



A new tableau model for representations of the special orthogonal group

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Abstract

We provide a new tableau model from which one can easily deduce the characters of finite-dimensional irreducible polynomial representations of the special orthogonal group $SO_n(\mathbb{C})$. This model originates from the representation theory of the q -quantum group (also known as the quantum symmetric pair coideal subalgebra) of type AI and is equipped with a combinatorial structure, which we call AI-crystal structure. This structure enables us to describe combinatorially the tensor product of an $SO_n(\mathbb{C})$ -module and a $GL_n(\mathbb{C})$ -module, and the branching from $GL_n(\mathbb{C})$ to $SO_n(\mathbb{C})$.

Keywords Tableau model · Crystal · Quantum group · Quantum symmetric pair

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1 Introduction

Combinatorial objects such as partitions, Young tableaux, and their variants have been effectively used to understand the representation theory of the symmetric group, the general linear group $GL_n = GL_n(\mathbb{C})$, the general linear algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$, and their variants. For example, the set of standard (resp., semistandard) Young tableaux of a fixed shape parametrizes a basis of the corresponding irreducible representation of the symmetric group (resp., GL_n and \mathfrak{gl}_n).

For a better understanding of a representation theory, it is quite useful to construct a concrete combinatorial model which uniformly models a certain class of representations. For example, in the representation theory of reductive groups and their Lie algebras, one may want a combinatorial model from which one can easily deduce the character of a representation under consideration.

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The semistandard Young tableau model for GL_n and \mathfrak{gl}_n is such a typical example. Let us see this in more detail. Let Par_n denote the set of partitions of length not greater than n . For each $\lambda \in \text{Par}_n$, let $\text{SST}_n(\lambda)$ denote the set of semistandard Young tableaux of shape λ with entries in $\{1, \dots, n\}$. To each semistandard Young tableau $T \in \text{SST}_n(\lambda)$, a weight $\text{wt}(T) \in \mathbb{Z}^n$ is assigned. Then, $\text{SST}_n(\lambda)$ models the finite-dimensional irreducible \mathfrak{gl}_n -module (equivalently, GL_n -module) $V^{\mathfrak{gl}_n}(\lambda)$ of highest weight λ in the sense that the character of $V^{\mathfrak{gl}_n}(\lambda)$ equals the generating function

$$\sum_{T \in \text{SST}_n(\lambda)} \mathbf{x}^{\text{wt}(T)} \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

of weights of $\text{SST}_n(\lambda)$, where

$$\mathbf{x}^{(a_1, \dots, a_n)} := x_1^{a_1} \cdots x_n^{a_n}.$$

The aim of this paper is to introduce a new tableau model for the finite-dimensional irreducible representations of integer highest weight for the special orthogonal algebra $\mathfrak{so}_n = \mathfrak{so}_n(\mathbb{C})$, equivalently, the finite-dimensional irreducible polynomial representations for the special orthogonal group. The isomorphism classes of such representations are parametrized by the set

$$X_{\mathfrak{so}_n, \text{int}}^+ := \begin{cases} \{v = (v_1, v_3, \dots, v_{2m-1}) \in \mathbb{Z}^m \mid v_1 \geq v_3 \geq \dots \geq v_{2m-1}\} & \text{if } n \text{ is odd,} \\ \{v = (v_1, v_3, \dots, v_{2m-1}) \in \mathbb{Z}^m \mid v_1 \geq v_3 \geq \dots \geq v_{2m-3} \geq |v_{2m-1}|\} & \text{if } n \text{ is even} \end{cases}$$

of dominant integer weights, where

$$m := \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

denotes the rank of \mathfrak{so}_n . It is convenient to introduce the following family of \mathfrak{so}_n -modules parametrized by Par_m . For each $\rho \in \text{Par}_m$, set

$$V^{\mathfrak{so}_n}(\rho) := \begin{cases} V^{\mathfrak{so}_n}(v_\rho) & \text{if } \ell(\rho) < \frac{n}{2}, \\ V^{\mathfrak{so}_n}(v_\rho^+) \oplus V^{\mathfrak{so}_n}(v_\rho^-) & \text{if } \ell(\rho) = \frac{n}{2}, \end{cases}$$

where

$$v_\rho := (\rho_1, \rho_2, \dots, \rho_{\ell(\rho)}, 0, \dots, 0), \quad v_\rho^\pm := (\rho_1, \rho_2, \dots, \rho_{m-1}, \pm \rho_m) \in X_{\mathfrak{so}_n, \text{int}}^+.$$

Let $\rho \in \text{Par}_m$ and $T \in \text{SST}_n(\rho)$. We say that T is an AI-tableau (the origin of the name will be clear later) if

$$t_{i,1}^c \leq t_{i,2} \text{ for all } 1 \leq i \leq d_2,$$

where $t_{i,j}$ denotes the (i, j) th entry of T , and

$$d_j := \max\{k \mid \rho_k \geq j\}, \quad \{t_{1,1}^c, \dots, t_{n-d_1,1}^c\} := \{1, \dots, n\} \setminus \{t_{1,1}, \dots, t_{d_1,1}\}.$$

For each $\rho \in \text{Par}_m$, set

$$\overline{\text{SST}}_n^{\text{AI}}(\rho) := \{T \in \text{SST}_n(\rho) \mid T \text{ is an AI-tableau}\}.$$

To each $T \in \overline{\text{SST}}_n^{\text{AI}}(\rho)$, a degree $\text{deg}(T) = (\text{deg}_1(T), \text{deg}_3(T), \dots, \text{deg}_{2m-1}(T)) \in \mathbb{Z}^m$ is assigned. Then, we can say that $\overline{\text{SST}}_n^{\text{AI}}(\rho)$ models the \mathfrak{so}_n -module $V^{\mathfrak{so}_n}(\rho)$ in the following sense.

Theorem A *Let $\rho \in \text{Par}_m$. Then, we have*

$$\text{ch } V^{\mathfrak{so}_n}(\rho) = \frac{1}{2^m} \sum_{T \in \overline{\text{SST}}_n^{\text{AI}}(\rho)} \sum_{\sigma_1, \sigma_3, \dots, \sigma_{2m-1} \in \{+, -\}} y_1^{\sigma_1 \text{deg}_1(T)} y_3^{\sigma_3 \text{deg}_3(T)} \dots y_{2m-1}^{\sigma_{2m-1} \text{deg}_{2m-1}(T)},$$

where the left-hand side denotes the character of the \mathfrak{so}_n -module $V^{\mathfrak{so}_n}(\rho)$.

Let us compare the already known tableau models, some of which can model irreducible representations of half-integer highest weight as well, with ours. King and El-Sharkaway [7] introduced the notion of orthogonal tableaux by investigating the branching from \mathfrak{so}_n to \mathfrak{so}_{n-2} . A weight is assigned to each orthogonal tableau. Then, they proved that the associated generating functions are irreducible characters of \mathfrak{so}_n . Koike and Terada [8] introduced another tableau model for \mathfrak{so}_n . Their tableaux need $3m$ (resp., $4m$) kinds of letters when n is odd (resp., even). Sundaram [16] (for \mathfrak{so}_{2m+1}), Proctor [14], and Okada [13] (for \mathfrak{so}_{2m}) constructed similar, but more or less simple, tableau models. Kashiwara and Nakashima [6] constructed totally different tableau model. Their model is equipped with a crystal structure, which originates from the representation theory of the quantum groups. Our new tableau model is, as a set, different from the other models above and is much simpler than models of King-El-Sharkaway, Koike-Terada, and Kashiwara-Nakashima; the constraints are only the ordinary semistandardness condition and an easily checked condition on the first two columns. Our character formula is a little bit involved compared to the other models, but it is still easy. What is unique to our model is a new combinatorial structure, which we call the AI-crystal structure. This structure is closely related to the representation theory of the ι quantum group (also known as the quantum symmetric pair coideal subalgebra) of type AI.

An ι quantum group is a certain right coideal subalgebra of a quantum group appearing in the theory of quantum symmetric pairs formulated by Letzter [10]. Among them, the ι quantum group of type AI, which has appeared in [2] earlier than [10], is the subalgebra U^ι of the quantum group $U = U_q(\mathfrak{gl}_n)$ associated to \mathfrak{gl}_n generated by

$$B_i := F_i + q^{-1} E_i K_i^{-1}, \quad i \in \{1, \dots, n-1\},$$

where $E_i, F_i, K_i^{\pm 1}, i \in \{1, \dots, n-1\}$ denote the Chevalley generators of \mathbf{U} . Under the classical limit $q \rightarrow 1$, it tends to the universal enveloping algebra $U(\mathfrak{so}_n)$. Here, \mathfrak{so}_n is embedded into \mathfrak{gl}_n as the Lie subalgebra generated by $b_i := f_i + e_i, i \in \{1, \dots, n-1\}$.

In [18], the crystal limit $q \rightarrow \infty$ of the action of B_i on a certain class of \mathbf{U}' -modules was defined. It is a linear operator \tilde{B}_i . Abstracting properties of \tilde{B}_i 's, we introduce the notion of AI-crystals. An AI-crystal is a set \mathcal{B} equipped with structure maps $\tilde{B}_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$ and $\text{deg}_i : \mathcal{B} \rightarrow \mathbb{Z}$ satisfying certain axioms. Then, it is shown that for each $\rho \in \text{Par}_m$, the set $\text{SST}_n^{\text{AI}}(\rho)$ admits an AI-crystal structure which can be thought of as the crystal limit of the \mathbf{U}' -module $V_q^{\mathfrak{so}_n}(\rho)$ (a q -analog of the \mathfrak{so}_n -module $V^{\mathfrak{so}_n}(\rho)$). This is the second achievement in an attempt to generalize the theory of crystal basis to ι quantum groups; the first example was given in [17] for quasi-split-type AIII_{2r} with asymptotic parameters. No general theory unifying these two crystal theory is known.

Thanks to this \mathbf{U}' -representation theoretic interpretation, it turns out that AI-crystals model not only the \mathfrak{so}_n -modules $V^{\mathfrak{so}_n}(\rho)$ but also the tensor product of an \mathfrak{so}_n -module and a \mathfrak{gl}_n -module, and the branching from \mathfrak{gl}_n to \mathfrak{so}_n .

Let $\lambda \in \text{Par}_n$ and $\rho \in \text{Par}_m$. Then, $\text{SST}_n^{\text{AI}}(\rho) \otimes \text{SST}_n(\lambda) := \text{SST}_n^{\text{AI}}(\rho) \times \text{SST}_n(\lambda)$ is equipped with an AI-crystal structure. This structure reflects the \mathfrak{so}_n -module structure of $V^{\mathfrak{so}_n}(\rho) \otimes V^{\mathfrak{gl}_n}(\lambda)$, which is the classical limit of the \mathbf{U}' -module structure of the corresponding tensor product module (recall that \mathbf{U}' is a right coideal of \mathbf{U}). Mimicking Schensted's insertion algorithm, we introduce an algorithm which tells us how the tensor product $\text{SST}_n^{\text{AI}}(\rho) \otimes \text{SST}_n(\lambda)$ decomposes into several copies of $\text{SST}_n^{\text{AI}}(\sigma)$'s, $\sigma \in \text{Par}_m$. Such insertion schemes for other tableau models have been invented in [9, 13, 14, 16].

As a special case of the tensor product modules, one can consider $V^{\mathfrak{gl}_n}(\lambda) = V^{\mathfrak{so}_n}(\emptyset) \otimes V^{\mathfrak{gl}_n}(\lambda)$ because $V^{\mathfrak{so}_n}$ is the trivial representation. Recall that $\text{SST}_n(\lambda)$ models $V^{\mathfrak{gl}_n}(\lambda)$. By the argument above, we see that $\text{SST}_n(\lambda) = \text{SST}_n^{\text{AI}}(\emptyset) \otimes \text{SST}_n(\lambda)$ is equipped with an AI-crystal structure. This structure reflects the \mathfrak{so}_n -module structure of $V^{\mathfrak{gl}_n}(\lambda)$. To each $T \in \text{SST}_n(\lambda)$, an AI-tableau $P^{\text{AI}}(T)$, called the P^{AI} -symbol of T , is assigned. For each $\rho \in \text{Par}_m$, the subset

$$\{T \in \text{SST}_n(\lambda) \mid \text{the shape of } P^{\text{AI}}(T) \text{ is } \rho\}$$

forms an AI-crystal isomorphic to the disjoint union of several copies of $\text{SST}_n^{\text{AI}}(\rho)$. In this way, we obtain an ‘‘irreducible decomposition’’ of $\text{SST}_n(\lambda)$ as an AI-crystal, which corresponds to a decomposition of $V^{\mathfrak{gl}_n}(\lambda)$ as an \mathfrak{so}_n -module.

This paper is organized as follows. In Sect. 2, we prepare notation concerning combinatorial objects and introduce the notion of AI-crystals and AI-tableaux, which play key roles in the construction of our new tableau model. In Sect. 3, we give a representation theoretic interpretation to combinatorial objects introduced in the previous section. Also, our main theorem is proved there. In Sect. 4, we develop an insertion scheme for our model. This enables us to understand the tensor product of an \mathfrak{so}_n -module and a \mathfrak{gl}_n -module and the branching from \mathfrak{gl}_n to \mathfrak{so}_n from an AI-crystal theoretic point of view.

Notation

Throughout this paper, we use the following notation:

- $\mathbb{Z}_{\geq 0}$: the set of nonnegative integers.
- \mathbb{Z}_{ev} : the set of even integers.
- \mathbb{Z}_{odd} : the set of odd integers.
- For $p \in \{\text{ev}, \text{odd}\}$, $\mathbb{Z}_{\geq 0, p} := \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_p$.
- For $a, b \in \mathbb{Z}$, $[a, b] := \{c \in \mathbb{Z} \mid a \leq c \leq b\}$.
- For $a, b \in \mathbb{Z}$ and $p \in \{\text{ev}, \text{odd}\}$, $[a, b]_p := [a, b] \cap \mathbb{Z}_p$.

2 Combinatorics

In this section, we prepare notation concerning combinatorial objects used throughout the paper. After reviewing the theory of crystals of type A, we introduce the notion of AI-crystals and AI-tableaux, which play key roles in the construction of our new tableau model for finite-dimensional \mathfrak{so}_n -modules of integer highest weight.

2.1 Tableaux

A partition is a non-increasing sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of positive integers; l is referred to as the length of λ , and is denoted by $\ell(\lambda)$. The empty sequence \emptyset is the unique partition of length 0. The size of a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is defined to be $\sum_{i=1}^l \lambda_i$, and is denoted by $|\lambda|$. Let Par_l denote the set of partitions of length not greater than l , and $\text{Par} := \bigcup_{l \geq 0} \text{Par}_l$ the set of partitions.

The Young diagram of shape $\lambda \in \text{Par}$ is the set

$$D(\lambda) := \{(i, j) \mid i \in [1, \ell(\lambda)], j \in [1, \lambda_i]\}.$$

For $\lambda, \mu \in \text{Par}$, we write $\lambda \triangleleft \mu$ if $D(\lambda) \subset D(\mu)$ and $|\mu| - |\lambda| = 1$.

Let A be a set. A Young tableau of shape $\lambda \in \text{Par}$ in alphabet A is a map

$$T : D(\lambda) \rightarrow A.$$

Unless otherwise stated, we always fix $n \geq 2$ and take $A = [1, n]$ as the alphabet. When considering Young tableaux, each element of the alphabet is referred to as a letter. For a Young tableau T of shape $\lambda \in \text{Par}$ and $(i, j) \in D(\lambda)$, the letter $T(i, j)$ is referred to as the (i, j) th entry of T . The partition λ is referred to as the shape of T , and is denoted by $\text{sh}(T)$. The size of T is defined to be $|\lambda|$, and is denoted by $|T|$. For a subset $A \subset [1, n]$, let $T|_A$ denote the map

$$T|_A : \{(i, j) \in D(\lambda) \mid T(i, j) \in A\} \rightarrow A; (i, j) \mapsto T(i, j).$$

Given $n \geq d_1 \geq d_2 \geq \dots \geq d_l > 0$ and $C_j = (c_{1,j}, c_{2,j}, \dots, c_{d_j,j}) \in [1, n]^{d_j}$, let $C_1 C_2 \dots C_l$ denote the Young tableau given by

$$(C_1 C_2 \dots C_l)(i, j) := c_{i,j}.$$

A sequence of letters is referred to as a word. Let \mathcal{W} denote the set of words, i.e.,

$$\mathcal{W} := \bigsqcup_{d \geq 0} [1, n]^d.$$

For two words $w_1, w_2 \in \mathcal{W}$, let $w_1 * w_2 \in \mathcal{W}$ denote the concatenation of them. The column reading of a Young tableau T is a word $\text{CR}(T) \in \mathcal{W}$ defined to be

$$(t_{d_1,1}, t_{d_1-1,1}, \dots, t_{1,1}) * (t_{d_2,2}, t_{d_2-1,2}, \dots, t_{1,2}) * \dots * (t_{d_{\lambda_1},\lambda_1}, t_{d_{\lambda_1}-1,\lambda_1}, \dots, t_{1,\lambda_1}),$$

where

$$d_j := \max\{i \mid \lambda_i \geq j\}$$

denotes the length of the j th column of $D(\lambda)$.

Example 2.1.1 Let $\lambda = (4, 2, 1)$, and

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 2 & 3 & & \\ \hline 4 & & & \\ \hline \end{array}.$$

Then, its column reading is

$$\text{CR}(T) = (4, 2, 1) * (3, 2) * (3) * (3) = (4, 2, 1, 3, 2, 3, 3).$$

A Young tableau is said to be semistandard if its entries weakly increase along the rows from left to right, and strongly increase along the columns from top to bottom. Let $\text{SST}_n(\lambda)$ denote the set of semistandard Young tableaux of shape $\lambda \in \text{Par}$. Note that $\text{SST}_n(\lambda) = \emptyset$ unless $\ell(\lambda) \leq n$.

A semistandard Young tableau of shape $\lambda \in \text{Par}$ in alphabet $[1, N]$ for some $N > 0$ is said to be standard if the assignment $(i, j) \mapsto T(i, j)$ is injective. Let $\text{ST}_N(\lambda)$ denote the set of standard Young tableaux of shape λ in alphabet $[1, N]$.

For a semistandard Young tableau $T \in \text{SST}_n(\lambda)$ and a letter $l \in [1, n]$, let $(T \leftarrow l)$ denote the semistandard Young tableau obtained by Schensted’s row insertion algorithm (see, e.g., [1, Chapter 7.1] for a precise definition).

Example 2.1.2 Let T be as in Example 2.1.1. Then, we have

$$(T \leftarrow 1) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline 4 & & & \\ \hline \end{array}, \quad (T \leftarrow 2) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & \\ \hline 4 & & & \\ \hline \end{array}, \quad (T \leftarrow 3) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 3 & & & \\ \hline 4 & & & & \\ \hline \end{array}.$$

Let us recall the Robinson–Schensted correspondence. Fix a word $w = (w_1, \dots, w_d) \in \mathcal{W}$. For each $k \in [0, d]$, define a semistandard Young tableau P^k inductively by

$$P^k := \begin{cases} \emptyset & \text{if } k = 0, \\ (P^{k-1} \leftarrow w_k) & \text{if } k > 0. \end{cases}$$

Let λ^k denote the shape of P^k . Note that we have $\lambda^{k-1} \triangleleft \lambda^k$. Define a standard Young tableau $Q^k \in \text{ST}_k(\lambda^k)$ inductively by $Q^0 := \emptyset$, and

$$Q^k(i, j) := \begin{cases} Q^{k-1}(i, j) & \text{if } (i, j) \in D(\lambda^{k-1}), \\ k & \text{if } (i, j) \in D(\lambda^k) \setminus D(\lambda^{k-1}) \end{cases}$$

for $k \in [1, d]$. The tableaux P^d and Q^d and the partition λ^d are referred to as the P-symbol, the Q-symbol, and the shape of w , and are denoted by $P(w)$, $Q(w)$, and $\text{sh}(w)$, respectively. The assignment

$$\text{RS} : \mathcal{W} \rightarrow \bigsqcup_{\lambda \in \text{Par}_n} \text{SST}_n(\lambda) \times \text{ST}_{|\lambda|}(\lambda); w \mapsto (P(w), Q(w))$$

is called the Robinson–Schensted correspondence. As is well known, RS is bijective.

