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Absolute points of correlations of $PG(4, q^n)$

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Abstract

The sets of the absolute points of (possibly degenerate) polarities of a projective space are well known. The sets of the absolute points of (possibly degenerate) correlations, different from polarities, of $\mathrm{PG}(2, q^n)$, have been completely determined by B.C. Kestenband in 11 papers from 1990 to 2014, for non-degenerate correlations and by D'haeseleer and Durante (Electron J Combin 27(2):2–32, 2020) for degenerate correlations. The sets of the absolute points of degenerate correlations, different from degenerate polarities, of a projective space $\mathrm{PG}(3, q^n)$ have been classified in (Donati and Durante in J Algebr Comb 54:109–133, 2021). In this paper, we consider the four dimensional case and completely determine the sets of the absolute points of degenerate correlations, different from degenerate polarities, of a projective space $\mathrm{PG}(4, q^n)$. As an application, we show that some of these sets are related to the Kantor's ovoid and to the Tits' ovoid of $Q(4, q^n)$ and hence also to the Tits' ovoid of $\mathrm{PG}(3, q^n)$.

Keywords Sesquilinear forms · Correlations · Ovoids

Mathematics Subject Classification 51E20 · 05B25

1 Introduction and preliminary results

1.1 Sesquilinear forms and correlations

Let V and W be two $\mathbb{F}$-vector spaces, where $\mathbb{F}$ is a field. A map $f : V \longrightarrow W$ is called *semilinear* or σ*-linear* if there exists an automorphism σ of $\mathbb{F}$ such that

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$$f(v+v') = f(v) + f(v') \quad \text{and} \quad f(av) = a^\sigma f(v)$$

for all vectors $v \in V$ and all scalars $a \in \mathbb{F}$. If σ is the identity map, then f is a usual linear map.Let V be an $\mathbb{F}$-vector space with finite dimension d. A map

$$\langle\ ,\ \rangle : (v, v') \in V \times V \longrightarrow \langle v, v' \rangle \in \mathbb{F}$$

is a *sesquilinear form* or a *semibilinear form* on V if it is a linear map on the first argument, and it is a σ-linear map on the second argument, that is:

$$\begin{aligned} \langle v+v', v'' \rangle &= \langle v, v'' \rangle + \langle v', v'' \rangle, \\ \langle v, v'+v'' \rangle &= \langle v, v' \rangle + \langle v, v'' \rangle, \\ \langle av, v' \rangle &= a\langle v, v' \rangle, \quad \langle v, av' \rangle = a^\sigma \langle v, v' \rangle, \end{aligned}$$

for all $v, v', v'' \in V$, $a \in \mathbb{F}$ and σ an automorphism of $\mathbb{F}$. If σ is the identity map, then $\langle\ ,\ \rangle$ is a usual bilinear form. If $\mathcal{B} = (e_1, e_2, \ldots, e_{d+1})$ is an ordered basis of V, then for $x, y \in V$ we have $\langle x, y \rangle = X^t A Y^\sigma$, where $A = (\langle e_i, e_j \rangle)$ is the *associated matrix* to the sesquilinear form with respect to the ordered basis $\mathcal{B}$; X and Y are the columns of the coordinates of x, y w.r.t. $\mathcal{B}$. The term *sesqui* comes from the Latin, and it means one and a half. For every subspace S of V, put

$$\begin{aligned} S^T &:= \{y \in V : \langle x, y \rangle = 0 \ \ \forall x \in S\}, \\ S^\perp &:= \{y \in V : \langle y, x \rangle = 0 \ \ \forall x \in S\}. \end{aligned}$$

Both S^T and $S^\perp$ are subspaces of V. The subspaces V^T and $V^\perp$ are called the *right* and the *left radical* of $\langle\ ,\ \rangle$ and will be also denoted by $\mathrm{Rad}_r(V)$ and $\mathrm{Rad}_l(V)$, respectively.

Proposition 1.1 *The right and the left radical of a sesquilinear form of a vector space V has the same dimension.*

A *non-degenerate* sesquilinear form $\langle\ ,\ \rangle$ has $V^\perp = V^T = \{0\}$.

Definition 1.2 A σ-sesquilinear form is *reflexive* if $\forall\ u, v \in V$:

$$\langle u, v \rangle = 0 \iff \langle v, u \rangle = 0.$$

Definition 1.3 Let V be an $\mathbb{F}$-vector space of dimension greater than two. A bijection $g : \mathrm{PG}(V) \longrightarrow \mathrm{PG}(V)$ is a *collineation* if g, together with g^{-1}, maps k-dimensional subspaces into k-dimensional subspaces. If V has dimension two, then a *collineation* is a map $\langle v \rangle \in \mathrm{PG}(V) \longrightarrow \langle f(v) \rangle \in \mathrm{PG}(V)$, induced by a bijective semilinear map $f : V \longrightarrow V$.

Theorem 1.4 (*Fundamental Theorem*) *Let V be an $\mathbb{F}$-vector space. Every collineation of* $\mathrm{PG}(V)$ *is induced by a bijective semilinear map $f : V \longrightarrow V$.*

In the sequel if S is a vector subspace of V, we will denote with the same symbol S the associated projective subspace of $\mathrm{PG}(V)$. If v is a nonzero vector of V, we denote by $\langle v \rangle$ a point of $\mathrm{PG}(V)$.

Definition 1.5 Let $f : V \longrightarrow V$ be a semilinear map, with $\mathrm{Ker} f \neq \{0\}$. The map

$$\langle v \rangle \in \mathrm{PG}(V) \setminus \mathrm{Ker} f \longrightarrow \langle f(v) \rangle \in \mathrm{PG}(V),$$

will be called a *degenerate collineation* of $\mathrm{PG}(V)$.

Definition 1.6 A (*degenerate*) *correlation* or (*degenerate*) *duality* of $\mathrm{PG}(d, \mathbb{F})$ is a (degenerate) collineation between $\mathrm{PG}(d, \mathbb{F})$ and its dual space $\mathrm{PG}(d, \mathbb{F})^*$.

Remark 1.7 A correlation of $\mathrm{PG}(d, \mathbb{F})$ can be seen as a bijective map of $\mathrm{PG}(d, \mathbb{F})$ that maps k-dimensional subspaces into $(d - 1 - k)$-dimensional subspaces reversing inclusion and preserving incidence.

A correlation of $\mathrm{PG}(d, \mathbb{F})$ applied twice gives a collineation of $\mathrm{PG}(d, \mathbb{F})$.

