



Superbinomial coefficients

Kathrin Maurischat¹ · Rainer Weissauer²

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Abstract

We investigate several families of polynomials that are related to certain Euler type summation operators. Being integer valued at integral points, they satisfy combinatorial properties and nearby symmetries, due to triangle recursion relations involving squares of polynomials. Some of these interpolate the Delannoy numbers. The results are motivated by and strongly related to our study of irreducible Lie supermodules, where dimension polynomials in many cases show similar features.

Keywords Lie supermodules · Triangle numbers · Integer values of polynomials · Summation operators · Delannoy numbers

Mathematics Subject Classification 05E05 · 05E10 · 17B10

1 Introduction

We are interested in a family of polynomials $p(n, x)$ that, for particular polynomials $a(n, x)$, satisfy recursion formulas of the form

$$p(n, x + 1) + 2p(n, x) + p(n, x - 1) = a(n, x)^2.$$

Evaluated at natural numbers $x = m$, these polynomials define integers $p(n, m)$ with interesting combinatorial properties and representation theoretic interpretations.

To illustrate this, let us consider the Pochhammer polynomials. Their values at integral points give the binomial coefficients for the classical Pascal triangle, which,

✉ Kathrin Maurischat
kathrin.maurischat@matha.rwth-aachen.de
Rainer Weissauer
weissauer@mathi.uni-heidelberg.de

¹ Lehrstuhl A für Mathematik, RWTH Aachen University, Templergraben 55, 52062 Aachen, Germany

² Mathematisches Institut, Heidelberg University, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany

among others, can be considered as dimensions of certain irreducible representations of the special linear group $SL(n)$. The isomorphism classes of finite dimensional irreducible representations of the special linear group $SL(n)$ over \mathbb{C} are described by their dominant weights λ , parameterized by integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0$. The symmetric powers $S^m(\mathbb{C}^n)$ of the n -dimensional standard representation \mathbb{C}^n of $SL(n)$ are the irreducible representations of dimension $\binom{m+n-1}{m-1}$ that are obtained for $\lambda_1 = m$ and $\lambda_i = 0, i \geq 2$. By an index shift, restoring the symmetry between n and m , for $n, m \geq 1$ we define

$$P_{cl}(n, m) := \dim(S^{m-1}(\mathbb{C}^n)) = \binom{(m-1) + (n-1)}{n-1}.$$

The numbers $P_{cl}(n, m)$ are values of rational polynomials $p_{cl}(n, x)$ of degree $n-1$ in the variable x . Together with the initial condition $p_{cl}(0, x) = 0$, they satisfy $p_{cl}(n, m) = p_{cl}(m, n) + \delta_{n1} \cdot \delta_{m0}$ for all integers $n, m \geq 0$. This almost symmetry condition and the initial condition uniquely characterize the polynomials $p_{cl}(n, x)$, such that $p_{cl}(1, x) = 1$ and $p_{cl}(n, x) = (x+n-2) \cdot \dots \cdot x / (n-1)!$ holds for all $n \geq 2$. Hence, $p_{cl}(n, m)$ coincides with $\dim(S^{m-1}(\mathbb{C}^n)) = \binom{m+n-2}{n-1}$ for $m, n \geq 1$. If we formally set $\dim(S^{-1}(\mathbb{C}^n)) = 0$, the dimension is given by $p_{cl}(n, m)$ for all $n, m \geq 0$, except for the case $(n, m) = (1, 0)$.

In analogy, here we consider polynomials $p(n, x)$ of degree $\leq 2(n-1)$ with $p(0, x) = 0$, such that the values $p(n, m) + (-1)^{m+n} \cdot n$ are symmetric for all $n, m \geq 0$. Imposing the additional conditions $p(n, x) = p(n, -x)$, this uniquely determines the polynomials $p(n, x)$. For $m \geq n$, we further define $P(n, m) = p(n, x)|_{x=m}$. Extended by symmetry $P(m, n) := P(n, m)$, these numbers (up to a simultaneous index shift of n, m by 1) will be called the superbinomial coefficients. We refer to Tables 1 and 2 for the first values of $p(n, m)$ and $P(n, m)$, respectively.

In order to relate these superbinomial coefficients $P(n, m)$ to the classical binomial coefficients, let us explain how they arise from the representation theory of the superlinear groups $SL(n|n)$. The isomorphism classes of finite dimensional irreducible representations are again described by dominant weights λ [7,9]. The superdimensions of these representations are zero unless λ is maximal atypical [6,10,13]. The maximal atypical λ are again parameterized by integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0$. For $\lambda_1 = m \geq 0$ and $\lambda_i = 0$ for $i \geq 2$ as before, let S^m denote the corresponding irreducible maximal atypical representations of $SL(n|n)$. Notice, the representations S^m are no longer isomorphic to the symmetric powers of some representations of $SL(n|n)$. However, for $m \geq 1$ we have for the dimension of the irreducible representation S^{m-1} of $SL(n|n)$

$$P(n, m) = \dim(S^{m-1}).$$

Observe the index shift, in analogy with the dimensions $P_{cl}(n, m)$ for the classical case. Next, in the relation between $p(n, m)$ and $P(n, m)$, the condition $m \geq n$ now really becomes significant. Indeed, for fixed n the dimensions do not depend on m in a polynomial way in the range $m < n$, as we briefly discuss eventually. Even more interestingly,

the polynomials $p(n, x)$ interpolate the superdimension $p(n, 0) = \text{sdim}(S^{m-1})$ at the point $x = 0 < n$, where $p(n, x)$ a priori does not have an obvious dimensional interpretation. This holds true in similar cases for more general $\text{SL}(n|n)$ modules; see [8]. For character and dimension formulas for irreducible Lie supermodules of $\text{SL}(n|n)$ in general, we refer to [2,4,9,12].

Returning to our main interest, in this note we focus on particular polynomial relations satisfied by the polynomials $p(n, m)$. The first type of relations is

$$p(n, m + 1) + 2 \cdot p(n, m) + p(n, m - 1) = A(n, m)^2 \tag{1}$$

for $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, where $A : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ is defined by

$$A(n, m) = \sum_{v=0}^{\min(m,n)} \binom{n}{v} \binom{n-1+m-v}{n-1}. \tag{2}$$

It is not hard to see that, for fixed n , there exist polynomials $a_2(n, x)$ of degree $2(n - 1)$ such that $A(n, m) = a_2(n, x)|_{x=m}$ holds for all integers $m \geq 1$. Let us define an Euler type summation operator E^2 by $E^2 f(x) = f(x + 1) + 2 \cdot f(x) + f(x - 1)$. Since this operator acts bijectively on the polynomial ring $\mathbb{Q}[x]$, the above relations characterize the polynomials $f(x) = p(n, x)$ as the unique solutions of the polynomial equations

$$E^2 f(x) = a_2(n, x)^2.$$

In Definition 3.1, we similarly give polynomials $a_1(m, x)$ satisfying $a_1(m, x)|_{x=n} = A(m, n)$. In Theorem 3.2, we show the first as well as second variable summation equations

$$p(n + 1, x) + 2p(n, x) + p(n - 1, x) = a_1(n, x)^2,$$

respectively,

$$p(n, x + 1) + 2p(n, x) + p(n, x - 1) = a_2(n, x)^2.$$

As polynomial identities, these equations hold for all x . In particular, $p(n, x)$ is of degree $2(n - 1)$ in x . This implies the polynomial identity $na_1(n, x) = xa_2(n, x)$ (Proposition 3.5), which amounts to the symmetry $nA(m, n) = mA(m, m)$. Besides the summation equations above, there also exists a summation relation of second kind

$$p(n, x) + p(n, x - 1) + p(n - 1, x) + p(n - 1, x - 1) = 2 \cdot d(n, x)^2. \tag{3}$$

Here, $d(n, x)$ turn out to be the Delannoy polynomials, defined in Sect. 4. However, we do not know any representation theoretic interpretations of the results in Sect. 4. This comes from the fact that such a result would relate representations of the groups $\text{SL}(n|n)$ for different n .