Example 2.1.3 Let $w = (4, 2, 3, 1, 3, 2)$. Then, P^k and Q^k , $k \in [0, 6]$ are as follows:

$$P^0 = \emptyset, \quad \begin{array}{|c|} \hline 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 4 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline \end{array} = P^6 = P(w),$$

$$Q^0 = \emptyset, \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline \end{array} = Q^6 = Q(w).$$

2.2 \mathfrak{gl} -Crystal

Combinatorics which have been introduced so far are closely related to the representation theory of the general linear algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ via the theory of crystals.

A \mathfrak{gl}_n -crystal (we omit the subscript “ n ” when there is no confusion) is a set \mathcal{B} equipped with structure maps

$$\tilde{E}_i, \tilde{F}_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}, \quad \varepsilon_i, \varphi_i : \mathcal{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad i \in [1, n - 1],$$

where 0 is a formal symbol, and

$$\text{wt} : \mathcal{B} \rightarrow \mathbb{Z}^n$$

satisfying certain axioms (see, e.g., [1, Definition 2.13]). Every \mathfrak{gl} -crystal appearing in this paper is a Stembridge crystal in the sense of [1, Chapter 4.2]. Among the axioms

of (Stembridge) crystals, what are particularly important for us are the following: Let $b, b' \in \mathcal{B}, i, j \in [1, n - 1]$. Then, the following hold.

- (1) $\tilde{F}_i b = b'$ if and only if $b = \tilde{E}_i b'$.
- (2) $\varepsilon_i(b) := \max\{k \mid \tilde{E}_i^k b \neq 0\}$.
- (3) $\varphi_i(b) := \max\{k \mid \tilde{F}_i^k b \neq 0\}$.
- (4) If $|i - j| > 1$ and $\tilde{E}_i b \neq 0$, then $\varepsilon_j(\tilde{E}_i b) = \varepsilon_j(b)$, $\varphi_j(\tilde{E}_i b) = \varphi_j(b)$, and $\tilde{E}_j \tilde{E}_i b = \tilde{E}_i \tilde{E}_j b$.
- (5) If $|i - j| = 1$ and $\tilde{E}_i b \neq 0$, then either $\varepsilon_j(\tilde{E}_i b) = \varepsilon_j(b)$ and $\varphi_j(\tilde{E}_i b) = \varphi_j(b) - 1$, or $\varepsilon_j(\tilde{E}_i b) = \varepsilon_j(b) + 1$ and $\varphi_j(\tilde{E}_i b) = \varphi_j(b)$.

Here, we set $\tilde{E}_i 0 = 0$ and $\tilde{F}_i 0 = 0$.

Let $\mathcal{B}_1, \mathcal{B}_2$ be \mathfrak{gl} -crystals. A morphism $\psi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ of \mathfrak{gl} -crystals is a map $\psi : \mathcal{B}_1 \sqcup \{0\} \rightarrow \mathcal{B}_2 \sqcup \{0\}$ such that

$$\begin{aligned} \psi(0) &= 0, \quad \psi(\tilde{E}_i b) = \tilde{E}_i \psi(b), \quad \psi(\tilde{F}_i b) = \tilde{F}_i \psi(b), \\ \varepsilon_i(\psi(b')) &= \varepsilon_i(b'), \quad \varphi_i(\psi(b')) = \varphi_i(b'), \quad \text{wt}(\psi(b')) = \text{wt}(b') \end{aligned}$$

for all $b, b' \in \mathcal{B}_1$ and $i \in [1, n - 1]$ such that $\psi(b') \neq 0$. A morphism of \mathfrak{gl} -crystals is said to be an isomorphism if the underlying map $\mathcal{B}_1 \sqcup \{0\} \rightarrow \mathcal{B}_2 \sqcup \{0\}$ is bijective.

The character $\text{ch}_{\mathfrak{gl}} \mathcal{B}$ of \mathcal{B} is a Laurent polynomial in n variables x_1, \dots, x_n given by

$$\text{ch}_{\mathfrak{gl}} \mathcal{B} := \sum_{b \in \mathcal{B}} \mathbf{x}^{\text{wt}(b)} \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

where $\mathbf{x}^{(a_1, \dots, a_n)} := x_1^{a_1} \dots x_n^{a_n}$.

The crystal graph of \mathcal{B} is a colored directed graph defined as follows. The vertex set is \mathcal{B} . For each $b, b' \in \mathcal{B}$ and $i \in [1, n - 1]$, there is an i -colored arrow from b to b' if and only if $b' = \tilde{F}_i(b)$.

Example 2.2.1 The set $\text{SST}_n(1)$ is equipped with a \mathfrak{gl} -crystal structure as follows:

$$\text{wt}(\boxed{j}) := \epsilon_j, \quad \tilde{E}_i \boxed{j} := \begin{cases} \boxed{j-1} & \text{if } j = i + 1, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{F}_i \boxed{j} := \begin{cases} \boxed{j+1} & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\epsilon_j := (\overbrace{0, \dots, 0}^{j-1}, 1, 0, \dots, 0) \in \mathbb{Z}^n.$$

Then, we have

$$\text{ch}_{\mathfrak{gl}} \text{SST}_n(1) = x_1 + x_2 + \dots + x_n.$$

The crystal graph of $SST_n(1)$ is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \dots \xrightarrow{n-1} \boxed{n}.$$

Let $\mathcal{B}_1, \mathcal{B}_2$ be \mathfrak{gl} -crystals. Then, $\mathcal{B}_1 \otimes \mathcal{B}_2 := \mathcal{B}_1 \times \mathcal{B}_2$ is equipped with a \mathfrak{gl} -crystal structure as follows (cf. [1, Section 2.3]): It is customary to denote $(b_1, b_2) \in \mathcal{B}_1 \otimes \mathcal{B}_2$ by $b_1 \otimes b_2$. For each $i \in [1, n - 1]$, we set

$$\begin{aligned} \text{wt}(b_1 \otimes b_2) &:= \text{wt}(b_1) + \text{wt}(b_2), \\ \tilde{F}_i(b_1 \otimes b_2) &:= \begin{cases} \tilde{F}_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2), \\ b_1 \otimes \tilde{F}_i(b_2) & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2), \end{cases} \\ \tilde{E}_i(b_1 \otimes b_2) &:= \begin{cases} \tilde{E}_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2), \\ b_1 \otimes \tilde{E}_i(b_2) & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2). \end{cases} \end{aligned}$$

Here, we set $0 \otimes b_2 = b_1 \otimes 0 = 0$. This definition is consistent with [1], and opposite to [5]. The tensor product of \mathfrak{gl} -crystals is associative;

$$(\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \mathcal{B}_3 = \mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3).$$

For each $d \geq 0$, we can construct a \mathfrak{gl} -crystal $SST_n(1)^{\otimes d}$ inductively by

$$SST_n(1)^{\otimes d} := SST_n(1)^{\otimes d-1} \otimes SST_n(1),$$

where, $SST_n(1)^{\otimes 0} = \{\emptyset\}$ with \mathfrak{gl} -crystal structure given by

$$\text{wt}(\emptyset) = (0, \dots, 0), \quad \tilde{E}_i(\emptyset) = 0 = \tilde{F}_i(\emptyset).$$

By associativity of the tensor product of \mathfrak{gl} -crystals, we see that

$$SST_n(1)^{\otimes d_1} \otimes SST_n(1)^{\otimes d_2} = SST_n(1)^{\otimes d_1+d_2}.$$

Recall that $\mathcal{W} = \bigsqcup_{d \geq 0} [1, n]^d$ denotes the set of words. By identifying each word $(w_1, w_2, \dots, w_d) \in \mathcal{W}$ with

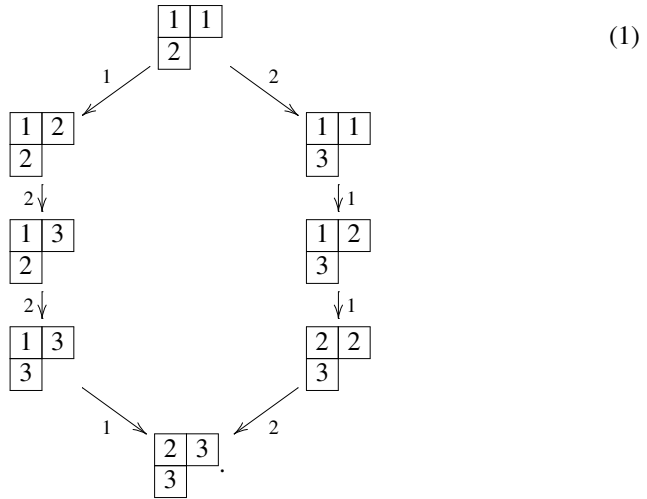
$$\boxed{w_1} \otimes \boxed{w_2} \otimes \dots \otimes \boxed{w_d} \in SST_n(1)^{\otimes d},$$

one can equip \mathcal{W} with a \mathfrak{gl} -crystal structure. Then, the concatenation of words is identical to the tensor product of \mathfrak{gl} -crystals.

Let $\lambda \in \text{Par}_n$. Then, $SST_n(\lambda)$ is equipped with a \mathfrak{gl} -crystal structure as follows: For each $T \in SST_n(\lambda)$ and $i \in [1, n - 1]$, we set

$$\text{wt}(T) := \text{wt}(\text{CR}(T)), \quad \tilde{E}_i T := P(\tilde{E}_i \text{CR}(T)), \quad \tilde{F}_i T := P(\tilde{F}_i \text{CR}(T)),$$

where we set $P(0) = 0$. For example, the crystal graph of $SST_3(2, 1)$ is as follows.



Then, the Robinson–Schensted correspondence

$$RS : \mathcal{W} \rightarrow \bigsqcup_{\lambda \in \text{Par}_n} SST_n(\lambda) \times ST_{|\lambda|}(\lambda); w \mapsto (P(w), Q(w))$$

is an isomorphism of \mathfrak{gl} -crystals (cf. [1, Section 8.3]), where the \mathfrak{gl} -crystal structure of $SST_n(\lambda) \times ST_{|\lambda|}(\lambda)$ is given by

$$\text{wt}(P, Q) := \text{wt}(P), \quad \tilde{E}_i(P, Q) := (\tilde{E}_i P, Q), \quad \tilde{F}_i(P, Q) := (\tilde{F}_i P, Q).$$

Given partitions $\lambda, \mu \in \text{Par}_n$ and tableaux $T \in SST_n(\lambda), S \in SST_n(\mu)$, define the P -symbol $P(T \otimes S)$ of $T \otimes S$ to be the P -symbol of $\text{CR}(T) * \text{CR}(S)$. For example, we have

$$P \left(\begin{array}{|c|c|c|} \hline 1 & & 3 \\ \hline 3 & & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline 4 & \\ \hline \end{array} \right) = P((3, 1, 3) * (4, 2, 1, 3, 2)) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline 3 & 4 & \\ \hline \end{array}.$$

For a later use, we put here an easy observation.

Lemma 2.2.2 *Let $k, l \in [1, n], 1 \leq i_1 < \dots < i_k \leq n$, and $1 \leq j_1 < \dots < j_l \leq n$, and set*

$$C_1 := P(i_k, \dots, i_1) = \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array}, \quad C_2 := P(j_l, \dots, j_1) = \begin{array}{|c|} \hline j_1 \\ \hline \vdots \\ \hline j_l \\ \hline \end{array},$$

and $\lambda := \text{sh}(P(C_1 \otimes C_2))$. Then, we have $\ell(\lambda) \geq k$. Furthermore, the following are equivalent:

- (1) $\ell(\lambda) = k$.
- (2) $k \geq l$ and $i_r \leq j_r$ for all $r \in [1, l]$.
- (3) $P(C_1 \otimes C_2) = C_1 C_2$.

Let \mathcal{B} be a gl-crystal and $b, b_1, b_2 \in \mathcal{B}$. We say that b_1 and b_2 are connected, or b_2 is connected to b_1 if we have

$$b_2 = \tilde{X}_{i_1} \tilde{X}_{i_2} \cdots \tilde{X}_{i_r} b_1$$

for some $\tilde{X}_{i_1}, \dots, \tilde{X}_{i_r} \in \{\tilde{E}_i, \tilde{F}_i \mid i \in [1, n - 1]\}$. The connected component $C(b)$ of \mathcal{B} containing b is defined by

$$C(b) := \{b' \in \mathcal{B} \mid b' \text{ is connected to } b\}.$$

We say that \mathcal{B} is connected if we have $\mathcal{B} = C(b)$ for some $b \in \mathcal{B}$.

Let $w, w_1, w_2 \in \mathcal{W}$. As is well known, w_1 and w_2 are connected if and only if their Q -symbols coincide. Hence, we have

$$C(w) = \{w' \in \mathcal{W} \mid Q(w') = Q(w)\}.$$

Under the Robinson–Schensted correspondence, this connected component corresponds to $SST_n(\lambda) \times \{Q(w)\} \simeq SST_n(\lambda)$.

2.3 AI-crystal

In this subsection, we introduce the notion of AI_{n-1} -crystals (we omit the subscript “ $n - 1$ ” when there is no confusion). From now on, we assume that $n \geq 3$, and set

$$m := \begin{cases} \frac{n}{2} & \text{if } n \in \mathbb{Z}_{\text{ev}}, \\ \frac{n-1}{2} & \text{if } n \in \mathbb{Z}_{\text{odd}}. \end{cases} \tag{2}$$

Definition 2.3.1 An AI-crystal is a set \mathcal{B} equipped with structure maps $\tilde{B}_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$ and $\text{deg}_i : \mathcal{B} \rightarrow \mathbb{Z}_{\geq 0}$, $i \in [1, n - 1]$ satisfying the following axioms: Let $b \in \mathcal{B}$ and $i, j \in [1, n - 1]$.

- (1) If $\tilde{B}_i b \neq 0$, then $\text{deg}_i(\tilde{B}_i b) = \text{deg}_i(b)$ and $\tilde{B}_i^2 b = b$.
- (2) If $\tilde{B}_i b \neq 0$ and $|i - j| = 1$, then $\text{deg}_j(\tilde{B}_i b) - \text{deg}_j(b) \in \{1, -1\}$.
- (3) If $\tilde{B}_i b \neq 0$ and $|i - j| > 1$, then $\text{deg}_j(\tilde{B}_i b) = \text{deg}_j(b)$.

Definition 2.3.2 Let $\mathcal{B}_1, \mathcal{B}_2$ be AI-crystals. A morphism $\psi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ of AI-crystals is a map $\psi : \mathcal{B}_1 \sqcup \{0\} \rightarrow \mathcal{B}_2 \sqcup \{0\}$ such that

$$\psi(0) = 0, \psi(\tilde{B}_i b) = \tilde{B}_i \psi(b), \text{deg}_i(\psi(b)) = \text{deg}_i(b)$$

for all $b \in \mathcal{B}_1$ and $i \in [1, n - 1]$. A morphism of AI-crystals is said to be an isomorphism if the underlying map is bijective.

Definition 2.3.3 Let \mathcal{B} be an AI-crystal. The AI-crystal graph of \mathcal{B} is a colored (non-directed) graph defined as follows. The vertex set is \mathcal{B} . For each $b, b' \in \mathcal{B}$ and $i \in [1, n - 1]$, there is an i -colored edge between b and b' if and only if $b' = \tilde{B}_i(b)$.

Definition 2.3.4 Let \mathcal{B} be an AI-crystal and $b, b_1, b_2 \in \mathcal{B}$. We say that b_1 and b_2 are connected, or b_2 is connected to b_1 if we have

$$b_2 = \tilde{B}_{i_1} \tilde{B}_{i_2} \cdots \tilde{B}_{i_r} b_1$$

for some $i_1, \dots, i_r \in [1, n - 1]$. We call

$$C^{AI}(b) := \{b' \in \mathcal{B} \mid b' \text{ is connected to } b\}$$

the connected component $C^{AI}(b)$ of \mathcal{B} containing b . We say that \mathcal{B} is connected if we have $\mathcal{B} = C^{AI}(b)$ for some $b \in \mathcal{B}$.

Example 2.3.5 $SST_n(1)$ is equipped with an AI-crystal structure such that

$$\text{deg}_i(\boxed{j}) = \begin{cases} 1 & \text{if } j \in \{i, i + 1\}, \\ 0 & \text{otherwise} \end{cases}, \quad \tilde{B}_i \boxed{j} = \begin{cases} \boxed{j-1} & \text{if } j = i + 1, \\ \boxed{j+1} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, its AI-crystal graph is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \dots \xrightarrow{n-1} \boxed{n}.$$

Proposition 2.3.6 Let \mathcal{B}_1 be an AI-crystal and \mathcal{B}_2 a gl-crystal. Then, $\mathcal{B}_1 \otimes \mathcal{B}_2 := \mathcal{B}_1 \times \mathcal{B}_2$ is equipped with an AI-crystal structure as follows:

$$\text{deg}_i(b_1 \otimes b_2) = \begin{cases} \text{deg}_i(b_1) - \varphi_i(b_2) + \varepsilon_i(b_2) & \text{if } \text{deg}_i(b_1) > \varphi_i(b_2), \\ \varepsilon_i(b_2) & \text{if } \varphi_i(b_2) - \text{deg}_i(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}}, \\ \varepsilon_i(b_2) + 1 & \text{if } \varphi_i(b_2) - \text{deg}_i(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}}. \end{cases}$$

$$\tilde{B}_i(b_1 \otimes b_2) = \begin{cases} \tilde{B}_i b_1 \otimes b_2 & \text{if } \text{deg}_i(b_1) > \varphi_i(b_2), \\ b_1 \otimes \tilde{E}_i b_2 & \text{if } \varphi_i(b_2) - \text{deg}_i(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}}, \\ b_1 \otimes \tilde{F}_i b_2 & \text{if } \varphi_i(b_2) - \text{deg}_i(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}}. \end{cases}$$

Proof Let us verify that $\mathcal{B}_1 \otimes \mathcal{B}_2$ satisfies the axioms of AI-crystal. During the proof, we use axioms of gl-crystal and AI-crystal without mentioning one by one.

- (1) Let $b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2$ and $i \in [1, n - 1]$ be such that $\tilde{B}_i(b_1 \otimes b_2) \neq 0$. We show that $\text{deg}_i(\tilde{B}_i(b_1 \otimes b_2)) = \text{deg}_i(b_1 \otimes b_2)$ and $\tilde{B}_i^2(b_1 \otimes b_2) = b_1 \otimes b_2$.

First, suppose that $\deg_i(b_1) > \varphi_i(b_2)$. In this case, we have

$$\tilde{B}_i(b_1 \otimes b_2) = \tilde{B}_i b_1 \otimes b_2.$$

Since $\deg_i(\tilde{B}_i b_1) = \deg_i(b_1) > \varphi_i(b_2)$, we obtain

$$\begin{aligned} \deg_i(\tilde{B}_i b_1 \otimes b_2) &= \deg_i(\tilde{B}_i b_1) - \varphi_i(b_2) + \varepsilon_i(b_2) \\ &= \deg_i(b_1) - \varphi_i(b_2) + \varepsilon_i(b_2) = \deg_i(b_1 \otimes b_2), \end{aligned}$$

and

$$\tilde{B}_i(\tilde{B}_i b_1 \otimes b_2) = \tilde{B}_i^2 b_1 \otimes b_2 = b_1 \otimes b_2,$$

as desired.

Next, suppose that $\varphi_i(b_2) - \deg_i(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}}$. In this case, we have

$$\tilde{B}_i(b_1 \otimes b_2) = b_1 \otimes \tilde{E}_i b_2.$$

Since $\varphi_i(\tilde{E}_i b_2) = \varphi_i(b_2) + 1$, we have $\varphi_i(\tilde{E}_i b_2) - \deg_i(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}}$. Hence, we obtain

$$\deg_i(b_1 \otimes \tilde{E}_i b_2) = \varepsilon_i(\tilde{E}_i b_2) + 1 = \varepsilon_i(b_2) = \deg_i(b_1 \otimes b_2),$$

and

$$\tilde{B}_i(b_1 \otimes \tilde{E}_i b_2) = b_1 \otimes \tilde{F}_i \tilde{E}_i b_2 = b_1 \otimes b_2,$$

as desired.

