X| = n$. Since

$$\frac{|\{g \in G \mid \varphi(g) \text{ is a derangement}\}|}{|G|} = \frac{|\{h \in H \mid h \text{ is a derangement}\}|}{|H|}$$

we define $\delta(G, X) = \delta(\varphi(G))$. Thus we shall henceforth consider *faithful* actions, and view G as a *subgroup* of $\mathrm{Sym}(X)$. Moreover, if $Y \to X$ is a surjection of G-sets, $\delta(G, Y) \geqslant \delta(G, X)$ holds, see [10, p. 3], which implies that when considering lower bounds for the proportion of derangements of a group (for any action), one can restrict to primitive actions.

Given a group A and a permutation group $B \leqslant \mathrm{Sym}(Y)$ the permutational wreath product $A \wr_Y B$ is defined as the split extension $A^Y \rtimes B$ where A^Y is the group of all functions $Y \to A$ with pointwise multiplication. The group $A^Y \rtimes B$ has underlying set $A^Y \times B$ and multiplication

$$(\alpha, b)(\beta, c) = (\alpha\beta^{b^{-1}}, bc) \qquad \text{where } \alpha, \beta \in A^Y, b, c \in B,$$

and $\beta^b\colon Y \to A$ is the map $y \mapsto \beta(yb^{-1})$. Following Cameron, Gewurz, and Merola [5], given permutation groups $A \leqslant \mathrm{Sym}(X)$ and $B \leqslant \mathrm{Sym}(Y)$ the direct product $A \times B$ has an *intransitive* action (**Ix**) and a *product* action (**Px**) and the wreath product $A \wr_Y B$ has an *imprimitive* action (**Iw**) and a *power* action (**Pw**), defined as follows. To avoid towers of exponents, we write the action of $g \in G$ on $x \in X$ as xg, and not x^g. Also $X \dot\cup Y$ denotes the disjoint union of X and Y.

- (**Ix**) $(a, b) \in A \times B$ acts on $X \dot\cup Y$ via $x(a, b) = xa$ if $x \in X$, and $y(a, b) = yb$ if $y \in Y$.
- (**Px**) $(a, b) \in A \times B$ acts on $(x, y) \in X \times Y$ via $(x, y)(a, b) = (xa, yb)$.
- (**Iw**) $(\alpha, b) \in A \wr_Y B$ acts on $(x, y) \in X \times Y$ via $(x, y)(\alpha, b) = (x\alpha(y), yb)$.
- (**Pw**) $(\alpha, b) \in A \wr_Y B$ acts on $\omega \in X^Y$ via $\omega(\alpha, b)\colon Y \to X, y \mapsto \omega(yb^{-1})\alpha(yb^{-1})$.

In the case that $Y = [n]$, we identify $\omega \in X^Y$ and $\alpha \in A^Y$ with n-tuples $(\omega_1, \dots, \omega_n)$, $(\alpha_1, \dots, \alpha_n)$ where each $\omega_i = \omega(i)$ and $\alpha_i = \alpha(i)$, respectively. The power action (**Pw**) of the base group A^Y coincides with (iterated) product action (**Px**). Most authors call (**Pw**) product action; although unconventional, we shall refer to (**Pw**) as power action to avoid confusion with (**Px**). If $A \leqslant \mathrm{Sym}(X)$ is primitive but not regular, $B \leqslant \mathrm{Sym}(Y)$ is transitive, $|Y|$ is finite, and $1 < |X|, |Y|$, then $A \wr_Y B \leqslant \mathrm{Sym}(X^Y)$ is primitive by [8, Lemma 2.7A], so (**Pw**) can suggest a *primitive/product/power wreath* product, so (**Pw**) is a good abbreviation. We change the subscript in $A \wr_Y B$ in favour of the notation $\wr_I$ or $\wr_P$.

Notation We abbreviate $(A \times B, X \dot\cup Y)$, $(A \times B, X \times Y)$, $(A \wr_Y B, X \times Y)$, and $(A \wr_Y B, X^Y)$ by $A \times_I B$, $A \times_P B$, $A \wr_I B$, and $A \wr_P B$, respectively.

We emphasise that there is a permutation isomorphism $(A \wr_I B) \wr_I C \cong A \wr_I (B \wr_I C)$, but associativity fails for $\wr_P$. Indeed $|(X^Y)^Z| = |X^{Y\times Z}| \neq |X^{(Y^Z)}|$ if $|Y| \geqslant 2, |Z| \geqslant 2$ and $|Y|^{|Z|} \neq 4$.

3 Sharply transitive groups

Let $G \leqslant \mathcal{S}_n$ be a subgroup that acts sharply t-transitively, that is, it acts regularly on the set of t-tuples with distinct entries in $[n]$. Hence $|G|$ equals the number $n!/(n-t)!$ of such tuples.

Theorem 3.1 *Let $n \geqslant 2$, $t \in [n]$, and $k \in \{0, \ldots, n\}$. If $G \leqslant \mathcal{S}_n$ is sharply t-transitive, then $\delta_n(G) = 1/|G|$, $\delta_k(G) = 0$ if $t \leqslant k < n$, and*

$$\delta_k(G) = \frac{1}{k!}\sum_{j=0}^{t-k-1} \frac{(-1)^j}{j!} + \frac{(n-t)!}{k!}\sum_{j=t-k}^{n-k} \frac{(-1)^j}{(n-k-j)!j!} \qquad \text{if } 0 \leqslant k < t.$$

Proof Note that $|\mathrm{Fix}_n(G)| = 1$ and $|\mathrm{Fix}_k(G)| = 0$ for $t \leqslant k < n$, so the claim is true for $k \geqslant t$. Suppose now that $k < t$. For a subset $K \subseteq [n]$ let $G_{(K)}$ be the elementwise stabiliser of K in G. Note that $|\mathrm{Der}(G, [n])| = |G| - |F|$, where $F = G_{(1)} \cup \cdots \cup G_{(n)}$. This number can be determined using a standard inclusion–exclusion argument (see also [2, Theorem 2.3]), and we obtain

$$\delta(G, [n]) = \sum_{j=0}^{t-1} \frac{(-1)^j}{j!} + (n-t)!\sum_{j=t}^{n} \frac{(-1)^j}{j!(n-j)!}. \tag{3}$$

The set K of fixed points of $g \in \mathrm{Fix}_k(G)$ has size k, and g induces a derangement on the complement $K' = [n] \setminus K$. We view $G_{(K)}$ as a sharply $(t-k)$-transitive permutation group of degree $|K'| = n-k$. Since $|G_{(K)}| = (n-k)!/(n-t)!$, we have

$$\begin{aligned} |\mathrm{Fix}_k(G)| &= \binom{n}{k}|G_{(K)}|\,\delta(G_{(K)}, K') = \frac{n!}{k!(n-t)!}\,\delta(G_{(K)}, K') \\ &= \frac{|G|}{k!}\,\delta(G_{(K)}, K'). \end{aligned} \tag{4}$$

Dividing by $|G|$, and using (3) with n and t replaced by $n-k$ and $t-k$ proves the claim. □

Remark 3.2 For an alternative proof of Theorem 3.1, let $\mathcal{Q}_G(x) = \sum_{\ell=0}^{n} a_\ell x^\ell/\ell!$ where a_ℓ denotes the number of orbits of G on ℓ-tuples of distinct points. If G is sharply t-transitive, then $a_\ell = \prod_{i=1}^{\ell-t}(n-\ell+i)$ if $t < \ell \leqslant n$, and $a_\ell = 1$ if $\ell \leqslant t$. We may deduce $\delta_k(G)$ from the polynomial identity $\mathcal{P}_G(x) = \sum_{k=0}^{n} \delta_k(G)x^k = \mathcal{Q}_G(x-1)$ see [2] or [3, Theorem 1.1].