Theorem 1.8 *Any* (*possibly degenerate*) *correlation of* $\mathrm{PG}(d, \mathbb{F})$, $d > 1$, *is induced by a σ-sesquilinear form of the underlying vector space $\mathbb{F}^{d+1}$. Conversely, every σ-sesquilinear form of $\mathbb{F}^{d+1}$ induces two* (*possibly degenerate*) *correlations of* $\mathrm{PG}(d, \mathbb{F})$. *The two correlations coincide if and only if the form $\langle\ ,\ \rangle$ is reflexive.*

Remark 1.9 A (possibly degenerate) correlation induced by a σ-sesquilinear form will be also called a σ-*correlation*. Sometimes a (degenerate) correlation whose associated form is bilinear is called *linear*.

Definition 1.10 A (degenerate) *polarity* is a (degenerate) correlation whose square is the identity.

If $\perp$ is a (possibly degenerate) polarity, then for every pair of points P and R the following holds:

$$P \in R^{\perp} \iff R \in P^{\perp}.$$

Proposition 1.11 *A* (*degenerate*) *correlation is a* (*degenerate*) *polarity if and only if the induced sesquilinear form is reflexive.*

The non-degenerate, reflexive σ-sesquilinear forms of a $(d+1)$-dimensional $\mathbb{F}$-vector space V have been classified (for a proof see, e.g., Theorem 3.6 in [1] or Theorem 6.3 and Proposition 6.4 in [2]).

In this paper, we will focus on degenerate non-reflexive σ-sesquilinear form of a five dimensional vector space over a finite field $\mathbb{F}_{q^n}$.

2 The σ-quadrics of PG(d, q^n)

Let V be an $\mathbb{F}$-vector space. If V is equipped with a sesquilinear form $\langle\ ,\ \rangle$ we may consider in $\mathrm{PG}(V)$ the set Γ of *absolute* points of the associated correlation that is the points X such that $X \in X^{\perp}$ (or equivalently $X \in X^{T}$). If A is the associated matrix to the σ-sesquilinear form $\langle\ ,\ \rangle$ w.r.t. an ordered basis of V, then the set Γ has equation $X^t A X^{\sigma} = 0$.

The definition of σ-quadrics of $\mathrm{PG}(d, q^n)$ has been first given in [10] (see also [4, 9]).

Definition 2.1 A *σ-quadric* of (d, q^n) is the set of the absolute points of a (possibly degenerate) σ-correlation, $\sigma \neq 1$, of $\mathrm{PG}(d, q^n)$. A σ-quadric of $\mathrm{PG}(2, q^n)$ will be called a *σ-conic*.

Proposition 2.2 *Let $\Gamma : X^t A X^\sigma = 0$ be a σ-quadric of $\mathrm{PG}(d, q^n)$. Every subspace S intersects Γ either in a σ-quadric of S, or it is contained in Γ.*

Proof Let S be an h-dimensional subspace of $\mathrm{PG}(d, q^n)$. We may assume, w.l.o.g. that $S : x_{h+2} = 0, \ldots, x_{d+1} = 0$. Let A' be the submatrix of A obtained by deleting the last $d - h$ rows and columns of A. If $A' = 0$, then $S \subset \Gamma$. If $A' \neq 0$, then $S \cap \Gamma$ is a σ-quadric of S. □

Regarding subspaces contained in σ-quadrics, in [9] the following has been proved.

Proposition 2.3 *If S_h is an h-dimensional subspace of $\mathrm{PG}(d, q^n)$ contained in a σ-quadric Γ with equation $X^t A X^\sigma = 0$, then $h \leq \left\lfloor d - \frac{\mathrm{rank}(A)}{2} \right\rfloor$. Moreover, there exists a σ-quadric with equation $X^t A X^\sigma = 0$, containing a subspace with dimension $\left\lfloor d - \frac{\mathrm{rank}(A)}{2} \right\rfloor$.*

We recall that σ-quadrics have been completely classified in $\mathrm{PG}(d, q^n)$ for $d \in \{1, 2\}$ (see [4]) and partially classified for $d = 3$ (see [9]). Here we will deal with the four dimensional case. As in [9], we will divide the σ-quadrics of $\mathrm{PG}(4, q^n)$ according to the rank of the associated matrix. We start with the rank 4 case.
For what follows, we can assume $\sigma \neq 1$. Let $V = \mathbb{F}_{q^n}^5$, let $\langle\ ,\ \rangle$ be a degenerate σ-sesquilinear form with associated (degenerate) correlations $\perp, T$ and let $\Gamma : X^t A X^\sigma = 0$ be the associated σ-quadric. We will denote by $L = V^\perp$ and $R = V^T$, the left and right radicals of $\langle\ ,\ \rangle$, respectively, seen as subspaces of $\mathrm{PG}(4, q^n)$ that will be called the *vertices* of Γ.

Proposition 2.4 [9, Proposition 2.3] *Let $\Gamma : X^t A X^\sigma = 0$ be a σ-quadric of $\mathrm{PG}(d, q^n)$.*

- *For every point $Y \in \Gamma \setminus R$, the hyperplane Y^T is the union of lines through Y either contained or 1-secant or 2-secant to Γ.*
- *For every point $Y \in \Gamma \setminus L$, the hyperplane $Y^\perp$ is the union of lines through Y either contained or 1-secant or 2-secant to Γ.*

Corollary 2.5 [9, Corollary 2.4] *Let $\Gamma : X^t A X^\sigma = 0$ be a σ-quadric of $\mathrm{PG}(d, q^n)$ and let $L = V^\perp$, $R = V^T$.*

- *For every point $Y \in L$, the set $Y^T \cap \Gamma$ is the union of lines through Y.*
- *For every point $Y \in R$, the set $Y^\perp \cap \Gamma$ is the union of lines through Y.*

Remark 2.6 Let $\Gamma : X^t A X^\sigma = 0$ be a σ-quadric of $\mathrm{PG}(d, q^n)$, $n \geq 2$, and let $L = V^\perp$ and $R = V^T$ be its vertices. For every point $Y \in L$, the hyperplane Y^T does not contain $(q + 1)$-secant lines through Y to Γ, and for every point $Y \in R$, the hyperplane $Y^\perp$ does not contain $(q + 1)$-secant lines through Y to Γ. Indeed, let $Y \in L$ and let Z be a point of Y^T. The line YZ has equation $X = \lambda Y + \mu Z$, $(\lambda, \mu) \in \mathrm{PG}(1, q^n)$; hence, $Y^T \cap \Gamma$ is determined by the solutions in (λ, μ) of the following equation:

$$Y^t A Y^\sigma \lambda^{\sigma+1} + Y^t A Z^\sigma \lambda \mu^\sigma + Z^t A Y^\sigma \lambda^\sigma \mu + Z^t A Z^\sigma \mu^{\sigma+1} = 0. \qquad (1)$$

In the previous equation, it is $Y^t A Y^\sigma = Y^t A Z^\sigma = 0$, since $Y \in L$ and $Z^t A Y^\sigma = 0$, since $Z \in Y^T$. Hence, Eq. (1) becomes $Z^t A Z^\sigma \mu = 0$ and two cases occur:

- If $Z^t A Z^\sigma = 0$, then the line YZ is contained in Γ.
- If $Z^t A Z^\sigma \neq 0$, then the line YZ intersects Γ exactly at the point Y.