Finally, suppose that $\varphi_i(b_2) - \deg_i(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}}$. In this case, we have

$$\tilde{B}_i(b_1 \otimes b_2) = b_1 \otimes \tilde{F}_i b_2.$$

Since $\varphi_i(\tilde{F}_i b_2) = \varphi_i(b_2) - 1$, we have $\varphi_i(\tilde{F}_i b_2) - \deg_i(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}}$. Hence, we obtain

$$\deg_i(b_1 \otimes \tilde{F}_i b_2) = \varepsilon_i(\tilde{F}_i b_2) = \varepsilon_i(b_2) + 1 = \deg_i(b_1 \otimes b_2),$$

and

$$\tilde{B}_i(b_1 \otimes \tilde{F}_i b_2) = b_1 \otimes \tilde{E}_i \tilde{F}_i b_2 = b_1 \otimes b_2,$$

as desired.

- (2) Let $b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2, i, j \in [1, n-1]$ be such that $|i-j| = 1$ and $\tilde{B}_i(b_1 \otimes b_2) \neq 0$. We show that $\deg_j(\tilde{B}_i(b_1 \otimes b_2)) - \deg_j(b_1 \otimes b_2) \in \{1, -1\}$.

Let us write $\tilde{B}_i(b_1 \otimes b_2) = b'_1 \otimes b'_2$ for some $b'_1 \in \{b_1, \tilde{B}_i b_1\} \setminus \{0\}$ and $b'_2 \in \{b_2, \tilde{E}_i b_2, \tilde{F}_i b_2\} \setminus \{0\}$. Then, exactly one of the following holds:

- $\deg_j(b'_1) - \deg_j(b_1) = 1, \varphi_j(b'_2) - \varphi_j(b_2) = 0,$ and $\varepsilon_j(b'_2) - \varepsilon_j(b_2) = 0.$
- $\deg_j(b'_1) - \deg_j(b_1) = -1, \varphi_j(b'_2) - \varphi_j(b_2) = 0,$ and $\varepsilon_j(b'_2) - \varepsilon_j(b_2) = 0.$
- $\deg_j(b'_1) - \deg_j(b_1) = 0, \varphi_j(b'_2) - \varphi_j(b_2) = -1,$ and $\varepsilon_j(b'_2) - \varepsilon_j(b_2) = 0.$
- $\deg_j(b'_1) - \deg_j(b_1) = 0, \varphi_j(b'_2) - \varphi_j(b_2) = 0,$ and $\varepsilon_j(b'_2) - \varepsilon_j(b_2) = 1.$
- $\deg_j(b'_1) - \deg_j(b_1) = 0, \varphi_j(b'_2) - \varphi_j(b_2) = 1,$ and $\varepsilon_j(b'_2) - \varepsilon_j(b_2) = 0.$
- $\deg_j(b'_1) - \deg_j(b_1) = 0, \varphi_j(b'_2) - \varphi_j(b_2) = 0,$ and $\varepsilon_j(b'_2) - \varepsilon_j(b_2) = -1.$

Therefore, we have either

$$\deg_j(b'_1) - \varphi_j(b'_2) \in \{\deg_j(b_1) - \varphi_j(b_2) \pm 1\} \text{ and } \varepsilon_j(b'_2) = \varepsilon_j(b_2)$$

or

$$\deg_j(b'_1) - \varphi_j(b'_2) = \deg_j(b_1) - \varphi_j(b_2) \text{ and } \varepsilon_j(b'_2) \in \{\varepsilon_j(b_2) \pm 1\}.$$

First, suppose that $\deg_j(b'_1) - \varphi_j(b'_2) = \deg_j(b_1) - \varphi_j(b_2) + 1.$ Then, we compute as

$$\begin{aligned} &\deg_j(b'_1 \otimes b'_2) \\ &= \begin{cases} \deg_j(b'_1) - \varphi_j(b'_2) + \varepsilon_j(b'_2) & \text{if } \deg_j(b'_1) > \varphi_j(b'_2), \\ \varepsilon_j(b'_2) & \text{if } \varphi_j(b'_2) - \deg_j(b'_1) \in \mathbb{Z}_{\geq 0, \text{ev}}, \\ \varepsilon_j(b'_2) + 1 & \text{if } \varphi_j(b'_2) - \deg_j(b'_1) \in \mathbb{Z}_{\geq 0, \text{odd}} \end{cases} \\ &= \begin{cases} \deg_j(b_1) - \varphi_j(b_2) + 1 + \varepsilon_j(b_2) & \text{if } \deg_j(b_1) \geq \varphi_j(b_2), \\ \varepsilon_j(b_2) & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}}, \\ \varepsilon_j(b_2) + 1 & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}} \setminus \{0\} \end{cases} \\ &= \begin{cases} \deg_j(b_1 \otimes b_2) + 1 & \text{if } \deg_j(b_1) \geq \varphi_j(b_2), \\ \deg_j(b_1 \otimes b_2) - 1 & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}}, \\ \deg_j(b_1 \otimes b_2) + 1 & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}} \setminus \{0\} \end{cases} \\ &\in \{\deg_j(b_1 \otimes b_2) \pm 1\}. \end{aligned}$$

Next, suppose that $\deg_j(b'_1) - \varphi_j(b'_2) = \deg_j(b_1) - \varphi_j(b_2) - 1.$ Then, we compute as

$$\begin{aligned} &\deg_j(b'_1 \otimes b'_2) \\ &= \begin{cases} \deg_j(b'_1) - \varphi_j(b'_2) + \varepsilon_j(b'_2) & \text{if } \deg_j(b'_1) > \varphi_j(b'_2), \\ \varepsilon_j(b'_2) & \text{if } \varphi_j(b'_2) - \deg_j(b'_1) \in \mathbb{Z}_{\geq 0, \text{ev}}, \\ \varepsilon_j(b'_2) + 1 & \text{if } \varphi_j(b'_2) - \deg_j(b'_1) \in \mathbb{Z}_{\geq 0, \text{odd}} \end{cases} \\ &= \begin{cases} \deg_j(b_1) - \varphi_j(b_2) - 1 + \varepsilon_j(b_2) & \text{if } \deg_j(b_1) > \varphi_j(b_2) + 1, \\ \varepsilon_j(b_2) & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}} \sqcup \{-1\}, \\ \varepsilon_j(b_2) + 1 & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}} \end{cases} \end{aligned}$$

$$= \begin{cases} \deg_j(b_1 \otimes b_2) - 1 & \text{if } \deg_j(b_1) > \varphi_j(b_2) + 1, \\ \deg_j(b_1 \otimes b_2) - 1 & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}} \sqcup \{-1\}, \\ \deg_j(b_1 \otimes b_2) + 1 & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}} \end{cases}$$

$$\in \{\deg_j(b_1 \otimes b_2) \pm 1\}.$$

Finally, suppose that $\deg_j(b'_1) - \varphi_j(b'_2) = \deg_j(b_1) - \varphi_j(b_2)$. Setting $a := \varepsilon_j(b'_2) - \varepsilon_j(b_2) \in \{\pm 1\}$, we compute as

$$\begin{aligned} &\deg_j(b'_1 \otimes b'_2) \\ &= \begin{cases} \deg_j(b'_1) - \varphi_j(b'_2) + \varepsilon_j(b'_2) & \text{if } \deg_j(b'_1) > \varphi_j(b'_2), \\ \varepsilon_j(b'_2) & \text{if } \varphi_j(b'_2) - \deg_j(b'_1) \in \mathbb{Z}_{\geq 0, \text{ev}}, \\ \varepsilon_j(b'_2) + 1 & \text{if } \varphi_j(b'_2) - \deg_j(b'_1) \in \mathbb{Z}_{\geq 0, \text{odd}} \end{cases} \\ &= \begin{cases} \deg_j(b_1) - \varphi_j(b_2) + \varepsilon_j(b_2) + a & \text{if } \deg_j(b_1) > \varphi_j(b_2), \\ \varepsilon_j(b_2) + a & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{ev}}, \\ \varepsilon_j(b_2) + a + 1 & \text{if } \varphi_j(b_2) - \deg_j(b_1) \in \mathbb{Z}_{\geq 0, \text{odd}} \end{cases} \\ &= \deg_j(b_1 \otimes b_2) + a \\ &\in \{\deg_j(b_1 \otimes b_2) \pm 1\}. \end{aligned}$$

Thus, our claim follows.

- (3) Let $b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2$, and $i, j \in [1, n - 1]$ be such that $|i - j| > 1$ and $\tilde{B}_i(b_1 \otimes b_2) \neq 0$. We show that $\deg_j(\tilde{B}_i(b_1 \otimes b_2)) = \deg_j(b_1 \otimes b_2)$. Let us write $\tilde{B}_i(b_1 \otimes b_2) = b'_1 \otimes b'_2$ for some $b'_1 \in \{b_1, \tilde{B}_i b_1\} \setminus \{0\}$ and $b'_2 \in \{b_2, \tilde{E}_i b_2, \tilde{F}_i b_2\} \setminus \{0\}$. Then, we have

$$\deg_j(b'_1) = \deg_j(b_1), \quad \varepsilon_j(b'_2) = \varepsilon_j(b_2), \quad \varphi_j(b'_2) = \varphi_j(b_2).$$

This implies that

$$\deg_j(b'_1 \otimes b'_2) = \deg_j(b_1 \otimes b_2),$$

as desired. □

Corollary 2.3.7 *Let \mathcal{B} be a \mathfrak{gl} -crystal. Then, \mathcal{B} is equipped with an AI-crystal structure as follows: For each $b \in \mathcal{B}$ and $i \in [1, n - 1]$, we set*

$$\deg_i(b) := \begin{cases} \varepsilon_i(b) & \text{if } \varphi_i(b) \in \mathbb{Z}_{\text{ev}}, \\ \varepsilon_i(b) + 1 & \text{if } \varphi_i(b) \in \mathbb{Z}_{\text{odd}}, \end{cases}$$

$$\tilde{B}_i b := \begin{cases} \tilde{E}_i b & \text{if } \varphi_i(b) \in \mathbb{Z}_{\text{ev}}, \\ \tilde{F}_i b & \text{if } \varphi_i(b) \in \mathbb{Z}_{\text{odd}}. \end{cases}$$

Proof Consider a \mathfrak{gl} -crystal $SST_n(\emptyset) = \{\emptyset\}$. Then, we have an isomorphism

$$SST_n(\emptyset) \otimes \mathcal{B} \rightarrow \mathcal{B}; \emptyset \otimes b \mapsto b$$

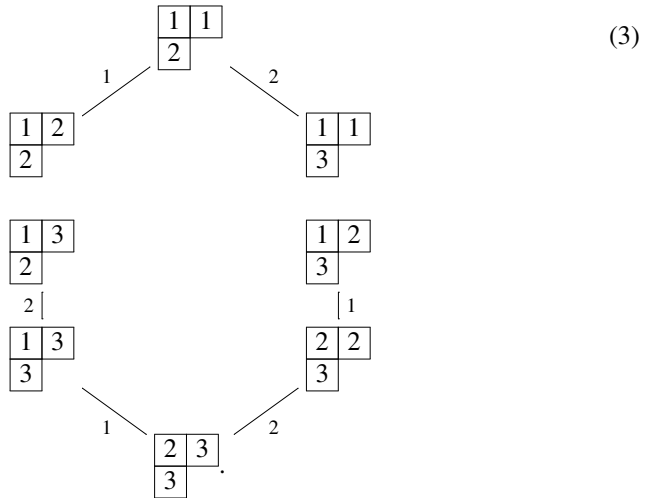
of \mathfrak{gl} -crystals. On the other hand, $SST_n(\emptyset)$ admits an AI-crystal structure given by

$$\tilde{B}_i(\emptyset) = 0, \quad \text{deg}_i(\emptyset) = 0.$$

Then, under the identification $SST_n(\emptyset) \otimes \mathcal{B} \simeq \mathcal{B}$, the AI-crystal structure on $SST_n(\emptyset) \otimes \mathcal{B}$ given by Proposition 2.3.6 is the same as the one on \mathcal{B} given by this corollary. Thus, the proof completes. \square

Remark 2.3.8 In the sequel, whenever we regard a \mathfrak{gl} -crystal as an AI-crystal, we assume that its AI-crystal structure is given by Corollary 2.3.7.

Example 2.3.9 The AI_2 -crystal graph of the \mathfrak{gl}_3 -crystal $SST_3(2, 1)$, whose crystal graph is given in equation (1), is as follows:



Proposition 2.3.10 Let \mathcal{B} be a \mathfrak{gl} -crystal, $b \in \mathcal{B}$, and $i, j \in [1, n - 1]$.

- (1) We have $\tilde{B}_i b = 0$ if and only if $\text{deg}_i(b) = 0$.
- (2) If $|i - j| > 1$ and $\tilde{B}_i b \neq 0$, then we have $\tilde{B}_j \tilde{B}_i b = \tilde{B}_i \tilde{B}_j b$.

Proof Let us prove the first assertion. Suppose that $\tilde{B}_i b = 0$. We show that $\varphi_i(b) \in \mathbb{Z}_{\text{ev}}$. Otherwise, we have

$$0 = \tilde{B}_i b = \tilde{F}_i b,$$

which implies $\varphi_i(b) = 0$. This is a contradiction. Hence, we obtain $\varphi_i(b) \in \mathbb{Z}_{\text{ev}}$, and consequently,

$$0 = \tilde{B}_i b = \tilde{E}_i b.$$

This shows that $\varepsilon_i(b) = 0$, and hence, we have

$$\deg_i(b) = \varepsilon_i(b) = 0.$$

Conversely, suppose that $\deg_i(b) = 0$. Then, we must have $\varphi_i(b) \in \mathbb{Z}_{\text{ev}}$; otherwise, $\deg_i(b) = \varepsilon_i(b) + 1 > 0$. Hence, we obtain

$$0 = \deg_i(b) = \varepsilon_i(b),$$

and consequently,

$$\tilde{B}_i b = \tilde{E}_i b = 0,$$

as desired. This completes the proof of the first assertion.

Now, let us prove the second assertion. Note that for each $X, Y \in \{E, F\}$, we have

$$\varphi_j(\tilde{X}_i b) = \varphi_j(b), \quad \varphi_i(\tilde{Y}_j b) = \varphi_i(b),$$

and

$$\tilde{Y}_j \tilde{X}_i b = \tilde{X}_i \tilde{Y}_j b.$$

Then, if we write $\tilde{B}_i b = \tilde{X}_i b$ for some $X \in \{E, F\}$, we compute as

$$\begin{aligned} \tilde{B}_j \tilde{B}_i b &= \tilde{B}_j \tilde{X}_i b \\ &= \begin{cases} \tilde{E}_j \tilde{X}_i b & \text{if } \varphi_j(b) \in \mathbb{Z}_{\text{ev}}, \\ \tilde{F}_j \tilde{X}_i b & \text{if } \varphi_j(b) \in \mathbb{Z}_{\text{odd}} \end{cases} \\ &= \begin{cases} \tilde{X}_i \tilde{E}_j b & \text{if } \varphi_j(b) \in \mathbb{Z}_{\text{ev}}, \\ \tilde{X}_i \tilde{F}_j b & \text{if } \varphi_j(b) \in \mathbb{Z}_{\text{odd}} \end{cases} \\ &= \tilde{X}_i \tilde{B}_j b \\ &= \begin{cases} \tilde{E}_i \tilde{B}_j b & \text{if } \varphi_i(b) \in \mathbb{Z}_{\text{ev}}, \\ \tilde{F}_i \tilde{B}_j b & \text{if } \varphi_i(b) \in \mathbb{Z}_{\text{odd}} \end{cases} \\ &= \tilde{B}_i \tilde{B}_j b. \end{aligned}$$

This implies the assertion. □

Proposition 2.3.11 *Let \mathcal{B}_1 be an AI-crystal, and $\mathcal{B}_2, \mathcal{B}_3$ be \mathfrak{gl} -crystals. Then, we have*

$$(\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \mathcal{B}_3 = \mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3).$$

Proof The proof is straightforward but long. Hence, we omit it. □

Remark 2.3.12 Let $\mathcal{B}_1, \mathcal{B}_2$ be \mathfrak{gl} -crystals. Then, we can equip $\mathcal{B}_1 \otimes \mathcal{B}_2$ with two AI-crystal structures; one is obtained by regarding \mathcal{B}_1 as an AI-crystal by means of Corollary 2.3.7 and then by taking tensor product, and the other is obtained by regarding the \mathfrak{gl} -crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$ as an AI-crystal by means of Corollary 2.3.7. These two structures are identical because the former is $(\text{SST}_n(\emptyset) \otimes \mathcal{B}_1) \otimes \mathcal{B}_2$, while the latter is $\text{SST}_n(\emptyset) \otimes (\mathcal{B}_1 \otimes \mathcal{B}_2)$.

Corollary 2.3.13 Let $\mathcal{B}_1, \mathcal{B}_2$ be AI-crystals, $\mathcal{B}_3, \mathcal{B}_4$ be \mathfrak{gl} -crystals, $\psi_1 : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ a morphism of AI-crystals, and $\psi_2 : \mathcal{B}_3 \rightarrow \mathcal{B}_4$ a morphism of \mathfrak{gl} -crystals. Then,

$$\psi_1 \otimes \psi_2 : \mathcal{B}_1 \otimes \mathcal{B}_3 \rightarrow \mathcal{B}_2 \otimes \mathcal{B}_4; b_1 \otimes b_2 \mapsto \psi_1(b_1) \otimes \psi_2(b_2)$$

is a morphism of AI-crystals. Furthermore, if both ψ_1 and ψ_2 are isomorphisms, then so is $\psi_1 \otimes \psi_2$.

Proof The assertion follows from Proposition 2.3.6. In fact, $\text{deg}_i(b_1 \otimes b_2)$ (resp., $\text{deg}_i(\psi_1(b_1) \otimes \psi_2(b_2))$) is determined by $\text{deg}_i(b_1), \varepsilon_i(b_2), \varphi_i(b_2)$ (resp., $\text{deg}_i(\psi(b_1)), \varepsilon_i(\psi(b_2)), \varphi_i(\psi(b_2))$), and \tilde{B}_i acts on $b_1 \otimes b_2$ (resp., $\psi_1(b_1) \otimes \psi_2(b_2)$) by either $\tilde{B}_i \otimes 1, 1 \otimes \tilde{E}_i$, or $1 \otimes \tilde{F}_i$ depending on $\text{deg}_i(b_1), \varepsilon_i(b_2), \varphi_i(b_2)$ (resp., $\text{deg}_i(\psi(b_1)), \varepsilon_i(\psi(b_2)), \varphi_i(\psi(b_2))$). \square

Definition 2.3.14 Let \mathcal{B} be an AI-crystal. Define its character $\text{ch}_{\text{AI}} \mathcal{B}$ to be a Laurent polynomial in m variables $y_1, y_3, \dots, y_{2m-1}$ given by

$$\frac{1}{2^m} \sum_{b \in \mathcal{B}} \sum_{\sigma_1, \sigma_3, \dots, \sigma_{2m-1} \in \{+, -\}} y_1^{\sigma_1 \text{deg}_1(b)} y_3^{\sigma_3 \text{deg}_3(b)} \dots y_{2m-1}^{\sigma_{2m-1} \text{deg}_{2m-1}(b)} \in \mathbb{Z}[y_1^{\pm 1}, y_3^{\pm 1}, \dots, y_{2m-1}^{\pm 1}].$$

Example 2.3.15 We have

$$\text{ch}_{\text{AI}_2} \text{SST}_3(1) = y_1 + 1 + y_1^{-1}, \quad \text{ch}_{\text{AI}_3} \text{SST}_4(1) = y_1 + y_1^{-1} + y_3 + y_3^{-1}.$$

Let us explain the meaning of the character of an AI-crystal \mathcal{B} . Assume that \mathcal{B} has the following property:

- (1) For each $b \in \mathcal{B}$ and $i \in [1, m]$, we have $\tilde{B}_{2i-1}(b) = 0$ if and only if $\text{deg}_{2i-1}(b) = 0$.
- (2) For each $b \in \mathcal{B}$ and $i \neq j \in [1, m]$, we have $\tilde{B}_{2i-1} \tilde{B}_{2j-1} b = \tilde{B}_{2j-1} \tilde{B}_{2i-1} b$.