Throughout, we define $d_0 = e_0 = 1$, and if $n \geqslant 1$ is an integer, then

$$d_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \text{and} \quad e_n = \sum_{k=n-1}^{n} \frac{(-1)^k}{k!} = \frac{(-1)^{n-1}(n-1)}{n!}.$$

Note that $\lim_{n\to\infty} e_n = 0$ and $\lim_{n\to\infty} d_n = e^{-1}$, where e denotes the Euler number. Also, if $n \geqslant 2$, then $\mathcal{S}_n$ is sharply $(n-1)$-transitive, and $\mathcal{A}_n$ is sharply $(n-2)$-transitive. The next result follows from Theorem 3.1 with a little algebra.

Corollary 3.3 *If* $n \geqslant 1$ *and* $k \in \{0, \ldots, n\}$, *then*

(a) $\delta_k(\mathcal{S}_n) = d_{n-k}/k!$,
(b) $\delta_k(\mathcal{A}_n) = (d_{n-k} + e_{n-k})/k!$, *and*
(c) $\delta_k(\mathcal{S}_n \setminus \mathcal{A}_n) = (d_{n-k} - e_{n-k})/k!$.

If k is fixed, then

$$\lim_{n\to\infty} \delta_k(\mathcal{S}_n) = \lim_{n\to\infty} \delta_k(\mathcal{A}_n) = \lim_{n\to\infty} \delta_k(\mathcal{S}_n \backslash \mathcal{A}_n) = e^{-1}/k!.$$

Remark 3.4 If $G \leqslant \mathcal{S}_n$ has s orbits on $[n]$, then the Orbit-Counting Lemma implies that

$$\frac{1}{s}\sum_{k=0}^{n} k|\mathrm{Fix}_k(G)| = |G| = \sum_{k=0}^{n} |\mathrm{Fix}_k(G)| \quad \text{and hence } \delta(G) = \sum_{k=1}^{n}\left(\frac{k}{s} - 1\right)\delta_k(G).$$

If $(G, [n])$ is a sharply t-transitive subgroup of $\mathcal{S}_n$, then $s = 1$ and $|G| = n!/(n-t)!$, and Equation (4) transforms the latter equation for $\delta(G)$ into the following recurrence relation

$$\delta(G, [n]) = \frac{(n-t)!}{n(n-2)!} + \sum_{k=2}^{t-1} \frac{1}{k(k-2)!}\delta(G_{([k])}, [n] \setminus [k]),$$

where $(G_{([k])}, [n] \setminus [k])$ is a $(t-k)$-transitive subgroup of $\mathrm{Sym}([n] \setminus [k]) \cong \mathcal{S}_{n-k}$.

4 Direct products

4.1 Product action

The following result generalises [2, Lemma 6.1].

Theorem 4.1 *For* $r > 0$ *and* $i \in [r]$ *let* $G_i \leqslant \mathrm{Sym}(X_i)$. *Let the subgroup* $G = G_1 \times \cdots \times G_r$ *of* $\mathrm{Sym}(X_1 \times \cdots \times X_r)$ *act via* (**Px**). *If* $C \subseteq G$ *is a subset of the form* $C = C_1 \times \cdots \times C_r$ *with each* $C_i \subseteq G_i$, *then*

$$\delta_k(C) = \sum_{i_1 \cdots i_r = k} \prod_{s=1}^{r} \delta_{i_s}(C_s) \quad \textit{and} \quad \delta(C) = 1 - \prod_{i=1}^{r}(1 - \delta(C_i)) \quad \textit{if } k = 0.$$

Proof Note that C is partitioned into sets $D_{i_1,\ldots,i_r} = \mathrm{Fix}_{i_1}(C_1) \times \cdots \times \mathrm{Fix}_{i_r}(C_r)$ for $i_1, \ldots, i_r \geqslant 0$, and every element in $D_{i_1,\ldots,i_r}$ has exactly $i_1 \cdots i_r$ fixed points. Thus we have

$$|\mathrm{Fix}_k(C)| = \sum_{i_1 \cdots i_r = k} \prod_{s=1}^{r} |\mathrm{Fix}_{i_s}(C_s)|$$

and dividing by $|C| = \prod_{s=1}^{r} |C_i|$ yields the first claim.

For $k = 0$, note that $(x_1, \dots, x_r) \in X_1 \times \cdots \times X_r$ is fixed by $(c_1, \dots, c_r) \in C$ if and only if $x_i c_i = x_i$ for each $i \in [r]$. Thus, $1 - \delta(C) = \prod_{i=1}^{r}(1 - \delta(C_i))$, as claimed. □

We now give examples of subgroups all having similar proportions of derangements.

Corollary 4.2 *Let $G_{m_1,\dots,m_r} \leqslant \mathcal{S}_{m_1} \times \cdots \times \mathcal{S}_{m_r}$ act via product action ($\mathbf{Px}$) where*

$$\mathcal{A}_{m_1} \times \cdots \times \mathcal{A}_{m_r} \trianglelefteq G_{m_1,\dots,m_r} \leqslant \mathcal{S}_{m_1} \times \cdots \times \mathcal{S}_{m_r}.$$

For the multi-indexed sequence $(G_{m_1,\dots,m_r})$ of subgroups, we have

$$\lim_{m_1,\dots,m_r \to \infty} \delta\left(G_{m_1,\dots,m_r}\right) = 1 - (1 - e^{-1})^r.$$

Proof Note that $G = G_{m_1,\dots,m_r}$ is a union of cosets of $A = \mathcal{A}_{m_1} \times \cdots \times \mathcal{A}_{m_r}$ of the form

$$C = A(c_1, \dots, c_r) = (\mathcal{A}_{m_1} c_1) \times \cdots \times (\mathcal{A}_{m_r} c_r).$$

For each i, either $\mathcal{A}_{m_i} c_i = \mathcal{A}_{m_i}$ or $\mathcal{A}_{m_i} c_i = \mathcal{S}_{m_i} \setminus \mathcal{A}_{m_i}$, so $\delta\left(\mathcal{A}_{m_i} c_i\right) = d_{m_i} \pm e_{m_i}$ by Corollary 3.3. Since each $\delta\left(\mathcal{A}_{m_i} c_i\right) \to e^{-1}$ for $m_i \to \infty$, Theorem 4.1 implies that $\delta\,(Ac) \to 1-(1-e^{-1})^r$ as $m_1, \dots, m_r \to \infty$. If G is a disjoint union $G = \bigcup_{c \in \mathscr{C}} Ac$, so $|G| = |A||\mathcal{C}|$, then

$$\delta\,(G) = \frac{1}{|G|} \sum_{c \in \mathscr{C}} |\mathrm{Der}(Ac)| = \frac{1}{|\mathscr{C}|} \sum_{c \in \mathscr{C}} \delta(Ac).$$

As shown above, $\sum_{c \in \mathscr{C}} \delta(Ac)$ converges to $|\mathscr{C}|(1 - (1 - e^{-1})^r)$, which implies the claim. □

4.2 Intransitive action

Counting derangements with an intransitive action is straightforward.

Theorem 4.3 *For $r > 0$ and $i \in [r]$ let $G_i \leqslant \mathrm{Sym}(X_i)$. Let the subgroup $G = G_1 \times \cdots \times G_r$ of $\mathrm{Sym}(X_1 \dot\cup \cdots \dot\cup X_r)$ act via ($\mathbf{Ix}$). If $C \subseteq G$ is a subset of the form $C = C_1 \times \cdots \times C_r$ with each $C_i \subseteq G_i$, then*

$$\delta_k\,(C) = \sum_{k_1+\cdots+k_r=k} \prod_{i=1}^{r} \delta_{k_i}\,(C_i) \qquad \textit{and hence} \qquad \delta(C) = \prod_{i=1}^{r} \delta(C_i).$$

Proof We have $(c_1, \ldots, c_r) \in \mathrm{Fix}_k(C)$ if and only if for each $i \in [r]$, there exists a k_i such that $c_i \in \mathrm{Fix}_{k_i}(C_i)$ and $k = k_1 + \cdots + k_r$. The result follows from

$$|\mathrm{Fix}_k(C)| = \sum_{k_1+\cdots+k_r=k} \prod_{i=1}^{r} |\mathrm{Fix}_{k_i}(C_i)|.$$

□

The following result gives subgroups with similar proportions of derangements. We omit the proof as it similar to that of Corollary 4.2 but using Theorem 4.3 instead of Theorem 4.1.

