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Abelian permutation groups with graphical representations

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Abstract
In this paper we characterize those automorphism groups of colored graphs and digraphs that are abelian as abstract groups. This is done in terms of basic permutation group properties. Using Schur's classical terminology, what we provide is characterizations of the classes of 2-closed and 2^*-closed abelian permutation groups. This is the first characterization concerning these classes since they were defined.

Keywords colored graph · Automorphism group · Permutation group · Abelian group · 2-closed

1 Introduction

This paper is motivated by the problem called by Babai [5, p.52] the *concrete representation problem*. Recall that König's problem for groups asked which finite groups are the automorphism groups of (simple) graphs. This question, in its abstract version, was quickly answered by Frucht who showed that every group is *isomorphic* to the automorphism group of some graph. The concrete version, of more combinatorial flavor, asks which finite *permutation groups* are the automorphism groups of graphs. This problem turns out to be much harder (see [5,10]). In the abstract version we look for a graph with an arbitrary number of vertices whose automorphism group is *isomorphic* to a given (abstract) group. In the concrete version we are given a permutation group (G, X) acting on a set X of elements and we are looking for a graph (X, E) with the same set X of vertices whose automorphisms *are precisely* the permutations in G.

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There are related active areas of research concerning graphical representations of abstract groups, Cayley graphs and automorphism groups of circulant graphs (see [4,5,9,26] for the beginnings, and [1,2,7,8,17,27–30] to mention just a few most recent results). All these are closely connected with the problem considered in this paper. Yet, we must emphasize that we approach the topic from a different point of view. Our main interest is in *permutation group* structure and classifying permutation groups by their natural representations as the automorphism groups of colored graphs. This relates directly to the direction of research suggested in Wielandt's [31]. Many natural applications of permutation groups, especially in fields outside mathematics, concern the ways they act rather than their group structure (cf. [21–23]). Moreover, we are particularly interested in intransitive permutation groups, as they reflect the way how graphs that fail to be vertex transitive are composed from transitive components. Now, while transitive permutation groups are understood pretty well (mainly due to research connected with the classification of finite simple groups), very little is known, in fact, about intransitive groups and their complex actions on different orbits. Note that most graphs are not vertex transitive and the action of the automorphism group on different orbits may be very different and relate closely to various graph properties.

It is easy to see that some permutation groups, like the alternating groups A_n on n elements or the groups C_n generated by the cyclic permutation $(1, 2, \dots, n)$, are not the groups of automorphisms of any graph (on n elements). For a long time there was no progress in the concrete version of König's problem. Only research on the so-called *Graphical Regular Representation* of groups (GRR) followed a more systematic approach to the problem. The final result of this extensive study by Godsil [9], even if it concerned representations of abstract groups, may be interpreted as the description of those *regular permutation groups* that are the automorphism groups of graphs. The analogous result for automorphism groups of directed graphs has been obtained by Babai [4].

The next natural class of permutation groups to study from this point of view is the class of cyclic permutation groups, that is, those generated by a single permutation. König's problem for this class turned out not so easy as it could seem at the first sight. After some partial results containing errors and wrong proofs ([24,25] corrected in [11]), the final result has been obtained only recently [15,16]. The full description turns out to consist of seven technical conditions concerning possible lengths of the orbits.

To obtain this description, we have applied the aforementioned Wielandt's approach to start from considering the invariance groups of families of binary relations rather than the automorphism groups of simple graphs. In the language of graphs these are the automorphism groups of (edge) colored graphs. This approach is more natural. So, first, in [15], we have obtained an easy-to-formulate result that a cyclic permutation group G is the automorphism group of a colored graph if and only if for every nontrivial orbit O of G there exists another orbit Q such that $\gcd(|O|, |Q|) \geq 3$. The proof yields also the result that if a cyclic permutation group is the automorphism group of a colored graph, then it is the automorphism group of a colored graph that uses at most 3 colors. Only then may one consider for which cyclic permutation groups 2 colors suffice, which turns out to have a rather technical solution (see [16]). There are many other results in the area showing that considering edge-colored graphs rather than simple graphs is the right approach (see [21]).

The next natural class to attack in the concrete version of König's problem is that of abelian permutation groups. The survey [26] reports a result by Zelikovskij [32] where the solution of König's problem for a large class of abelian permutation groups (namely, those whose order is not divisible by 2, 3 or 5) is provided. There is no English translation of [32], so the survey quotes only the English summary. The restriction means that the lengths of the orbits must be relatively prime to 30, and its apparent aim is to avoid technical complications. Worse, the result as stated is false. In [14] we demonstrate a counterexample and point out the false algebraic assumption used by Zelikovskij in the proof.

In this paper we make the first step to find a correct characterization of the abelian permutation groups that are the automorphism groups of graphs. As before, we start from characterizing those abelian permutation groups that are the automorphism groups of colored graphs and digraphs. Again, it turns out that this can be done in a quite nice way in terms of basic properties of permutation groups.

Our characterizations use a technical notion specific to intransitive permutation groups. For a given permutation group G we say that a permutation σ is 2-orbit compatible with G if for every pair of orbits O and Q of G there is a permutation $\sigma' \in G$ such that σ and σ' have identical actions on $O \cup Q$. The group G is *2-orbit-closed* if every permutation that is 2-orbit-compatible with G belongs to G. Every transitive permutation group, or having just two orbits, is trivially 2-orbit-closed, but permutation groups containing more than 2 orbits may not be.