If $Y \in R$, the result follows in a similar way.

3 σ-Quadrics of rank 4 in PG(4, q^n)

Let $\Gamma : X^t A X^\sigma = 0$ be a σ-quadric of PG$(4, q^n)$ associated with a σ-sesquilinear form $\langle\ ,\ \rangle$. In this section, we assume that $\mathrm{rk}(A) = 4$. Therefore, the radicals $V^\perp$ and $V^\top$ are one-dimensional vector subspaces of V, so they are points of PG$(4, q^n)$. We distinguish several cases:

1) $V^\perp \neq V^\top$. We may assume w.l.o.g. that the point $R = (1, 0, 0, 0, 0)$ is the right radical and the point $L = (0, 0, 0, 0, 1)$ is the left radical. It follows that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\Gamma : (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{43}x_4)x_2^\sigma + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4)x_3^\sigma + (a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4)x_4^\sigma + (a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4)x_5^\sigma = 0$. The degenerate collineation

$$\top : Y \in \mathrm{PG}(4, q^n) \setminus \{R\} \mapsto X^t A Y^\sigma = 0 \in \mathrm{PG}(4, q^n)^*$$

associated with the sesquilinear form maps points into hyperplanes through the point L. Points that are on a common line through R are mapped into the same hyperplanes through L. Therefore, $\top$ induces a collineation $\Phi : \mathcal{S}_R \longrightarrow \mathcal{S}_L^*$. Let

$$\mathcal{S}_R = \{l_{\alpha,\beta,\gamma,\delta} : (\alpha, \beta, \gamma, \delta) \in \mathrm{PG}(3, q^n)\},$$

where

$$l_{\alpha,\beta,\gamma,\delta} : \begin{cases} x_1 = \lambda \\ x_2 = \mu\alpha \\ x_3 = \mu\beta \\ x_4 = \mu\gamma \\ x_5 = \mu\delta \end{cases}, (\lambda, \mu) \in \mathrm{PG}(1, q^n)$$

and

$$\mathcal{S}_L^* = \{\Sigma_{\alpha',\beta',\gamma',\delta'} : (\alpha', \beta', \gamma', \delta') \in \mathrm{PG}(3, q^n)\},$$

where

$$\Sigma_{\alpha',\beta',\gamma',\delta'} : \alpha' x_1 + \beta' x_2 + \gamma' x_3 + \delta' x_4 = 0.$$

The collineation Φ is given by $\Phi(l_{\alpha,\beta,\gamma,\delta}) = \Sigma_{\alpha',\beta',\gamma',\delta'}$, with

$$(\alpha', \beta', \gamma', \delta')^t = A'\big[(\alpha, \beta, \gamma, \delta)^t\big]^\sigma,$$

where A' is the matrix obtained by A by deleting the last row and the first column. Note that $|A'| \neq 0$ since $\mathrm{rk}(A) = 3$. It is easy to see that Γ is the set of points of intersection of corresponding elements under the collineation Φ. If $Y = (y_1, y_2, y_3, y_4, y_5)$ is a point of $\Gamma \setminus \{R\}$, then the tangent hyperplane to Γ at the point Y is the hyperplane $\Sigma_Y = Y^\top$ with equation $X^t A Y^\sigma = 0$. It follows that for every point Y of $\Gamma \setminus \{R\}$ the hyperplane Σ_Y contains the point L. The tangent hyperplane $\Sigma_L = L^\top$ to Γ at the point L is the hyperplane with equation $X^t A L^\sigma = 0$, that is:

$$\Sigma_L : a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4 = 0.$$

We again distinguish some cases.

i) First assume that Σ_L contains the line RL. It follows that, w.l.o.g., we may put $\Sigma_L : x_4 = 0$. Hence, $a_{15} = a_{25} = a_{35} = 0$ and we can put $a_{45} = 1$, obtaining $\Gamma : (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4)x_2^\sigma + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4)x_3^\sigma + (a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4)x_4^\sigma + x_4 x_5^\sigma = 0$. With this assumption, the collineation Φ maps the line RL into the hyperplane Σ_L. Consider now the star $\mathcal{S}_{R,\Sigma_L}$ of lines through R in Σ_L. We distinguish two cases.

i.1) Suppose that Φ maps the lines of $\mathcal{S}_{R,\Sigma_L}$ into the hyperplanes through the line RL. In this case, we can assume that Φ maps the line $x_3 = x_4 = x_5 = 0$ into the hyperplane $x_2 = 0$, the line $x_2 = x_4 = x_5 = 0$ into the hyperplane $x_3 = 0$ and the line $x_2 = x_3 = x_5 = 0$ into the hyperplane $x_1 = 0$ obtaining

$$\Gamma : a x_1 x_4^\sigma + b x_2^{\sigma+1} - c x_3^{\sigma+1} + x_4 x_5^\sigma = 0.$$

We can assume that Γ contains the points $(0, 1, 0, 1, -1)$ and $(1, 0, 0, 1, -1)$ obtaining $a = b = 1$. By Corollary 2.5, we know that $R^\perp \cap \Gamma$ is the union of lines through R. Since $R^\perp$ has equation $x_4 = 0$, then a line $l_{\alpha,\beta,\gamma,\delta}$ through R is contained in Γ if, and only if, $\alpha^{\sigma+1} - c\beta^{\sigma+1} = 0$. Hence, the number of lines through R contained in Γ, different from the line RL, depends on the cardinality of the set $\{l_{x,1,0,y} \in \mathcal{S}_R : x^{\sigma+1} = c\}$, and this is given by $q^n|\{x \in \mathbb{F}_{q^n} : x^{\sigma+1} = c\}|$. Moreover, the number of lines through L contained in Γ, different from the line RL, is equal to $q^n|\{x \in \mathbb{F}_{q^n} : x^{\sigma+1} = c\}|$. Indeed, let $\mathcal{S}_L = \{t_{\alpha,\beta,\gamma,\delta} : (\alpha, \beta, \gamma, \delta) \in$