Although the connection of the polynomials $p(n, x)$ and their combinatorial properties to representation theory is not the primary focus of this paper, our interest arose while searching for alternatives to the combinatorial character formulas [2,9], in order to obtain and understand dimension formulas. In fact, for atypical irreducible representations [7] these character formulas are not entirely satisfying, because the known formulas turn out to be “rather intricate and difficult to apply” [4]. We therefore looked for a different approach to dimension formulas: In [8], we conjectured that for all irreducible maximal atypical representations of weight λ that are attached to a fixed basic weight λ_{basic} (see [6]), the dimensions of the representations depend on the coefficients of the weight λ in a polynomial way. Notice that, for fixed n , the number of different basic weights λ_{basic} is finite and given by the Catalan number C_n . Hence, this conjecture predicts the existence of C_n (usually different) dimension polynomials for fixed n . For example, the highest weights of the representations S^k are basic in the sense above if and only if $0 \leq k \leq n - 1$. For their basic weights λ_{basic} , consider the set of isomorphism classes of the irreducible maximal atypical representations of weight λ attached to λ_{basic} . These sets are singletons for $k < n - 1$, represented by S^k . For $k = n - 1$, on the other hand, this set is infinite and its representatives are the representations S^m for $m \geq k$. For $k < n - 1$, the associated dimension polynomials are constant and equal to $P(n, k)$. However, for $k = n - 1$, the dimension polynomial is our polynomial $p(n, x)$ above. Hence, our results on the superbinomial coefficients suggest subtle relations between the C_n different dimension polynomials. Furthermore, it could be that the Euler type summation equations, discussed here, are special cases of dimension formulas for certain indecomposable, but not necessarily irreducible modules. A special type of such indecomposable modules are the Kac modules. It is well known that the knowledge of Jordan Hölder constituents of Kac modules is strongly related to the character and dimension formulas. In [8], we also work with other types of indecomposable modules not directly related to Kac modules. We therefore suggest not to ignore other indecomposable modules that are different from Kac modules.

2 A family of polynomials

Proposition 2.1 *There is a unique family of polynomials*

$$\{p(n, x) \in \mathbb{Q}[x] \mid n \in \mathbb{N}_0\}$$

satisfying the following properties.

- (i) $p(0, x) = 0$.
- (ii) *Degree condition:* The degree is $\deg_x p(n, x) \leq 2(n - 1)$ for all $n \geq 1$.
- (iii) *Parity:* $p(n, x) = p(n, -x)$ holds for all $n \in \mathbb{N}_0$.
- (iv) *Symmetry:* The function $f(n, m) = p(n, m) + (-1)^{m+n} \cdot n$ is a symmetric function on $\mathbb{N}_0 \times \mathbb{N}_0$, i.e., $f(m, n) = f(n, m)$ holds.

Proof of Proposition 2.1 We show that the properties (i)–(iv) uniquely define the polynomials $p(n, x)$ by recursion. For $n = 0$, the polynomial $p(0, x) = 0$ is fixed by

property (i). For $n = 1$, by (ii) we know $p(1, x) = c$ is a constant polynomial. By (iv),

$$p(1, 0) + (-1)^{1+0} \cdot 1 = p(0, 1) + (-1)^{0+1} \cdot 0,$$

so $c = 1$. Assuming $p(k, x)$ to be constructed for $0 \leq k \leq n$, we obtain by property (iv) the following values of $p(n + 1, x)$

$$p(n + 1, k) = p(k, n + 1) + (-1)^{n+k}(n + 1 - k).$$

Using (iii), we find $p(n + 1, -k) = p(n + 1, k)$. We thus have fixed the values $p(n + 1, x)$ at the $2n + 1$ places $x \in \{-n, \dots, 0, \dots, n\}$. But by (ii), the degree of $p(n + 1, x)$ is at most $2n$. Hence, $p(n + 1, x)$ is the unique interpolation polynomial of degree $2n$ for the values above. \square

For example, condition (iv) together with (i) implies

$$p(n, 0) = (-1)^{n-1} \cdot n \quad \text{and} \quad p(n, 1) = 1 + (-1)^n(n - 1).$$

In particular,

$$p(0, x) = 0,$$

$$p(1, x) = 1,$$

$$p(2, x) = 4x^2 - 2,$$

$$p(3, x) = 4x^4 - 8x^2 + 3,$$

$$p(4, x) = \frac{16}{9}x^6 - \frac{56}{9}x^4 + \frac{112}{9}x^2 - 4,$$

$$p(5, x) = \frac{4}{9}x^8 - \frac{16}{9}x^6 + \frac{92}{9}x^4 - \frac{152}{9}x^2 + 5,$$

$$p(6, x) = \frac{16}{225}x^{10} - \frac{8}{45}x^8 + \frac{848}{225}x^6 - \frac{592}{45}x^4 + \frac{1612}{75}x^2 - 6,$$

$$p(7, x) = \frac{16}{2025}x^{12} + \frac{32}{2025}x^{10} + \frac{596}{675}x^8 - \frac{7984}{2025}x^6 + \frac{34696}{2025}x^4 - \frac{5872}{225}x^2 + 7,$$

$$p(8, x) = \frac{64}{99225}x^{14} + \frac{32}{4725}x^{12} + \frac{64}{405}x^{10} - \frac{46384}{99225}x^8 + \frac{27968}{4725}x^6 - \frac{41312}{2025}x^4 + \frac{339392}{11025}x^2 - 8.$$

The proof of Proposition 2.1 shows that, for all $n \geq 0$, the values $p(n, k)$ for $k = -n, \dots, n$ are integers. So by the almost symmetry (iv), for every integer $j > 0$, the value

$$p(n, n + j) = p(n + j, n) + (-1)^j \cdot j$$

is integral. This proves

Corollary 2.2 *Let $p(n, x)$ be the polynomials in Proposition 2.1. Then, the map p defined on $\mathbb{N}_0 \times \mathbb{N}_0$ by $p(n, m)$ is integer valued.*

Let n be a natural number. For integers $0 \leq \mu \leq n - 1$, put

$$\mu^* = n - 1 - \mu.$$

For integers $0 \leq v, \mu \leq n - 1$, define the polynomials

$$t(v, \mu, n; x) = \prod_{k=1}^{\mu} (x + v - \mu + k) \cdot \prod_{l=1}^v (x - 1 - \mu + l).$$

Proposition 2.3 *The polynomials $p(0, x) = 0$ and*

$$p(n, x) = \sum_{v, \mu=0}^{n-1} \frac{t(v, \mu, n; x) \cdot t(\mu^*, v^*, n; x)}{v!v^*!\mu!\mu^*!}$$

for $n > 0$ satisfy the properties of Proposition 2.1.