For example, an AI-subcrystal of a \mathfrak{gl} -crystal admits this property (see Proposition 2.3.10). Set $\bar{\mathcal{L}} := \mathbb{C}\mathcal{B}$, and extend the maps \tilde{B}_i to linear operators on $\bar{\mathcal{L}}$. Note that the operators $\tilde{B}_1, \tilde{B}_3, \dots, \tilde{B}_{2m-1}$ pairwise commute. We say that a vector $u \in \bar{\mathcal{L}}$ is a weight vector of weight $\nu = (\nu_1, \nu_3, \dots, \nu_{2m-1}) \in \mathbb{Z}^m$ if it satisfies the following:

- (1) u is a linear combination of $b \in \mathcal{B}$ such that $\text{deg}_{2i-1}(b) = |\nu_{2i-1}|$ for all $i \in [1, m]$.
- (2) $\tilde{B}_{2i-1}(u) = \begin{cases} u & \text{if } \nu_{2i-1} > 0, \\ 0 & \text{if } \nu_{2i-1} = 0, \\ -u & \text{if } \nu_{2i-1} < 0, \end{cases}$

Let $\overline{\mathcal{L}}_\nu$ denote the subspace of weight vectors of weight ν . Then, $\overline{\mathcal{L}}$ admits the weight space decomposition

$$\overline{\mathcal{L}} = \bigoplus_{\nu \in \mathbb{Z}^m} \overline{\mathcal{L}}_\nu.$$

In fact, if we take a complete set \mathcal{B}' of representatives for \mathcal{B}/\sim with respect to the equivalence relation given by

$$b_1 \sim b_2 \text{ if and only if } b_2 \in \{\tilde{B}_{2i_1-1} \tilde{B}_{2i_2-1} \cdots \tilde{B}_{2i_r-1} b_1 \mid r \in [0, m], 1 \leq i_1 < \cdots < i_r \leq m\},$$

then the set

$$\{(1 + \sigma_1 \tilde{B}_1)(1 + \sigma_3 \tilde{B}_3) \cdots (1 + \sigma_{2m-1} \tilde{B}_{2m-1})b \mid b \in \mathcal{B}', \deg_{2i-1}(b) = |\nu_{2i-1}| \text{ for all } i \in [1, m]\},$$

where $\sigma_{2i-1} \in \{+, -\}$ denotes the signature of ν_{2i-1} , forms a basis of $\overline{\mathcal{L}}_\nu$. Therefore, we have

$$\text{ch}_{\text{AI}} \mathcal{B} = \sum_{\nu \in \mathbb{Z}^m} (\dim \overline{\mathcal{L}}_\nu) \mathbf{y}^\nu,$$

where

$$\mathbf{y}^{(\nu_1, \nu_3, \dots, \nu_{2m-1})} := y_1^{\nu_1} y_3^{\nu_3} \cdots y_{2m-1}^{\nu_{2m-1}}.$$

2.4 K-matrices

In this subsection, we introduce a family of isomorphisms of AI-crystals. They are closely related to K -matrices appearing in the representation theory of ι quantum group of type AI.

Let $k \in [0, n]$ and consider a \mathfrak{gl} -crystal $\text{SST}_n(1^k)$, where

$$1^k := \begin{cases} \overbrace{(1, \dots, 1)}^k & \text{if } k \neq 0, \\ \emptyset & \text{if } k = 0. \end{cases}$$

For each $1 \leq j_1 < \cdots < j_k \leq n$, set

$$u_{j_1, \dots, j_k} := \begin{cases} \begin{matrix} \boxed{j_1} \\ \vdots \\ \boxed{j_k} \end{matrix} & \text{if } k \neq 0, \\ \emptyset & \text{if } k = 0. \end{cases}$$

Then, we have

$$\text{SST}_n(1^k) = \begin{cases} \{u_{j_1, \dots, j_k} \mid 1 \leq j_1 < \dots < j_k \leq n\} & \text{if } k \neq 0, \\ \{\emptyset\} & \text{if } k = 0. \end{cases}$$

The AI-crystal structure of $\text{SST}_n(1^k)$ can be easily described as follows.

Lemma 2.4.1 *Let $1 \leq j_1 < \dots < j_k \leq n$ and $i \in [1, n - 1]$. Then, we have*

$$\text{deg}_i(u_{j_1, \dots, j_k}) = \begin{cases} 1 & \text{if } |\{j_1, \dots, j_k\} \cap \{i, i + 1\}| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{B}_i u_{j_1, \dots, j_k} = \begin{cases} u_{j_1, \dots, j_{l-1}, i, j_{l+1}, \dots, j_k} & \text{if } j_{l-1} < i = j_l - 1 \text{ for some } l \in [1, k], \\ u_{j_1, \dots, j_{l-1}, i+1, j_{l+1}, \dots, j_k} & \text{if } j_l + 1 = i + 1 < j_{l+1} \text{ for some } l \in [1, k], \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4.2 For each $k \in [0, n]$, define a map $K = K^{(k)} : \text{SST}_n(1^k) \rightarrow \text{SST}_n(1^{n-k})$ by

$$K(u_{j_1, \dots, j_k}) = u_{j_1^c, \dots, j_{n-k}^c},$$

where $\{j_1^c, \dots, j_{n-k}^c\} = [1, n] \setminus \{j_1, \dots, j_k\}$.

Proposition 2.4.3 *Let $k \in [0, n]$. Then, $K^{(k)} : \text{SST}_n(1^k) \rightarrow \text{SST}_n(1^{n-k})$ is an isomorphism of AI-crystals with inverse $K^{(n-k)} : \text{SST}_n(1^{n-k}) \rightarrow \text{SST}_n(1^k)$.*

Proof It is clear from the definition that $K^{(k)}$ is a bijection with inverse $K^{(n-k)}$. For each $1 \leq j_1 < \dots < j_k \leq n$ and $i \in [1, n - 1]$, by Lemma 2.4.1, we see that

$$K(\tilde{B}_i u_{j_1, \dots, j_k}) = \tilde{B}_i K(u_{j_1, \dots, j_k}),$$

and

$$\text{deg}_i(K(u_{j_1, \dots, j_k})) = \text{deg}_i(u_{j_1, \dots, j_k}).$$

Thus, the proof completes. □

Corollary 2.4.4 *Let \mathcal{B} be a \mathfrak{gl} -crystal. Then, for each $k \in [0, n]$, the map*

$$K \otimes 1 : \text{SST}_n(1^k) \otimes \mathcal{B} \rightarrow \text{SST}_n(1^{n-k}) \otimes \mathcal{B}; \quad b_1 \otimes b_2 \mapsto K(b_1) \otimes b_2$$

is an isomorphism of AI-crystals.

Proof The assertion follows from Proposition 2.4.3 and Corollary 2.3.13. □

Let $\lambda \in \text{Par}_n$. For each $j \in [1, \lambda_1]$, let d_j denote the length of the j th column of $D(\lambda)$. By the definition of the \mathfrak{gl} -crystal structure of $\text{SST}_n(\lambda)$, there exists an embedding

$$\text{SST}_n(\lambda) \hookrightarrow \text{SST}_n(1^{d_1}) \otimes \text{SST}_n(1^{d_2}) \otimes \cdots \otimes \text{SST}_n(1^{d_{\lambda_1}})$$

of a \mathfrak{gl} -crystal which sends $T = C_1 \cdots C_{\lambda_1}$ to $C_1 \otimes \cdots \otimes C_{\lambda_1}$, where C_j denotes the j th column of T . Define a new semistandard Young tableau $K_1(T)$ by

$$K_1(T) := P(K(C_1) \otimes C_2 \otimes \cdots \otimes C_{\lambda_1}).$$

Then, the following are immediate from the definition of K_1 and Corollary 2.4.4.

Proposition 2.4.5 *Let $\lambda \in \text{Par}_n$ and $T \in \text{SST}_n(\lambda)$. Then, we have*

$$|K_1(T)| - |T| = n - 2\ell(\lambda).$$

Proposition 2.4.6 *Let $\lambda \in \text{Par}_n$ and $T \in \text{SST}_n(\lambda)$. Then, for each $i \in [1, n - 1]$, we have*

$$\tilde{B}_i(K_1(T)) = K_1(\tilde{B}_i T), \quad \text{deg}_i(K_1(T)) = \text{deg}_i(T).$$

Here, we set $K_1(0) := 0$.

2.5 AI-tableaux

In this subsection, we introduce AI-tableaux, which are central objects in this paper. They provide us many concrete examples of AI-crystals which are not \mathfrak{gl} -crystals.

Definition 2.5.1 Let $\lambda \in \text{Par}_n$ and $T \in \text{SST}_n(\lambda)$. We say that T satisfies the AI-condition, or T is an AI-tableau, if it satisfies the following two conditions:

- (1) $d_1 \leq m$ (see equation (2) for the definition of m).
- (2) $t_{i,1}^c \leq t_{i,2}$ for all $i \in [1, d_2]$, where $\{t_{1,1}^c, \dots, t_{n-d_1,1}^c\} = [1, n] \setminus \{t_{1,1}, \dots, t_{d_1,1}\}$.

Here, d_j denotes the length of the j th column of $D(\lambda)$, and $t_{i,j}$ denotes the (i, j) th entry of T . For each $\lambda \in \text{Par}_n$, let $\text{SST}_n^{\text{AI}}(\lambda)$ denote the set of semistandard Young tableaux of shape λ satisfying the AI-condition.

Remark 2.5.2 By Lemma 2.2.2, the second condition for AI-tableaux is equivalent to saying that if we write $T = C_1 C_2 \cdots C_{\lambda_1}$, where C_j denotes the j th column of T , then

$$K_1(T) = K(C_1) C_2 \cdots C_{\lambda_1}.$$

Remark 2.5.3 If T satisfies the AI-condition for some n , then so does for all $n' \geq n$. However, AI-condition depends on n , in general. For example, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an AI-tableau when $n \geq 4$, but not when $n = 3$.

Remark 2.5.4 It is clear that $\text{SST}_n^{\text{AI}}(\lambda) = \emptyset$ unless $\ell(\lambda) \leq m$.

Example 2.5.5

(1) Let $n \geq 3, \lambda = (l), l \geq 0$. Then, $T \in \text{SST}_n(l)$ is an AI-tableau if and only if T does not begin with $\begin{smallmatrix} \boxed{1} & \boxed{1} \end{smallmatrix}$. For example,

$$\text{SST}_3^{\text{AI}}(2) = \{ \begin{smallmatrix} \boxed{1} & \boxed{2} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} \end{smallmatrix}, \begin{smallmatrix} \boxed{2} & \boxed{2} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{2} & \boxed{3} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{3} & \boxed{3} \\ \boxed{2} \end{smallmatrix} \}$$

(2) Let $n \geq 4, \lambda = (l_1, l_2), l_1 \geq l_2 > 0$. Then, $T \in \text{SST}_n(l_1, l_2)$ is an AI-tableau if and only if the following conditions are satisfied:

- (a) The first row of T does not begin with $\begin{smallmatrix} \boxed{1} & \boxed{1} \end{smallmatrix}$.
- (b) The second row of T does not begin with $\begin{smallmatrix} \boxed{3} & \boxed{3} \end{smallmatrix}$.
- (c) The first two rows of T do not begin with $\begin{smallmatrix} \boxed{1} & \boxed{2} \\ \boxed{2} \end{smallmatrix}$.

For example,

$$\text{SST}_4^{\text{AI}}(2, 1) = \{ \begin{smallmatrix} \boxed{1} & \boxed{2} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{2} \\ \boxed{4} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{4} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{4} \\ \boxed{2} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{4} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{4} \\ \boxed{4} \end{smallmatrix}, \begin{smallmatrix} \boxed{2} & \boxed{2} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{2} & \boxed{2} \\ \boxed{4} \end{smallmatrix}, \begin{smallmatrix} \boxed{2} & \boxed{3} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{2} & \boxed{3} \\ \boxed{4} \end{smallmatrix}, \begin{smallmatrix} \boxed{2} & \boxed{4} \\ \boxed{3} \end{smallmatrix}, \begin{smallmatrix} \boxed{2} & \boxed{4} \\ \boxed{4} \end{smallmatrix}, \begin{smallmatrix} \boxed{3} & \boxed{3} \\ \boxed{4} \end{smallmatrix}, \begin{smallmatrix} \boxed{3} & \boxed{4} \\ \boxed{4} \end{smallmatrix} \}$$

Lemma 2.5.6 Let $\lambda \in \text{Par}_n$ and $T \in \text{SST}_n(\lambda)$. Then, there exists $r \geq 0$ such that $K_1^r(T)$ is an AI-tableau.

Proof We prove by induction on $|T|$. When T is an AI-tableau, there is nothing to prove. Suppose that T is not an AI-tableau. Set $d_1 := \ell(\lambda)$. Then, we have two possibilities; (1) $d_1 > m$ or (2) $d_1 \leq m$ and the second condition in Definition 2.5.1 fails. If we are in the first case, then by Proposition 2.4.5, we have

$$|K_1(T)| - |T| = n - 2d_1 \leq n - 2(m + 1) < 0.$$

Hence, the assertion follows from our induction hypothesis.

Now, assume that we are in the second case. Then, by Remark 2.5.2 and Lemma 2.2.2, the length of $\text{sh}(K_1(T))$ must be $n - d_1 + \alpha$ for some $\alpha > 0$. Using Proposition 2.4.5, we compute as

$$\begin{aligned} |K_1^2(T)| - |T| &= (|K_1^2(T)| - |K_1(T)|) + (|K_1(T)| - |T|) \\ &= (n - 2(n - d_1 + \alpha)) + (n - 2d_1) \\ &= -2\alpha < 0. \end{aligned}$$

Hence, the assertion follows from our induction hypothesis. This completes the proof. □

Lemma 2.5.7 Let $\rho \in \text{Par}_m, T \in \text{SST}_n^{\text{AI}}(\rho)$. Then, we have $K_1^2(T) = T$. Furthermore, $K_1(T)$ is an AI-tableau if and only if $\ell(\rho) = \frac{n}{2}$.

Proof For each $j \in [1, \lambda_1]$, let d_j denote the length of the j th column of $D(\lambda)$, and C_j the j th column of T . By Remark 2.5.2, we have

$$K_1(T) = K(C_1)C_2 \cdots C_{\lambda_1}. \tag{4}$$

Since T is a semistandard Young tableau, by Lemma 2.2.2, we see that

$$K_1^2(T) = K^2(C_1)C_2 \cdots C_{\lambda_1} = C_1C_2 \cdots C_{\lambda_1} = T. \tag{5}$$

This implies the first assertion. Combining Remark 2.5.2 and equation (5), we see that $K_1(T)$ is an AI-tableau if and only if $\ell(\text{sh}(K_1(T))) \leq m$. Since $\ell(\text{sh}(K_1(T))) = n - \ell(\lambda)$ by equation (4), and since $\ell(\lambda) \leq m$, the second assertion follows. This completes the proof. \square

Definition 2.5.8 Let $\lambda \in \text{Par}_n$ and $T \in \text{SST}_n(\lambda)$. The P^{AI} -symbol of T is an AI-tableau $P^{\text{AI}}(T)$ given by

$$P^{\text{AI}}(T) := K_1^r(T),$$

where $r := \min\{s \mid K_1^s(T) \text{ is an AI-tableau}\}$.

Example 2.5.9 Let $n = 4$ and $T = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \notin \text{SST}_4^{\text{AI}}(2, 2)$. Then, we compute as

$$K_1 \left(\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \right) = P \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \notin \text{SST}_4^{\text{AI}}(2, 1, 1),$$

$$K_1 \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) = P \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \in \text{SST}_4^{\text{AI}}(2).$$

Hence, we obtain $P^{\text{AI}}(T) = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$.

Remark 2.5.10 $P^{\text{AI}}(T)$ depends on n , in general. For example, we have

$$P^{\text{AI}} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{cases} \begin{bmatrix} 3 \end{bmatrix} & \text{if } n = 3, \\ \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \text{if } n > 3. \end{cases}$$

Proposition 2.5.11 Let $\rho \in \text{Par}_m$. Then, $\text{SST}_n^{\text{AI}}(\rho)$ is an AI-subcrystal of $\text{SST}_n(\rho)$.

Proof Let $T \in \text{SST}_n^{\text{AI}}(\rho)$ and $i \in [1, n - 1]$ be such that $\tilde{B}_i T \neq 0$. It suffices to show that $\tilde{B}_i T \in \text{SST}_n^{\text{AI}}(\rho)$. Assume contrary. Then, by the proof of Lemma 2.5.6, we have

$$|P^{\text{AI}}(\tilde{B}_i T)| < |\tilde{B}_i T| = |T|.$$

On the other hand, if we write $P^{\text{AI}}(\tilde{B}_i T) = K_1^r(\tilde{B}_i T)$ for some $r > 0$, then we have

$$P^{\text{AI}}(\tilde{B}_i T) = \tilde{B}_i K_1^r(T)$$

by Proposition 2.4.6. Hence, we compute as

$$|P^{\text{AI}}(\tilde{B}_i T)| = |\tilde{B}_i K_1^r(T)| = |K_1^r(T)| \geq |T|,$$

where the last inequality follows from Lemma 2.5.7. Thus, we obtain a contradiction. Hence, the proof completes. \square

Proposition 2.5.12 *The map*

$$P^{\text{AI}} : \bigsqcup_{\lambda \in \text{Par}_n} \text{SST}_n(\lambda) \rightarrow \bigsqcup_{\rho \in \text{Par}_m} \text{SST}_n^{\text{AI}}(\rho); T \mapsto P^{\text{AI}}(T)$$

is a morphism of AI-crystals.

Proof Let $\lambda \in \text{Par}_n, T \in \text{SST}_n(\lambda), i \in [1, n - 1]$. Let $r \geq 0$ be the minimum integer such that $P^{\text{AI}}(T) = K_1^r(T)$. By Proposition 2.4.6, we see that

$$\tilde{B}_i P^{\text{AI}}(T) = K_1^r(\tilde{B}_i T), \quad \text{deg}_i(P^{\text{AI}}(T)) = \text{deg}_i(T).$$

Hence, it suffices to show that

$$P^{\text{AI}}(\tilde{B}_i T) = K_1^r(\tilde{B}_i T).$$

First, suppose that $\tilde{B}_i T = 0$. Since K_1^r is a morphism of AI-crystals, our claim follows immediately.

Next, suppose that $\tilde{B}_i T \neq 0$. Let $r' \geq 0$ be the minimum integer such that

$$P^{\text{AI}}(\tilde{B}_i T) = K_1^{r'}(\tilde{B}_i T).$$

By Proposition 2.5.11, we see that $\tilde{B}_i P^{\text{AI}}(T)$, which equals $K_1^r(\tilde{B}_i T)$, is an AI-tableau. By the minimality of r' , we have $r' \leq r$. Let us show that $r' = r$. If $r' < r$, then

$$K_1^{r'}(T) = \tilde{B}_i^2 K_1^{r'}(T) = \tilde{B}_i K_1^{r'}(\tilde{B}_i T) = \tilde{B}_i P^{\text{AI}}(\tilde{B}_i T).$$

Here, the first equality holds because $\tilde{B}_i T \neq 0$, and hence,

$$\tilde{B}_i K_1^{r'}(T) = K_1^{r'}(\tilde{B}_i T) \neq 0.$$

Since $P^{\text{AI}}(\tilde{B}_i T)$ is an AI-tableau, so is $\tilde{B}_i P^{\text{AI}}(\tilde{B}_i T)$, which equals $K_1^{r'}(T)$. This contradicts the minimality of r . Thus, we obtain $r' = r$, and hence,

$$P^{\text{AI}}(\tilde{B}_i T) = K_1^r(\tilde{B}_i T),$$

as desired. \square

2.6 Low rank examples

In this subsection, we investigate the AI-crystal structures of $SST_n^{AI}(\rho)$, $\rho \in \text{Par}_m$ in the case when $n = 3, 4$.