$\mathrm{PG}(3, q^n)\}$ be the set of lines through L, where

$$t_{\alpha,\beta,\gamma,\delta} : \begin{cases} x_1 = \lambda\alpha \\ x_2 = \lambda\beta \\ x_3 = \lambda\gamma \\ x_4 = \lambda\delta \\ x_5 = \mu \end{cases} , (\lambda, \mu) \in \mathrm{PG}(1, q^n).$$

By Corollary 2.5, $\Sigma_L \cap \Gamma$ is the union of lines through L. Then a line $t_{\alpha,\beta,\gamma,\delta}$ is contained in Γ if, and only if, $\beta^{\sigma+1} - c\gamma^{\sigma+1} = 0$. This yields that the number of lines through L contained in Γ, different from the line RL, depends on the cardinality of the set $\{t_{x,y,1,0} \in \mathcal{S}_L : y^{\sigma+1} = c\}$. The number of solutions of the equation $x^{\sigma+1} = c$ is either 0, 1, 2 or $q + 1$ depending upon q even or odd and n even or odd. We distinguish several cases:

- If q is even and n is even, then there are either 0 or 1 or $q + 1$ solutions giving either 0 or q^n or $(q + 1)q^n$ lines through R (and hence through L) contained in Γ.
- If q is even and n is odd, then there is a unique solution of the equation giving q^n lines through R (and through L) contained in Γ.
- If q is odd and n is even, then there are either 0 or $q + 1$ solutions of the equation giving either 0 or $(q + 1)q^n$ lines through R (and through L) contained in Γ.
- If q is odd and n is odd, then there are either 0 or 2 solutions of the equation giving either 0 or $2q^n$ lines through R (and through L) contained in Γ.

In these cases, we will call the set Γ either an *elliptic* or a q^n-*parabolic* or a $2q^n$-*hyperbolic* or a $(q + 1)q^n$-*hyperbolic* σ-*quadric* with *collinear vertex points* R and L according to the number of lines through R (different from the line RL) contained in Γ is either 0 or q^n or $2q^n$ or $(q + 1)q^n$. Now, let l a line through R. If $l \notin \mathcal{S}_{R,\Sigma_L}$ then $\Phi(l) \not\supset RL$ and so $l \cap \Phi(l)$ is a point. If $l \in \mathcal{S}_{R,\Sigma_L}$ then $l \cap \Phi(l)$ is either the point R or the line l. Recalling that Γ contains the line RL, we get $|\Gamma| = q^{3n} + q^n \cdot q^n |\{x \in \mathbb{F}_{q^n} : x^{\sigma+1} = c\}| + q^n + 1$. If q is even and n is even, put $d = (q^n - 1, q^m + 1)$.

Theorem 3.1 *Let Γ be a degenerate elliptic σ-quadric of* $\mathrm{PG}(4, q^n)$ *with collinear vertex points R and L. Then, Γ has canonical equation $\Gamma : x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0$, with c a nonsquare if q is odd and $c^{(q^n-1)/d} \neq 1$ if q is even and n is even. Moreover, $|\Gamma| = q^{3n} + q^n + 1$ and Γ contains only the line RL.*

Theorem 3.2 *Let Γ be a degenerate q^n-parabolic σ-quadric of* $\mathrm{PG}(4, q^n)$ *with collinear vertex points R and L. Then, q is even and Γ has canonical equation $\Gamma : x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0$, where the equation $x^{\sigma+1} = c$ has a unique solution. Moreover, $|\Gamma| = q^{3n} + q^{2n} + q^n + 1$ and Γ contains q^n lines through R and q^n lines through L (beside RL).*

Theorem 3.3 *Let* Γ *be a degenerate* $2q^n$*-hyperbolic* σ*-quadric of* $\mathrm{PG}(4, q^n)$ *with collinear vertex points R and L. Then q and n are odd and* Γ *has canonical equation* $\Gamma : x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0$*, where* $x^{\sigma+1} = c$ *has exactly two solutions. Moreover,* $|\Gamma| = q^{3n} + 2q^{2n} + q^n + 1$ *and* Γ *contains* $2q^n$ *lines through R and* $2q^n$ *lines through L* (*beside* RL).

Theorem 3.4 *Let* Γ *be a degenerate* $(q+1)q^n$*-hyperbolic* σ*-quadric of* $\mathrm{PG}(4, q^n)$ *with collinear vertex points R and L. Then, n is even and* Γ *has canonical equation* $\Gamma : x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0$*,* $x^{\sigma+1} = c$ *has exactly* $q+1$ *solutions. Moreover,* $|\Gamma| = q^{3n} + (q+1)q^{2n} + q^n + 1$ *and* Γ *contains* $(q+1)q^n$ *lines through R and* $(q+1)q^n$ *lines through L* (*beside* RL).

i.2) Now, suppose that Φ does not map the lines of $\mathcal{S}_{R,\Sigma_L}$ into the hyperplanes through the line RL. In this case, there exists a hyperplane Σ containing RL such that the lines of the star $\mathcal{S}_{R,\Sigma}$ are mapped, under Φ, into the hyperplanes through RL. Hence, there is another line through R (together with RL) contained in Γ. In this case, we may assume that $\Sigma : x_3 = 0$ and Φ maps the line $x_3 = x_4 = x_5 = 0$ into the hyperplane $x_2 = 0$, the line $x_2 = x_4 = x_5 = 0$ into the hyperplane $x_1 = 0$, and the line $x_2 = x_3 = x_5 = 0$ into the hyperplane $x_3 = 0$. Hence,

$$\Gamma : -ax_2^{\sigma+1} + bx_1x_3^\sigma + cx_3x_4^\sigma + x_4x_5^\sigma = 0.$$