It is not difficult to observe that each automorphism group of an edge-colored directed graph is necessarily 2-orbit-closed. In this paper, we prove that for abelian groups this condition is also sufficient: an abelian permutation group A is the automorphism group of an edge-colored directed graph if and only if A is 2-orbit-closed (Theorem 3). Moreover, we prove that an abelian permutation group A is the automorphism group of an edge-colored (simple) graph if and only if A is 2-orbit-closed and satisfies an additional condition concerning groups induced by A on its orbits (Theorem 2). Our main tool in proving these results is the subdirect sum decomposition of intransitive groups, which is recalled for convenience of the reader in Sect. 2.1.

2 Terminology

For standard notions and terminology of permutation groups see, e.g., [6]. We use the notation (G, X) to denote a permutation group G acting on a finite set X. Permutation groups are considered up to permutation isomorphism, i.e., two groups that are permutation isomorphic (in the sense of [6, p. 17]) are treated as identical. In particular we usually assume that $X = \{1, 2, \ldots, n\}$, and by S_n and A_n we denote the full *symmetric group* and the *alternating group*, respectively, acting on X. By C_n we denote the subgroup of S_n generated by the cyclic permutation $(1, 2, \ldots, n)$. By I_n we denote the *trivial* permutation group acting on n elements, that is, the subgroup of S_n containing the identity permutation only.

A *k-colored digraph* $\Gamma = (X, \gamma)$ is a set X (of vertices) with a function $\gamma : X \times X \to \{0, 1, \ldots, k-1\}$. If γ is a function from the unordered pairs $P_2(X)$ of the points of X to $\{0, 1, \ldots, k-1\}$, then Γ is called a *k-colored graph*. They may be viewed as the complete digraph or the complete graph, respectively, on a set X, whose edges are colored with at most k different colors. In the case $k = 2$, they may be identified with simple digraphs and graphs, with edges colored 1, and nonedges colored 0.

The i-th *degree* of a vertex $x \in X$, denoted $d_i(x)$, is the number of edges in color i incident with x. The k-tuple $(d_0(x), \ldots, d_{k-1}(x))$ is referred to as the k-tuple of color degrees of x. In colored digraphs we may distinguish also outdegrees and indegrees, and the corresponding k-tuples.

A permutation σ of X is an *automorphism* of $\Gamma = (X, \gamma)$, if it preserves the colors of edges in Γ. Obviously, each automorphism preserves also the k-tuples of color degrees of vertices. The automorphisms of Γ form a permutation group, which is denoted $\mathrm{Aut}(\Gamma)$. We say also that Γ *represents* (graphically) the permutation group $\mathrm{Aut}(\Gamma)$. We note that not every permutation group is representable by a colored graph or digraph. For example, the alternating group A_n is not representable for any $n > 3$ (neither by a colored graph nor by a digraph). The cyclic group C_n is not representable by a colored graph (for any $n > 2$), but it is representable by a (2-colored) digraph.

Given a permutation group (G, X), by $\mathrm{Orb}(G, X) = \mathrm{Orb}(G)$ we denote the colored digraph in which two edges have the same color if and only if they belong to the same orbit of G in its action on $X \times X$ (i.e., *orbital*). Similarly, by $\mathrm{Orb}^*(G, X) = \mathrm{Orb}^*(G)$ we denote the colored graph in which two edges have the same color if and only if they belong to the same orbit of G in its action on $P_2(X)$.

It is easy to see that $\mathrm{Aut}(\mathrm{Orb}^*(G)) \supseteq \mathrm{Aut}(\mathrm{Orb}(G)) \supseteq G$ as groups of permutations over X. The first group is called the 2^*-*closure* of G, while the second group is called the 2-*closure* of G. When a permutation group happens to be equal to its 2^*-closure or 2-closure, then it is called 2^*-closed or 2-closed, respectively. These groups are the largest permutation groups with the given set of orbits on $P_2(X)$ or on $X \times X$, respectively (see [5,31]; according to Wielandt [31] the notion of 2-closure as a tool in the study of permutation groups was introduced by I. Schur.

If G is not 2^*-closed, then G is not the automorphism group of any (colored) graph. Otherwise, there may be various colored graphs Γ such that $G = \mathrm{Aut}(\Gamma)$. Yet, each such graph can be obtained from $\mathrm{Orb}^*(G)$ by identifying some colors. In particular, if $G = \mathrm{Aut}(\Gamma)$ for a simple graph Γ, then Γ can be obtained from $\mathrm{Orb}^*(G)$ by identifying some colors with 1 (corresponding to edges) and other colors with 0 (nonedges).

We define $GR(k)$ to be the class of all permutation groups that are automorphism groups of colored graphs using at most k colors. The union $GR = \bigcup_{k \geq 1} GR(k)$ is just the class of 2^*-closed permutation groups. Similarly, we define $DGR(k)$ as the class of all permutation groups that are automorphism groups of colored digraphs using at most k colors. The union $DGR = \bigcup_{k \geq 1} DGR(k)$ is just the class of 2-closed permutation groups.