First, assume that $n = 3$ and consider $SST_3^{AI}(l)$, $l \geq 0$. For each $a \in \{1, 2\}$ and $b \in [0, l - 1]$, set

$$T_{a,b} := \boxed{a \ 2 \ 2 \ \cdots \ 2 \ 3 \ 3 \ \cdots \ 3} \in SST_3^{AI}(l),$$

where b is the number of 3's. Also, set

$$T_l := \boxed{3 \ 3 \ \cdots \ 3} \in SST_3^{AI}(l).$$

Then, we have

$$SST_3^{AI}(l) = \{T_{a,b} \mid a \in \{1, 2\}, b \in [0, l - 1]\} \sqcup \{T_l\},$$

and hence,

$$|SST_3^{AI}(l)| = 2l + 1. \tag{6}$$

Lemma 2.6.1 *Let $n = 3$ and $l \geq 0$. Then, the AI_2 -crystal $SST_3^{AI}(l)$ is connected.*

Proof From definitions, we obtain

$$\begin{aligned} \tilde{B}_1 T_{a,b} &= T_{a',b}, \\ \tilde{B}_2 T_{a,b} &= \begin{cases} 0 & \text{if } l - \delta_{a,1} - b \in \mathbb{Z}_{\text{ev}} \text{ and } b = 0, \\ T_{a,b-1} & \text{if } l - \delta_{a,1} - b \in \mathbb{Z}_{\text{ev}} \text{ and } b > 0, \\ T_{a,b+1} & \text{if } l - \delta_{a,1} - b \in \mathbb{Z}_{\text{odd}} \text{ and } b < l - 1, \\ T_l & \text{if } l - \delta_{a,1} - b \in \mathbb{Z}_{\text{odd}} \text{ and } b = l - 1, \end{cases} \\ \tilde{B}_1 T_l &= 0, \\ \tilde{B}_2 T_l &= T_{2,l-1}, \end{aligned}$$

where $a' \in \{1, 2\} \setminus \{a\}$. Then, we see that

$$\tilde{B}_2 \tilde{B}_1 T_{1,l-1} = \tilde{B}_2 T_{2,l-1} = T_l,$$

and

$$\tilde{B}_2 \tilde{B}_1 T_{a',b} = \tilde{B}_2 T_{a,b} = T_{a,b+1}$$

for all $b \in [0, l - 2]$, where $a = 1$ if $l - b \in \mathbb{Z}_{\text{ev}}$ and $a = 2$ otherwise, and $a' \in \{1, 2\} \setminus \{a\}$. These show that each $T \in SST_3^{AI}(\rho)$ is connected to T_l . Therefore, $SST_3^{AI}(\rho)$ is connected. □

Next, assume that $n = 4$ and consider $\text{SST}_4^{\text{AI}}(l)$, $l \geq 0$. For each $a \in \{1, 2\}$, $c \in [0, l - 1]$, and $b \in [0, l - c - 1]$, set

$$T_{a,b,c} := \boxed{a \ 2 \ 2 \ \cdots \ 2 \ 3 \ 3 \ \cdots \ 3 \ 4 \ 4 \ \cdots \ 4} \in \text{SST}_4^{\text{AI}}(l),$$

where b and c are the numbers of 3's and 4's, respectively. Also, for each $c \in [0, l]$, set

$$T_c := \boxed{3 \ 3 \ \cdots \ 3 \ 4 \ 4 \ \cdots \ 4} \in \text{SST}_4^{\text{AI}}(l),$$

where c is the number of 4's. Then, we have

$$\text{SST}_4^{\text{AI}}(l) = \{T_{a,b,c} \mid a \in \{1, 2\}, c \in [0, l - 1], b \in [0, l - c - 1]\} \sqcup \{T_c \mid c \in [0, l]\},$$

and hence,

$$|\text{SST}_4^{\text{AI}}(l)| = \sum_{c=0}^{l-1} 2(l - c) + (l + 1) = (l + 1)^2. \tag{7}$$

Lemma 2.6.2 *Let $n = 4$ and $l \geq 0$. Then, the AI_3 -crystal $\text{SST}_4^{\text{AI}}(l)$ is connected.*

Proof From definitions, we obtain

$$\begin{aligned} \tilde{B}_1 T_{a,b,c} &= T_{a',b,c}, \\ \tilde{B}_2 T_{a,b,c} &= \begin{cases} 0 & \text{if } l - \delta_{a,1} - b - c \in \mathbb{Z}_{\text{ev}} \text{ and } b = 0, \\ T_{a,b-1,c} & \text{if } l - \delta_{a,1} - b - c \in \mathbb{Z}_{\text{ev}} \text{ and } b > 0, \\ T_{a,b+1,c} & \text{if } l - \delta_{a,1} - b - c \in \mathbb{Z}_{\text{odd}} \text{ and } b < l - c - 1, \\ T_c & \text{if } l - \delta_{a,1} - b - c \in \mathbb{Z}_{\text{odd}} \text{ and } b = l - c - 1, \end{cases} \\ \tilde{B}_3 T_{a,b,c} &= \begin{cases} 0 & \text{if } b \in \mathbb{Z}_{\text{ev}} \text{ and } c = 0, \\ T_{a,b+1,c-1} & \text{if } b \in \mathbb{Z}_{\text{ev}} \text{ and } c > 0, \\ T_{a,b-1,c+1} & \text{if } b \in \mathbb{Z}_{\text{odd}}, \end{cases} \\ \tilde{B}_1 T_c &= 0, \\ \tilde{B}_2 T_c &= \begin{cases} 0 & \text{if } l - c = 0, \\ T_{2,l-c-1,c} & \text{if } l - c > 0, \end{cases} \\ \tilde{B}_3 T_c &= \begin{cases} 0 & \text{if } l - c \in \mathbb{Z}_{\text{ev}} \text{ and } c = 0, \\ T_{c-1} & \text{if } l - c \in \mathbb{Z}_{\text{ev}} \text{ and } c > 0, \\ T_{c+1} & \text{if } l - c \in \mathbb{Z}_{\text{odd}}. \end{cases} \end{aligned}$$

where $a' \in \{1, 2\} \setminus \{a\}$. We show that each $T_{a,b,c}$ and T_c are connected to $T_{1,0,0}$ by induction on c . When $c = 0$, our claim follows from Lemma 2.6.1. Assume that $c > 0$. Then, by Lemma 2.6.1, we see that $T_{a,b,c}$ is connected to T_c . Then, we have

$\tilde{B}_3(T_c) = T_{c-1}$ (resp., $\tilde{B}_3\tilde{B}_2T_c = T_{2,l-c,c-1}$) if $l - c$ is even (resp., $l - c$ is odd). Hence, our induction hypothesis implies that T_c is connected to $T_{1,0,0}$. Thus, the proof completes. \square

Finally, assume that $n = 4$ and consider $SST_4^{\text{AI}}(l_1, l_2)$, $l_1 \geq l_2 > 0$. For each $a \in \{1, 2\}$, $c \in [0, l_1 - l_2]$, and $b \in [0, l_1 - c - 1]$, set

$$T_{a,b,c} := \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline a & 2 & 2 & \cdots & 2 & 3 & 3 & \cdots & 3 & 4 & 4 & \cdots & 4 \\ \hline 4 & 4 & \cdots & 4 & & & & & & & & & \\ \hline \end{array} \in SST_4(l_1, l_2),$$

where b and c are the numbers of 3's and 4's in the first row, respectively. Also, for each $c \in [0, l_1 - l_2]$, set

$$T_c := \begin{array}{|c|c|c|c|c|c|c|c|} \hline 3 & 3 & \cdots & \cdots & 3 & 4 & 4 & \cdots & 4 \\ \hline 4 & 4 & \cdots & 4 & & & & & \\ \hline \end{array} \in SST_4(l),$$

where c is the number of 4's in the first row. Then, we have

$$SST_4^{\text{AI}}(l_1, l_2) = \{T_{a,b,c}, K_1(T_{a,b,c}) \mid a \in \{1, 2\}, c \in [0, l_1 - l_2], b \in [0, l_1 - c - 1]\} \sqcup \{T_c, K_1(T_c) \mid c \in [0, l_1 - l_2]\},$$

and hence,

$$|SST_4^{\text{AI}}(l_1, l_2)| = 2 \left(\sum_{c=0}^{l_1-l_2} 2(l_1 - c) + (l_1 - l_2 + 1) \right) = 2(l_1 - l_2 + 1)(l_1 + l_2 + 1). \tag{8}$$

Note that we have

$$K_1(T_{a,b,c}), K_1(T_c) \notin \{T_{a',b',c'} \mid a' \in \{1, 2\}, c' \in [0, l_1 - l_2], b' \in [0, l_1 - c' - 1]\} \sqcup \{T_{c'} \mid c' \in [0, l_1 - l_2]\}$$

because the $(2, 1)$ th entries of $K_1(T_{a,b,c})$ and $K_1(T_c)$ are not 4, while those of $T_{a',b',c'}$ and $T_{c'}$ are 4.

Lemma 2.6.3 *Let $n = 4$ and $l_1 \geq l_2 > 0$. Then, the AI_3 -crystal $SST_4^{\text{AI}}(l_1, l_2)$ is connected.*

Proof By definitions, we obtain

$$\begin{aligned} \tilde{B}_1 T_{a,b,c} &= T_{a',b,c}, \\ \tilde{B}_2 T_{a,b,c} &= \begin{cases} 0 & \text{if } l_1 - \delta_{a,1} - b - c \in \mathbb{Z}_{\text{ev}} \text{ and } b = 0, \\ T_{a,b-1,c} & \text{if } l_1 - \delta_{a,1} - b - c \in \mathbb{Z}_{\text{ev}} \text{ and } b > 0, \\ T_{a,b+1,c} & \text{if } l_1 - \delta_{a,1} - b - c \in \mathbb{Z}_{\text{odd}} \text{ and } b < l_1 - c - 1, \\ T_c & \text{if } l_1 - \delta_{a,1} - b - c \in \mathbb{Z}_{\text{odd}} \text{ and } b = l_1 - c - 1, \end{cases} \\ \tilde{B}_3 T_{a,b,c} &= \begin{cases} K_1(T_{a',b,c}) & \text{if } b < l_2, \\ 0 & \text{if } b - l_2 \in \mathbb{Z}_{\geq 0, \text{ev}} \text{ and } c = 0, \\ T_{a,b+1,c-1} & \text{if } b - l_2 \in \mathbb{Z}_{\geq 0, \text{ev}} \text{ and } c > 0, \\ T_{a,b-1,c+1} & \text{if } b - l_2 \in \mathbb{Z}_{\geq 0, \text{odd}}, \end{cases} \\ \tilde{B}_1 T_c &= 0, \\ \tilde{B}_2 T_c &= T_{2,l_1-c-1,c}, \\ \tilde{B}_3 T_c &= \begin{cases} 0 & \text{if } l_1 - l_2 - c \in \mathbb{Z}_{\text{ev}} \text{ and } c = 0, \\ T_{c-1} & \text{if } l_1 - l_2 - c \in \mathbb{Z}_{\text{ev}} \text{ and } c > 0, \\ T_{c+1} & \text{if } l_1 - l_2 - c \in \mathbb{Z}_{\text{odd}}. \end{cases} \end{aligned}$$

where $a' \in \{1, 2\} \setminus \{a\}$. We show that each $T_{a,b,c}$ and T_c are connected to $T_{1,0,0}$ by induction on c . If this is the case, then the assertion follows from the fact that K_1 commutes with \tilde{B}_i 's and that $K_1(T_{1,0,0}) = \tilde{B}_1 \tilde{B}_3 T_{1,0,0}$ as verified from computation above. When $c = 0$, our claim follows from Lemma 2.6.1. Assume that $c > 0$. By Lemma 2.6.1, we see that $T_{a,b,c}$ is connected to T_c . Then, we have $\tilde{B}_3 T_c = T_{c-1}$ (resp., $\tilde{B}_3 \tilde{B}_2 T_c = T_{2,l_1-c,c-1}$) if $l_1 - l_2 - c$ is even (resp., if $l_1 - l_2 - c$ is odd). Hence, our induction hypothesis implies that T_c is connected to $T_{1,0,0}$. Thus, the proof completes. \square

3 Representation theoretic interpretation

In this section, we show that the characters of $\text{SST}_n^{\text{AI}}(\rho)$, $\rho \in \text{Par}_m$ coincide with the characters of certain \mathfrak{so}_n -modules. To do so, we need results from the representation theory of the ι quantum group of type AI obtained in [18].

3.1 Representation theoretic interpretation of \mathfrak{gl} -crystals

Let us briefly review the finite-dimensional representation theory of the general linear algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$. The \mathfrak{gl}_n is realized as the Lie algebra of $n \times n$ matrices. For each $1 \leq i, j \leq n$, let $E_{i,j}$ denote the matrix unit with entry 1 at (i, j) -position. Let $e_i := E_{i,i+1}$, $f_i := E_{i+1,i}$, $i \in \{1, \dots, n-1\}$ denote the Chevalley generators.

Let M be a finite-dimensional \mathfrak{gl}_n -module. Then, it admits a weight space decomposition

$$M = \bigoplus_{\lambda=(\lambda_1, \dots, \lambda_n) \in X_{\mathfrak{gl}_n}} M_\lambda, \quad M_\lambda := \{m \in M \mid E_{i,i}m = \lambda_i m \text{ for all } i \in \{1, \dots, n\}\},$$

where $X_{\mathfrak{gl}_n} := \mathbb{Z}^n$ denotes the weight lattice for \mathfrak{gl}_n . The character $\text{ch}_{\mathfrak{gl}_n} M$ of M is a Laurent polynomial defined by

$$\text{ch}_{\mathfrak{gl}_n} M := \sum_{\lambda \in X_{\mathfrak{gl}_n}} (\dim M_\lambda) \mathbf{x}^\lambda \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

where

$$\mathbf{x}^{(\lambda_1, \dots, \lambda_n)} := x_1^{\lambda_1} \dots x_n^{\lambda_n}.$$

The finite-dimensional \mathfrak{gl}_n -modules are completely reducible, and the isomorphism classes of finite-dimensional irreducible \mathfrak{gl}_n -modules are parametrized by the set

$$X_{\mathfrak{gl}_n}^+ := \{\lambda = (\lambda_1, \dots, \lambda_n) \in X_{\mathfrak{gl}_n} \mid \lambda_1 \geq \dots \geq \lambda_n\}$$

of dominant integral weights. For $\lambda \in X_{\mathfrak{gl}_n}^+$, let $V^{\mathfrak{gl}_n}(\lambda)$ denote the corresponding \mathfrak{gl}_n -module, i.e., the finite-dimensional irreducible \mathfrak{gl}_n -module of highest weight λ .

The assignment

$$\text{Par}_n \rightarrow X_{\mathfrak{gl}_n, \geq 0}^+ := \{\lambda = (\lambda_1, \dots, \lambda_n) \in X_{\mathfrak{gl}_n}^+ \mid \lambda_n \geq 0\}; \quad \lambda \mapsto (\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, \dots, 0)$$

is bijective. In this way, we often identify these two sets. In particular, for a partition $\lambda \in \text{Par}_n$, we understand $\lambda_i = 0$ for $i > \ell(\lambda)$. As is well known, the \mathfrak{gl} -crystal $\text{SST}_n(\lambda)$ models $V^{\mathfrak{gl}_n}(\lambda)$ in the following sense:

$$\text{ch}_{\mathfrak{gl}_n} V^{\mathfrak{gl}_n}(\lambda) = \text{ch}_{\mathfrak{gl}_n} \text{SST}_n(\lambda).$$

3.2 Representation theory of the q quantum group of type A

Let $\mathfrak{so}_n = \mathfrak{so}_n(\mathbb{C})$ denote the special orthogonal algebra. It is realized as the Lie subalgebra of \mathfrak{gl}_n generated by $b_i := f_i + e_i, i \in \{1, \dots, n - 1\}$ (cf. [18, Section 4.1]). Recall the integer m from equation (2). Note that it is the rank of \mathfrak{so}_n .

Let M be a finite-dimensional \mathfrak{so}_n -module. Then, it admits a weight space decomposition

$$M = \bigoplus_{\nu=(\nu_1, \nu_3, \dots, \nu_{2m-1}) \in X_{\mathfrak{so}_n}} M_\nu, \quad M_\nu := \{m \in M \mid b_{2i-1}m = \nu_{2i-1}m \text{ for all } i \in \{1, \dots, m\}\},$$

where $X_{\mathfrak{so}_n} := \mathbb{Z}^m \sqcup (\frac{1}{2} + \mathbb{Z})^m$ denotes the weight lattice for \mathfrak{so}_n . Note that $b_1, b_3, \dots, b_{2m-1}$ generate a Cartan subalgebra of \mathfrak{so}_n , and the sets $\{b_1 - b_3, b_3 - b_5, \dots, b_{2m-3} - b_{2m-1}, 2b_{2m-1}\}$ and $\{b_1 - b_3, b_3 - b_5, \dots, b_{2m-3} - b_{2m-1}, b_{2m-3} + b_{2m-1}\}$ form the simple coroots when n is odd and when n is even, respectively (cf. [18, Section 4.1]). A weight $\nu \in X_{\mathfrak{so}_n}$ is said to be an integer weight if $\nu \in \mathbb{Z}^m$. Let $X_{\mathfrak{so}_n, \text{int}}$ denote the set of integer weights. The character $\text{ch}_{\mathfrak{so}_n} M$ of M is a Laurent polynomial defined by

$$\text{ch}_{\mathfrak{so}_n} M := \sum_{\nu \in X_{\mathfrak{so}_n}} (\dim M_\nu) \mathbf{y}^\nu \in \mathbb{Z}[y_1^{\pm \frac{1}{2}}, y_3^{\pm \frac{1}{2}}, \dots, y_{2m-1}^{\pm \frac{1}{2}}],$$

where

$$\mathbf{y}^{(v_1, v_3, \dots, v_{2m-1})} := y_1^{v_1} y_3^{v_3} \cdots y_{2m-1}^{v_{2m-1}}.$$

The finite-dimensional \mathfrak{so}_n -modules are completely reducible, and the isomorphism classes of irreducible finite-dimensional \mathfrak{so}_n -modules are parametrized by the set

$$X_{\mathfrak{so}_n}^+ := \begin{cases} \{(v_1, v_3, \dots, v_{2m-1}) \in X_{\mathfrak{so}_n} \mid v_1 \geq \dots \geq v_{2m-3} \geq |v_{2m-1}|\} & \text{if } n \in \mathbb{Z}_{\text{ev}}, \\ \{(v_1, v_3, \dots, v_{2m-1}) \in X_{\mathfrak{so}_n} \mid v_1 \geq \dots \geq v_{2m-1} \geq 0\} & \text{if } n \in \mathbb{Z}_{\text{odd}} \end{cases}$$

of dominant integral weights. For $\nu \in X_{\mathfrak{so}_n}^+$, let $V^{\mathfrak{so}_n}(\nu)$ denote the corresponding \mathfrak{so}_n -module, that is, the irreducible module of highest weight ν . Set $X_{\mathfrak{so}_n, \text{int}}^+ := X_{\mathfrak{so}_n}^+ \cap X_{\mathfrak{so}_n, \text{int}}$.

For each $\rho \in \text{Par}_m$, set

$$V^{\mathfrak{so}_n}(\rho) := \begin{cases} V^{\mathfrak{so}_n}(\nu_\rho) & \text{if } \ell(\rho) < \frac{n}{2}, \\ V^{\mathfrak{so}_n}(\nu_\rho^+) \oplus V^{\mathfrak{so}_n}(\nu_\rho^-) & \text{if } \ell(\rho) = \frac{n}{2}, \end{cases}$$

where

$$\nu_\rho := (\rho_1, \rho_2, \dots, \rho_{\ell(\rho)}, 0, \dots, 0), \quad \nu_\rho^\pm := (\rho_1, \rho_2, \dots, \rho_{m-1}, \pm \rho_m) \in X_{\mathfrak{so}_n, \text{int}}^+.$$

The rest of this section is devoted to showing that $\text{SST}_n^{\text{Al}}(\rho)$ models $V^{\mathfrak{so}_n}(\rho)$.