Corollary 4.4 *Let $G_{m_1,\ldots,m_r} \leqslant \mathcal{S}_{m_1} \times \cdots \times \mathcal{S}_{m_r}$ act via intransitive action* (**Ix**) *where*

$$\mathcal{A}_{m_1} \times \cdots \times \mathcal{A}_{m_r} \trianglelefteqslant G_{m_1,\ldots,m_r} \leqslant \mathcal{S}_{m_1} \times \cdots \times \mathcal{S}_{m_r}.$$

The multi-indexed sequence $(G_{m_1,\ldots,m_r})$ satisfies $\lim_{m_1,\ldots,m_r\to\infty} \delta\left(G_{m_1,\ldots,m_r}\right) = e^{-r}$.

5 Wreath products $A \wr_I B$ with imprimitive action

In this section we find a formula for $\delta_k(A \wr_I B)$, which is the first formula in Theorem 1.1, and we prove the density result Theorem 1.2.

Theorem 5.1 *Let $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$. Then*

$$\delta_k(A \wr_I B) = \sum_{\ell=0}^{n} \delta_\ell(B) \sum_{j_1+\cdots+j_\ell=k} \prod_{r=1}^{\ell} \delta_{j_r}(A) \quad \textit{and hence} \quad \delta(A \wr_I B) = \sum_{\ell=0}^{n} \delta_\ell(B)\delta(A)^\ell.$$

Proof An element $(\alpha, b) \in A \wr_I B$ fixes $(x, y) \in [m] \times [n]$ if and only if $x = x\alpha(y)$ and $y = yb$. If $y_1, \ldots, y_\ell$ are the fixed points of b (so $b \in \mathrm{Fix}_\ell(B)$) and $x_{i,1}, \ldots, x_{i,j_i}$ are the fixed points of $\alpha(y_i)$ (so $\alpha(y_i) \in \mathrm{Fix}_{j_i}(A)$), then each $(x_{i,s}, y_i)$ is a fixed point of (α, b), and every fixed point of (α, b) has this form. No constraints are imposed upon $\alpha(y)$ for $y \in [n] \setminus \{j_1, \ldots, j_\ell\}$. The total number of fixed points is $k = j_1 + \cdots + j_\ell$, hence $(\alpha, b) \in \mathrm{Fix}_k(A \wr_I B)$. This shows the following, and dividing by $|A \wr_I B| = |A|^n|B|$ proves the claim:

$$|\mathrm{Fix}_k(A \wr_I B)| = \sum_{\ell=0}^{n} |\mathrm{Fix}_\ell(B)| \sum_{j_1+\cdots+j_\ell=k} |A|^{n-\ell} \prod_{r=1}^{\ell} |\mathrm{Fix}_{j_r}(A)|.$$ □

Corollary 5.2 *If $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$, then*

$$\delta_1(A \wr_I B) = \delta_1(A)\mathcal{P}'_B(\delta(A))$$

where $\mathcal{P}'_B(x) = \sum_{\ell=1}^{n} \ell\delta_\ell(B)x^{\ell-1}$ is the derivative of $\mathcal{P}_B(x)$.

The polynomial $\mathcal{P}_B(x)$ can be difficult to compute precisely, but in some cases can be easy to approximate. For example, when B is transitive and $\delta(B)$ is small, then $\mathcal{P}_B(x)$ is convex for $x \in [0, 1]$, that is, cup shaped, and slightly larger than x by the following lemma.

Lemma 5.3 *If* $B \leqslant \mathcal{S}_n$*, then* $\mathcal{P}_B(x)$ *is convex. If* B *is transitive, then* $0 \leqslant \mathcal{P}_B(x) - x \leqslant \mathcal{P}_B(0)$ *holds for* $0 \leqslant x \leqslant 1$.

Proof The claim is trivially true if $n = 1$, so let $n \geqslant 2$ and write $f(x) = \mathcal{P}_B(x) - x$. Since $\mathcal{P}_B(x)$ has non-negative coefficients and highest term $x^n/|B|$, the second derivative satisfies $f''(x) = \mathcal{P}''_B(x) > 0$ for $0 < x \leqslant 1$, so both $f(x)$ and $\mathcal{P}_B(x)$ are convex on $[0, 1]$. By [2, Theorem 4.6(4)], we have $\mathcal{P}'_B(x) = \mathcal{P}_H(x)$ where $H \leqslant \mathcal{S}_{n-1}$ is the stabiliser of n in B acting on $[n - 1]$. This implies $\mathcal{P}'_B(1) = 1$ and $\mathcal{P}'_B(x) \leqslant 1$ for $x \in [0, 1]$. Thus, $f(1) = f'(1) = 0$, and convexity implies that $f(x)$ lies above the tangent line $y = 0$ at $(1, 0)$, that is, $0 \leqslant f(x)$ on the domain $[0, 1]$. Since $f'(x) = \mathcal{P}'_B(x) - 1 \leqslant 0$ on $[0, 1]$, the function $f(x)$ is decreasing on $[0, 1]$. This proves the last claim $f(x) \leqslant f(0) = \mathcal{P}_B(0)$. □

We now prove Theorem 1.2, employing a similar argument as in [2, Theorem 4.13].

Proof of Theorem 1.2 Fix $B \leqslant \mathcal{S}_n$. The map $\mathcal{P}_B\colon [0, 1] \to (0, \infty)\colon z \mapsto \sum_{k=0}^{n} \delta_k(B) z^k$ satisfies $\delta(A \wr_I B) = \mathcal{P}_B(\delta(A))$ by Theorem 2.1(a). Note that $\mathcal{P}_B$ is a continuous increasing polynomial function with $\mathcal{P}_B(0) = \delta(B)$ and $\mathcal{P}_B(1) = 1$, so $\delta(B) \leqslant \delta(A \wr_P B) < 1$ for all $A \leqslant \mathcal{S}_m$. Letting $m \geqslant 1$ and A vary over the primitive subgroups of $\mathcal{S}_m$, it follows from Theorem 2.1(c) that the values of $\delta(A \wr_P B)$ are dense in the interval $[\delta(B), 1]$. This proves the first claim.

For the second claim, fix $A \leqslant \mathcal{S}_m$. If $B \leqslant \mathcal{S}_n$ is imprimitive, then B is transitive and $\delta(A \wr_I B) = \mathcal{P}_B(\delta(A)) \geqslant \delta(A)$ by Lemma 5.3, so the values of $\delta(A \wr_I B)$ (with B imprimitive) lie in $[\delta(A), 1]$. Let q be a prime power, and let $C_q = \mathrm{AGL}_1(q) \leqslant \mathcal{S}_q$ act naturally on q affine points; we also abbreviate $C = C_q$. An easy calculation shows that

$$\mathcal{P}_C(x) = \frac{1}{q} + \frac{(q-2)x}{q-1} + \frac{x^q}{q(q-1)},$$

see [2, Example 4.4]. Let $B_r = B_{r,q}$ be the imprimitive wreath product $C \wr_I \cdots \wr_I C$ of r copies of C. We show that the closure of the set $\{\delta(A \wr_I B_{r,q}) \mid r \geqslant 0, q \text{ prime power}\}$ is the interval $[\delta(A), 1]$.