While this seems pretty natural topic in the area of graphs and permutation groups not much has been done so far in it. The reason is that, on the one hand, the topic turned out to be rather hard, and on the other hand, the main stream of research in permutation groups was focused so far on delivering tools for the classification of finite simple groups, and this restricted research to transitive groups.

2.1 Subdirect sum decomposition

A natural tool in the study of intransitive permutation groups is the subdirect sum of permutation groups. Given two permutation groups $G \leq S_n$ and $H \leq S_m$, the *direct sum* $G \oplus H$ is the permutation group on $\{1, 2, \ldots, n+m\}$ defined as the set of permutations $\pi = (\sigma, \tau)$ such that

$$\pi(i) = \begin{cases} \sigma(i), & \text{if } i \leq n \\ n + \tau(i-n), & \text{otherwise.} \end{cases}$$

Thus, in $G \oplus H$, permutations of G and H act independently in a natural way on a disjoint union of the base sets of the summands.

We introduce the notion of the *subdirect sum* following [13] (and the notion of *intransitive product* in [20]). Let $H_1 \triangleleft; G_1 \leq S_n$ and $H_2 \triangleleft\ G_2 \leq S_m$ be permutation groups such that H_1 and H_2 are normal subgroups of G_1 and G_2, respectively. Suppose, in addition, that factor groups G_1/H_1 and G_2/H_2 are (abstractly) isomorphic and $\phi : G_1/H_1 \to G_2/H_2$ is the isomorphism mapping. Then, by

$$G = G_1[H_1] \oplus_\phi G_2[H_2]$$

we denote the subgroup of $G_1 \oplus G_2$ consisting of all permutations (σ, τ), $\sigma \in G_1$, $\tau \in G_2$ such that $\phi(\sigma H_1) = \tau H_2$. Each such group will be called a *subdirect sum* of G_1 and G_2, and denoted briefly $G_1 \oplus_\phi G_2$ (in this notation the normal subgroups H_1 and H_2 are assumed to be specified in the full description of the isomorphism ϕ).

If $H_1 = G_1$, then G_1/H_1 is a trivial (one-element) group, and consequently, G_2/H_2 must be trivial, which means that we have also $H_2 = G_2$. Then, $G = G_1 \oplus G_2$ is the usual direct sum, with ϕ being the mapping between one-element sets. In such special case the subdirect sum $\oplus_\phi$ will be referred to as *trivial*. In the case when $G_1 = G_2 = G$ and $H_1 = H_2$ is the trivial one-element subgroup of G, and $\phi : G \to G$ is the identity mapping, the subdirect sum is nontrivial. In this case we call the resulting sum a *parallel sum* (permutation isomorphic groups G_1 and G_2 act in a parallel manner on their orbits) and denote it briefly $G||G$. For example, the cyclic group generated by the permutation $\sigma = (1, 2, 3)(4, 5, 6)$ is permutation isomorphic to $C_3||C_3$.

The main fact established in [20] is that every intransitive group has the form of a subdirect sum, and its components can be easily described. Let G be an intransitive group acting on a set $X = X_1 \cup X_2$ in such a way that X_1 and X_2 are disjoint fixed blocks for G. Let G_1 and G_2 be restrictions of G to the sets X_1 and X_2, respectively (they are called also *constituents*). Let $H_1', H_2' \leq G$ be the subgroups fixing pointwise X_2 and X_1, respectively. Let $H_1 \leq G_1$ and $H_2 \leq G_2$ be the restrictions of H_1' and H_2' to X_1 and X_2, respectively. Then we have

Theorem 1 [20, Theorem 4.1] *If G is a permutation group as described above, then*

a) *H_1 and H_2 are normal subgroups of G_1 and G_2, respectively,*

b) *the factor groups* G_1/H_1 *and* G_2/H_2 *are abstractly isomorphic, and*

$$G = G_1[H_1] \oplus_\phi G_2[H_2],$$

where ϕ *is the isomorphism of the factor groups.*

3 Preliminary results

First, we establish the representability of *regular* abelian permutation groups and some other groups connected with the automorphism groups of Cayley graphs. Here, we make use of known results on the so-called *Cayley index* of abelian groups.

Recall that each regular permutation group may be viewed as the action of an abstract group G on itself given by left multiplication. In such a case we have $X = G$, and the resulting permutation group will be denoted by (G, G), or simply by G, if it is clear from the context that we mean the corresponding regular permutation group. In particular, we use standard notation Z_k^m and $Z_k^m \times Z_r^s$ for abstract abelian groups to denote also corresponding permutation groups obtained by the regular action of these groups on themselves (in particular, C_n and Z_n denote here the same permutation group).

Given an abstract group G, by $\mathrm{Cay}(G)$ we denote the complete directed colored Cayley graph, that is one with all nontrivial elements as generators defining different colors. Observe that $\mathrm{Cay}(G) = \mathrm{Orb}(G, G)$. By $\mathrm{Cay}^*(G)$ we denote the complete undirected colored graph obtained from $\mathrm{Cay}(G)$ by identifying colors corresponding to g and g^{-1} for every nontrivial $g \in G$, and removing the loops. Again, $\mathrm{Cay}^*(G) = \mathrm{Orb}^*(G, G)$.