Assuming that Γ contains the points $(0, 1, 0, 1, 1)$, $(0, 0, -1, 1, 1)$, $(-1, 0, 1, 1, 0)$, we get $a = b = c = 1$. Since $R^\perp$ has equation $x_3 = 0$; then, a line $l_{\alpha,\beta,\gamma,\delta}$ through R is contained in Γ if, and only if, $\alpha^{\sigma+1} = \gamma\delta^\sigma$. Observe that if $\alpha = 0$, then either $\gamma = 0$, which gives the line RL, or $\delta = 0$, which gives the line $l_{0,0,1,0}$. So, the number of lines through R contained in Γ, different from the lines RL and $l_{0,0,1,0}$, depends on the cardinality of the set $\{l_{1,0,x,y} \in \mathcal{S}_R : xy^\sigma = 1\}$. A pair (x, y) is a solution of $xy^\sigma = 1$ if, and only if, $y = x^{-\sigma^{-1}}$. Hence, there are $q^n - 1$ solutions of the equation giving $q^n - 1$ lines through R contained in Γ. We will call the set Γ a non-degenerate *parabolic* σ*-quadric* with *collinear vertex points* R and L. The following holds:

Theorem 3.5 *Let* Γ *be a non-degenerate parabolic* σ*-quadric of* $\mathrm{PG}(4, q^n)$ *with collinear vertex points R and L. Then,* Γ *has canonical equation* $\Gamma : -x_2^{\sigma+1} + x_1x_3^\sigma + x_3x_4^\sigma + x_4x_5^\sigma = 0$*. Moreover,* $|\Gamma| = q^{3n} + q^{2n} + q^n + 1$ *and* Γ *contains* q^n *lines through R and* q^n *lines through L* (*beside* RL).

ii) Next, assume that Σ_L does not contain the line RL (or equivalently Φ does not map the line RL into a hyperplane through the line RL). W.l.o.g. we may put $\Sigma_L : x_1 = 0$. In this case, there is a hyperplane through R (not containing L), say Σ_R, such that the star of lines through R in Σ_R is mapped, under Φ, into the hyperplanes through RL. We may assume that $\Sigma_R : x_5 = 0$. Hence, Φ maps the lines $l_{\alpha,\beta,\gamma,0}$ into the hyperplanes $\Sigma_{0,\beta',\gamma',\delta'}$ so we may assume that Φ maps the line $l_{1,0,0,0}$ into the hyperplanes $\Sigma_{0,1,0,0}$, the line $l_{0,1,0,0}$ into the hyperplanes $\Sigma_{0,0,1,0}$, and the line $l_{0,0,1,0}$ into the hyperplanes $\Sigma_{0,0,0,1}$. Hence, the points of Γ satisfy the equation

$$\Gamma : ax_2^{\sigma+1} + bx_3^{\sigma+1} + cx_4^{\sigma+1} + x_1x_5^\sigma = 0.$$

Assuming, w.l.o.g., that the point $(-1, 1, 0, 0, 1)$ belongs to Γ we obtain $a = 1$. Since $R^{\perp}$ has equation $x_5 = 0$, then a line $l_{\alpha,\beta,\gamma,\delta}$ through R is contained in Γ if, and only if, $\alpha^{\sigma+1} + b\beta^{\sigma+1} + c\gamma^{\sigma+1} = 0$. Hence, the numbers of lines through R contained in Γ depend on the number of points of the Kestenband σ-conic of $\mathrm{PG}(2, q^n)$ (see [11–21]) given by the equation $x^{\sigma+1} + by^{\sigma+1} + cz^{\sigma+1} = 0$. We distinguish several cases:

- If q is odd and n is odd, then there are $q^n + 1$ points giving $q^n + 1$ lines through R (and through L) contained in Γ.
- If q is even and n is odd, then there are $q^n + 1$ points giving $q^n + 1$ lines through R (and through L) contained in Γ.
- If n is even, then there are either $q^n+1+(-q)^{n/2+1}(q-1)$ or $q^n+1+(-q)^{n/2}(q-1)$ or $q^n + 1 - 2(-q)^{n/2}$ points giving either $q^n + 1 + (-q)^{n/2+1}(q - 1)$ or $q^n + 1 + (-q)^{n/2}(q - 1)$ or $q^n + 1 - 2(-q)^{n/2}$ lines through R (and through L) contained in Γ.

In these cases, we will call the set Γ either *of type* 1 or *of type* 2 or *of type* 3 or *of type* 4 with *vertex points* R and L according to the number of lines through R contained in Γ is either q^n+1 or $q^n+1+(-q)^{n/2+1}(q-1)$ or $q^n+1+(-q)^{n/2}(q-1)$ or $q^n + 1 - 2(-q)^{n/2}$. Now, let l a line through R. If l is not contained in Σ_R, then $\Phi(l) \not\supset RL$ and so $l \cap \Phi(l)$ is a point. If l is contained in Σ_R, then $l \cap \Phi(l)$ is either the point R or the line l. Recalling that Γ does not contain the line RL, we get $|\Gamma| = q^{3n} + q^n|\{(x, y, z) \in (2, q^n) : x^{\sigma+1} + by^{\sigma+1} + cz^{\sigma+1} = 0\}| + 1$.

Theorem 3.6 *Let Γ be a non-degenerate σ-quadric of type* 1 *of* $\mathrm{PG}(4, q^n)$ *with vertex points R and L. Then, q is odd and n is either odd or even and Γ has canonical equation $\Gamma : x_2^{\sigma+1} + x_3^{\sigma+1} + x_4^{\sigma+1} + x_1x_5^{\sigma} = 0$, where the Kestenband σ-conic of* $\mathrm{PG}(2, q^n)$ *given by the equation $x^{\sigma+1}+y^{\sigma+1}+z^{\sigma+1} = 0$ has q^n+1 points. Moreover, $|\Gamma| = q^{3n} + q^{2n} + q^n + 1$ and Γ contains exactly $q^n + 1$ lines through R and exactly $q^n + 1$ lines through L.*

Theorem 3.7 *Let Γ be a non-degenerate σ-quadric of type* 2 *of* $\mathrm{PG}(4, q^n)$ *with vertex points R and L. Then, n is even and Γ has canonical equation $\Gamma : x_2^{\sigma+1} + x_3^{\sigma+1} + x_4^{\sigma+1} + x_1x_5^{\sigma} = 0$, where the Kestenband σ-conic of* $\mathrm{PG}(2, q^n)$ *given by the equation $x^{\sigma+1}+y^{\sigma+1}+z^{\sigma+1} = 0$ has $q^n+1+(-q)^{n/2+1}(q-1)$ points. Moreover, $|\Gamma| = q^{3n}+q^{2n}+q^n+q^n(-q)^{n/2+1}(q-1)+1$ and Γ contains exactly $q^n+1+(-q)^{n/2+1}(q-1)$ lines through R and exactly $q^n + 1 + (-q)^{n/2+1}(q - 1)$ lines through L.*