The recurrence $\mathcal{P}_{B_r}(x) = \mathcal{P}_{B_{r-1}}(\mathcal{P}_C(x))$ holds by [2, Theorem 4.6(8)] since $B_r = C \wr_I B_{r-1}$. The function $\mathcal{P}_{B_{r-1}}(x)$ is increasing, and since $\mathcal{P}_C(x) > \alpha + \beta x$ for $\alpha = \frac{1}{q}$ and $\beta = \frac{q-2}{q-1}$, we have $\mathcal{P}_{B_r}(x) > \mathcal{P}_{B_{r-1}}(\alpha + \beta x)$. An induction on r now shows that $\mathcal{P}_{B_r}(x) > \alpha(1 + \beta + \cdots + \beta^{r-1}) + \beta^r x$, and therefore $\mathcal{P}_{B_r}(x) > \alpha \frac{\beta^r - 1}{\beta - 1} = (1 - \frac{1}{q})(1 - \beta^r)$ for all $x \in [0, 1]$. Now choose $q > 1/\varepsilon$, so that $1 - \frac{1}{q} > 1 - \varepsilon$, and then choose r such that $(1 - \frac{1}{q})(1 - \beta^r) > 1 - \varepsilon$.

Let $c_r = \delta(A \wr_I B_r)$. It follows from Theorem 2.1(a) that $\delta(A \wr_I B_r) = \mathcal{P}_{B_r}(\delta(A))$ and the previous paragraph shows that $c_r > 1 - \varepsilon$. As C is transitive, $0 \leqslant \mathcal{P}_C(x) - x \leqslant \mathcal{P}_C(0)$ for $x \in [0, 1]$ by Lemma 5.3, and so $0 \leqslant \mathcal{P}_C(x) - x \leqslant \frac{1}{q} < \varepsilon$. Set $c_0 = \delta(A)$

and $c_i = \mathcal{P}_C(c_{i-1})$. Then $0 \leqslant c_i - c_{i-1} < \varepsilon$ holds for $1 \leqslant i \leqslant r$ and $1 - \varepsilon < c_r < 1$. The claim now follows since each $c_i = \delta(A \wr_I C \wr_I \cdots \wr_I C) = \delta(A \wr_I B_{i,q})$ with $B_{i,q}$ imprimitive. □

In the special case that $B = \mathcal{S}_m$, we obtain the following explicit formula.

Corollary 5.4 *Setting $B = \mathcal{S}_n$ in Theorem 5.1 gives*

$$\delta(A \wr_I \mathcal{S}_n) = \sum_{\ell=0}^{n} \frac{\delta(A)^\ell}{\ell!} \sum_{i=0}^{n-\ell} \frac{(-1)^i}{i!} = \sum_{i=0}^{n} \frac{(-1)^i}{i!} \sum_{\ell=0}^{n-i} \frac{\delta(A)^\ell}{\ell!}.$$

Hence $\lim_{n\to\infty} \delta(A \wr_I \mathcal{S}_n) = e^{\delta(A)-1}$, *and so* $\lim_{n,m\to\infty} \delta(\mathcal{S}_m \wr_I \mathcal{S}_n) = e^{e^{-1}-1}$.

Proof The first displayed formula follows from Theorem 5.1 since $|\delta_\ell(\mathcal{S}_n)| = d_{n-\ell}/\ell!$ by Corollary 3.3. Interchanging summations gives the second formula. The first limit now follows, and the second follows from $\lim_{m\to\infty} \delta(\mathcal{S}_m) = e^{-1}$ by Corollary 3.3. □

6 Wreath products $A \wr_P B$ with power action (Pw)

Given $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$, we study $A \wr_P B$ acting with power action on the set $\Omega = [m]^{[n]}$. Recall that the elements of $A \wr_P B$ are (α, b) with $\alpha \in A^{[n]}$ and $b \in B$, and that we sometimes write $\alpha \in A^{[n]}$ and $\omega \in \Omega$ as $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\omega = (\omega_1, \ldots, \omega_n)$, respectively. Using (**Pw**), the element $(\alpha, b) \in A \wr_P B$ acts on $\omega \in \Omega$ via

$$\omega(\alpha, b) \colon [n] \to [m], \quad y \mapsto \omega(yb^{-1})\alpha(yb^{-1}).$$

We start with a brief discussion of the numbers $\left[{G \atop \ell}\right]$ defined in Equation (2). If $G = \mathcal{S}_n$, then $\left[{G \atop \ell}\right] = \left[{n \atop \ell}\right]$ is the (unsigned) Stirling number $\left[{n \atop \ell}\right]$ of the first kind, see [12, Section 6.1] for details. It is easy to see that for $n \geqslant 1$ and $G \leqslant \mathcal{S}_n$ we have

$$\left[{n \atop 1}\right] = (n-1)!, \quad \left[{n \atop n-1}\right] = \binom{n}{2}, \quad \left[{n \atop n}\right] = 1 \quad \text{and} \quad \sum_{\ell=0}^{n} \left[{G \atop \ell}\right] = |G|.$$

Example 6.1 If $C_n = \langle (1, 2, \ldots, n) \rangle \leqslant \mathcal{S}_n$, then $\left[{C_n \atop d}\right] = \phi(n/d)$ if $d \mid n$, and 0 otherwise, where ϕ denotes Euler's ϕ-function, see [2, Lemma 5.6]: the disjoint cycle decomposition of $(1, 2, \ldots, n)^k$ consists of n/d cycles each of length $d = \gcd(n, k)$. Such an element has order n/d, and there are $\phi(n/d)$ such elements in B. In particular, $\sum_{d|n} \phi(n/d) = n$.

For $\mathcal{A}_n$, we observe an alternating behaviour: $\left[{\mathcal{A}_n \atop \ell}\right] = 0$ if $\ell \not\equiv n \bmod 2$, and $\left[{\mathcal{A}_n \atop \ell}\right] = \left[{n \atop \ell}\right]$ otherwise. This follows from the next lemma.

Lemma 6.2 *If $G \leqslant \mathcal{S}_n$, then $\left[{G \atop \ell}\right] = \left[{G \cap \mathcal{A}_n \atop \ell}\right]$ if $\ell \equiv n \bmod 2$, and $\left[{G \atop \ell}\right] = \left[{G \cap (\mathcal{S}_n \setminus \mathcal{A}_n) \atop \ell}\right]$ otherwise.*

Proof Let $g \in G$ have cycle decomposition $g = g_1 \cdots g_\ell$. Note that $|g_1|+\cdots+|g_\ell| = n$, and g_i has sign -1 if and only if $|g_i|$ is even. We can assume $g_1, \ldots, g_s$ have sign -1 and $g_{s+1}, \ldots, g_\ell$ have sign 1, so $g \in \mathcal{A}_n$ if and only if s is even. The cycles $g_1, \ldots, g_s$ involve an even number $e = |g_1| + \cdots + |g_s|$ of points, and the remaining $\ell - s$ cycles involve $n - e$ points. Each of $g_{s+1}, \ldots, g_\ell$ has odd length, which forces $n - e \equiv \ell - s \bmod 2$. Thus, if $n \equiv \ell \bmod 2$, then $s \equiv e \equiv 0 \bmod 2$, and $g \in \mathcal{A}_n$. If $n \not\equiv \ell \bmod 2$, then $s \not\equiv e \equiv 0 \bmod 2$, and $g \in \mathcal{S}_n \setminus \mathcal{A}_n$. The claim now follows. □

In particular, if $G \leqslant \mathcal{S}_n$ but $\mathcal{A}_n \not\leqslant G$, then $H = G \cap \mathcal{A}_n$ has index 2 in G, which implies that half of all elements lie in H (and these elements all have cycle number congruent to n modulo 2) and the other half of elements lie in $G \setminus H$ (and these elements all have cycle number congruent to $n+1$ modulo 2). This is summarised in the following lemma.