Now, given a set S of nontrivial elements of G (i.e., different from the identity), by $\mathrm{Cay}^*(G; S)$ we define the colored graph obtained from $\mathrm{Cay}^*(G)$ by identifying all colors not in S. To admit further identifications, let Π be a partition of S. Then by $\mathrm{Cay}^*(G; \Pi)$ we define the colored graph obtained from $\mathrm{Cay}^*(G; S)$ by identifying the colors in each block of Π. In our notation applied below, Π is written simply by listing its blocks, a block is written in the square brackets, and in the case of a one-element block, brackets are omitted. To make notation as compact as possible we adopt the convention that Π contains only one representative of each pair $\{g, g^{-1}\}$. In addition, we assume that there are nontrivial pairs g, g^{-1} not in S, and all elements not in S get color 0 in diagrams represented by nonedges. Then, the graph $\mathrm{Cay}^*(G; \Pi)$ is the complete directed graph whose edges are colored with exactly $|\Pi| + 1$ colors.

The following lemma presents the colored graphs whose automorphism groups are (Z_2^k, Z_2^k) for $k = 2, 3, 4$. (The k-tuples of elements of Z_2^k are denoted below by corresponding strings of 0's and 1's.)

Lemma 1 *Each of the following colored graphs represents the regular action of its defining group:*

(i) $\mathrm{Cay}^*(Z_2^2; 10, 01)$,
(ii) $\mathrm{Cay}^*(Z_2^3; 100, 010, 001)$,
(iii) $\mathrm{Cay}^*(Z_2^4; 1000, [0100, 1010], [0010, 0001])$

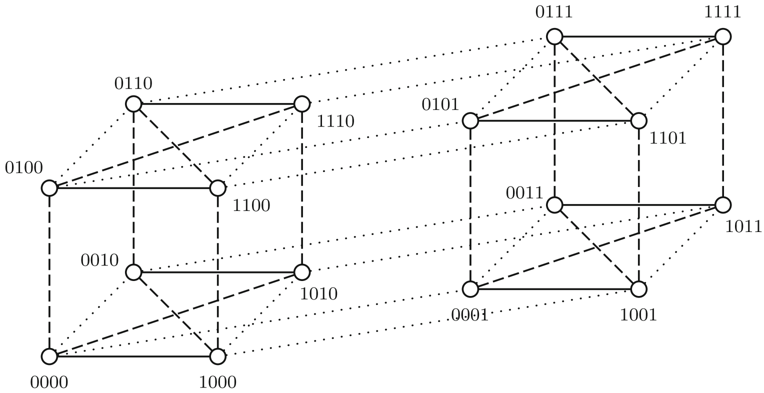

Fig. 1 Cay$^*(Z_2^4$; 1000, [0100, 1010], [0010, 0001])

Proof We consider the case (iii). Denote the graph by Γ. It is pictured in Fig. 1. (Solid, dashed, and dotted lines correspond to colors 1000, 0100, and 0010, respectively). We will speak correspondingly of solid, dashed, and dotted neighbors.

Since Γ is obtained from a Cayley graph on Z_2^4 by identifying colors, it follows that its automorphism group Aut(Γ) contains the regular action of Z_2^4 (which in this notation is given by addition). We need only to prove that Aut(Γ) contains no other permutation. Let us consider the stabilizer A_0 of the vertex 0000 in Aut(Γ). As the latter is transitive, it is enough to show that A_0 is trivial.

Since the only solid neighbor of 0000 is 1000, A_0 fixes 1000 as well. Further, the only dashed neighbor of 0000 that is a dotted neighbor of 1000 is 1010, while the only solid neighbor of the latter is 0010. Thus the four vertices with coordinates $x0y0$ are fixed. Considering their dashed neighbors, we see that also each vertex with coordinates $x1y0$ must be fixed. It follows that the vertices of the cube $xyz0$ are individually fixed. Considering their dotted neighbors, we conclude that the same holds for the cube $xyz1$, which completes the proof.

The cases (i) and (ii) are easier and are left to the reader as an exercise. □

Now, recall that an abelian permutation group (A, X) is transitive if and only if it is regular. It follows that a transitive abelian permutation group A can be identified with the regular action of A (considered as an abstract group) on itself (cf. [6]). In this case, we have a special permutation on A defined by $\alpha : x \rightarrow x^{-1}$ called the *involution*. (For properties and a very special role of this permutation see, e.g., [7,19]). It is easy to observe that the involution preserves the colors of the edges in Cay$^*(A)$. This leads to the well-known fact:

Lemma 2 *Le A be a regular abelian permutation group, and α its involution. If Γ is a colored graph such that* Aut(Γ) $\supseteq A$*, then* $\alpha \in$ Aut(Γ).

This is so since Cay$^*(A) =$ Orb(A, A), and Γ needs to be a graph obtained from Orb(A, A) by identification of colors.

It follows from this lemma that generally a regular abelian permutation group A does not belong to GR, except for the case when α is trivial (the identity permutation).