Proof of Proposition 2.3 By definition, condition (i) of Proposition 2.1 is satisfied. For the summands of $p(n, x)$, we have, for all v, μ ,

$$\deg_x t(v, \mu, n; x) \cdot t(\mu^*, v^*, n; x) = v + \mu + v^* + \mu^* = 2(n - 1).$$

So the same holds true for $p(n, x)$. This implies property (ii). Obviously,

$$t(v, \mu, n; -x) = (-1)^{v+\mu} \cdot t(\mu, v, n; x),$$

so condition (iii) follows

$$p(n, -x) = \sum_{\mu, v=0}^{n-1} (-1)^{2(n-1)} \frac{t(\mu, v, n; x) \cdot t(v^*, \mu^*, n; x)}{v!v^*!\mu!\mu^*!} = p(n, x).$$

In order to prove (iv), which is trivial for $n = m$, we assume $m > n$. Substituting $\mu \mapsto n - 1 - \mu$, we write

$$p(n, m) = \sum_{v^*, \mu^*=0}^{m-1} \frac{t(v, \mu^*, n; m) \cdot t(\mu, v^*, n; m)}{v!v^*!\mu!\mu^*!}.$$

Notice that, for $m \leq \mu^*$, the value

$$t(v, \mu^*, n; m) = (m + v) \cdots (m + v - \mu^* + 1) \cdot (m + v - \mu^* - 1) \cdots (m - \mu^*)$$

Table 1 Values $p(n, m)$ of the polynomials $p : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{Z}$

| n | m | | | | | | | | |
|-----|-----|----|-----|--------|---------|-----------|------------|-------------|---------------|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | -2 | 2 | 14 | 34 | 62 | 98 | 142 | 194 | 254 |
| 3 | 3 | -1 | 35 | 255 | 899 | 2303 | 4899 | 9215 | 15,875 |
| 4 | -4 | 4 | 60 | 900 | 5884 | 24,196 | 75,324 | 194,820 | 441,340 |
| 5 | 5 | -3 | 101 | 2301 | 24,197 | 151,805 | 676,197 | 2,376,701 | 7,031,301 |
| 6 | -6 | 6 | 138 | 4902 | 75,322 | 676,198 | 4,160,778 | 19,475,142 | 74,307,834 |
| 7 | 7 | -5 | 199 | 9211 | 194,823 | 2,376,699 | 19,475,143 | 118,493,179 | 573,785,095 |
| 8 | -8 | 8 | 248 | 15,880 | 441,336 | 7,031,304 | 74,307,832 | 573,785,096 | 3,465,441,272 |

is zero unless $m + v - \mu^* = 0$, where the value is $(-1)^v v! \mu^*!$. We obtain

$$p(n, m) = \sum_{v^*, \mu^*=0}^{m-1} \frac{t(v, \mu^*, n; m) \cdot t(\mu, v^*, n; m)}{v! v^*! \mu! \mu^*!} + \sum_{v+\mu=n-1-m} (-1)^{v+\mu}.$$

Substituting $i = n - 1 - v$ and $j = n - 1 - \mu$, the first sum becomes

$$\sum_{i, j=0}^{m-1} \frac{t(i, m - 1 - j, m; n) t(j, m - 1 - i, m; n)}{i!(m - 1 - i)! j!(m - 1 - j)!} = p(m, n).$$

So, for $m > n$ we obtain $p(n, m) = p(m, n) + (-1)^{m+n-1} (n - m)$. Hence, condition (iv) of Proposition 2.1 holds for all integers $m, n > 0$. □

Definition 2.4 Let $m \in \mathbb{N}$. For integers α and β , we define the natural numbers

$$D_m(\alpha + 1, \beta) = \begin{cases} \frac{m}{\alpha + \beta + 1} \binom{m+\alpha}{\alpha} \binom{m-1}{\beta} & \text{if } \alpha \geq 0 \text{ and } 0 \leq \beta \leq m - 1 \\ 0 & \text{else} \end{cases}.$$

Remark 2.5 i) Property (ii) of Proposition 2.1 can be sharpened to become

(ii') $\deg_x p(n, x) = 2(n - 1)$ for all $n > 0$.

ii) Fixing the first variable, the function $p(n, x)$ is polynomial in x by definition. By property (iv) of Proposition 2.1, the values $p(n, m)$ are nearly symmetric

$$p(m, n) = p(n, m) + (-1)^{m+n} (n - m).$$

Hence, for fixed $m \in \mathbb{N}$, the function $p(n, m)$ is almost a polynomial of degree $2(m - 1)$ in the first variable n .

iii) For integers $m > 0$, there is a convenient presentation of $t(v, \mu, n; m)$,

$$\frac{t(v, \mu, n; m)}{v! \mu!} = \begin{cases} \frac{m}{m+v-\mu} \binom{m+v}{v} \binom{m-1}{\mu} & \text{if } \mu \neq m + v \\ (-1)^v & \text{if } \mu = m + v \end{cases}.$$

In particular, for integers $m \geq n > 0$, we obtain

$$p(n, m) = \sum_{v, \mu=0}^{n-1} \frac{m^2}{(m + v - \mu)^2} \binom{m + v}{v} \binom{m - 1}{\mu} \binom{m + (n - 1 - \mu)}{n - 1 - \mu} \binom{m - 1}{n - 1 - v}.$$

Equivalently, using Definition 2.4, for integers $m \geq n > 0$,

$$p(n, m) = \sum_{v, \mu=0}^{n-1} D_m(v + 1, m - 1 - \mu) \cdot D_m((n - 1 - \mu) + 1, m - 1 - (n - 1 - v)).$$

Table 2 Values $P(n, m)$ of the function $P : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{Z}$

| n | m | | | | | | | | |
|-----|-----|---|-----|--------|---------|-----------|------------|-------------|---------------|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 14 | 34 | 62 | 98 | 142 | 194 | 254 |
| 3 | 0 | 1 | 34 | 255 | 899 | 2303 | 4899 | 9215 | 15,875 |
| 4 | 0 | 1 | 62 | 899 | 5884 | 24,196 | 75,324 | 194,820 | 441,340 |
| 5 | 0 | 1 | 98 | 2303 | 24,196 | 151,805 | 676,197 | 2,376,701 | 7,031,301 |
| 6 | 0 | 1 | 142 | 4899 | 75,324 | 676,197 | 4,160,778 | 19,475,142 | 74,307,834 |
| 7 | 0 | 1 | 194 | 9215 | 194,820 | 2,376,701 | 19,475,142 | 118,493,179 | 573,785,095 |
| 8 | 0 | 1 | 254 | 15,875 | 441,340 | 7,031,301 | 74,307,834 | 573,785,095 | 3,465,441,272 |

Hence, the values $p(n, m)$ are natural numbers for all integers $m \geq n$.