Lemma 6.3 *If $G \leqslant \mathcal{S}_n$ and $\mathcal{A}_n \not\leqslant G$, then* $\sum_{i=1}^{\lfloor n/2 \rfloor} \begin{bmatrix} G \\ 2i \end{bmatrix} = \frac{|G|}{2} = \sum_{i=1}^{\lceil n/2 \rceil} \begin{bmatrix} G \\ 2i-1 \end{bmatrix}$.

6.1 Formula for $\delta_k(A \wr_P B)$ in Theorem 1.1

We start with a definition.

Definition 6.4 Let $b \in \mathcal{S}_n$ have disjoint cycle decomposition $b = b_1 \cdots b_\ell$ (including trivial cycles) and let $b_i = (y_i, y_i b, \ldots, y_i b^{k_i - 1})$ be a k_i-cycle whose smallest element is y_i. If $\alpha \in A^{[n]}$, then the *b_i-product* of α is $b_i(\alpha) = \alpha(y_i)\alpha(y_i b) \cdots \alpha(y_i b^{k_i-1}) \in A$.

Theorem 6.5 *Let $G = A \wr_P B \leqslant \mathcal{S}_{m^n}$ where $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$.*

(a) *Let $b \in B$ with disjoint cycle decomposition $b = b_1 \cdots b_\ell$. Then*

$$\delta_k(A^{[n]}b) = \sum_{j_1 \cdots j_\ell = k} \prod_{r=1}^{\ell} \delta_{j_r}(A);$$

if $k = 0$, then $\delta(A^{[n]}b) = 1 - (1 - \delta(A))^\ell$.

(b) *We have*

$$\delta_k(A \wr_P B) = \frac{1}{|B|} \sum_{\ell=1}^{n} \begin{bmatrix} B \\ \ell \end{bmatrix} \sum_{j_1 \cdots j_\ell = k} \prod_{r=1}^{\ell} \delta_{j_r}(A);$$

if $k = 0$, then

$$\delta(A \wr_P B) = 1 - \frac{1}{|B|} \sum_{\ell=1}^{n} \begin{bmatrix} B \\ \ell \end{bmatrix} (1 - \delta(A))^\ell.$$

Proof (a) Suppose each cycle b_i is defined as $b_i = (y_i, y_i b, \ldots, y_i b^{k_i-1})$ where y_i is the smallest element in the cycle. Suppose $(\alpha, b) \in G$ fixes some $\omega \in \Omega$. Then

$\omega(yb^{-1})\alpha(yb^{-1}) = \omega(y)$ for all $y \in [n]$ by (**Pw**). Letting $y = y_i b^{j+1}$ shows that for each i and j we have

$$\omega(y_i b^j)\alpha(y_i b^j) = \omega(y_i b^{j+1});$$

in particular, $\omega(y_i)\alpha(y_i)\alpha(y_i b)\cdots\alpha(y_i b^{k_i-1}) = \omega(y_i)$, and so $\omega(y_i)$ is a fixed point of $b_i(\alpha)$. Moreover, the converse is true, and ω is a fixed point of (α, b) if and only if each $\omega(y_i)$ is a fixed point of $b_i(\alpha)$, and $\omega(y_i b^{j+1})$ is defined as $\omega(y_i)\alpha(y_i)\alpha(y_i b)\cdots\alpha(y_i b^j)$. If the fixed points of $b_i(\alpha)$ are $x_{i,1}, \ldots, x_{i,j_i}$ (so $b_i(\alpha) \in \mathrm{Fix}_{j_i}(A)$), then (α, b) has exactly $j_1 \cdots j_\ell$ fixed points. Thus, for b as above, the number of $\alpha \in A^{[n]}$ for which (α, b) has exactly k fixed points is

$$\begin{aligned} &|\mathrm{Fix}_k(A^{[n]}b)| \\ &= \sum_{j_1\cdots j_\ell = k} |\{\alpha \in A^{[n]} \mid \text{ each } b_i(\alpha) = \alpha(y_i)\alpha(y_i b)\cdots\alpha(y_i b^{k_i-1}) \in \mathrm{Fix}_{j_i}(A)\}| \end{aligned}$$

For each i, we have $b_i(\alpha) \in \mathrm{Fix}_{j_i}(A)$ if and only if $\alpha(y_i) = c(\alpha(y_i b)\cdots\alpha(y_i b^{k_i-1}))^{-1}$ where $c \in \mathrm{Fix}_{j_i}(A)$ and $\alpha(y_i b), \ldots, \alpha(y_i b^{k_i-1}) \in A$ are arbitrary. Thus,

$$|\mathrm{Fix}_k(A^{[n]}b)| = \sum_{j_1\cdots j_\ell = k} |A|^{n-\ell} \prod_{r=1}^{\ell} |\mathrm{Fix}_{j_r}(A)|,$$

and dividing by $|A^{[n]}b| = |A|^n$ proves the first claim.

The claim for $k = 0$ follows via induction on ℓ. Indeed, if $\ell = 1$, then the claim is true by the above display. For $\ell > 1$, we split the sum into the cases $j_1 = 0$ and $j_1 \neq 0$, giving

$$\delta(A^{[n]}b) = \delta(A) \sum_{j_2,\ldots,j_\ell \geqslant 0} \prod_{r=2}^{\ell} \delta_{j_r}(A) + \sum_{j_1=1}^{m} \sum_{j_2\cdots j_\ell = 0} \prod_{r=1}^{\ell} \delta_{j_r}(A).$$

The first summand simplifies as follows

$$\delta(A) \sum_{j_2,\ldots,j_\ell \geqslant 0} \prod_{r=2}^{\ell} \delta_{j_r}(A) = \delta(A) \prod_{r=2}^{\ell} \sum_{j_r=0}^{m} \delta_{j_r}(A) = \delta(A) \prod_{r=2}^{\ell} 1 = \delta(A)$$

and the second summand simplifies as

$$\sum_{j_1=1}^{m} \sum_{j_2\cdots j_\ell = 0} \prod_{r=1}^{\ell} \delta_{j_r}(A) = \sum_{j_1=1}^{n} \delta_{j_1}(A) \sum_{j_2\cdots j_\ell = 0} \prod_{r=2}^{\ell} \delta_{j_r}(A) = (1 - \delta(A)) \sum_{j_2\cdots j_\ell = 0} \prod_{r=2}^{\ell} \delta_{j_r}(A)$$

with $\sum_{j_2\cdots j_\ell=0}\prod_{r=2}^{\ell}\delta_{j_r}(A) = 1-(1-\delta(A))^{\ell-1}$ by the induction hypothesis. Together,

$$\delta(A^{[n]}b)=\delta(A)+(1-\delta(A))(1-(1-\delta(A))^{\ell-1})=1-(1-\delta(A))^{\ell},$$

as claimed.
(b) Observe that both formulas in (a) depend only on k, and the number ℓ of cycles in the cycle decomposition of $b\in B$. Therefore

$$|\mathrm{Fix}_k(A\wr_P B)|=\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}|A|^n\sum_{j_1\cdots j_\ell=k}\prod_{r=1}^{\ell}\delta_{j_r}(A),$$

and dividing by $|A\wr_P B|=|A|^n|B|$ yields the first formula of (b). The formula for $k=0$ also follows from (a) together with the fact that $\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}=|B|$. □

Corollary 6.6 *If $A\leqslant\mathcal{S}_m$ and $B\leqslant\mathcal{S}_n$, then $\delta_1(A\wr_P B)=\mathcal{C}_B(\delta_1(A))$ where $\mathcal{C}_B(x)=\frac{1}{|B|}\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}x^{\ell}$.*

6.2 Derangements of $A\wr_P\mathcal{S}_n$

If $c_n(g)$ denotes the number of cycles of $g\in\mathcal{S}_n$, then the formula of Theorem 2.1(b) can be written as

$$\delta(A\wr_P B)=1-\frac{1}{|B|}\sum_{b\in B}(1-\delta(A))^{c_n(b)}.$$

For $B=C_n=\langle(1,2,\ldots,n)\rangle\leqslant\mathcal{S}_n$, we obtain the following by Example 6.1 and Theorem 2.1(b).