Theorem 3.8 *Let Γ be a non-degenerate σ-quadric of type* 3 *of* $\mathrm{PG}(4, q^n)$ *with vertex points R and L. Then, n is even and Γ has canonical equation $\Gamma : x_2^{\sigma+1} + x_3^{\sigma+1} + cx_4^{\sigma+1} + x_1x_5^{\sigma} = 0$, where the Kestenband σ-conic of* $\mathrm{PG}(2, q^n)$ *given by the equation $x^{\sigma+1} + y^{\sigma+1} + cz^{\sigma+1} = 0$ has $q^n + 1 + (-q)^{n/2}(q - 1)$ points, with $c \notin \{x^{q+1} : x \in \mathbb{F}_{q^n}\}$. Moreover, $|\Gamma| = q^{3n} + q^{2n} + q^n + q^n(-q)^{n/2}(q - 1) + 1$ and Γ contains exactly $q^n+1+(-q)^{n/2}(q-1)$ lines through R and exactly $q^n+1+(-q)^{n/2}(q-1)$ lines through L.*

Theorem 3.9 *Let Γ be a non-degenerate σ-quadric of type* 4 *of* $\mathrm{PG}(4, q^n)$ *with vertex points R and L. Then, n is even and Γ has canonical equation $\Gamma : x_2^{\sigma+1} + bx_3^{\sigma+1} +$*

$cx_4^{\sigma+1} + x_1 x_5^\sigma = 0$, *where the Kestenband* σ*-conic of* $\mathrm{PG}(2, q^n)$ *given by the equation* $x^{\sigma+1} + by^{\sigma+1} + cz^{\sigma+1} = 0$ *has* $q^n + 1 - 2(-q)^{n/2}$ *points, with* b, c, $b/c \notin \{x^{q+1} : x \in \mathbb{F}_{q^n}\}$. *Moreover,* $|\Gamma| = q^{3n} + q^{2n} + q^n - 2q^n(-q)^{n/2} + 1$ *and* Γ *contains exactly* $q^n + 1 - 2(-q)^{n/2}$ *lines through* R *and exactly* $q^n + 1 - 2(-q)^{n/2}$ *lines through* L.

2) $V^\perp = V^\top$. We may assume w.l.o.g. that the point $R = L = (1, 0, 0, 0, 0)$ is both the left radical and the right radicals. It follows that, in this case, Γ is a cone with vertex the point R projecting a σ-quadric of rank 4 in a hyperplane not through R. Indeed, since the matrix A has rank four with first column and last row equal to 0, by choosing a hyperplane not through the point R, e.g., $\Sigma : x_1 = 0$, we get that the set $\Gamma \cap \Sigma$ is a σ-quadric of the hyperplane Σ with associated matrix of rank 4.

Proposition 3.10 *Let* $\Gamma : X^t A X^\sigma = 0$ *be a* σ*-quadric of* $\mathrm{PG}(4, q^n)$, *with* $\mathrm{rk}(A) = 4$. *The set* Γ *is one of the following:*

- *a degenerate either elliptic or* q^n*-parabolic or* $2q^n$*-hyperbolic or* $(q + 1)q^n$*-hyperbolic* σ*-quadric with two collinear vertex points;*
- *a non-degenerate parabolic* σ*-quadric with two collinear vertex points;*
- *a non-degenerate* σ*-quadric either of type* 1 *or of type* 2 *or of type* 3 *or of type* 4 *with two vertex points;*
- *a cone with vertex a point* V *projecting a* σ*-quadric of rank* 4 *in a hyperplane* Σ, *with* $V \notin \Sigma$.

4 σ-Quadrics of rank 3 in PG(4, q^n)

In this section, a σ-quadric Γ of $\mathrm{PG}(4, q^n)$ will have equation $X^t A X^\sigma = 0$ with $\mathrm{rk}(A) = 3$. Hence, $\dim V^\perp = \dim V^\top = 2$ so right and left radicals in $\mathrm{PG}(4, q^n)$ are two lines r and l. We distinguish three cases:

1) $r \cap l = \emptyset$. We may assume w.l.o.g. that $r : x_3 = x_4 = x_5 = 0$ and $l : x_1 = x_2 = x_3 = 0$. Then:

$$\Gamma : (a_{13}x_1 + a_{23}x_2 + a_{33}x_3)x_3^\sigma + (a_{14}x_1 + a_{24}x_2 + a_{34}x_3)x_4^\sigma$$
$$+(a_{15}x_1 + a_{25}x_2 + a_{35}x_3)x_5^\sigma = 0.$$

Let

$$\mathcal{P}_r = \{\pi_{a,b,c} : (a, b, c) \in \mathrm{PG}(2, q^n)\},$$

where

$$\pi_{a,b,c} = \begin{cases} x_1 = \lambda \\ x_2 = \mu \\ x_3 = \gamma a \\ x_4 = \gamma b \\ x_5 = \gamma c \end{cases}, (\lambda, \mu, \gamma) \in \mathrm{PG}(2, q^n)$$

and

$$\mathcal{S}_l = \{\Sigma_{a,b,c} : (a, b, c) \in \mathrm{PG}(2, q^n)\}, \text{ where } \Sigma_{a,b,c} : ax_1 + bx_2 + cx_3 = 0.$$

The σ-quadric Γ is the set of points of $\mathrm{PG}(4, q^n)$ of intersection of corresponding elements under a collineation $\Phi : \mathcal{P}_r \longrightarrow \mathcal{S}_l$. Let Σ_{rl} be the hyperplane spanned by the lines r and l, it follows that:

$$\Sigma_{rl} : x_3 = 0.$$

Let π the plane through r s.t. $\Sigma_{rl} = \Phi(\pi)$. We distinguish two cases.