The numbers $P(n, m)$: In general, for integers $m, n > 0$, let us define the numbers

$$P(n, m) = \sum_{v, \mu=0}^{\min\{n-1, m-1\}} D_m(v + 1, m - 1 - \mu) \cdot D_m((n - 1 - \mu) + 1, m - 1 - (n - 1 - v)).$$

Hence, $p(n, m) = P(n, m)$ holds for $m \geq n > 0$, whereas for $m < n$ we obtain

$$p(n, m) = P(n, m) + (-1)^{m+n-1}(n - m).$$

On the other hand, we know $p(n, m) = p(m, n) + (-1)^{m+n-1}(n - m)$ by property (iv). For $m < n$, we obtain

$$P(n, m) = p(m, n) = P(m, n).$$

Hence, the numbers $P(n, m)$ are symmetric.

3 Summation operators

Consider the Eulerian summation operator E acting on functions f by

$$Ef(x) = f\left(x + \frac{1}{2}\right) + f\left(x - \frac{1}{2}\right).$$

On monomials, E acts by $E(x^n) = \sum_{k=0}^n \binom{n}{k} x^{n-k} 2^{-k} (1 + (-1)^k)$. The preimages $E^{-1}(x^n)$ are unique by recursion, starting with $E^{-1}(0) = 0$ and $E^{-1}(1) = \frac{1}{2}$. This

shows that E is bijective on polynomial rings over fields of characteristic $\neq 2$. Furthermore, the summation operator E^2 is also well defined on functions $f : \mathbb{Z} \rightarrow \mathbb{C}$. We have

$$E^2 f(x) = f(x + 1) + 2 \cdot f(x) + f(x - 1).$$

For integers k , consider the following polynomials of degree k with $\begin{bmatrix} x \\ 0 \end{bmatrix} = 1$ and

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{x(x - 1) \cdots (x - k + 1)}{k!} \quad \text{for } k \geq 1.$$

For all integers $n \geq 0$, their values coincide with the binomial coefficients $\binom{n}{k} = \begin{bmatrix} n \\ k \end{bmatrix}$.

Definition 3.1 For all $n \in \mathbb{N}$, we define polynomials of degree n

$$a_1(n, x) = \sum_{\nu=0}^n \begin{bmatrix} x - 1 + \nu \\ \nu \end{bmatrix} \begin{bmatrix} x \\ n - \nu \end{bmatrix},$$

and for all $m \in \mathbb{N}$ we define polynomials of degree $m - 1$

$$a_2(m, x) = \sum_{\nu=0}^m \binom{m}{\nu} \begin{bmatrix} x - \nu + m - 1 \\ m - 1 \end{bmatrix}.$$

Theorem 3.2 Let $p : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{Z}$ be the function defined in Corollary 2.2. Let E_1 and E_2 be the summation operators in the first and second variable, respectively. Consider the function $A : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ given by (Table 3)

$$E_1^2 p(n, m) = p(n + 1, m) + 2 \cdot p(n, m) + p(n - 1, m) = A(m, n)^2. \tag{4}$$

Then, we also have

$$E_2^2 p(n, m) = p(n, m + 1) + 2 \cdot p(n, m) + p(n, m - 1) = A(n, m)^2. \tag{5}$$

Further, $A(n, m)$ defines the numbers [11, A266213] resp. the transposed array of the numbers [11, A122542]. They satisfy the tribonacci identities¹

$$A(n, m) = A(n - 1, m) + A(n, m - 1) + A(n - 1, m - 1).$$

¹ Indeed $A(n, m) = D(n, m) - D(n, m - 1)$ holds for the Delannoy numbers $D(n, m)$ (our remarks on Delannoy numbers following the proof of Proposition 4.3). Since the $D(n, m)$ satisfy the tribonacci identities [1], hence also the $A(n, m)$ do.

Table 3 Values $A(n, m)$ of the function $A : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$

| n | m | | | | | | | | |
|-----|-----|----|-----|-----|------|------|--------|--------|---------|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | 1 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 |
| 3 | 1 | 6 | 18 | 38 | 66 | 102 | 146 | 198 | 258 |
| 4 | 1 | 8 | 32 | 88 | 192 | 360 | 608 | 952 | 1408 |
| 5 | 1 | 10 | 50 | 170 | 450 | 1002 | 1970 | 3530 | 5890 |
| 6 | 1 | 12 | 72 | 292 | 912 | 2364 | 5336 | 10,836 | 20,256 |
| 7 | 1 | 14 | 98 | 462 | 1666 | 4942 | 12,642 | 28,814 | 59,906 |
| 8 | 1 | 16 | 128 | 688 | 2816 | 9424 | 27,008 | 68,464 | 157,184 |

Further, they coincide with evaluations of the polynomials in Definition 3.1,

$$a_1(m, n) = A(n, m) \quad , \quad a_2(n, m) = A(n, m).$$

Let $p(n, x)$ be the family of polynomials of Proposition 2.1 defining the numbers $p(n, m)$. Then, for all integers $n > 0$,

$$E_1^2 p(n, x) = p(n + 1, x) + 2 \cdot p(n, x) + p(n - 1, x) = a_1(n, x)^2 \tag{6}$$

holds. Furthermore, for all integers $m > 0$, we have

$$E_2^2 p(m, x) = p(m, x + 1) + 2 \cdot p(m, x) + p(m, x - 1) = a_2(m, x)^2. \tag{7}$$

Before we prove Theorem 3.2, we deduce some consequences. Consider the symmetric numbers $P(n, m)$ defined in Remark 2.5(iii), which can be written as

$$P(n, m) = \sum_{\nu, \mu=0}^{\min(n-1, m-1)} \frac{n^2}{(m + \nu - \mu)^2} \binom{m + \nu}{\nu} \binom{m - 1}{\mu} \binom{n + m - 1 - \mu}{n - 1 - \mu} \binom{m - 1}{n - 1 - \nu}. \tag{8}$$

For $m > n$, they satisfy $P(n, m + i) = p(n, m + i)$ for $i = -1, 0, 1$. Similarly, by Proposition 2.1(iv), for $m < n$, they satisfy $P(n, m + i) = p(n, m + i) + (-1)^{m+i+n}(n - m - i)$ for $i = -1, 0, 1$. And for $m = n$, we obtain $P(n, n + i) = p(n, n + i)$ for $i = 0, 1$, and $P(n, n + 1) = p(n, n + 1) - 1$. By Theorem 3.2, this implies the following corollary.