Corollary 6.7 *Let $A\leqslant\mathcal{S}_m$ and let $C_n\leqslant\mathcal{S}_n$ be generated by an n-cycle. Then*

$$\delta(A\wr_P C_n)=1-\frac{1}{n}\sum_{d|n}\phi(d)(1-\delta(A))^{n/d},$$

where ϕ denotes Euler's ϕ-function.

Corollary 6.8 *If $m,n\geqslant 2$ and $A\leqslant\mathcal{S}_m$, then*

$$\delta(A\wr_P\mathcal{S}_n)=1-\prod_{\ell=1}^{n}\left(1-\frac{\delta(A)}{\ell}\right)\quad\text{and hence}\quad\lim_{n\to\infty}\delta(A\wr_P\mathcal{S}_n)=1.$$

We have $1-\frac{1}{n^{\delta(A)}}\leqslant\delta(A\wr_P\mathcal{S}_n)$, and if $A\leqslant\mathcal{S}_m$ is 2-transitive, then $\delta(A\wr_P\mathcal{S}_n)\leqslant 1-\frac{1}{n+1}$.

Proof Let $A \wr_P \mathcal{S}_n$ act on $[m]^{[n]}$ via (**Pw**). The identity $\sum_{\ell=1}^{n} \left[{n \atop \ell}\right] x^\ell = \prod_{\ell=1}^{n}(x+\ell-1)$, see [12, Table 264], and Theorem 2.1(b) yield

$$\delta(A \wr_P \mathcal{S}_n) = 1 - \frac{1}{|\mathcal{S}_n|}\sum_{\ell=1}^{n}\left[{\mathcal{S}_n \atop \ell}\right](1-\delta(A))^\ell = 1 - \frac{1}{n!}\sum_{\ell=1}^{n}\left[{n \atop \ell}\right](1-\delta(A))^\ell$$
$$= 1 - \prod_{\ell=1}^{n}\frac{(1-\delta(A))+\ell-1}{\ell} = 1 - \prod_{\ell=1}^{n}\left(1-\frac{\delta(A)}{\ell}\right).$$

This proves the first claim. Moreover, $\lim_{n\to\infty}\delta(A \wr_P \mathcal{S}_n) = 1$ follows from $\prod_{\ell=1}^{\infty}\left(1-\frac{\delta(A)}{\ell}\right) = 0$, which is a consequence of $\sum_{\ell=1}^{\infty}\delta(A)/\ell = \infty$, see [18, Theorem 2.2.2].

We now bound $\delta(A \wr_P \mathcal{S}_n)$. It follows from [18, Eq. (2.2.2)] that $1-\prod_{\ell=1}^{n}(1-\frac{\delta(A)}{\ell}) \geqslant 1-e^{-\delta(A)H_n}$ where $H_n = \sum_{\ell=1}^{n}\frac{1}{\ell}$. Now $H_n > \log(n)$ yields $1-\prod_{\ell=1}^{n}(1-\frac{\delta(A)}{\ell}) \geqslant 1-e^{-\delta(A)\log(n)} = 1-n^{-\delta(A)}$, as desired. For the last claim, let $A \leqslant \mathcal{S}_m$ be 2-transitive. By (1), it suffices to show that $G = A \wr_P \mathcal{S}_n$ has rank $r = n+1$. We now prove this fact. Let C be the stabiliser in A of the point m. Then $|A : C| = m$ and hence the stabiliser of the point $[m, m, \ldots, m] \in [m]^{[n]}$ is $D = C \wr_P \mathcal{S}_n$. Since $|G : D| = m^n$ we see that G is transitive on $[m]^{[n]}$. Consider the orbits of D on $[m]^{[n]}$. If $\omega = (\omega_1, \ldots, \omega_n) \in [m]^{[n]}$, then by 2-transitivity we may choose $\gamma \in C^{[n]}$ such that $\omega_i\gamma_i = m$ if $\omega_i = m$, and $\omega_i\gamma_i = 1$ otherwise, for every $i \in [n]$. If $\omega_i = m$ for exactly j indices $i \in [n]$, then we may choose $(\gamma, b) \in C \wr_P \mathcal{S}_n$ such that $\omega(\gamma, b) = (1, \ldots, 1, m, \ldots, m)$ with exactly j copies of m and $n-j$ copies of 1. Thus, there are $n+1$ orbits in $[m]^{[n]}$ under the action of D, one for each $j \in \{0, 1, \ldots, n\}$, and therefore G has rank $n+1$, as claimed. □

Remark 6.9 The proof of Corollary 6.8 can be modified to prove that $\lim_{n\to\infty}\delta(A \wr_P \mathcal{A}_n) = 1$ for all $A \leqslant \mathcal{S}_m$ with $m \geqslant 2$ by using $\left[{n \atop \ell}\right] = \left[{\mathcal{S}_n \atop \ell}\right] \geqslant \left[{\mathcal{A}_n \atop \ell}\right]$ in the above display.

6.3 Density in $[\delta(A), 1]$: proof of Theorem 1.3

We now prove Theorem 1.3.

Proof of Theorem 1.3 Suppose first that $B \leqslant \mathcal{S}_n$ is fixed. Theorem 2.1(b) yields $\delta(A \wr_P B) = f(\delta(A))$ where $f(x) = 1-\mathcal{C}_B(1-x)$. Since $\mathcal{C}_B(x)$ is continuous and increasing on $[0, 1]$, the same is true for $f(x)$. The set $\{\delta(A) \mid A \text{ is primitive}\}$ is dense in $[0, 1]$ by Theorem 2.1(c), so the same is true for the set $\{f(\delta(A)) \mid A \text{is primitive}\}$. This proves the first claim as $f(0) = 0$ and $f(1) = 1$.

Suppose next that $A \leqslant \mathcal{S}_m$ is fixed. For each $B \leqslant \mathcal{S}_n$ we have $\mathcal{C}_B(x) \leqslant \frac{1}{|B|}\sum_{\ell=1}^{n}\left[{B \atop \ell}\right]x = x$ for $x \in [0, 1]$. It follows from $0 \leqslant \mathcal{C}_B(x) \leqslant x \leqslant 1$ that $0 \leqslant x \leqslant 1-\mathcal{C}_B(1-x) \leqslant 1$ and hence by Theorem 2.1(b) that $\delta(A \wr_P B) = 1-\mathcal{C}_B(1-\delta(A)) \geqslant \delta(A)$.

Let $p_1 < p_2 < \cdots$ be a sequence of primes. Set $n = p_1 \cdots p_r$ and let Z_r be the cyclic and regular subgroup $Z_r \leqslant \mathcal{S}_n$ generated by an n-cycle. Let $z = 1-\delta(A)$.