This is exactly the case, when $A = Z_2^n$ for some $n \geq 0$. It is well known that for $n \geq 5$, Z_2^n is representable as the automorphism group of a simple (Cayley) graph (see [18], or claim 1.2 in [19]). Combining this with Lemma 1 we have

Lemma 3 *Let A be a regular abelian permutation group. If $A = Z_2^n$ for some $n \geq 0$, then $A \in GR(4)$; otherwise, $A \notin GR$.*

We note that Z_2^3 requires 4 colors, in the sense that there exists no k-colored graph with $k < 4$ whose automorphism group is Z_2^3. The proof of this fact is rather tedious, but one may also check this with a help of computer. We mention it, because it means that the number 4 in the results of this paper cannot be lowered.

The permutation group generated by left translations of a regular abelian group A and its involution α plays a special role in this paper. We denote it by $A^+ = \langle A, \alpha \rangle$. We note that if α is nontrivial, then A^+ is nonabelian. Nevertheless we need knowledge about the representability of such groups, and to establish it, we apply Theorem 1 in [19].

Lemma 4 *If A is a regular abelian permutation group, then $A^+ \in GR(2)$, except for the following groups: $A = Z_2^2, Z_2^3, Z_2^4, Z_4 \times Z_2, Z_4 \times Z_2^2, Z_3^2, Z_3^3$, or Z_4^2. In any case, $A^+ \in GR(4)$.*

Proof The first claim follows from [19, Theorem 1] combined with the remark 1.2 preceding this theorem (which adds to the list of exceptions Z_2^2). For the second claim, the three first cases follow by Lemma 1. For the remaining cases the following five 4-colored graphs of the form $\mathrm{Cay}^*(A, P)$ have the automorphism group equal to A^+:

$\mathrm{Cay}^*(Z_4 \times Z_2; 10, 01)$, $\mathrm{Cay}^*(Z_4 \times Z_2^2; 100, 010, 001])$, $\mathrm{Cay}^*(Z_3^2; 10, 01, 11)$, $\mathrm{Cay}^*(Z_3^3; 010, [001, 100], [110, 101])$, $\mathrm{Cay}^*(Z_4^2; 10, 01, 13)$.

Checking this fact for each of the five graphs is routine, and similar to checking the case (iii) in the proof of Lemma 1. □

We note that $\mathrm{Cay}^*(Z_3^2; 10, 01, 11)$ (pictured in the left-hand side of Fig. 4) is a *unique* 4-colored graph (in the sense of [12]) with the unique automorphism group $(Z_3^2)^+$. This means that if a colored graph has the automorphism group $(Z_3^2)^+$, then it is color-isomorphic to $\mathrm{Cay}^*(Z_3^2; 10, 01, 11)$ (i.e., it can be obtained from the latter by suitable renaming vertices and colors). In particular, the number 4 in this lemma cannot be lowered.

We have also two exceptional intransitive abelian permutation groups whose representability (from the point of view of our proof) needs to be established directly. They are two nontrivial subgroups of the direct sum $Z_3^2 \oplus Z_3^2$.

Lemma 5 *Let A be a nontrivial subgroup of $Z_3^2 \oplus Z_3^2$ of the form*

$$A = Z_3^2[H] \oplus_\phi Z_3^2[H],$$

such that $H = Z_3$ or $H = I_9$. Then $A \in GR(4)$.

Proof First consider the case when H is a subgroup isomorphic to Z_3. Note that in this case the decomposition formula above describes A uniquely up to permutation

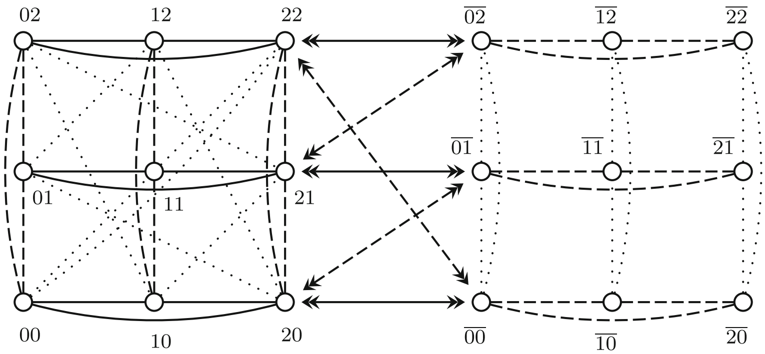

Fig. 2 $\mathrm{Aut}(\Gamma) = Z_3^2[Z_3] \oplus_\phi Z_3^2[Z_3]$

isomorphism. Indeed, each subgroup Z_3 of Z_3^2 may be treated as one of the summands of suitably presented Z_3^2, and the isomorphism ϕ between groups isomorphic to Z_3 is unique up to renaming generators of Z_3.