a) First assume that Σ_{rl} contains the plane π.
W.l.o.g., we may put $\pi : x_3 = x_4 = 0$. Then, $\Phi(\pi_{0,0,1}) = \Sigma_{0,0,1}$. We may assume that Φ maps the plane $\pi_{1,0,0}$ into the hyperplane $\Sigma_{1,0,0}$, the plane $\pi_{0,1,0}$ into the hyperplane $\Sigma_{0,1,0}$ and the plane $\pi_{1,1,1}$ into the hyperplane $\Sigma_{1,1,1}$, and hence, a canonical equation of Γ in this case is given by

$$\Gamma : x_1x_3^\sigma + x_2x_4^\sigma + x_3x_5^\sigma = 0.$$

The set Γ is the union of the plane π and $q^{2n} + q^n$ lines. Let P be a point of the plane $\pi_{a,b,c} \in \mathcal{P}_r$, then $P = (\lambda, \mu, \gamma a, \gamma b, \gamma c)$. Observe that P belongs to l if, and only if, $\lambda = \mu = a = 0$ and $\gamma \neq 0$. It follows that the plane $\pi_{a,b,c}$ is skew with l if, and only if, $a \neq 0$. Therefore, Γ contains q^n lines which are transversal with r and l, and q^{2n} which are incident with r and skew with l. Hence, Γ has $q^{3n} + 2q^{2n} + q^n + 1$ points. We will call this set a degenerate *hyperbolic σ-quadric* with *skew vertex lines* r and l.

b) Now assume that Σ_{rl} does not contain the plane π.
It follows that, w.l.o.g., we may put $\pi : x_4 = x_5 = 0$. Then, $\Phi(\pi_{1,0,0}) = \Sigma_{0,0,1}$. We may assume that Φ maps the plane $\pi_{0,1,0}$ into the hyperplane $\Sigma_{0,1,0}$, the plane $\pi_{0,0,1}$ into the hyperplane $\Sigma_{1,0,0}$ and the plane $\pi_{1,1,1}$ into the hyperplane $\Sigma_{1,1,1}$, and hence, a canonical equation of Γ in this case is given by

$$\Gamma : x_3^{\sigma+1} + x_2x_4^\sigma + x_1x_5^\sigma = 0.$$

The set Γ is the union of $q^{2n} + q^n + 1$ lines whose $q^n + 1$ lines are transversal with r and l, and q^{2n} lines are incident with r and skew with l. Observing that $\pi \cap \Sigma_{rl} = r$, it follows that Γ has $q^{3n} + q^{2n} + q^n + 1$ points. We will call this set a non-degenerate *parabolic σ-quadric* with *skew vertex lines* r and l.

2) $r \cap l = \{V\}$ is a point. We may assume w.l.o.g. that $r : x_3 = x_4 = x_5 = 0$ and $l : x_2 = x_3 = x_4 = 0$. In this case, the σ-quadric Γ is a cone with vertex the point V. Since the matrix A has rank three with first two columns and first and last rows equal to 0, by choosing a hyperplane not through the point V, e.g., $\Sigma : x_1 = 0$, we get that the set $\Gamma \cap \Sigma$ is a σ-quadric of the hyperplane Σ with associated matrix of rank three and

two (collinear or not) vertex points given by $R = r \cap \Sigma$ and $L = l \cap \Sigma$. It follows that Γ is a cone with vertex the point V projecting a σ-quadric of rank 3 in a hyperplane not through V with two (collinear or not) vertex points.

3) $r = l$. We may assume w.l.o.g. that $r = l : x_3 = x_4 = x_5 = 0$. It follows that, in this case, Γ is a cone with vertex the line r. Since the matrix A has rank three with first two columns and first two rows equal to 0, by choosing a plane not through the line r, e.g., $\pi : x_1 = x_2 = 0$, we get that the set $\Gamma \cap \pi$ is a σ-conic of the plane π with associated matrix of rank three. Hence, it is a Kestenband σ-conic of π. It follows that Γ is a cone with vertex the line r projecting a Kestenband σ-conic in a plane not through r. In particular, if $n = 2$ and $\sigma^2 = 1$, then Γ is a Hermitian cone with vertex the line r.

Proposition 4.1 *Let $\Gamma : X^t A X^\sigma = 0$ be a σ-quadric of* $\mathrm{PG}(4, q^n)$, *with* $\mathrm{rk}(A) = 3$. *The set Γ is one of the following:*

- *a degenerate hyperbolic σ-quadric with skew vertex lines r and l;*
- *a non-degenerate parabolic σ-quadric with skew vertex lines r and l;*
- *a cone with vertex a point V projecting a σ-quadric of rank 3 in a hyperplane Σ with two (collinear or not) vertex points, with $V \notin \Sigma$;*
- *a cone with vertex a line v projecting a Kestenband σ-conic of a plane π not through v.*

5 σ-Quadrics of rank 2 in $\mathrm{PG}(4, q^n)$

In this section, a σ-quadric Γ of $\mathrm{PG}(4, q^n)$ will have equation $X^t A X^\sigma = 0$ with $\mathrm{rk}(A) = 2$. Hence, $\dim V^\perp = \dim V^\top = 3$ so right and left radicals in $\mathrm{PG}(4, q^n)$ are two planes π_R and π_L. We distinguish three cases:

1) $\pi_R \cap \pi_L = \{V\}$ is a point. We may assume w.l.o.g. that $\pi_R : x_4 = x_5 = 0$ and $\pi_L : x_1 = x_2 = 0$. It follows that, in this case, Γ is a cone with vertex the point V. Since the matrix A has rank two with first three columns and last three rows equal to 0, by choosing a hyperplane not through the point V, e.g., $\Sigma : x_3 = 0$, we get that the set $\Gamma \cap \Sigma$ is σ-quadric of the hyperplane Σ with associated matrix of rank 2 and vertices the lines

$$r : x_3 = x_4 = x_5 = 0 \text{ and } l : x_1 = x_2 = x_3 = 0.$$

Hence, it is a σ-quadric of pseudoregulus type of Σ with skew vertex lines r and l (see [7, 9]). It follows that Γ is a cone with vertex the point V projecting a σ-quadric of pseudoregulus type in a hyperplane not through V with skew vertex lines.

2) $\pi_R \cap \pi_L = t$ is a line. We may assume w.l.o.g. that $\pi_R : x_4 = x_5 = 0$, $\pi_L : x_3 = x_4 = 0$. In this case, the σ-quadric Γ is a cone with vertex the line t projecting a (degenerate or not) C_F^m-set (see [5, 6, 8]) in a plane not through t. Indeed, let π

a plane not through the line t and let $A = \pi_R \cap \pi$, $B = \pi_L \cap \pi$. It follows that $\Gamma \cap \pi$ is a set of points of π generated by a collineation between the pencils of lines of π with center the points A and B induced by the collineation between the pencils of hyperplanes $\mathcal{P}_{\pi_R}$ and $\mathcal{P}_{\pi_L}$ that is associated with Γ.