When $r = 1$, and $n = p_1$ we have

$$\frac{1}{n}\sum_{d|n}\phi(d)z^{n/d} = \left(1 - \frac{1}{p_1}\right)z + \frac{1}{p_1}z^{p_1},$$

by Example 6.1. Since $0 < z \leqslant 1$, Corollary 6.7 shows that

$$\lim_{p_1\to\infty}\delta(A \wr_P Z_1) = \lim_{p_1\to\infty}\left(1 - \left(1 - \frac{1}{p_1}\right)z - \frac{1}{p_1}z^{p_1}\right) = 1 - z = \delta(A).$$

Let $q = p_{r+1}$ be a prime with $p_r < q$. We consider $Z_r \leqslant \mathcal{S}_n$ and $Z_{r+1} \leqslant \mathcal{S}_{nq}$ where $n = p_1 \cdots p_r$ and calculate the difference $|\delta(A \wr_P Z_{r+1}) - \delta(A \wr_P Z_r)|$. Since $\phi(nq) = (q-1)\phi(n)$, Corollary 6.7 shows the following

$$\begin{aligned}\delta(A \wr_P Z_{r+1}) &= 1 - \frac{1}{nq}\left(\sum_{d|n}\phi(d)z^{nq/d} + \sum_{d|n}\phi(dq)z^{nq/dq}\right)\\ &= 1 - \frac{1}{nq}\left(\sum_{d|n}\phi(d)z^{n/d}\left((z^{n/d})^{q-1} + (q-1)\right)\right)\\ &= \delta(A \wr_P Z_r) + D(n,q),\end{aligned}$$

where

$$D(n,q) = \frac{1}{nq}\left(\sum_{d|n}\phi(d)z^{n/d}\left(1 - (z^{n/d})^{q-1}\right)\right).$$

Since $0 < 1-(z^{n/d})^{q-1} < 1$ we have $0 < D(n,q) < \frac{1}{q}\delta(A \wr_P Z_r)$, and so $D(n,q) \to 0$ as $q \to \infty$. In conclusion, the sequence $(\delta(A \wr_P Z_r))_{r\geqslant 1}$ is strictly increasing and can be arranged to start arbitrarily close to $\delta(A)$ with arbitrarily small step size $D(n,q)$.

It remains to show that this sequence converges to 1. We focus on Z_r and for $\Delta \subseteq [r]$ define $p(\Delta) = \prod_{i\in\Delta} p_i$ and $\Delta' = [r]\setminus\Delta$. Since $p([r]) = p(\Delta)p(\Delta') = n$, we have

$$\begin{aligned}\delta(A \wr_P Z_r) &= 1 - \frac{1}{n}\sum_{d|n}\phi(d)z^{n/d} = 1 - \frac{1}{p([r])}\sum_{\Delta\subseteq[r]}\phi(p(\Delta))z^{p(\Delta')}\\ &= 1 - \frac{\phi(p([r]))}{p([r])}z - \sum_{\substack{\Delta\subseteq[r]\\ \Delta\neq[r]}}\frac{\phi(p(\Delta))}{p(\Delta)}\frac{z^{p(\Delta')}}{p(\Delta')}\\ &\geqslant 1 - \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right)z - \sum_{\substack{\Delta\subseteq[r]\\ \Delta\neq[r]}}\frac{z^{p(\Delta')}}{p(\Delta')} > 1 - \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right)z - \sum_{i\geqslant p_1}\frac{z^i}{i}.\end{aligned}$$

The Taylor series $\sum_{i=1}^{\infty} z^i/i$ converges to $-\log(1-z)$ since $|z|<1$. Thus, for any given $\varepsilon>0$ we can choose p_1 large enough so that $\sum_{i\geqslant p_1} z^i/i < \varepsilon/2$. Moreover, $\prod_{p \text{ prime}}(1-\frac{1}{p})$ diverges to 0 by [18, Theorem 2.2.2] since $\sum_{p \text{ prime}} 1/p$ diverges. Thus, we can choose r large enough so that $\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) < \varepsilon/2z$. The claim now follows from

$$\delta(A \wr_P Z_r) > 1 - \frac{\varepsilon}{2z}z - \frac{\varepsilon}{2} = 1-\varepsilon. \qquad \square$$

6.4 Proof of Theorem 1.4

Proof of Theorem 1.4 Let T be a transversal for C_m in A_m, so that $\mathcal{T} = T^{[n]}$ is a transversal for $C_m^{[n]}$ in $A_m^{[n]}$. Every element in G_m can be written as $(\beta\tau, b)$ where $\beta \in C_m^{[n]}$, $\tau \in \mathcal{T}$, and $b \in B$. We fix (τ, b) and determine the proportion $\delta(C_m^{[n]}(\tau, b))$ in the coset $C_m^{[n]}(\tau, b)$. Write $\tau = (\tau_1, \dots, \tau_n)$ and let $b = b_1 \cdots b_\ell$ be the disjoint cycle decomposition of b. Let

$$D_\tau(b) = D_\tau(b_1) \cup \cdots \cup D_\tau(b_\ell) \quad \text{where} \quad D_\tau(b_i) = \{\alpha \in C_m^{[n]} \mid b_i(\alpha\tau) \in \mathrm{Der}(\mathcal{S}_m)\}$$

with $b_i(\alpha\tau)$ as in Definition 6.4. Arguing as in the proof of Theorem 6.5(a), it follows that the number of $\alpha \in C_m^{[n]}$ for which $(\alpha\tau, b)$ is a derangement in $C_m^{[n]}(\tau, b)$ is $|D_\tau(b)|$. In the next paragraph we now show that if b has exactly ℓ cycles, then

$$\lim_{m\to\infty} \delta(C_m^{[n]}(\tau, b)) = 1 - (1-\delta_0)^\ell,$$

independent of τ.

Before computing this number via inclusion–exclusion, we focus on one of the cycles b_i and count $|D_\tau(b_i)|$. To simplify notation, we conjugate by an element of $\mathcal{S}_m \wr_P \mathcal{S}_n$ (which preserves derangements and non-derangements), so we can assume that $b_i = (1, 2, \dots, k_i)$. The b_i-product of $\alpha\tau$ with $\alpha \in C_m^{[n]}$ now is

$$b_i(\alpha\tau) = \alpha_1\tau_1\alpha_2\tau_2\cdots\alpha_{k_i}\tau_{k_i} = \alpha_1\left(\alpha_2^{\tau_1^{-1}}\alpha_3^{(\tau_1\tau_2)^{-1}}\cdots\alpha_{k_i}^{(\tau_1\cdots\tau_{k_i})^{-1}}\right)\tau_1\tau_2\cdots\tau_{k_i},$$

which is a derangement in $C_m\tau_1\cdots\tau_{k_i}$ if and only if

$$\alpha_1 = x\left(\alpha_2^{\tau_1^{-1}}\alpha_3^{(\tau_1\tau_2)^{-1}}\cdots\alpha_{k_i}^{(\tau_1\cdots\tau_{k_i})^{-1}}\right)^{-1}$$

with $x \in C_m$ such that $x\tau_1\cdots\tau_{k_i} \in \mathrm{Der}(C_m\tau_1\cdots\tau_{k_i})$ and $\alpha_j \in C_m$ for each $j \in [n]\backslash\{1\}$. Since $b_i(\tau) = \tau_1\cdots\tau_{k_i}$, we have $|\mathrm{Der}_\tau(b_i)| = |\mathrm{Der}(C_m^{[n]}(\tau, b_i))| = |\mathrm{Der}(C_m b_i(\tau))||C_m|^{n-1}$. Hence

$$\frac{|\mathrm{Der}_\tau(b_i)|}{|C_m|^n} = \frac{|\mathrm{Der}(C_m b_i(\tau))|}{|C_m|} \to \delta_0 \quad \text{as } m \to \infty.$$

Since the cycles $b_1, \dots, b_\ell$ have disjoint support, the previous argument shows that for distinct elements $i_1, \dots, i_j \in [\ell]$, we have

$$|D_\tau(b_{i_1}) \cap \cdots \cap D_\tau(b_{i_j})| = \left(\prod_{k=1}^{j} |\mathrm{Der}(C_m b_{i_k}(\tau))|\right) |C_m|^{n-j} = \left(\prod_{k=1}^{j} \frac{|\mathrm{Der}(C_m b_{i_k}(\tau))|}{|C_m|}\right) |C_m|^{n},$$

and hence

$$\frac{|D_\tau(b_{i_1}) \cap \cdots \cap D_\tau(b_{i_j})|}{|C_m|^n} = \left(\prod_{k=1}^{j} \frac{|\mathrm{Der}(C_m b_{i_k}(\tau))|}{|C_m|}\right) \to \delta_0^j \quad \text{as } m \to \infty.$$