Corollary 3.3 For all integers $m, n > 0$, the symmetric numbers $P(n, m)$ satisfy the summation equations

$$P(n, m + 1) + 2 \cdot P(n, m) + P(n, m - 1) = A(n, m)^2 \quad \text{for } m \neq n,$$

and

$$P(n, n + 1) + 2 \cdot P(n, n) + P(n, n - 1) + 1 = A(n, n)^2 \quad \text{for } m = n.$$

Theorem 3.4 *Let S^{m-1} be the irreducible $\mathrm{SL}(n|n)$ Lie supermodule of maximal atypical weight $(m - 1, 0, \dots, 0 | 0, \dots, 0, 1 - m)$. For $n, m \geq 1$, its dimension is given by the number (8):*

$$\dim(S^{m-1}) = P(n, m).$$

Proof of Theorem 3.4 We use the indecomposable $\mathrm{SL}(n|n)$ -module

$$\mathbb{A}_{S^m} = \mathrm{Sym}^m(k^{n|n}) \otimes \Lambda^m(k^{n|n})^\vee,$$

whose dimension is the product of the dimensions of its tensor factors. This immediately implies $\dim(\mathbb{A}_{S^m}) = A(n, m)^2$. By [5, Lemma 4.1], for all $m \geq 1$ we have the following decomposition in the Grothendieck ring of representations

$$\mathbb{A}_{S^m} \sim S^{m-2} + S^m + 2 \cdot S^{m-1}, \quad (9)$$

for $m \neq n$, and for $m = n$ we have

$$\mathbb{A}_{S^n} \sim S^{n-2} + S^n + 2 \cdot S^{n-1} + 1. \quad (10)$$

Formally set $S^{-1} = 0$. The summation equations for the dimensions are

$$\dim(S^m) + 2 \cdot \dim(S^{m-1}) + \dim(S^{m-2}) = A(n, m)^2 \quad \text{for } m \neq n,$$

and

$$\dim(S^n) + 2 \cdot \dim(S^{n-1}) + \dim(S^{n-2}) + 1 = A(n, n)^2 \quad \text{for } m = n.$$

By Corollary 3.3, for fixed n , the numbers $P(n, m)$ are also solutions of these equations. Any solution of these recursion equations is of the form $P(n, m) + (-1)^m(c_1(n)m + c_0(n))$ for certain constants c_1 and c_0 . This holds for $\dim(S^{m-1})$ with constants $c_1 = c_0 = 0$, due to the initial conditions $\dim(S^{-1}) = 0 = P(n, 0)$ and $\dim(S^0) = 1 = P(n, 1)$. \square

We now list some of the polynomials $a_1(n, x)$ and $a_2(m, x)$:

$$\begin{aligned} a_1(1, x) &= 2x, & a_2(1, x) &= 2, \\ a_1(2, x) &= 2x^2, & a_2(2, x) &= 4x, \\ a_1(3, x) &= \frac{4}{3}x \left(x^2 + \frac{1}{2}\right), & a_2(3, x) &= 4 \left(x^2 + \frac{1}{2}\right), \\ a_1(4, x) &= \frac{2}{3}x^2(x^2 + 2), & a_2(4, x) &= \frac{8}{3}x(x^2 + 2), \end{aligned}$$

$$\begin{aligned}
 a_1(5, x) &= \frac{4}{15}x \left(x^4 + 5x^2 + \frac{3}{2}\right), & a_2(5, x) &= \frac{4}{3} \left(x^4 + 5x^2 + \frac{3}{2}\right), \\
 a_1(6, x) &= \frac{4}{45}x^2 \left(x^4 + 10x^2 + \frac{23}{2}\right), & a_2(6, x) &= \frac{8}{15}x \left(x^4 + 10x^2 + \frac{23}{2}\right), \\
 a_1(7, x) &= \frac{8}{315}x \left(x^6 + \frac{35}{2}x^4 + 49x^2 + \frac{45}{4}\right), & a_2(7, x) &= \frac{8}{45} \left(x^6 + \frac{35}{2}x^4 + 49x^2 + \frac{45}{4}\right), \\
 a_1(8, x) &= \frac{2}{315}x^2(x^2 + 6)(x^4 + 22x^2 + 22), & a_2(8, x) &= \frac{16}{315}x(x^2 + 6)(x^4 + 22x^2 + 22).
 \end{aligned}$$

The polynomials $a_1(n, x)$ and $a_2(m, x)$ satisfy the following properties.

Proposition 3.5 (a) *For all $n > 0$, there is an identity of polynomials*

$$n \cdot a_1(n, x) = x \cdot a_2(n, x).$$

In particular, $nA(m, n) = mA(n, m)$ is a symmetric function on \mathbb{N}^2 .

(b) *The polynomial*

$$a_2(m, x) = \frac{2^m}{(m-1)!} \cdot x^{m-1} + \dots + \left(1 + (-1)^{m-1}\right)$$

of degree $(m-1)$ is even or odd. Its value at $x = 1$ is $a_2(m, 1) = 2m$.

Proof of Proposition 3.5 The ν -th summand of the sum defining $a_1(n, -x)$ is

$$\frac{1}{\nu!(n-\nu)!}(-x-1+\nu) \cdots (-x+1)(-x) \cdot (-x)(-x-1) \cdots (-x-n+\nu+1).$$

This equals

$$\frac{(-1)^\nu \cdot x}{n} \cdot \binom{n}{\nu} \left[\begin{matrix} x - \nu + n - 1 \\ n - 1 \end{matrix} \right],$$

which up to the factor $\frac{(-1)^\nu x}{n}$ is the ν -th summand of $a_2(n, x)$. This implies $(-1)^\nu n \cdot a_1(n, -x) = x \cdot a_2(n, x)$. Property (iii) of Proposition 2.1, i.e. $p(n, x) = p(n, -x)$, is inherited by the images $E_1^2 p(n, x) = a_1(n, x)^2$ under the summation operator E_1^2 . So $a_1(n, x)$ is either even or odd, depending on whether its degree is even or odd. Part (a) follows. Accordingly, $a_2(n, -x) = (-1)^{n-1} a_2(n, x)$ is even or odd. The leading term of the polynomial $\left[\begin{matrix} x \\ m-1 \end{matrix} \right]$ is $\frac{1}{(m-1)!} x^{n-1}$. So the leading term of $a_2(m, x)$ is

$$\frac{x^{m-1}}{(m-1)!} \cdot \sum_{\nu=0}^m \binom{m}{\nu} = \frac{2^m}{(m-1)!} \cdot x^{m-1}.$$

The evaluation of $a_2(m, x)$ at $x = 0$ only has constituents from $\nu = 0$ and m , so we obtain $a_2(m, 0) = 1 + (-1)^{m-1}$. Similarly, at $x = 1$ only the terms for $\nu = 0$ and 1 are nonzero, and we obtain $a_2(m, 1) = 2m$. □

In the rest of this section, we prove Theorem 3.2. We start with a lemma.

Lemma 3.6 Define the numbers

$$a(v, \mu, n, m) = \binom{m-1+v}{v} \binom{m}{n-v} \binom{m-1+n-\mu}{n-\mu} \binom{m}{\mu}. \quad (11)$$

For $m > n$, they satisfy

$$a(v, \mu, n, m) = b(v, \mu, n, m),$$

where $b(v, \mu, n, m)$ abbreviates the following sum of four terms

$$\begin{aligned} & D_m(v+1, m-1-\mu) \cdot D_m(n-\mu+1, m-1-n+v) \\ & + (1-\delta_{v,n}) \cdot (1-\delta_{\mu,n}) \cdot D_m(v+1, m-1-\mu) \cdot D_m(n-\mu, m-n+v) \\ & + (1-\delta_{v,0}) \cdot (1-\delta_{\mu,0}) \cdot D_m(v, m-\mu) \cdot D_m(n-\mu+1, m-1-n+v) \\ & + (1-\delta_{v,0}) \cdot (1-\delta_{\mu,0}) \cdot (1-\delta_{v,n}) \cdot (1-\delta_{\mu,n}) \\ & \cdot D_m(v, m-\mu) \cdot D_m(n-\mu, m-n+v). \end{aligned}$$