We construct a graph Γ as a suitable composition of two graphs of the form $\mathrm{Cay}^*(Z_2^3; \Pi)$. The first component of Γ (corresponding to the first orbit of A) is $\mathrm{Cay}^*(Z_2^3; 01, 11, 10)$, and the second one (corresponding to the second orbit of A) is $\mathrm{Cay}^*(Z_2^3; 10, 01)$ (we assume here that the colors $1, 2, 3$ are assigned to edges in accordance with the position on the list, so in particular, the edge $(00, 01)$ in the first graph has the same color 1 as the edge $(00, 10)$ in the second graph). To describe the colors of edges between the two components, we assume that the pairs in the second component are denoted with overline (thus, the first component consists of pairs xy, where $x, y \in \{0, 1, 2\}$, and the second component consists of analogous overlined pairs $\overline{xy}$). Then, we put the color 3 for the edge $(00, \overline{00})$ and the color 1 for the edge $(00, \overline{01})$. This is done under assumption that $\mathrm{Aut}(\Gamma) \supseteq A$, where the first Z_3 subgroup in the decomposition $A = Z_3^2[Z_3] \oplus_\phi Z_3^2[Z_3]$ is equal to $(Z_3, \{00, 10, 20\})$, and the second to $(Z_3, \{\overline{00}, \overline{10}, \overline{20}\})$. This assumption forces the colors for other edges in those orbitals of A that contain the mentioned edges. The remaining edges are colored 0. Note that there is no edge of color 2 between the components.

The graph Γ is illustrated in Fig. 2. The dashed, dotted, and solid lines correspond to colors $1, 2,$ and 3, respectively. To make the drawing more readable, we have applied the convention that each line between components ending with double arrows corresponds to nine edges in the given color joining each vertex in the horizontal line pointed out by the arrows in the left component with each vertex in the horizontal line pointed out by the arrows in the right component.

Thus we have $\mathrm{Aut}(\Gamma) \supseteq A = Z_3^2[Z_3] \oplus_\phi Z_3^2[Z_3]$, where ϕ is the natural isomorphism between Z_3-subgroups. Moreover, $\mathrm{Aut}(\Gamma)$ preserves the orbits of A, since the quadruples of color degrees in the first component are different from that in the second component.

Now, consider the stabilizer S_0 of point 00 in $\mathrm{Aut}(\Gamma)$. To get the equality above, it is enough to show that the cardinality $|S_0| = 3$. Because of the dotted edges (color 2) coming out from 00, S_0 fixes the set $\{11, 22\}$. We show that, actually, S_0 fixes individually points in this pair. Suppose, to the contrary, that S_0 contains an automorphism σ transposing 11 and 22.

Then, σ transposes sets $\{10, 11, 12\}$ and $\{20, 21, 22\}$, which are disjoint triangles in color 3 (solid line) in the first component. Due to edges in color 3 between the components, σ transposes also the sets: $\{\overline{01}, \overline{11}, \overline{21}\}$ and $\{\overline{02}, \overline{12}, \overline{22}\}$, which are the triangles in color 1 (dashed lines) in the second component. Similarly, due to edges in color 1 between the components, σ transposes the sets: $\{\overline{00}, \overline{10}, \overline{20}\}$ and $\{\overline{02}, \overline{12}, \overline{22}\}$, which contradicts the previous claim.

Thus, the set $\{00, 11, 22\}$ remains individually fixed in S_0. It is now easy to see that each point in the first component remains fixed under S_0. Consequently, (because of solid edges between the two components) each triangle in color 1 in the second component remains fixed, and (due to the edges colored 2 in the second component), for every automorphism, the permutation in one of these triangles determines permutations in other triangles. Thus, the cardinality of the stabilizer of 00 in $\mathrm{Aut}(\Gamma)$ is equal to 3, proving the lemma for $H = Z_3$.

Consider now the case when $H = I_9$ is the trivial subgroup of Z_3^2. Then A is the parallel sum $A = Z_3^2||Z_3^2$. In this case we construct a graph Γ as a combination of $\mathrm{Cay}^*(Z_3^2; 10, 01, 11)$ and $\mathrm{Cay}^*(Z_3^2; 01, 10)$. In addition, each edge of the form $(xy, \overline{xy})$ is colored 1, each edge of the form $(xy, \overline{(x+1)y})$, where addition is modulo 3, is colored 2, and the other edges are colored 0. The proof that $\mathrm{Aut}(\Gamma) = Z_3^2||Z_3^2$ is similar to the first case (but simpler), so we leave it to the reader. □

In fact, the parallel sum $Z_3^2||Z_3^2$ is known to belong to $GR(2)$. A suitable construction is contained in the proof of the main result in [3].

4 The structure of abelian permutation groups

From now on (A, X) denotes an abelian permutation group on a fixed set X, with orbits $X_1, \ldots, X_r$. Then, by $A_i = A|_{X_i}$ we denote the restriction of A to X_i, by A_i^j the restriction of the pointwise stabilizer of the orbit X_j to the orbit X_i, and by A_i^* the restriction of the pointwise stabilizer of the set $V \setminus X_i$ to the orbit X_i.

Two orbits X_i and X_j, $i \neq j$, are called *adjacent* if the factor group A_i/A_i^j is not an elementary abelian 2-group. We note that this relation is symmetric. Indeed, the restriction B of A to $X_i \cup X_j$ can be presented as $B = A_i[A_i^j] \oplus_\phi A_j[A_j^i]$. This means, in particular, that A_i/A_i^j is isomorphic to A_j/A_j^i, which implies the claim. Accordingly, an orbit X_i of A will be called *isolated*, if it is not adjacent to any orbit X_j, $j \neq i$.