Inclusion-exclusion now shows that

$$\frac{|\mathrm{Der}(C_m^{[n]}(\tau, b))|}{|C_m|^n} = \frac{1}{|C_m|^n} \left|\bigcup_{i=1}^{\ell} D_\tau(b_i)\right| = \sum_{j=1}^{\ell} (-1)^{j-1} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant \ell} \frac{|D_\tau(b_{i_1}) \cap \cdots \cap D_\tau(b_{i_j})|}{|C_m|^n}. \tag{5}$$

Taking the limit as $m \to \infty$ gives

$$\lim_{m\to\infty} \frac{|\mathrm{Der}(C_m^{[n]}(\tau, b))|}{|C_m|^n} = \sum_{j=1}^{\ell} (-1)^{j-1} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant \ell} \delta_0^j = \sum_{j=1}^{\ell} (-1)^{j-1} \binom{\ell}{j} \delta_0^j = 1 - (1-\delta_0)^\ell. \tag{6}$$

In summary, if b has exactly ℓ cycles, then $\lim_{m\to\infty} \delta(C_m^{[n]}(\tau, b)) = 1 - (1-\delta_0)^\ell$, independent of τ and the precise structure of the ℓ cycles of b.

By assumption, $\pi_n(G_m) = B \leqslant \mathcal{S}_n$ for each m. The kernel of the restriction of π_n to G_m is the normal subgroup $U_m = G_m \cap \mathcal{S}_m^{[n]} = G_m \cap A_m^{[n]}$. In particular, $G_m/U_m \cong B$ by the Isomorphism Theorem. By assumption, $C_m^{[n]} \trianglelefteqslant U_m$. It follows that for every $b \in B$ with exactly ℓ cycles, there exist $|U_m|$ elements in G_m of the form (β, b) with $\beta \in A_m^{[n]}$. Moreover, there exist $|U_m|/|C_m|^n$ coset representatives of $C_m^{[n]}$ in G_m of this form, that is, there are $|U_m/C_m^{[n]}|$ choices for τ. Recall that B has exactly $\left[{B \atop \ell}\right]$ elements with precisely ℓ cycles, and let $b(\ell)$ be a fixed element with this property (and let $b(\ell)$ be arbitrary if $\left[{B \atop \ell}\right] = 0$). By Equation (6), the value of the limit $\lim_{m\to\infty} |\mathrm{Der}(C_m^{[n]}(1, b(\ell)))|/|C_m|^n$ is independent of the precise structure of $b(\ell)$

and the choice of $\tau = 1$. Thus, using the equations $|G_m| = |B||U_m|$, $\frac{1}{|B|}\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix} = 1$, and $\mathcal{C}_B(x) = \frac{1}{|B|}\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}x^\ell$ we obtain

$$\begin{aligned}\lim_{m\to\infty}\delta(G_m) &= \lim_{m\to\infty}\frac{|\mathrm{Der}(G_m)|}{|G_m|}\\ &= \lim_{m\to\infty}\frac{1}{|G_m|}\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}\frac{|U_m|}{|C_m|^n}|\mathrm{Der}(C_m^{[n]}(1, b(\ell)))|\\ &= \frac{1}{|G_m|}\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}|U_m|(1-(1-\delta_0)^\ell)\\ &= 1 - \mathcal{C}_B(1-\delta_0).\end{aligned}$$

□

Corollary 6.10 *Fix $n \geqslant 2$ and let $G_1, G_2, \ldots$ be a sequence of subgroups where each G_m satisfies $\mathcal{A}_m^{[n]} \trianglelefteqslant G_m \leqslant \mathcal{S}_m \wr_P \mathcal{S}_n$ and $\pi_n(G_m) = B \leqslant \mathcal{S}_n$ is independent of m. Then*

$$\lim_{m\to\infty}\delta(G_m) = 1 - \frac{1}{|B|}\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}(1-e^{-1})^\ell > e^{-1}.$$

Proof The first equality follows from Theorem 1.4 by choosing $C_m = \mathcal{A}_m$ and $A_m = \mathcal{S}_m$, and the observation that $\lim_{m\to\infty}\delta(\mathcal{A}_m) = \lim_{m\to\infty}\delta(\mathcal{S}_m \setminus \mathcal{A}_m) = e^{-1}$ by Corollary 3.3(b,c), so we set $\delta_0 = e^{-1}$ in Theorem 1.4. The inequality follows from $\sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}(1-e^{-1})^\ell < \sum_{\ell=1}^{n}\begin{bmatrix}B\\ \ell\end{bmatrix}(1-e^{-1}) = (1-e^{-1})|B|$. □

A modification of the proof of Theorem 1.4 also shows the following.

Corollary 6.11 *Suppose $A \leqslant \mathcal{S}_m$ and $B \leqslant \mathcal{S}_n$ and let $C \trianglelefteqslant A$. Let $\delta_L, \delta_U \in [0, 1]$ such $\delta_L \leqslant \delta(Ca) \leqslant \delta_U$ for all $a \in A$. Suppose $G \leqslant \mathcal{S}_m \wr_P \mathcal{S}_n$ satisfies $C^{[n]} \trianglelefteqslant G \leqslant A \wr_P B$ and $B = \pi_n(G)$ is the image of the natural projection $\pi_n \colon \mathcal{S}_m \wr_P \mathcal{S}_n \to \mathcal{S}_n$. We have*

$$1 - x - y \leqslant \delta(G) \leqslant 1 - x + y$$

where $x = \frac{1}{2}\big(\mathcal{C}_B(1-\delta_U) + \mathcal{C}_B(1-\delta_L)\big)$ and $y = \frac{1}{2}\big(\mathcal{C}_B(1+\delta_U) - \mathcal{C}_B(1+\delta_L)\big)$.

Proof Starting as in the proof of Theorem 1.4, this time we estimate

$$|C|^n\delta_L^j \leqslant |D_\tau(b_{i_1}) \cap \cdots \cap D_\tau(b_{i_j})| \leqslant |C|^n\delta_U^j.$$

If $b \in B$ has ℓ cycles, this yields the following lower bound in Equation (5):

$$\begin{aligned}\frac{|\mathrm{Der}(C^{[n]}(\tau,b))|}{|C|^n} &= \sum_{j=1}^{\ell}(-1)^{j-1}\sum_{1\leqslant i_1<\cdots<i_j\leqslant \ell}\frac{|D_\tau(b_{i_1})\cap\cdots\cap D_\tau(b_{i_j})|}{|C|^n}\\ &\geqslant 1-\sum_{j\geqslant 0\text{ even}}\binom{\ell}{j}\delta_U^j+\sum_{j\geqslant 0\text{ odd}}\binom{\ell}{j}\delta_L^j\\ &= 1-\frac{(1+\delta_U)^\ell+(1-\delta_U)^\ell}{2}+\frac{(1+\delta_L)^\ell-(1-\delta_L)^\ell}{2}.\end{aligned}$$

Using this in the last equations of the proof of Theorem 1.4 yields

$$\delta(G)\geqslant 1-\frac{1}{2}\big(\mathcal{C}_B(1-\delta_U)+\mathcal{C}_B(1-\delta_L)+\mathcal{C}_B(1+\delta_U)-\mathcal{C}_B(1+\delta_L)\big).$$

The upper bound follows analogously by swapping the subscripts U and L. □

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