Let $\lambda \in \text{Par}_n$ and consider $\text{SST}_n(\lambda)$. Set

$$\overline{\mathcal{L}}(\lambda) := \mathbb{C}\text{SST}_n(\lambda),$$

and extend the operators $\tilde{E}_i, \tilde{F}_i, \tilde{B}_i$ on $\text{SST}_n(\lambda)$ to linear operators on $\overline{\mathcal{L}}(\lambda)$. As we have seen in the end of Subsection 2.3, the space $\overline{\mathcal{L}}(\lambda)$ admits an \mathfrak{so}_n -weight space decomposition

$$\overline{\mathcal{L}}(\lambda) = \bigoplus_{\nu \in X_{\mathfrak{so}_n, \text{int}}} \overline{\mathcal{L}}(\lambda)_\nu.$$

This decomposition corresponds to the weight space decomposition of $V^{\mathfrak{gl}_n}(\lambda)$ regarded as an \mathfrak{so}_n -module: For each $\nu \in X_{\mathfrak{so}_n, \text{int}}$, it holds that

$$\dim V^{\mathfrak{gl}_n}(\lambda)_\nu = \dim \overline{\mathcal{L}}(\lambda)_\nu.$$

Hence, we have

$$\text{ch}_{\mathfrak{so}_n} V^{\mathfrak{gl}_n}(\lambda) = \text{ch}_{\text{AI}} \text{SST}_n(\lambda).$$

From this, we can say that the AI-crystal $\text{SST}_n(\lambda)$ models the \mathfrak{so}_n -module $V^{\mathfrak{gl}_n}(\lambda)$.

In [18], certain linear operators on $\overline{\mathcal{L}}(\lambda)$ are defined via the representation theory of the ι quantum group of type AI. They are denoted by \tilde{X}_j, \tilde{Y}_j with $j \in \tilde{I}$ (in [18], \tilde{I} is denoted by \tilde{I}_ξ), where

$$\tilde{I} := \{(i, +), (i, -) \mid i \in [1, n - 2]_{\text{ev}}\} \sqcup [1, n - 1]_{\text{ev}}.$$

These operators are defined locally in the following sense: \tilde{X}_2, \tilde{Y}_2 (resp., $\tilde{X}_{2, \pm}, \tilde{Y}_{2, \pm}$) are defined in terms of $\tilde{E}_i, \tilde{F}_i, i \in \{1, 2\}$ (resp., $i \in \{1, 2, 3\}$). And $\tilde{X}_i, \tilde{Y}_i, i \in [1, n - 1]_{\text{ev}}$ (resp., $\tilde{X}_{i, \pm}, \tilde{Y}_{i, \pm}, i \in [1, n - 2]_{\text{ev}}$) are defined in the same way as \tilde{X}_2, \tilde{Y}_2 (resp., $\tilde{X}_{2, \pm}, \tilde{Y}_{2, \pm}$) with the role of $\{1, 2\}$ (resp., $\{1, 2, 3\}$) replaced by $\{i - 1, i\}$ (resp., $\{i - 1, i, i + 1\}$).

A nonzero weight vector $v \in \overline{\mathcal{L}}(\lambda)_\nu$ is said to be a highest weight vector of weight $\nu \in X_{\mathfrak{so}_n, \text{int}}^+$ if $\tilde{X}_j v = 0$ for all $j \in \tilde{I}$. For a highest weight vector v of weight ν , set

$$\tilde{Y}v := \mathbb{C}\{\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_r} v \mid r \geq 0, j_1, \dots, j_r \in \tilde{I}\}.$$

This space corresponds to an irreducible \mathfrak{so}_n -submodule of $V^{\mathfrak{gl}_n}(\lambda)$ isomorphic to $V^{\mathfrak{so}_n}(\nu)$. Similarly, for a vector $u \in \overline{\mathcal{L}}(\lambda)$, set

$$\tilde{B}u := \mathbb{C}\{\tilde{B}_{i_1} \cdots \tilde{B}_{i_r} u \mid r \geq 0, i_1, \dots, i_r \in [1, n - 1]\}.$$

By definitions of \tilde{Y}_j 's and \tilde{B}_i 's, for each highest weight vector v , we see that

$$\tilde{B}v \subset \tilde{Y}v.$$

Also, $\tilde{Y}v$ admits an \mathfrak{so}_n -weight space decomposition

$$\tilde{Y}v = \bigoplus_{\xi \in X_{\mathfrak{so}_n, \text{int}}} (\tilde{Y}v \cap \overline{\mathcal{L}}(\lambda)_\xi),$$

and it holds that

$$\dim(\tilde{Y}v \cap \overline{\mathcal{L}}(\lambda)_\xi) = \dim V^{\mathfrak{so}_n}(\nu)_\xi.$$

For each $\nu \in X_{\mathfrak{so}_n, \text{int}}^+$, choose a basis $\{v_1^\nu, \dots, v_{m_\nu}^\nu\}$ of the subspace $H(\nu) \subset \overline{\mathcal{L}}(\lambda)$ consisting of highest weight vectors of weight ν . Then, we have

$$\overline{\mathcal{L}}(\lambda) = \bigoplus_{\nu \in X_{\mathfrak{so}_n, \text{int}}^+} \bigoplus_{k=1}^{m_\nu} \tilde{Y} v_k^\nu.$$

By observation above, this corresponds to an irreducible decomposition of $V^{\mathfrak{gl}_n}(\lambda)$ as an \mathfrak{so}_n -module.

A basis of the subspace $H(\nu)$ can be found as follows (see [18] for detail).

Definition 3.2.1 Let \mathcal{B} be an AI-crystal. An element $b \in \mathcal{B}$ is said to be a singular element of degree $\rho \in \text{Par}_m$ if it satisfies the following:

- (1) $\text{deg}_{2i-1}(b) = \rho_i$ for all $i \in [1, m]$.
- (2) $\text{deg}_{2i}(b) = 0$ for all $i \in [1, m]$ such that $2i < n$.
- (3) $\text{deg}_{2i+1}((\tilde{B}_{2i-1} \tilde{B}_{2i})^{\rho_i+1} b) = 0$ for all $i \in [1, m]$ such that $2i + 1 < n$.

Let $\text{Sing}(\mathcal{B}, \rho)$ denote the set of singular elements of degree ρ .

Let $\rho \in \text{Par}_m$, and set

$$\text{Sing}(\lambda, \rho) := \text{Sing}(\text{SST}_n(\lambda), \rho).$$

For each $S \in \text{Sing}(\lambda, \rho)$, set

$$h(S) := (1 + \tilde{B}_1)(1 + \tilde{B}_3) \cdots (1 + \tilde{B}_{2m-3})S.$$

Then, when $\ell(\rho) < \frac{n}{2}$ (resp., $\ell(\rho) = \frac{n}{2}$), the vector $h(S)$ (resp., $h(S)_\pm := (1 \pm \tilde{B}_{2m-1})h(S)$) is a highest weight vector of weight

$$\begin{aligned} \nu &:= (\rho_1, \rho_2, \dots, \rho_{\ell(\rho)}, 0, \dots, 0) \\ (\text{resp., } \nu_\pm &:= (\rho_1, \rho_2, \dots, \rho_{m-1}, \pm \rho_m)). \end{aligned}$$

Furthermore, $\{h(S) \mid S \in \text{Sing}(\lambda, \rho)\}$ (resp., $\{h(S)_+ \mid S \in \text{Sing}(\lambda, \rho)\}$) forms a basis of $H(\nu)$ (resp., $H(\nu_+)$). For each $S, S' \in \text{Sing}(\lambda, \rho)$, we have $h(S) = h(S')$ if and only if $S = S'$ (resp., $h(S)_+ = h(S')_+$ if and only if $S' = \tilde{B}_1 \tilde{B}_3 \cdots \tilde{B}_{2m-1} S$). When $\ell(\rho) = \frac{n}{2}$, we have $h(S)_- = -h(S')_-$ if and only if $S' = \tilde{B}_1 \tilde{B}_3 \cdots \tilde{B}_{2m-1} S$.

Now, it is convenient to set $\text{Sing}'(\lambda, \rho)$ to be $\text{Sing}(\lambda, \rho)$ if $\ell(\rho) < \frac{n}{2}$, and to be a complete set of representatives for $\text{Sing}(\lambda, \rho)/\sim$ with respect to the equivalence relation given by

$$S \sim S' \text{ if and only if } S' = \tilde{B}_1 \tilde{B}_3 \cdots \tilde{B}_{2m-1} S$$

if $\ell(\rho) = \frac{n}{2}$. Then, from discussion above, we obtain

$$\overline{\mathcal{L}}(\lambda) = \bigoplus_{\rho \in \text{Par}_m} \bigoplus_{S \in \text{Sing}'(\lambda, \rho)} \tilde{Y} h(S). \tag{9}$$

Furthermore, for each $S \in \text{Sing}'(\lambda, \rho)$, the subspace $\tilde{Y}h(S)$ corresponds to an \mathfrak{so}_n -submodule of $V^{\mathfrak{gl}_n}(\lambda)$ isomorphic to $V^{\mathfrak{so}_n}(\rho)$.

Lemma 3.2.2 *Let $S \in \text{Sing}(\lambda, \rho)$. Let m' denote the maximal integer such that $2m' < n$ and $\rho_{m'} \neq 0$. Then, we have*

$$\tilde{B}_1 \tilde{B}_2^2 \tilde{B}_3 \tilde{B}_4^2 \cdots \tilde{B}_{2m'-1} \tilde{B}_{2m'}^2 h(S) = S.$$

Consequently, we have

$$\tilde{B}S = \tilde{B}h(S).$$

Proof By the definition of m' , we have

$$h(S) = (1 + \tilde{B}_1)(1 + \tilde{B}_3) \cdots (1 + \tilde{B}_{2m'-1})S,$$

and hence,

$$\begin{aligned} \tilde{B}_1 \tilde{B}_2^2 \tilde{B}_3 \tilde{B}_4^2 \cdots \tilde{B}_{2m'-1} \tilde{B}_{2m'}^2 h(S) &= \tilde{B}_1 \tilde{B}_2^2 (1 + \tilde{B}_1) \tilde{B}_3 \\ &\tilde{B}_4^2 (1 + \tilde{B}_3) \cdots \tilde{B}_{2m'-1} \tilde{B}_{2m'}^2 (1 + \tilde{B}_{2m'-1})S. \end{aligned}$$

Here, we used Proposition 2.3.10 (2). Therefore, it suffices to show that

$$\tilde{B}_{2i-1} \tilde{B}_{2i}^2 (1 + \tilde{B}_{2i-1})S = S$$

for all $i \in [1, m']$.

Let $i \in [1, m']$. Then, we have $\text{deg}_{2i}(S) = 0$ and $\text{deg}_{2i-1}(S) \neq 0$. By Proposition 2.3.10 (1), we have $\tilde{B}_{2i}S = 0$ and $\tilde{B}_{2i-1}S \neq 0$, and hence,

$$\tilde{B}_{2i-1} \tilde{B}_{2i}^2 (1 + \tilde{B}_{2i-1})S = \tilde{B}_{2i-1} \tilde{B}_{2i}^2 \tilde{B}_{2i-1}S.$$

By Definition 2.3.1 (2), we must have

$$\text{deg}_{2i}(\tilde{B}_{2i-1}S) = 1,$$

and hence,

$$\tilde{B}_{2i}^2 \tilde{B}_{2i-1}S = \tilde{B}_{2i-1}S.$$

Therefore, we have

$$\tilde{B}_{2i-1} \tilde{B}_{2i}^2 \tilde{B}_{2i-1}S = \tilde{B}_{2i-1}^2 S = S,$$

as desired. This completes the proof. □

3.3 Connectedness and the character of $SST_n^{\text{AI}}(\rho)$

In this subsection, we show that the AI-crystal $SST_n^{\text{AI}}(\rho)$ is connected, and its character coincides with $\text{ch } V^{\text{so}_n}(\rho)$ for each $\rho \in \text{Par}_m$.

Let $\rho \in \text{Par}_m$. Define $T_\rho \in SST_n^{\text{AI}}(\rho)$ by

$$T_\rho(i, j) = \begin{cases} a_{2i-1} & \text{if } j = 1, \\ 2i & \text{if } j > 1, \end{cases}$$

where $a_{2i-1} \in \{2i - 1, 2i\}$ is such that

$$\rho_i - \delta_{a_{2i-1}, 2i-1} - \delta_{a_{2i+1}, 2i+1} \in \mathbb{Z}_{\text{ev}}.$$

Here, we set $a_{2\ell(\rho)+1} = 2\ell(\rho) + 2$. Note that such a_{2i-1} 's are uniquely determined.

Example 3.3.1 Let $\rho \in \text{Par}_m$.

(1) If $\ell(\rho) = 1$, then

$$T_\rho = \boxed{a \ 2 \ 2 \ \cdots \ 2}, \quad a := \begin{cases} 2 & \text{if } \rho_1 \in \mathbb{Z}_{\text{ev}}, \\ 1 & \text{if } \rho_1 \in \mathbb{Z}_{\text{odd}}. \end{cases}$$

(2) If $\ell(\rho) = 2$, then

$$T_\rho = \begin{array}{cccccc} a & 2 & 2 & \cdots & 2 & 2 & \cdots & 2 \\ b & 4 & 4 & \cdots & 4 & & & \end{array},$$

$$b := \begin{cases} 4 & \text{if } \rho_2 \in \mathbb{Z}_{\text{ev}}, \\ 3 & \text{if } \rho_2 \in \mathbb{Z}_{\text{odd}}, \end{cases} \quad a := \begin{cases} 2 & \text{if } \rho_1 - \delta_{b,3} \in \mathbb{Z}_{\text{odd}}, \\ 1 & \text{if } \rho_1 - \delta_{b,3} \in \mathbb{Z}_{\text{ev}}. \end{cases}$$

Lemma 3.3.2 Let $\rho \in \text{Par}_m$ be such that $\ell(\rho) = \frac{n}{2}$. Then, we have

$$K_1(T_\rho) = \tilde{B}_1 \tilde{B}_3 \cdots \tilde{B}_{2m-1} T_\rho.$$

Proof By Remark 2.5.2, $K_1(T_\rho)$ is obtained from T_ρ by applying K to the first column. Since the $(i, 1)$ th entry of T_ρ is either $2i - 1$ or $2i$ for each $i \in [1, m]$, we have

$$K_1(T_\rho)(i, j) = \begin{cases} a'_{2i-1} & \text{if } j = 1, \\ 2i & \text{if } j > 1, \end{cases}$$

where $a'_{2i-1} \in \{2i - 1, 2i\} \setminus \{a_{2i-1}\}$. Now, the assertion is easily verified. □

Lemma 3.3.3 Let $\rho, \sigma \in \text{Par}_m$ and $S \in SST_n^{\text{AI}}(\sigma) \cap \text{Sing}(\sigma, \rho)$. Then, we have $\sigma = \rho$, and either $S = T_\rho$ or $S = K_1(T_\rho)$.

Proof Let us show by induction on $l \in [1, m]$ that in S , the letters $2l - 1$ and $2l$ can appear only in the l th row. Let $l \in [1, m]$. By our induction hypothesis, it holds that $\text{sh}(S|_{[1, 2(l-1)]}) = (\sigma_1, \sigma_2, \dots, \sigma_{l-1})$. In particular, the l th row or below consists of letters in $[2l - 1, n]$.

First, we show that the l th row consists of only $2l - 1$ and $2l$. Assume contrary that $k > 2l$ appears in the l th row. If k is odd, then it must hold that $\varepsilon_{k-1}(S) > 0$, and hence $\text{deg}_{k-1}(S) > 0$, which contradicts Definition 3.2.1 (2). On the other hand, if k is even, then it must hold that $\varepsilon_{k-1}(\tilde{B}_{k-2}\tilde{B}_{k-3})^{\text{deg}_{k-1}(S)} > 0$ (note that \tilde{B}_{k-2} and \tilde{B}_{k-3} do not change the entries other than $k - 3, k - 2, k - 1$), and hence, $\text{deg}_{k-1}(\tilde{B}_{k-2}\tilde{B}_{k-3})^{\text{deg}_{k-1}(S)} > 0$, which contradicts Definition 3.2.1 (3). Thus, we see that the l th row of S consists of only $2l - 1$ and $2l$.

Next, we show that

$$S|_{\{2l-1, 2l\}} = \boxed{s_l \mid 2l \mid 2l \mid \cdots \mid 2l},$$

where $s_l := S(l, 1) \in \{2l - 1, 2l\}$. By the argument above, we have

$$S|_{\{2l-1, 2l\}} = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2l - 1 & 2l - 1 & \cdots & \cdots & 2l - 1 & 2l & 2l & \cdots & 2l \\ \hline 2l & 2l & \cdots & 2l, & & & & & \\ \hline \end{array}$$

where the first row consists of σ_l boxes. By our induction hypothesis, for each $l' < l$, we have $s_{l'} := S(l', 1) \in \{2l' - 1, 2l'\}$. Hence, the $(l', 1)$ th entry of $K_1(S)$ is the unique letter $s'_{l'}$ in $\{2l' - 1, 2l'\} \setminus \{s_{l'}\}$. Now, suppose that $s_{l+1} := S(l + 1, 1) \neq 2l$. Then, by observation above, the $(l, 1)$ th entry s'_l of $K_1(S)$ is the unique letter in $\{2l - 1, 2l\} \setminus \{s_l\}$. Since S is an AI-tableau, we have

$$s'_l \leq S(l, 2) \leq \cdots \leq S(l, \rho_l).$$

This, together with the semistandardness condition on the l th row

$$s_l \leq S(l, 2) \leq \cdots \leq S(l, \rho_l),$$

implies that $S(l, 2) \geq 2l$. Therefore, we must have

$$S|_{\{2l-1, 2l\}} = \boxed{s_l \mid 2l \mid 2l \mid \cdots \mid 2l},$$

as desired.

It remains to show that $s_{l+1} \neq 2l$. If $s_{l+1} = 2l$, then it must hold that $s_l = 2l - 1$, and consequently,

$$s'_l > 2l.$$

Since S is an AI-tableau and the l th row consists of $2l - 1$ and $2l$, this implies that $\sigma_l = 1$. Therefore, the l th row and below of S is of the form

$$\begin{array}{|c|} \hline 2l-1 \\ \hline 2l \\ \hline s_{l+2} \\ \hline \vdots \\ \hline s_{\ell(\sigma)} \\ \hline \end{array} \tag{10}$$

This shows that $\text{deg}_{2l-1}(S) = 0$. Since $\rho_l = \text{deg}_{2l-1}(S)$ and ρ is a partition, we obtain $\rho = (\rho_1, \rho_2, \dots, \rho_{l-1})$. This implies that $\text{deg}_i(S) = 0$ for all $i \in [2l - 1, n - 1]$. From (10), we see that $\text{deg}_{2l}(S) = 0$ if and only if $s_{l+2} = 2l + 1$. Proceeding in this way, we must have $s_{l+k} = 2l - 1 + k$ for all $k \in [1, n - 2l + 1]$, which is impossible because $l + (n - 2l + 1) = n - l + 1 > m \geq \ell(\sigma)$. Thus, our claim follows.

So far, we have obtained that

$$S(i, j) = \begin{cases} s_i & \text{if } j = 1, \\ 2i & \text{if } j > 1 \end{cases}$$

for some $s_i \in \{2i - 1, 2i\}$. From this, one can easily see that

$$\text{deg}_{2i-1}(S) = \sigma_i$$

for all $i \in [1, m]$. On the other hand, since $\text{deg}_{2i-1}(S) = \rho_i$, we obtain

$$\sigma = \rho.$$

In order to complete the proof, we need to determine s_l for all $l \in [1, \ell(\rho)]$. First, suppose that $\ell(\rho) < \frac{n}{2}$. In this case, $2\ell(\rho) \in [1, n - 1]$, and hence, we must have

$$\text{deg}_{2\ell(\rho)}(S) = 0.$$

This is equivalent to that

$$\rho_{\ell(\rho)} - \delta_{s_{\ell(\rho)}, 2\ell(\rho)-1} \in \mathbb{Z}_{\text{ev}}.$$

Similarly, the constraint that $\text{deg}_{2(\ell(\rho)-1)}(S) = 0$ is equivalent to that

$$\rho_{\ell(\rho)-1} - \delta_{s_{\ell(\rho)-1}, 2\ell(\rho)-3} - \delta_{s_{\ell(\rho)}, 2\ell(\rho)-1} \in \mathbb{Z}_{\text{ev}}.$$

Proceeding in this way, we see that

$$\rho_l - \delta_{s_l, 2l-1} - \delta_{s_{l+1}, 2l+1} \in \mathbb{Z}_{\text{ev}},$$

for all $l \in [1, \ell(\rho)]$, where we set $s_{\ell(\rho)+1} := 2\ell(\rho) + 2$. This implies that

$$S = T_\rho,$$

as desired.