Let us recall that a permutation σ preserving orbits of A is called 2-orbit-compatible with the permutation group A, if for each pair of orbits X_i and X_j, $i \neq j$, the restriction of σ to $X_i \cup X_j$ belongs to the restriction of the group A to $X_i \cup X_j$. The group A is 2-*orbit-closed* if every permutation that is 2-orbit-compatible with A belongs to A. The 2-orbit-closure of A, denoted $\bar{A}$, is the group consisting of all permutations 2-orbit-compatible with A. Obviously, $\bar{A}$ has the same orbits as A. Moreover, it has the same restrictions $(\bar{A})_i = A_i$, $(\bar{A})_i^j = A_i^j$, $(\bar{A})_i^* = A_i^*$. In particular, we have the following.

Lemma 6 *Let A be a permutation group. Then the following hold*

1. *A is abelian if and only if $\bar{A}$ is abelian.*
2. *An orbit X_i is isolated in A if and only if X_i is isolated in $\bar{A}$.*

The notion of 2-orbit closure arises naturally, when one considers automorphism groups of colored graphs and digraphs. All these groups are obviously 2-orbit closed. It is enough to observe that a colored graph (or digraph) Γ has exactly two kinds of edges with regard to its automorphism group: those joining vertices within an orbit of the group and those joining vertices between two different orbits. It is easily seen that a permutation 2-orbit-compatible with $\mathrm{Aut}(\Gamma)$ preserves the colors of all edges. The following is an obvious property of 2-*orbit-closed* groups.

Lemma 7 *Let A and B be 2-orbit-closed permutation groups acting on the same set X and having the same orbits. If for any two orbits O and Q, the restriction $A|_{O\cup Q} = B|_{O\cup Q}$, then $A = B$.*

We have also the following crucial characterization.

Lemma 8 *If A is a 2-orbit-closed abelian permutation group, then for every orbit X_i of A, X_i is isolated in A if and only if A_i/A_i^* is an elementary abelian 2-group.*

Proof First observe that for each $i \le r$, $A_i^* \subseteq \bigcap_{j\neq i} A_i^j$. We show that for 2-orbit-closed groups the converse inclusion holds, as well. Indeed, suppose that $\tau \in A_i^j$ for each $j \neq i$. It follows, that for each $j \neq i$, there is a permutation $\sigma_j \in A$, such that its restriction to $X_i \cup X_j$ is equal to τ extended to X_j by fixing all points in X_j. Consequently, the permutation σ whose restriction to X_i is equal to τ and fixing all points in $X \setminus X_i$ is 2-orbit-compatible with A and therefore belongs to A. Whence, $\tau \in A_i^*$, as required.

Now, we prove our claim by contraposition. Suppose that A_i/A_i^* is not an elementary abelian 2-group, that is, it has an element xA_i^* of order > 2. This is equivalent to that $x^2 \notin A_i^*$, which means (by what proved above) that there is $j \neq i$ such that $x^2 \notin A_i^j$. The latter is equivalent to A_i/A_i^j has an element xA_i^* of order > 2, that is, A_i/A_i^j is not an elementary abelian 2-group. This means that X_i is not isolated. These equivalences yield the required result. □

Note that the factor group A_i/A_i^* is an abstract group; we do not define any action of this group. It plays a special role in our main result below.

5 Characterization of 2*-closed abelian permutation groups

Using definitions formulated at the beginning of Sect. 4, we state our main result.

Theorem 2 *Let A be a nontrivial abelian permutation group. Then A is the automorphism group of a colored graph if and only if the following conditions hold*

1. *A is* 2*-orbit-closed, and*
2. *for every orbit X_i of A, if the factor group A_i/A_i^* is an elementary abelian* 2*-group, then so is A_i.*

Note that the factor group A_i/A_i^* is an elementary abelian 2-group (as abstract group) if and only if it is isomorphic to Z_2^n for some $n \geq 0$. In turn, the permutation group A_i (being transitive) is an elementary abelian 2-group if and only if it is permutation isomorphic to the regular action of Z_2^n for some $n \geq 0$. Note that this includes trivial cases with $n = 0$. (There are also other permutation groups that are elementary abelian 2-groups, but they are not transitive, and do not apply in our theorem).

The proof of Theorem 2 consists of a number of lemmas. We keep the notation of the previous section. First we prove the "only if" part of the theorem.

Lemma 9 *If an abelian permutation group $A \in GR$, then A satisfies conditions* (1) *and* (2) *of Theorem 5.1.*

Proof As we have already noted before Lemma 7, condition (1) obviously holds. For (2), let Γ be a colored graph with $\mathrm{Aut}(\Gamma) = A$, and suppose that A_i/A_i^* is isomorphic to Z_2^m for some $m \geq 0$. Since A_i is abelian and transitive on X_i, it acts regularly on X_i. Therefore X_i may be identified with A_i and the action of A_i with the regular action on itself. In particular, A_i^* may be considered as a subset of X_i.

For each pair of elements $x, y \in A_i^*$, there is a permutation $\sigma \in A$ moving x into y and fixing all the elements outside X_i. Because of commutativity, σ does the same with any pair tx and ty, where $t \in A_i$ is treated as a permutation on A_i. It follows that the cosets of A_i/A_i^* have the same property: for each pair of elements x, y in the same coset, there is a permutation $\sigma \in A$ moving x into y and fixing all the elements outside X_i. It follows that for every pair of such elements x, y, and every element $z \notin X_i$, the edges xz and yz in Γ have the same color.