Proof of Lemma 3.6 Using the definition of $D_m(\alpha+1, \beta)$ (see 2.4), the lemma follows by straight forward calculations, distinguishing the cases v, μ equal to 0, n , or generic. We exemplify this in the generic case $0 < v, \mu < n$. The sum $b(v, \mu, n, m)$ here is

$$\begin{aligned} & \frac{m^2}{(m+v-\mu)^2} \binom{m+v}{v} \binom{m-1}{m-1-\mu} \binom{m+n-\mu}{n-\mu} \binom{m-1}{m-1-n+v} \\ & + \frac{m^2}{(m+v-\mu)^2} \binom{m+v}{v} \binom{m-1}{m-1-\mu} \binom{m+n-1-\mu}{n-1-\mu} \binom{m-1}{m-n+v} \\ & + \frac{m^2}{(m+v-\mu)^2} \binom{m+v-1}{v-1} \binom{m-1}{m-\mu} \binom{m+n-\mu}{n-\mu} \binom{m-1}{m-1-n+v} \\ & + \frac{m^2}{(m+v-\mu)^2} \binom{m+v-1}{v-1} \binom{m-1}{m-\mu} \binom{m+n-1-\mu}{n-1-\mu} \binom{m-1}{m-n+v}. \end{aligned}$$

Summing the first and the second lines as well as the third and fourth, we obtain

$$\begin{aligned} & \frac{m}{(m+v-\mu)} \left[\binom{m+v}{v} \binom{m-1}{\mu} + \binom{m+v-1}{v-1} \binom{m-1}{m-\mu} \right] \\ & \binom{m}{n-v} \binom{m+n-1-\mu}{n-\mu}, \end{aligned}$$

which simplifies to $a(v, \mu, n, m)$. \square

Proof of Theorem 3.2 We show (6) at the integer points $x = m > n$. Then, both, $E_1^2 p(n, x)$ and $a_1(n, x)^2$ being polynomials, coincide. Let $m > n$ be an integer. By

changing the summation index, we obtain

$$a_1(n, m)^2 = \left(\sum_{\nu=0}^n \binom{m-1+\nu}{n-1} \binom{m}{n-\nu} \right) \cdot \left(\sum_{\mu=0}^n \binom{m-1+n-\mu}{m-1} \binom{m}{\mu} \right).$$

Hence, $a_1(n, m)^2 = \sum_{\nu, \mu=0}^n a(\nu, \mu, n, m)$ for the numbers (11). On the other hand, using the notation of Remark 2.5(iii),

$$\begin{aligned} E_1^2 p(n, m) &= \sum_{\nu, \mu=0}^n D_m(\nu+1, m-1-\mu) \cdot D_m(n-\mu+1, m-1-n+\nu) \\ &+ 2 \cdot \sum_{\nu, \mu=0}^{n-1} D_m(\nu+1, n-1-\mu) \cdot D_m(n-1-\mu+1, m-1-(n-1)+\nu) \\ &+ \sum_{\nu, \mu=0}^{n-2} D_m(\nu+1, m-1-\mu) \cdot D_m(n-2-\mu+1, m-1-(n-2)+\nu). \end{aligned}$$

Index shifts $\nu \mapsto \nu+1, \mu \mapsto \mu+1$ in the last sum and in one of the second sums yield

$$\begin{aligned} E_1^2 p(n, m) &= \sum_{\nu, \mu=0}^n D_m(\nu+1, m-1-\mu) \cdot D_m(n-\mu+1, m-1-n+\nu) \\ &+ \sum_{\nu, \mu=0}^{n-1} D_m(\nu+1, m-1-\mu) \cdot D_m(n-\mu, m-n+\nu) \\ &+ \sum_{\nu, \mu=1}^n D_m(\nu, m-\mu) \cdot D_m(n-\mu+1, m-1-n+\nu) \\ &+ \sum_{\nu, \mu=1}^{n-1} D_m(\nu, m-\mu) \cdot D_m(n-1-\mu+1, m-n+\nu). \end{aligned}$$

Using this, equation (6) now follows from Lemma 3.6. The identity $a_1(n, m) = a_2(m, n)$ for integers $m, n > 0$ becomes obvious after substituting $\nu \mapsto n-\nu$ in $a_1(n, m)$, noticing that in both sums the summands are actually zero for $\nu > \min(m, n)$. Since $a_2(n, m) = A(n, m)$ by (2), formula (4) follows from (6). Then, equation (7) holds for all integers $x = n > 0$ and also for all $m > 0$ due to the symmetry relation satisfied by $p(n, m)$. Exchanging m and n , we obtain (5). Therefore, both being polynomials of degree $2(m-1)$ in x , $a_2(m, x)^2$ and $E_2^2 p(m, x)$ coincide. \square

4 Mixed summation operators

Proposition 4.1 *There exists a unique family of polynomials*

$$\{d(n, x) \in \mathbb{Q}[x] \mid n \in \mathbb{N}\}$$

satisfying the following properties:

- (i) For all n , the degree of the polynomial $d(n, x)$ is $n - 1$.
- (ii) For all n , the leading coefficient of $d(n, x)$ is $\frac{2^{n-1}}{(n-1)!}$.
- (iii) The function $d(n, m)$ is symmetric on $\mathbb{N} \times \mathbb{N}$, i.e., $d(n, m) = d(m, n)$.

Proof The conditions imply $d(1, x) = 1$. By recursion, let $d(m, x)$ be determined for all $1 \leq m < n$. By (i) and (ii), the polynomial $d(n, x)$ has degree $n - 1$ and a fixed leading coefficient. By (iii), the identities $d(n, m) = d(m, n)$ determine $n - 1$ equations for the $n - 1$ missing coefficients of $d(n, x)$. So they determine $d(n, x)$ uniquely. \square

We formally put $d(0, x) = 0$ and obtain

$$\begin{aligned} d(1, x) &= 1, \\ d(2, x) &= 2\left(x - \frac{1}{2}\right), \\ d(3, x) &= 2\left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right), \\ d(4, x) &= \frac{4}{3}\left(x - \frac{1}{2}\right)\left(\left(x - \frac{1}{2}\right)^2 + \frac{5}{4}\right), \\ d(5, x) &= \frac{2}{3}\left(\left(x - \frac{1}{2}\right)^4 + \frac{7}{2}\left(x - \frac{1}{2}\right)^2 + \frac{9}{16}\right), \\ d(6, x) &= \frac{4}{15}\left(x - \frac{1}{2}\right)\left(\left(x - \frac{1}{2}\right)^4 + \frac{15}{2}\left(x - \frac{1}{2}\right)^2 + \frac{89}{16}\right), \\ d(7, x) &= \frac{4}{45}\left(\left(x - \frac{1}{2}\right)^2 + \frac{9}{4}\right)\left(\left(x - \frac{1}{2}\right)^4 + \frac{23}{2}\left(x - \frac{1}{2}\right)^2 + \frac{25}{16}\right), \\ d(8, x) &= \frac{8}{315}\left(x - \frac{1}{2}\right)\left(\left(x - \frac{1}{2}\right)^6 + \frac{91}{4}\left(x - \frac{1}{2}\right)^4 + \frac{1519}{16}\left(x - \frac{1}{2}\right)^2 + \frac{3429}{64}\right). \end{aligned}$$

Proposition 4.2 *The polynomials $d(n, x)$ defined by Proposition 4.1 have the following properties for all $n \geq 1$:*

- (a) *Tribonacci identity:* $d(n, x) = d(n, x - 1) + d(n - 1, x - 1) + d(n - 1, x)$.
- (b) $d(n, x)$ in $x = m$ is the Delannoy number $D(n - 1, m - 1)$; see [11, A008288].
- (c) *Functional equation:* $d(n, 1 - x) = (-1)^{n-1}d(n, x)$.