Next, suppose that $\ell(\rho) = \frac{n}{2}$. In a similar way to above, we see that

$$\rho_l - \delta_{s_l, 2l-1} - \delta_{s_{l+1}, 2l+1} \in \mathbb{Z}_{\text{ev}},$$

for all $l \in [1, \ell(\rho) - 1]$. This implies that we have $S = T_\rho$ if $\rho_m - \delta_{s_m, n-1} \in \mathbb{Z}_{\text{ev}}$, or $S = K_1(T_\rho)$ otherwise. Thus, the proof completes. \square

Proposition 3.3.4 *Let $\lambda \in \text{Par}_n$, $\rho \in \text{Par}_m$, and $S \in \text{Sing}(\lambda, \rho)$. Then, we have either $P^{\text{AI}}(S) = T_\rho$ or $P^{\text{AI}}(S) = K_1(T_\rho)$.*

Proof Let $\sigma \in \text{Par}_m$ denote the shape of $P^{\text{AI}}(S)$. Then, we have

$$P^{\text{AI}}(S) \in \text{SST}_n^{\text{AI}}(\sigma) \cap \text{Sing}(\sigma, \rho).$$

Then, the assertion follows from Lemma 3.3.3. \square

Lemma 3.3.5 *Let $\lambda \in \text{Par}_n$, $\rho \in \text{Par}_m$, and $S \in \text{Sing}(\lambda, \rho)$. Then, we have $\tilde{B}S = \tilde{Y}h(S)$.*

Proof Since we know $\tilde{B}S = \tilde{B}h(S) \subset \tilde{Y}h(S)$ (by Lemma 3.2.2) and $h(S) \in \tilde{B}S$, it suffices to show that $\tilde{Y}_j(\tilde{B}S) \subset \tilde{B}S$ for all $j \in \tilde{I}$. Furthermore, since \tilde{Y}_j 's are defined locally (see Subsection 3.2), it suffices to prove for the case when $n = 3$ and $j = 2$, and when $n = 4$ and $j = (2, \pm)$.

Let $(n, j) \in \{(3, 2), (4, (2, \pm))\}$. By Proposition 3.3.4 and Lemmas 2.6.1–2.6.3, we see that

$$C^{\text{AI}}(S) \rightarrow \text{SST}_n^{\text{AI}}(\rho); T \mapsto P^{\text{AI}}(T)$$

is a surjective morphism of AI-crystals. This implies that

$$\dim \tilde{B}S \geq |\text{SST}_n^{\text{AI}}(\rho)|$$

since we have $\tilde{B}S = \mathbb{C}C^{\text{AI}}(S)$. On the other hand, since $\tilde{B}S \subset \tilde{Y}h(S)$, we have

$$\dim \tilde{B}S \leq \dim \tilde{Y}h(S) = \dim V^{\text{so}_n}(\rho) = |\text{SST}_n^{\text{AI}}(\rho)|.$$

The last equality follows from equations (6)–(8). Therefore, we obtain $\dim \tilde{B}S = \dim \tilde{Y}h(S)$, and hence,

$$\tilde{B}S = \tilde{Y}h(S).$$

Since the right-hand side is closed under \tilde{Y}_j , so is the left-hand side. Thus, the proof completes. \square

Theorem 3.3.6 *Let $\rho \in \text{Par}_m$. Then, the following hold.*

- (1) $\text{SST}_n^{\text{Al}}(\rho)$ is connected.
- (2) $\text{ch}_{\text{Al}} \text{SST}_n^{\text{Al}}(\rho) = \text{ch } V^{\text{so}_n}(\rho)$.
- (3) Suppose that $\ell(\rho) < \frac{n}{2}$. Then, we have $\text{ch}_{\text{Al}} \text{SST}_n^{\text{Al}}(\rho) = \text{ch}_{\text{so}_n} V^{\text{so}_n}(v_\rho)$.
- (4) Suppose that $\ell(\rho) = \frac{n}{2}$. Then, $\{T + K_1(T) \mid T \in \text{SST}_n^{\text{Al}}(\rho)\}$ is a connected Al-crystal, and $\text{ch}_{\text{Al}} \text{SST}_n^{\text{Al}}(\rho) = \text{ch}_{\text{so}_n} V^{\text{so}_n}(v_\rho^+)$.

Proof Let us prove the first assertion. By equation (9) and Lemma 3.3.5, we see that

$$\bar{\mathcal{L}}(\rho) = \bigoplus_{\sigma \in \text{Par}_m} \bigoplus_{S \in \text{Sing}'(\rho, \sigma)} \mathbb{C}C^{\text{Al}}(S).$$

This implies that for each $T \in \text{SST}_n(\rho)$, there exists a unique $\sigma \in \text{Par}_m$ and $S \in \text{Sing}'(\rho, \sigma)$ such that $T \in C^{\text{Al}}(S)$. In particular, since $\text{SST}_n^{\text{Al}}(\rho)$ is closed under \tilde{B}_i 's, each $T \in \text{SST}_n^{\text{Al}}(\rho)$ is connected to an element of $\text{SST}_n^{\text{Al}}(\rho) \cap \text{Sing}'(\rho, \sigma)$ for some $\sigma \in \text{Par}_m$. By Lemma 3.3.3, we see that

$$|\text{SST}_n^{\text{Al}}(\rho) \cap \text{Sing}'(\rho, \sigma)| \leq \delta_{\rho, \sigma}.$$

Therefore, each $T \in \text{SST}_n^{\text{Al}}(\rho)$ is connected to the element of $\text{SST}_n^{\text{Al}}(\rho) \cap \text{Sing}'(\rho, \rho)$. This implies that $\text{SST}_n^{\text{Al}}(\rho)$ is connected.

Next, Let $S \in \text{SST}_n^{\text{Al}}(\rho) \cap \text{Sing}'(\rho, \rho)$. From Lemma 3.3.5 and the first assertion, we have

$$\tilde{Y}h(S) = \tilde{B}S = \mathbb{C}\text{SST}_n^{\text{Al}}(\rho).$$

Then, the second assertion is clear from discussion after equation (9).

The third assertion is clear from the second assertion and the definitions.

Finally, assume that $\ell(\rho) = \frac{n}{2}$. Then, that $\{T + K_1(T) \mid T \in \text{SST}_n^{\text{Al}}(\rho)\}$ is a connected Al-crystal follows from the facts that K_1 is an automorphism of Al-crystal on $\text{SST}_n^{\text{Al}}(\rho)$, and that $\text{SST}_n^{\text{Al}}(\rho)$ is connected. The assertion concerning characters follows from the fact that

$$h(S) + h(K_1(S)) = h(T_\rho) + h(\tilde{B}_1 \tilde{B}_3 \cdots \tilde{B}_{2m-1} T_\rho) = 2h(T_\rho)_+$$

is a highest weight vector of weight ν , where $S \in \text{SST}_n^{\text{Al}}(\rho) \cap \text{Sing}'(\rho, \rho)$ (see also Lemma 3.3.2). Thus, the proof completes. \square

Remark 3.3.7 During the proof of Theorem 3.3.6, we obtained

$$\text{SST}_n^{\text{Al}}(\rho) \cap \text{Sing}(\rho, \sigma) = \begin{cases} \emptyset & \text{if } \sigma \neq \rho, \\ \{T_\rho\} & \text{if } \sigma = \rho \text{ and } \ell(\rho) < \frac{n}{2}, \\ \{T_\rho, K_1(T_\rho)\} & \text{if } \sigma = \rho \text{ and } \ell(\rho) = \frac{n}{2}. \end{cases}$$

4 Robinson–Schensted-type correspondence

In this section, we generalize the Robinson–Schensted correspondence to the setting of AI-crystals. This tells us how various AI-crystals decompose into their connected components.

4.1 Insertion scheme

Let $\lambda \in \text{Par}_n$. Then, the map

$$\text{SST}_n(\lambda) \otimes \text{SST}_n(1) \rightarrow \bigsqcup_{\substack{\mu \in \text{Par}_n \\ \lambda \triangleleft \mu}} \text{SST}_n(\mu); T \otimes \square \mapsto (T \leftarrow l)$$

is an isomorphism of \mathfrak{gl} -crystals. The aim of this subsection is to provide an AI-crystal analogue of this isomorphism, which will play a central role when generalizing the Robinson–Schensted correspondence.

Let $\rho \in \text{Par}_m$ and consider the AI-crystal $\text{SST}_n^{\text{AI}}(\rho) \otimes \text{SST}_n(1)$. In order to analyze its structure, let us recall the following fact, which is the special case of [5, Lemma 7].

Lemma 4.1.1 *Let $v \in X_{\mathfrak{so}_n, \text{int}}^+$. For each $k \in [1, m]$, set*

$$\epsilon_{2k-1} := (\overbrace{0, \dots, 0}^{k-1}, 1, 0, \dots, 0) \in X_{\mathfrak{so}_n, \text{int}}.$$

Then, we have

$$V^{\mathfrak{so}_n}(v) \otimes V^{\mathfrak{so}_n}(\epsilon_1) \simeq \bigoplus_{\xi} V^{\mathfrak{so}_n}(v + \xi),$$

where ξ runs through $X_{\mathfrak{so}_n, \text{int}}$ satisfying the following:

- (1) $V^{\mathfrak{so}_n}(\epsilon_1)_{\xi} \neq 0$.
- (2) $V^{\mathfrak{so}_n}(\epsilon_1)_{\xi + (1 + v_{2i-1} - v_{2i+1})(\epsilon_{2i-1} - \epsilon_{2i+1})} = 0$ for all $i \in [1, m - 1]$.
- (3) If $n \in \mathbb{Z}_{\text{ev}}$, then $V^{\mathfrak{so}_n}(\epsilon_1)_{\xi + (1 + v_{2m-3} + v_{2m-1})(\epsilon_{2m-3} + \epsilon_{2m-1})} = 0$.
- (4) If $n \in \mathbb{Z}_{\text{odd}}$, then $V^{\mathfrak{so}_n}(\epsilon_1)_{\xi + (1 + 2v_{2m-1})\epsilon_{2m-1}} = 0$.

In terms of AI-crystals, this lemma can be rewritten as follows.

Lemma 4.1.2 *Let $\rho \in \text{Par}_m$.*

(1) *Suppose that $n \in \mathbb{Z}_{\text{ev}}$ and $\rho_m \neq 1$. Then, we have*

$$\text{SST}_n^{\text{AI}}(\rho) \otimes \text{SST}_n(1) \simeq \bigsqcup_{\substack{\sigma \in \text{Par}_m \\ \rho \triangleleft \sigma \text{ or } \sigma \triangleleft \rho}} \text{SST}_n^{\text{AI}}(\sigma).$$

(2) Suppose that $n \in \mathbb{Z}_{\text{ev}}$ and $\rho_m = 1$. Then, we have

$$\text{SST}_n^{\text{Al}}(\rho) \otimes \text{SST}_n(1) \simeq \bigsqcup_{\substack{\sigma \in \text{Par}_m \setminus \text{Par}_{m-1} \\ \rho \triangleleft \sigma \text{ or } \sigma \triangleleft \rho}} \text{SST}_n^{\text{Al}}(\sigma) \sqcup \text{SST}_n^{\text{Al}}(\rho')^2,$$

where $\rho' := (\rho_1, \rho_2, \dots, \rho_{m-1})$.

(3) Suppose that $n \in \mathbb{Z}_{\text{odd}}$ and $\rho_m = 0$. Then, we have

$$\text{SST}_n^{\text{Al}}(\rho) \otimes \text{SST}_n(1) \simeq \bigsqcup_{\substack{\sigma \in \text{Par}_m \\ \rho \triangleleft \sigma \text{ or } \sigma \triangleleft \rho}} \text{SST}_n^{\text{Al}}(\sigma).$$

(4) Suppose that $n \in \mathbb{Z}_{\text{odd}}$ and $\rho_m \neq 0$. Then, we have

$$\text{SST}_n^{\text{Al}}(\rho) \otimes \text{SST}_n(1) \simeq \bigsqcup_{\substack{\sigma \in \text{Par}_m \\ \rho = \sigma, \rho \triangleleft \sigma, \text{ or } \sigma \triangleleft \rho}} \text{SST}_n^{\text{Al}}(\sigma).$$

Now, let us investigate the connected components of $\text{SST}_n^{\text{Al}}(\rho) \otimes \text{SST}_n(1)$. Let $T \in \text{SST}_n^{\text{Al}}(\rho)$ and $l \in [1, n]$. Let $\sigma \in \text{Par}_m$ denote the shape of $P^{\text{Al}}(T \leftarrow l)$. Then, by Proposition 2.5.12 and Theorem 3.3.6 (1), the connected component containing $T \otimes \sqcup$ is isomorphic to $\text{SST}_n^{\text{Al}}(\sigma)$. This observation, together with Lemma 4.1.2, implies the following: Unless $n \in \mathbb{Z}_{\text{ev}}$ and $\sigma = \rho'$, we have

$$C^{\text{Al}}(T \otimes \sqcup) = \{T' \otimes \sqcup \mid \text{sh}(P^{\text{Al}}(T' \leftarrow l')) = \sigma\}.$$

Hence, let us consider the case when $n \in \mathbb{Z}_{\text{ev}}$ and $\sigma = \rho'$ (this can happen only when $\rho_m = 1$). In this case, the set

$$\{T' \otimes \sqcup \mid \text{sh}(P^{\text{Al}}(T' \leftarrow l')) = \rho'\}$$

consists of exactly two connected components, and both of them are isomorphic to $\text{SST}_n^{\text{Al}}(\rho')$. The following lemma describes one of these connected components.

Lemma 4.1.3 *Let $n \in \mathbb{Z}_{\text{ev}}$ and $\rho \in \text{Par}_m$ be such that $\rho_m = 1$. Set $\rho' := (\rho_1, \dots, \rho_{m-1})$, and $\rho'' := (\rho_1, \dots, \rho_m, 1)$. Then, the set*

$$\{T \otimes \sqcup \in \text{SST}_n^{\text{Al}}(\rho) \otimes \text{SST}_n(1) \mid \text{sh}(P^{\text{Al}}(T \leftarrow l)) = \rho' \text{ and } \text{sh}(T \leftarrow l) = \rho''\}$$

is a connected component of $\text{SST}_n^{\text{Al}}(\rho) \otimes \text{SST}_n(1)$ isomorphic to $\text{SST}_n^{\text{Al}}(\rho')$.

Proof By Lemma 4.1.2 (2), there are exactly two elements $T_1 \otimes \sqcup_1, T_2 \otimes \sqcup_2 \in \text{SST}_n^{\text{Al}}(\rho) \otimes \text{SST}_n(1)$ such that $P^{\text{Al}}(T_i \leftarrow l_i) = T_{\rho'}, i = 1, 2$. Set $\sigma_i := \text{sh}(T_i \leftarrow l_i)$. Since the operators $\tilde{B}_j, j \in [1, n - 1]$ preserve semistandard tableaux, we have

$$\text{sh}(T' \leftarrow l') = \sigma_i$$

for all $T' \otimes \square \in C^{\text{AI}}(T_i \otimes \square_i)$.

Now, we show that exactly one of σ_1, σ_2 is equal to ρ'' . To do so, it suffices to prove that there is a unique tableau $T'' \in \text{SST}_n(\rho'')$ such that $P^{\text{AI}}(T'') = T_{\rho'}$. One can easily verify that $K(T_{\rho'})$ is such a tableau. To prove the uniqueness, let T'' be such a tableau. Let $C_j, j \in [1, \rho_1]$ denote the j th column of T'' . Since $|C_1| = m + 1 > m$, we have $|K_1(T'')| \leq |T''| - 2 = |\rho| - 1 = |\rho'| = |P^{\text{AI}}(T'')|$. Hence, by Lemma 2.2.2, we have

$$P^{\text{AI}}(T'') = K_1(T'') = P(K(C_1) \otimes C_2 \otimes \cdots \otimes C_{\rho_1}).$$

Since $P^{\text{AI}}(T'') = T_{\rho'}$, the length of the first column of $P^{\text{AI}}(T'')$ is $m - 1$. On the other hand, we have $|K(C_1)| = m - 1$. Hence, it must hold that

$$P(K(C_1) \otimes C_2 \otimes \cdots \otimes C_{\rho_1}) = K(C_1)C_2 \cdots C_{\rho_1}.$$

Therefore, we obtain

$$K(C_1)C_2 \cdots C_{\rho_1} = T_{\rho'}.$$

This implies that $T'' = C_1C_2 \cdots C_{\rho_1}$ is obtained from $T_{\rho'}$ by applying K to the first column. In particular, T'' is uniquely determined. Thus, the proof completes. \square

Combining Lemmas 4.1.2 and 4.1.3, we obtain the following.

Proposition 4.1.4 *Let $\rho \in \text{Par}_m$.*

(1) *Suppose that $n \in \mathbb{Z}_{\text{ev}}$ and $\rho_m \neq 1$. Then, the map*

$$\text{SST}_n^{\text{AI}}(\rho) \otimes \text{SST}_n(1) \rightarrow \bigsqcup_{\substack{\sigma \in \text{Par}_m \\ \rho \triangleleft \sigma \text{ or } \sigma \triangleleft \rho}} \text{SST}_n^{\text{AI}}(\sigma); T \otimes \square \mapsto P^{\text{AI}}(T \leftarrow l)$$

is an isomorphism of AI-crystals.

(2) *Suppose that $n \in \mathbb{Z}_{\text{ev}}$ and $\rho_m = 1$. Then, the map*

$$\text{SST}_n^{\text{AI}}(\rho) \otimes \text{SST}_n(1) \rightarrow \bigsqcup_{\substack{\sigma \in \text{Par}_m \setminus \text{Par}_{m-1} \\ \rho \triangleleft \sigma \text{ or } \sigma \triangleleft \rho}} \text{SST}_n^{\text{AI}}(\sigma) \sqcup (\text{SST}_n^{\text{AI}}(\rho') \times \{+, -\})$$

$$T \otimes \square \mapsto \begin{cases} P^{\text{AI}}(T \leftarrow l) & \text{if } \ell(\text{sh}(P^{\text{AI}}(T \leftarrow l))) = m, \\ (P^{\text{AI}}(T \leftarrow l), +) & \text{if } \ell(\text{sh}(P^{\text{AI}}(T \leftarrow l))) < m \text{ and } \ell(\text{sh}(T \leftarrow l)) > m, \\ (P^{\text{AI}}(T \leftarrow l), -) & \text{if } \ell(\text{sh}(P^{\text{AI}}(T \leftarrow l))) < m \text{ and } \ell(\text{sh}(T \leftarrow l)) = m \end{cases}$$

is an isomorphism of AI-crystals, where $\rho' := (\rho_1, \rho_2, \dots, \rho_{m-1})$.

(3) *Suppose that $n \in \mathbb{Z}_{\text{odd}}$ and $\rho_m = 0$. Then, the map*

$$\text{SST}_n^{\text{AI}}(\rho) \otimes \text{SST}_n(1) \rightarrow \bigsqcup_{\substack{\sigma \in \text{Par}_m \\ \rho \triangleleft \sigma \text{ or } \sigma \triangleleft \rho}} \text{SST}_n^{\text{AI}}(\sigma); T \otimes \square \mapsto P^{\text{AI}}(T \leftarrow l)$$

is an isomorphism of AI-crystals.

(4) Suppose that $n \in \mathbb{Z}_{\text{odd}}$ and $\rho_m \neq 0$. Then, the map

$$\text{SST}_n^{\text{AI}}(\rho) \otimes \text{SST}_n(1) \rightarrow \bigsqcup_{\substack{\sigma \in \text{Par}_m \\ \rho = \sigma, \rho \triangleleft \sigma, \text{ or } \sigma \triangleleft \rho}} \text{SST}_n^{\text{AI}}(\sigma); T \otimes \square \mapsto P^{\text{AI}}(T \leftarrow l)$$

is an isomorphism of AI-crystals.