3) $\pi_R = \pi_L$. We may assume w.l.o.g. that $\pi_R = \pi_L : x_4 = x_5 = 0$. In this case, the σ-quadric is a cone with vertex the plane π_R over a σ-quadric of a line skew with π_R. That is, Γ is either just the plane π_R or a hyperplane through π_R or a pair of distinct hyperplanes through π_R or $q + 1$ hyperplanes through π_R forming an $\mathbb{F}_q$-subpencil of hyperplanes through π_R.

Proposition 5.1 *Let $\Gamma : X^t A X^\sigma = 0$ be a σ-quadric of* $\mathrm{PG}(4, q^n)$, *with* $\mathrm{rk}(A) = 2$. *The set Γ is one of the following:*

- *a cone with vertex a point V projecting a σ-quadric of pseudoregulus type in a hyperplane Σ with skew vertex lines, with $V \notin \Sigma$;*
- *a cone with vertex a line v projecting a (possibly degenerate) C_F^m-set of a plane π, with $v \cap \pi = \emptyset$;*
- *a cone with vertex a plane π projecting a σ-quadric of a line v, with $v \cap \pi = \emptyset$ (hence either just the plane π or one, two or $q + 1$ hyperplanes through π).*

6 σ-Quadrics of rank 1 in PG(4, q^n)

In this section, a σ-quadric Γ of $\mathrm{PG}(4, q^n)$ will have equation $X^t A X^\sigma = 0$ with $\mathrm{rk}(A) = 1$. Hence, $\dim V^\perp = \dim V^\top = 4$ so left and right radicals in $\mathrm{PG}(4, q^n)$ are hyperplanes. We distinguish two cases:

- $V^\perp \neq V^\top$. We may assume that $\Sigma_R : x_5 = 0$ is the right radical and $\Sigma_L : x_1 = 0$ is the left radical. Hence, $\Gamma : x_1 x_5^\sigma = 0$ that is the union of two different hyperplanes.
- $V^\perp = V^\top$. We may assume that $\Sigma_R = \Sigma_L : x_5 = 0$ is both the left and the right radical. Hence, $\Gamma : x_5^{\sigma+1} = 0$ that is a hyperplane of $\mathrm{PG}(4, q^n)$.

7 σ-Quadrics of PG(4, q) and ovoids of $Q(4, q)$

In this final section, we will show, as an application of σ-quadrics of $\mathrm{PG}(4, q)$, that two of the known ovoids of $Q(4, q)$ can be obtained as intersection of a suitable σ-quadric with $Q(4, q)$.

An *ovoid* of $Q(4, q)$ is a set of $q^2 + 1$ points no two collinear on the quadric. Let $Q(4, q) : x_1 x_5 + x_2 x_4 + x_3^2 = 0$. Any ovoid of $Q(4, q)$ can be written in the following way:

$$\mathcal{O}(f) = \{(0, 0, 0, 0, 1)\} \cup \{(1, x, y, f(x, y), -y^2 - x f(x, y)) : x, y \in \mathbb{F}_q\}$$

for some function $f(x, y)$. They are rare objects and, beside the classical example given by an elliptic quadric, only three classes are known for q odd, one class for q even and a sporadic example for $q = 3^5$. They have been studied since the end of the 1980s also because of their connections with many other important and well studied objects such as semifield flocks of a three-dimensional quadratic cone, ovoids of $\mathrm{PG}(3, q)$, eggs of finite projective spaces, translation generalized quadrangles, rank 2 commutative semifields, etc. Here we present the two classes of ovoids related to σ-quadrics of $\mathrm{PG}(4, q)$. Let n be a non-square of $\mathbb{F}_q$, $q = p^h$, q odd and $h > 1$, and let $\sigma \neq 1$ be an automorphism of $\mathbb{F}_q$, then the set $\mathcal{O}(f_1)$ with $f_1(x, y) = -nx^\sigma$ is an ovoid of $Q(4, q)$ and it is called *Kantor ovoid*. If $q = 2^{2h+1}$ and $\sigma = 2^{h+1}$, then $\mathcal{O}(f_2)$ with $f_2(x, y) = x^{\sigma+1} + y^\sigma$ is an ovoid of $Q(4, q)$, and it is called *Tits ovoid*. The following holds:

Proposition 7.1 *Let n be a non-square of $\mathbb{F}_q$, $q = p^h$, q odd and $h > 1$, and let $\sigma \neq 1$ be an automorphism of $\mathbb{F}_q$. The σ-quadric Γ of rank 2 of $(4, q)$ given by the equation $x_4x_1^\sigma + nx_1x_2^\sigma = 0$ meets the quadric $Q(4, q) : x_1x_5 + x_2x_4 + x_3^2 = 0$ in the union of a Kantor ovoid and a quadratic cone contained in a hyperplane of $\mathrm{PG}(4, q)$.*

Proof Start observing that $\Gamma = \{x_1 = 0\} \cup \{(1, x, y, -nx^\sigma, z) : x, y, z \in \mathbb{F}_q\}$.
First let $P = (1, x, y, -nx^\sigma, z)$ be a point in $\Gamma \setminus \{x_1 = 0\}$. It follows that P belongs to $Q(4, q)$ if, and only if, $z = -y^2 + nx^{\sigma+1}$.
Now observe that the intersection $\{x_1 = 0\} \cap Q(4, q)$ is given by

$$\begin{cases} x_1 = 0 \\ x_2x_4 + x_3^2 = 0, \end{cases}$$

that is a quadratic cone of the hyperplane $x_1 = 0$.

Proposition 7.2 *Let $q = 2^{2h+1}$ and let $\sigma = 2^{h+1}$. The σ-quadric Γ of rank 3 of $\mathrm{PG}(4, q)$ given by the equation $x_4x_1^\sigma + x_2^{\sigma+1} + x_1x_3^\sigma = 0$ meets the quadric $Q(4, q) : x_1x_5 + x_2x_4 + x_3^2 = 0$ in the union of a Tits ovoid and the line $x_1 = x_2 = x_3 = 0$.*

Proof Start observing that

$$\Gamma = \{x_1 = x_2 = x_3 = 0\} \cup \{(1, x, y, x^{\sigma+1} + y^\sigma, z) : x, y, z \in \mathbb{F}_q\}.$$

First let $P = (1, x, y, x^{\sigma+1} + y^\sigma, z)$ be a point in $\Gamma \setminus \{x_1 = x_2 = x_3 = 0\}$. It follows that P belongs to $Q(4, q)$ if, and only if, $z = y^2 + x^{\sigma+2} + xy^\sigma$. Now observe that the quadric $Q(4, q)$ contains the lines $\{x_1 = x_2 = x_3 = 0\}$. □

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