Proof of Proposition 4.2 (a) The family of polynomials $v(0, x) = 0$ and, for $n \geq 1$,

$$v(n, x) = d(n, x - 1) + d(n - 1, x - 1) + d(n - 1, x)$$

Table 4 Delannoy array $D(n - 1, m - 1)$ of the function $d : \mathbb{N} \times \mathbb{Z} \rightarrow \mathbb{Z}$

| n | m | | | | | | | | | | | |
|-----|-------|-------|------|------|-----|----|---|----|-----|-----|------|------|
| | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | -11 | -9 | -7 | -5 | -3 | -1 | 1 | 3 | 5 | 7 | 9 | 11 |
| 3 | 61 | 41 | 25 | 13 | 5 | 1 | 1 | 5 | 13 | 25 | 41 | 61 |
| 4 | -231 | -129 | -63 | -25 | -7 | -1 | 1 | 7 | 25 | 63 | 129 | 231 |
| 5 | 681 | 321 | 129 | 41 | 9 | 1 | 1 | 9 | 41 | 129 | 321 | 681 |
| 6 | -1683 | -681 | -231 | -61 | -11 | -1 | 1 | 11 | 61 | 231 | 681 | 1683 |
| 7 | 3653 | 1289 | 377 | 85 | 13 | 1 | 1 | 13 | 85 | 377 | 1289 | 3653 |
| 8 | -7183 | -2241 | -575 | -113 | -15 | -1 | 1 | 15 | 113 | 575 | 2241 | 7183 |

obviously satisfies the properties of Proposition 4.1. Hence, $v(n, x) = d(n, x)$.

(b) Notice that the Delannoy numbers $D(n - 1, m - 1)$ are determined by the tribonacci identity and the conditions $D(n - 1, 0) = D(0, m - 1) = 1$, in agreement with $d(1, x) = 1$ and the symmetry property Proposition 4.1(iii).

(c) Notice that, by definition,

$$b(n, x) := (-1)^{n-1}d(n, 1 - x) - d(n, x)$$

satisfies $b(n, 1 - x) = (-1)^{n-1}b(n, x)$. It suffices to show

$$b(n, -x) = (-1)^n \cdot b(n, x). \tag{12}$$

Because, since in this case $b(n, 1 + x) = -b(n, x)$, and by considering the leading term, we obtain $b(n, x) = 0$ and hence (c). Equation (12) is obvious for $n = 0$. For $n \geq 1$, we compute

$$\begin{aligned} (-1)^n b(n, x) - b(n, -x) &= (-1)^n (d(n, 1 + x) - d(n, x)) \\ &\quad - (d(n, 1 - x) - d(n, -x)). \end{aligned} \tag{13}$$

Using part (a) for $1 + x$, respectively, $-x$, instead of x , the right-hand side of (13) becomes

$$\begin{aligned} &(-1)^n (d(n, 1 + x) + d(n - 1, x)) - (d(n - 1, -x) + d(n, 1 - x)) \\ &= b(n - 1, -x) - (-1)^{n-1}b(n - 1, x). \end{aligned}$$

In here, the last term is zero by induction hypothesis. So (12) is proved in general. \square

The proof of Proposition 4.2(c) essentially amounts to extending the Delannoy array (Table 4) from $\mathbb{N} \times \mathbb{N}$ to non-positive m by a reflection along the line $x = \frac{1}{2}$. Thereby, it introduces signs so that the tribonnaci rule of Proposition 4.2(a) remains valid.

Proposition 4.3 For all $n \in \mathbb{N}$, the polynomials $d(n, x)$ of Proposition 4.1 and the polynomials $p(n, x)$ of Proposition 2.1 satisfy the polynomial identities

$$p(n, x) + p(n, x - 1) + p(n - 1, x) + p(n - 1, x - 1) = 2 \cdot d(n, x)^2.$$

Proof of Proposition 4.3 It is easy to see that, for $n \in \mathbb{N}$, there is a unique family of polynomials $q(n, x)$ in $\mathbb{Q}[x]$ satisfying the following properties:

- (i) For all n , the degree of the polynomial is $\deg_x q(n, x) = 2(n - 1)$.
- (ii) For all n , the leading coefficient of $q(n, x)$ is $\frac{2^{2n-1}}{(n-1)!^2}$.
- (iii) $q(n, x) = q(n, 1 - x)$ holds for all $n \in \mathbb{N}$.
- (iv) The function $q(n, m)$ is symmetric on $\mathbb{N} \times \mathbb{N}$, i.e., $q(n, m) = q(m, n)$.

Obviously, the family $2d(n, x)^2$ satisfies these axioms. Hence, $q(n, x) = 2d(n, x)^2$.

The polynomials $p(n, x)$ satisfy the properties of Proposition 2.1. By Remark 2.5(i), we know $\deg_x p(n, x) = 2(n - 1)$. Hence, $p(n, x) + p(n, x - 1) + p(n - 1, x) + p(n - 1, x - 1)$ satisfies the preceding properties (i), (iii), and (iv). By Proposition 2.3, the leading coefficient of $p(n, x)$ can easily be determined to be

$$\sum_{v, \mu=0}^{n-1} \frac{1}{v!(n-1-v)!\mu!(n-1-\mu)!} = \left(\frac{2^{n-1}}{(n-1)!} \right)^2.$$

Therefore, the leading coefficient of $p(n, x) + p(n, x - 1) + p(n - 1, x) + p(n - 1, x - 1)$ is twice this number, and therefore equal to $\frac{2^{2n-1}}{(n-1)!^2}$. So we have found a second solution of $q(n, x)$. The proposition follows. □

Remarks on the Delannoy numbers $D(n, m)$. It is known that

$$D(n, m) = \sum_{k=0}^{\min(n,m)} 2^k \binom{n}{k} \binom{m}{k} = \sum_{k=0}^{\min(n,m)} \binom{n}{k} \binom{m+n-k}{n},$$

with generating series $\sum_{n,m} D(n, m)x^n y^m = \frac{1}{1-x-y-xy}$. See, e.g., [11,14,15]. This in particular implies

$$d(n, x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \left[\begin{matrix} x-k+n-2 \\ n-1 \end{matrix} \right],$$

and relates the numbers $A(n, m)$ and the Delannoy numbers $D(n, m)$ by the identity

$$A(n, m) = D(n, m) - D(n, m - 1),$$

or equivalently, $A(n, m) = D(n - 1, m) + D(n - 1, m - 1)$. Indeed, these relations are equivalent to

$$a_2(n - 1, x) = d(n, x + 1) - d(n, x),$$

and the latter follows from $\begin{bmatrix} x - k + n - 1 \\ n - 1 \end{bmatrix} - \begin{bmatrix} x - k + n - 2 \\ n - 1 \end{bmatrix} = \begin{bmatrix} x - k + n - 2 \\ n - 2 \end{bmatrix}$.
 From this, we obtain the alternative presentation

$$A(n, m) = \sum_{k=1}^{\min(n,m)} 2^k \binom{n}{k} \binom{m-1}{k-1}.$$

For the relation of $A(n, m)$ with diamond numbers, see [15].