4.2 Robinson–Schensted-type correspondence

Given a word $w = (w_1, \dots, w_d) \in \mathcal{W}$, define its P^{AI} -symbol $P^{\text{AI}}(w)$ by

$$P^{\text{AI}}(w) := P^{\text{AI}}(P(w)).$$

For each $k \in [0, d]$, set

$$P^{\text{AI},k} := P^{\text{AI}}(w_1, \dots, w_k)$$

and

$$\rho^k := \text{sh}(P^{\text{AI},k}).$$

Also, define $Q^{\text{AI},k}$ to be the pair $(Q_1^{\text{AI},k}, Q_2^{\text{AI},k})$ of a standard tableau $Q_1^{\text{AI},k} \in \text{ST}_k(\rho^k)$ and a set $Q_2^{\text{AI},k}$ of subsets of $([1, k] \setminus \{Q_1^{\text{AI},k}(i, j) \mid (i, j) \in D(\rho^k)\}) \sqcup \{+, -\}$ inductively as follows (cf. [15, Section 8]). First, set $Q^{\text{AI},0} := (\emptyset, \emptyset)$. Next, note that we have either $\rho^k = \rho^{k-1}$, $\rho^{k-1} \triangleleft \rho^k$, or $\rho^k \triangleleft \rho^{k-1}$. Suppose that $\rho^k = \rho^{k-1}$. Then, we set

$$Q_1^{\text{AI},k} := Q_1^{\text{AI},k-1}, \quad Q_2^{\text{AI},k} := Q_2^{\text{AI},k-1} \sqcup \{\{k\}\}.$$

Next, suppose that $\rho^{k-1} \triangleleft \rho^k$. Then, we set

$$Q_1^{\text{AI},k}(i, j) := \begin{cases} Q_1^{\text{AI},k-1}(i, j) & \text{if } (i, j) \in D(\rho^{k-1}), \\ k & \text{if } (i, j) \in D(\rho^k) \setminus D(\rho^{k-1}), \end{cases} \quad Q_2^{\text{AI},k} := Q_2^{\text{AI},k-1}.$$

Finally, suppose that $\rho^k \triangleleft \rho^{k-1}$. Then, we define $Q_1^{\text{AI},k}$ to be the unique tableau of shape ρ^k such that $(Q_1^{\text{AI},k} \leftarrow l) = Q_1^{\text{AI},k-1}$ for some $l \in [1, k - 1]$ (such $Q_1^{\text{AI},k}$ and l can be computed by the inverse of the insertion algorithm), and

$$Q_2^{\text{AI},k} := \begin{cases} Q_2^{\text{AI},k-1} \sqcup \{\{l, k, +\}\} & \text{if } \ell(\rho^{k-1}) = \frac{n}{2} > \ell(\rho^k) \text{ and } \ell(\text{sh}(P^{\text{AI},k-1} \leftarrow w_k)) > m, \\ Q_2^{\text{AI},k-1} \sqcup \{\{l, k, -\}\} & \text{if } \ell(\rho^{k-1}) = \frac{n}{2} > \ell(\rho^k) \text{ and } \ell(\text{sh}(P^{\text{AI},k-1} \leftarrow w_k)) = m, \\ Q_2^{\text{AI},k-1} \sqcup \{\{l, k\}\} & \text{otherwise.} \end{cases}$$

Now, define the Q^{AI} -symbol $Q^{\text{AI}}(w) = (Q^{\text{AI}}(w)_1, Q^{\text{AI}}(w)_2)$ and the AI-shape $\text{sh}^{\text{AI}}(w)$ of w to be $(Q_1^{\text{AI},d}, Q_2^{\text{AI},d})$ and ρ^d , respectively.

Example 4.2.1

(1) Let $n = 4$ and $w = (1, 1, 4, 2, 1, 1, 1)$. Then, $(P^{\text{AI}}(w), Q^{\text{AI}}(w))$ is calculated as follows: The rightmost tableaux of the first and second row are $P^{\text{AI}}(w)$ and $Q^{\text{AI}}(w)_1$, respectively, and the sets in the third row are the elements of $Q^{\text{AI}}(w)_2$.

$$\begin{array}{cccccccc}
 \emptyset & \boxed{1} & \emptyset & \boxed{4} & \boxed{\frac{2}{4}} & \boxed{3} & \boxed{\frac{1}{3}} & \boxed{3} \\
 \emptyset & \boxed{1} & \emptyset & \boxed{3} & \boxed{\frac{3}{4}} & \boxed{4} & \boxed{\frac{4}{6}} & \boxed{6} \\
 & & \{1, 2\} & & \{3, 5, +\} & & \{4, 7, -\} &
 \end{array}$$

(2) Let $n = 5$ and $w = (1, 1, 4, 2, 1)$. Then, $(P^{\text{AI}}(w), Q^{\text{AI}}(w))$ is calculated as follows:

$$\begin{array}{ccccccc}
 \emptyset & \boxed{1} & \emptyset & \boxed{4} & \boxed{\frac{2}{4}} & \boxed{\frac{3}{5}} & \\
 \emptyset & \boxed{1} & \emptyset & \boxed{3} & \boxed{\frac{3}{4}} & \boxed{\frac{3}{4}} & \\
 & & \{1, 2\} & & \{5\} & &
 \end{array}$$

The Q^{AI} -symbols for various words are characterized by the notion of \mathfrak{so}_n -oscillating tableaux, which we now define.

Definition 4.2.2 Let $\rho \in \text{Par}_m$ and $d \geq 0$. An \mathfrak{so}_n -oscillating tableau of shape ρ and length d is a sequence $\boldsymbol{\rho} = ((\rho^0, s^0), (\rho^1, s^1), \dots, (\rho^d, s^d))$ of pairs $(\rho^k, s^k) \in \text{Par}_m \times \{0, +, -\}$ satisfying the following:

- (1) $\rho^0 = \emptyset, \rho^d = \rho$, and either $\rho^k = \rho^{k-1}, \rho^{k-1} \triangleleft \rho^k$, or $\rho^k \triangleleft \rho^{k-1}$.
- (2) If $\rho^k = \rho^{k-1}$ for some k , then $n \in \mathbb{Z}_{\text{odd}}$ and $\ell(\rho^k) = m$.
- (3) If $s^k \in \{+, -\}$ for some k , then $n \in \mathbb{Z}_{\text{ev}}$
- (4) $s^k \in \{+, -\}$ if and only if $\ell(\rho^{k-1}) = m$, and $\ell(\rho^k) = m - 1$.

Let $\text{OT}_n(\rho)$ denote the set of \mathfrak{so}_n -oscillating tableaux of shape ρ , and set

$$\text{OT}_n := \bigsqcup_{\rho \in \text{Par}_m} \text{OT}_n(\rho).$$

Given an \mathfrak{so}_n -oscillating tableau $\boldsymbol{\rho} = ((\rho^0, s^0), (\rho^1, s^1), \dots, (\rho^d, s^d))$ of shape ρ and length d , we associate a pair $Q(\boldsymbol{\rho}) = (Q_1, Q_2)$ of a standard tableau $Q_1 \in \text{ST}_d(\rho)$ and a set Q_2 of subsets of $([1, d] \setminus \{Q_1(i, j) \mid (i, j) \in D(\boldsymbol{\rho})\}) \sqcup \{+, -\}$ as follows. When $d = 0$, define $Q(\boldsymbol{\rho}) = (\emptyset, \emptyset)$. Now, assume that $d > 0$ and set

$\rho' := ((\rho^0, s^0), (\rho^1, s^1), \dots, (\rho^{d-1}, s^{d-1}))$ and $Q(\rho') = (Q'_1, Q'_2)$. Then, $Q(\rho)$ is defined as follows:

- (1) If $\rho^d = \rho^{d-1}$, then $Q_1 = Q'_1$ and $Q_2 = Q'_2 \sqcup \{\{d\}\}$.
- (2) If $\rho^{d-1} \triangleleft \rho^d$, then

$$Q_1(i, j) = \begin{cases} Q'_1(i, j) & \text{if } (i, j) \in D(\rho^{d-1}), \\ d & \text{if } (i, j) \in D(\rho^d) \setminus D(\rho^{d-1}), \end{cases}$$

and $Q_2 = Q'_2$.

- (3) If $\rho^d \triangleleft \rho^{d-1}$ and $s^d = 0$, then $Q'_1 = (Q_1 \leftarrow l)$ for some uniquely determined $l \in [1, d - 1]$, and $Q_2 = Q'_2 \sqcup \{\{l, d\}\}$.
- (4) If $\rho^d \triangleleft \rho^{d-1}$ and $s^d \in \{+, -\}$, then $Q'_1 = (Q_1 \leftarrow l)$ for some uniquely determined $l \in [1, d - 1]$, and $Q_2 = Q'_2 \sqcup \{\{l, d, s^d\}\}$.

As before, the tableau Q'_1 and letter l in items (3) and (4) above can be computed by the inverse of the insertion algorithm.

Example 4.2.3 Set $\rho := (1)$, and

$$\rho := ((\emptyset, 0), ((1), 0), (\emptyset, 0), ((1), 0), ((1, 1), 0), ((1), +), ((1, 1), 0), ((1), -))$$

Then, we have $\rho \in \text{OT}_4(\rho)$, and $Q(\rho)$ coincides with $Q^{\text{Al}}(w)$ in Example 4.2.1 (1).

Lemma 4.2.4 *The assignment $\rho \mapsto Q(\rho)$ is injective.*

Proof Let $\rho, \sigma \in \text{OT}_n$ be such that $Q(\rho) = Q(\sigma)$. Set

$$(Q_1, Q_2) := Q(\rho), \quad \rho := \text{sh}(Q_1).$$

First of all, we see that the shapes of ρ and σ coincide; both are ρ .

From the definition of $Q(\rho)$, we see that the length of ρ is the maximal integer appearing in $Q(\rho)$. In other words, the length of ρ is determined by $Q(\rho)$. Therefore, ρ and σ have the same length, say d .

We prove that $\rho = \sigma$ by induction on d ; the case when $d = 0$ is clear. Assume that $d > 0$, and set $\rho' = ((\rho^0, s^0), (\rho^1, s^1), \dots, (\rho^{d-1}, s^{d-1}))$. Then, we have

$$(Q(\rho'), s^d) = \begin{cases} ((Q_1, Q_2 \setminus \{\{d\}\}), 0) & \text{if } \{d\} \in Q_2, \\ ((Q_1|_{[1, d-1]}, Q_2), 0) & \text{if } d \in Q_1, \\ (((Q_1 \leftarrow l), Q_2 \setminus \{\{l, d\}\}), 0) & \text{if } \{l, d\} \in Q_2 \text{ for some } l < d, \\ (((Q_1 \leftarrow l), Q_2 \setminus \{\{l, d, \pm\}\}), \pm) & \text{if } \{l, d, \pm\} \in Q_2 \text{ for some } l < d. \end{cases}$$

This implies that $Q(\rho')$ and s^d are determined by $Q(\rho)$. By our induction hypothesis, ρ' is determined by $Q(\rho')$. Hence, ρ is determined by $Q(\rho)$, which implies $\rho = \sigma$, as desired. This completes the proof. □

In this way, we often identify ρ with $Q(\rho)$. Now, the following is immediate from the results obtained so far.

Theorem 4.2.5 *Let $n \geq 3$. Then, the assignment*

$$RS^{AI} : \mathcal{W} \rightarrow \bigsqcup_{\rho \in \text{Par}_m} SST_n^{AI}(\rho) \times OT_n(\rho); w \mapsto (P^{AI}(w), Q^{AI}(w))$$

is an isomorphism of AI-crystals.

4.3 Branching rule

We have obtained two isomorphisms of AI-crystals:

$$RS : \mathcal{W} \rightarrow \bigsqcup_{\lambda \in \text{Par}_n} SST_n(\lambda) \times ST_{|\lambda|}(\lambda),$$

and

$$RS^{AI} : \mathcal{W} \rightarrow \bigsqcup_{\rho \in \text{Par}_m} SST_n^{AI}(\rho) \times OT_n(\rho).$$

Let $\lambda \in \text{Par}_n$, $\rho \in \text{Par}_m$, $(P, Q) \in SST_n(\lambda) \times ST_{|\lambda|}(\lambda)$, $(P', Q') \in SST_n^{AI}(\rho) \times OT_n(\rho)$ be such that

$$(P', Q') = RS^{AI} \circ RS^{-1}(P, Q).$$

Since $SST_n^{AI}(\rho)$ is connected, for each $P'' \in SST_n^{AI}(\rho)$, there exist $i_1, \dots, i_r \in [1, n - 1]$ such that

$$P'' = \tilde{B}_{i_1} \cdots \tilde{B}_{i_r} P'.$$

Therefore, we obtain

$$(P'', Q') = \tilde{B}_{i_1} \cdots \tilde{B}_{i_r}(P', Q') = RS^{AI} \circ RS^{-1}(\tilde{B}_{i_1} \cdots \tilde{B}_{i_r} P, Q).$$

This implies that Q is independent of P' and uniquely determined by Q' . Let us say Q is the Q -symbol of Q' , and write it as $Q(Q')$. Hence, there exists a map

$$Q : \bigsqcup_{\rho \in \text{Par}_m} OT_n(\rho) \rightarrow \bigsqcup_{\lambda \in \text{Par}_n} ST_{|\lambda|}(\lambda); Q' \mapsto Q(Q').$$

Theorem 4.3.1 *Let $\lambda \in \text{Par}_n$. Then, the map*

$$\begin{aligned} SST_n(\lambda) &\rightarrow \bigsqcup_{\rho \in \text{Par}_m} SST_n^{AI}(\rho) \times \{Q' \in OT_n(\rho) \mid Q(Q') = T(\lambda)\}; \\ T &\mapsto (P^{AI}(T), Q^{AI}(\text{CR}(T))) \end{aligned}$$

is an isomorphism of AI-crystals, where $T(\lambda) \in \text{ST}_{|\lambda|}(\lambda)$ is given by

$$T(\lambda)(i, j) := \sum_{k=1}^{j-1} d_k + i,$$

and d_k denotes the length of the k th column of $D(\lambda)$.

Proof The assignment $T \mapsto (P^{\text{AI}}(T), Q^{\text{AI}}(\text{CR}(T)))$ factors

$$\text{SST}_n(\lambda) \xrightarrow{\text{CR}} \mathcal{W} \xrightarrow{\text{RS}^{\text{AI}}} \bigsqcup_{\rho \in \text{Par}_m} \text{SST}_n^{\text{AI}}(\rho) \times \text{OT}_n(\rho).$$

Hence, it suffices to show that

$$\{Q^{\text{AI}}(\text{CR}(T)) \mid T \in \text{SST}_n(\lambda)\} = \bigsqcup_{\rho \in \text{Par}_m} \{Q' \in \text{OT}_n(\rho) \mid Q(Q') = T(\lambda)\}$$

Let $T \in \text{SST}_n(\lambda)$. Since $Q(\text{CR}(T)) = T(\lambda)$, we see that

$$Q(Q^{\text{AI}}(\text{CR}(T))) = T(\lambda)$$

for all $T \in \text{SST}_n(\lambda)$. On the other hand, if $Q' \in \text{OT}_n(\rho)$ is such that $Q(Q') = T(\lambda)$, then there exist $w \in \mathcal{W}$ and $P \in \text{SST}_n^{\text{AI}}(\rho)$ such that

$$P^{\text{AI}}(w) = P, \quad Q^{\text{AI}}(w) = Q', \quad \text{and} \quad Q(w) = T(\lambda).$$

This shows that if we set $T := P(w)$, then we have $\text{CR}(T) = w$, and hence,

$$Q^{\text{AI}}(\text{CR}(T)) = Q^{\text{AI}}(w) = Q'.$$

This completes the proof. □

Corollary 4.3.2 *Let $\lambda \in X_{\mathfrak{gl}_n, \geq 0}^+$ and $\nu \in X_{\mathfrak{so}_n, \text{int}}^+$. Then, we have*

$$V^{\mathfrak{gl}_n}(\lambda) \simeq \bigoplus_{\nu \in X_{\mathfrak{so}_n, \text{int}}^+} V^{\mathfrak{so}_n}(\nu)^{\oplus [\lambda : \nu]},$$

where

$$[\lambda : \nu] := \#\{Q' \in \text{OT}_n(\nu_1, \nu_3, \dots, \nu_{2m-3}, |\nu_{2m-1}|) \mid Q(Q') = T(\lambda)\}.$$

Example 4.3.3 Let $n = 3, \lambda = (2, 1)$. Then, the \mathfrak{so}_n -oscillating tableaux of length $|\lambda| = 3$ are the following (we omit the second factors as they are all 0):

$$\begin{aligned} Q'_1 &= (\emptyset, (1), (2), (3)), \\ Q'_2 &= (\emptyset, (1), (2), (2)), \\ Q'_3 &= (\emptyset, (1), (2), (1)), \\ Q'_4 &= (\emptyset, (1), (1), (2)), \\ Q'_5 &= (\emptyset, (1), (1), (1)), \\ Q'_6 &= (\emptyset, (1), (1), \emptyset), \\ Q'_7 &= (\emptyset, (1), \emptyset, (1)). \end{aligned}$$

We have

$$\begin{aligned} (\boxed{1|2|2}, Q'_1) &\xrightarrow{(RS^{Al})^{-1}} (1, 2, 2) \xrightarrow{RS} (\boxed{1|2|2}, \boxed{1|2|3}), \\ (\boxed{1|2}, Q'_2) &\xrightarrow{(RS^{Al})^{-1}} (2, 3, 2) \xrightarrow{RS} \left(\boxed{\frac{2}{3}|2}, \boxed{\frac{1}{3}|2}\right), \\ (\boxed{1}, Q'_3) &\xrightarrow{(RS^{Al})^{-1}} (2, 2, 1) \xrightarrow{RS} \left(\boxed{\frac{1}{2}|2}, \boxed{\frac{1}{3}|2}\right), \\ (\boxed{1|2}, Q'_4) &\xrightarrow{(RS^{Al})^{-1}} (3, 2, 2) \xrightarrow{RS} \left(\boxed{\frac{2}{3}|2}, \boxed{\frac{1}{2}|3}\right), \\ (\boxed{1}, Q'_5) &\xrightarrow{(RS^{Al})^{-1}} (2, 1, 2) \xrightarrow{RS} \left(\boxed{\frac{1}{2}|2}, \boxed{\frac{1}{2}|3}\right), \\ (\emptyset, Q'_6) &\xrightarrow{(RS^{Al})^{-1}} (3, 2, 1) \xrightarrow{RS} \left(\boxed{\frac{1}{3}|2}, \boxed{\frac{1}{3}|2}\right), \\ (\boxed{1}, Q'_7) &\xrightarrow{(RS^{Al})^{-1}} (1, 1, 1) \xrightarrow{RS} (\boxed{1|1|1}, \boxed{1|2|3}). \end{aligned}$$

Therefore, we obtain

$$[\lambda; (3)] = 0, [\lambda; (2)] = 1, [\lambda; (1)] = 1, [\lambda; \emptyset] = 0.$$

Compare this result with the AI-crystal graph of $SST(\lambda)$ in equation (3).

Corollary 4.3.2 gives a combinatorial cancellation-free branching rule from \mathfrak{gl}_n to \mathfrak{so}_n ; the multiplicity $[\lambda : \mu]$ is equal to the number of \mathfrak{so}_n -oscillating tableaux satisfying certain conditions. Jang and Kwon [4] proved that the multiplicity $[\lambda : \nu]$ is equal to the number of Littlewood–Richardson tableaux satisfying certain conditions. Their description of the multiplicities generalizes famous Littlewood’s branching rule [11, 12]. It would be interesting to compare their multiplicity set with ours.

Jagenteufel [3] found an algorithm which gives a bijection from $OT_n(\rho)$ (with n odd) to the set of pairs of a standard Young tableau of shape λ in alphabet $[1, |\lambda|]$ and an orthogonal Littlewood–Richardson tableau associated to λ, μ (in [3], \mathfrak{so}_n -oscillating tableaux are called vacillating tableaux). This algorithm may be used to make the assignment $Q' \mapsto Q(Q')$ more explicit.

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