5 Central values

Using the summation operator $Ef(x) = f(x + \frac{1}{2}) + f(x - \frac{1}{2})$, Proposition 4.3 can be reformulated as

$$2 \cdot d(n, x + \frac{1}{2})^2 = E(p(n, x) + p(n - 1, x)). \tag{14}$$

This suggests that our families of polynomials have interesting properties at half-integral places. We illustrate this by determining their values at $x = \frac{1}{2}$.

Proposition 5.1 (a) *For the polynomials $a_1(n, x)$ and $a_2(n, x)$, the values at $x = \frac{1}{2}$ are given by*

$$a_1\left(2m, \frac{1}{2}\right) = a_1\left(2m + 1, \frac{1}{2}\right) = \begin{bmatrix} m - \frac{1}{2} \\ m \end{bmatrix},$$

respectively,

$$a_2\left(n, \frac{1}{2}\right) = 2n \cdot a_1\left(n, \frac{1}{2}\right).$$

(b) *For all $m \in \mathbb{N}_0$, define the rational number*

$$r(m) = \sum_{k=0}^m \left[\begin{matrix} k - \frac{1}{2} \\ k \end{matrix} \right]^2 = \sum_{k=0}^m \left(\frac{1}{2^{2k}} \binom{2k}{k} \right)^2.$$

Then, the values at $x = \frac{1}{2}$ of the polynomials $P(n, x)$ of Proposition 2.1 are given by the recursion formula

$$p\left(2m + 1, \frac{1}{2}\right) = r(m) = (-1) \cdot p\left(2m + 2, \frac{1}{2}\right). \tag{15}$$

(c) *The values at $x = \frac{1}{2}$ of the polynomials $d(n, x)^2$ are*

$$d\left(2m, \frac{1}{2}\right) = 0, \text{ respectively, } d\left(2m + 1, \frac{1}{2}\right) = \begin{bmatrix} m - \frac{1}{2} \\ m \end{bmatrix}.$$

By the set of initial values of Proposition 5.1, and by the tribonacci identities satisfied by $a_2(n, x)$ and $d(n, x)$, we obtain recursion formulas for their values at $x = \frac{2k+1}{2}$. Identity (7) then gives a recursion for the values $p(n, \frac{2k+1}{2})$.

Proof of Proposition 5.1 (a) We first notice that

$$\left[\begin{matrix} m - \frac{1}{2} \\ m \end{matrix} \right] = \frac{1}{2^{2m}} \binom{2m}{m} = (-1)^m \left[\begin{matrix} -\frac{1}{2} \\ m \end{matrix} \right].$$

So the generating series of $\left[\begin{matrix} m - \frac{1}{2} \\ m \end{matrix} \right]$ is given by the Taylor series

$$\sum_{m=0}^{\infty} x^m \left[\begin{matrix} m - \frac{1}{2} \\ m \end{matrix} \right] = \sum_{m=0}^{\infty} (-x)^m \left[\begin{matrix} -\frac{1}{2} \\ m \end{matrix} \right] = (1 - x)^{-\frac{1}{2}},$$

whereas

$$(1 + x)^{\frac{1}{2}} = \sum_{m=0}^{\infty} x^m \left[\begin{matrix} \frac{1}{2} \\ m \end{matrix} \right].$$

By Cauchy product expansion, we obtain the generating series for $a_1(n, \frac{1}{2})$,

$$\sqrt{\frac{1+x}{1-x}} = \sum_{n=0}^{\infty} x^n \sum_{\nu=0}^n \left[\begin{matrix} \frac{1}{2} \\ \nu \end{matrix} \right] \left[\begin{matrix} n - \nu - \frac{1}{2} \\ n - \nu \end{matrix} \right] = \sum_{n=0}^{\infty} x^n a_1\left(n, \frac{1}{2}\right).$$

On the other hand, from

$$\frac{1}{\sqrt{1-x^2}} = \sum_{m=0}^{\infty} x^{2m} \left[\begin{matrix} m - \frac{1}{2} \\ m \end{matrix} \right]$$

we obtain

$$\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}} = \sum_{m=0}^{\infty} (x^{2m} + x^{2m+1}) \left[\begin{matrix} m - \frac{1}{2} \\ m \end{matrix} \right].$$

Comparing coefficients, the first identity of part (a) is proved. For the second one, recall $na_1(n, x) = xa_2(n, x)$ from Proposition 3.5.

(b) From the list of $p(n, x)$ following Proposition 2.1, we obtain $p(1, \frac{1}{2}) = 1 = r(0) = -p(2, \frac{1}{2})$ and $p(3, \frac{1}{2}) = \frac{5}{4} = r(1) = p(4, \frac{1}{2})$. By induction, assuming (15) holds true for all $m < M$, and by (6),

$$p\left(2M + 1, \frac{1}{2}\right) + 2p\left(2M, \frac{1}{2}\right) + p\left(2M - 1, \frac{1}{2}\right) = a_1\left(2M, \frac{1}{2}\right)^2,$$

we conclude

$$p\left(2M + 1, \frac{1}{2}\right) = \left[\begin{matrix} M - \frac{1}{2} \\ M \end{matrix} \right]^2 + 2r(M - 1) - r(M - 1) = r(M).$$

Similarly, we deduce $p(2M + 2, \frac{1}{2}) = -r(M)$.

(c) By (14), we obtain

$$2 \cdot d\left(n, \frac{1}{2}\right)^2 = p\left(n, \frac{1}{2}\right) + p\left(n, -\frac{1}{2}\right) + p\left(n - 1, \frac{1}{2}\right) + p\left(n - 1, -\frac{1}{2}\right).$$

Since the polynomials $p(n, x)$ are even functions, we get

$$d\left(n, \frac{1}{2}\right)^2 = p\left(n, \frac{1}{2}\right) + p\left(n - 1, \frac{1}{2}\right).$$

By part (b), this is zero in case $n = 2m$, whereas in case $n = 2m + 1$ we obtain

$$d\left(2m + 1, \frac{1}{2}\right)^2 = r(m) - r(m - 1) = \left[\begin{matrix} m - \frac{1}{2} \\ m \end{matrix} \right]^2.$$

In particular, by interpolating the Delannoy numbers, the $d(2m + 1, x)$ are positive. This implies part (c). Notice that $d(2m, \frac{1}{2}) = 0$ follows from Proposition 4.2(c). \square

Remark. The polynomials $a_1(n, x)$ and other polynomial interpretations of the number families in the proof above occur as Mellin transforms of Laguerre functions in [3]. By [3, Thm 4], their zeros lie on the line $x = \frac{1}{2}$.

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Data availability All data needed to verify the statements of this article, or on which this article relies on, are included or cited in this manuscript.

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