We observe that for each $x \in X_i$, x^{-1} is in the same coset as x. Indeed, since $A_i/A_i^* \cong Z_2^m$, for cosets we have $xA_i^*xA_i^* = A_i^*$, and by commutativity, $x^2A_i^* = A_i^*$; hence $xA_i^* = x^{-1}A_i^*$, as required. Thus, we infer that the edges xz and $x^{-1}z$ in Γ have the same color, for every element $z \notin X_i$.

We proceed to show the involution α in $X_i = A_i$ treated as a permutation of X (fixing all elements $x \notin X_i$) preserves the colors of edges in Γ. Indeed, by what established above, it preserves the colors of all edges in Γ that have at most one end in X_i. On the other hand, by Lemma 2 we know that α preserves the colors of the edges within X_i, which proves the claim.

Consequently, $\alpha \in A_i$. Since A_i is regular, it means that α must be trivial, that is, $x = x^{-1}$ for all $x \in A_i$. It follows that A_i is an elementary abelian 2-group, proving the lemma. □

The proof of the "if" part is by induction on the number of orbits in A. Below, we establish the result for two orbits. Note that in this case A is trivially 2-orbit-closed.

Lemma 10 *If A is a nontrivial abelian permutation group with two orbits and A satisfies condition* (2) *of the theorem, then $A \in GR(4)$.*

Proof Let O and Q be the orbits of A. Let $A = B[B'] \oplus_\phi C[C']$ be the subdirect decomposition of A with regard to O and Q. Let $O_1, \ldots, O_r$ be the partition of the orbit O into orbits of B'. Since the actions of B on O is regular (as B is abelian and transitive), the factor group B/B' acts on the set of orbits $O_1, \ldots, O_r$ in a regular way. The same is true of the action of C/C' on the set of orbits $Q_1, \ldots, Q_s$ of C' on Q, since B/B' and C/C' are isomorphic, $r = s$. It follows also that $\phi : B/B' \to C/C'$ establishes a one-to-one correspondence between the orbits $O_1, \ldots, O_r$ and $Q_1, \ldots, Q_r$ so that the action of B/B' on $O_1, \ldots, O_r$ is equivalent to the action of C/C' on $Q_1, \ldots, Q_r$. After suitable renumbering we may assume that $\phi(O_i) = Q_i$ for all $i = 1, \ldots, r$. Moreover, we may assume that the orbits O_i are identified with the cosets of B/B', the orbits Q_i are identified with the cosets of C/C', and $B' = O_1$ and $C' = Q_1$.

We construct a 4-colored graph on the set $O \cup Q$ of vertices. It consists of two parts: Γ_1 on the set of vertices O, and Γ_2 on the set of vertices Q. We take Γ_i to be a 4-colored graph given by Lemma 4, such that $\mathrm{Aut}(\Gamma_1) = \langle B, \beta \rangle$ and $\mathrm{Aut}(\Gamma_2) = \langle C, \gamma \rangle$, where β and γ are corresponding involutions.

For the edges joining O and Q we put colors as follows. First, for each $i = 1, \ldots, r$, and for all $y \in O_i$ and all $z \in Q_i$, the edge yz is colored 1. These edges reflect the one-to-one correspondence between cosets. They guarantee that the regular action on cosets is *parallel*: if (σ, τ) is a permutation on $O \cup Q$ preserving the set of these edges (where σ permutes O, and τ permutes Q), then $\sigma(O_i) = O_j$ implies $\tau(Q_i) = Q_j$ for all $i, j \le r$. Therefore, in the remaining part of the construction we assume that these edges are the only edges between O and Q colored 1. Note that this guarantees also that, if $\mathrm{Aut}(\Gamma) \subseteq B \oplus C$, then $\mathrm{Aut}(\Gamma) = B[B'] \oplus_\phi C[C'] = A$. So, it remains only to prove that $\mathrm{Aut}(\Gamma) \subseteq B \oplus C$. (This construction will be referred further as *joining cosets in parallel manner*).

Now the construction differs depending on whether the orbits O and Q are adjacent or not. First we consider the case of adjacent orbits, and define the set of edges between O and Q colored 2. They are chosen to prevent involutions in O and Q.

Since B/B' is not isomorphic to Z_2^n, it has an element xB' of order greater than 2. If we would have $xB' = x^{-1}B'$, then $x^2B' = B'$, a contradiction. Hence x and x^{-1} lie in different cosets. We color all the edges between B and $\phi(xB)$ with the color 2. Moreover, to make sure that $\mathrm{Aut}(\Gamma) \supseteq A$, we put color 2 for all edges between yB and $\phi(yxB)$ for any $y \in B$. The remaining edges between X and O are colored 0. Thus, since $x^{-1}B \neq xB$, the edges between B and $\phi(x^{-1}B)$ have color 0, while the edges between B and $\phi(xB)$ have color 2. This ensures that the involution γ does not preserve colors of the edges, and similarly, β does not, either.

Figure 3 illustrates the case for $B = Z_8$, $B' = Z_2$, $C = Z_{12}$, and $C' = Z_3$; solid lines correspond to color 1, while dotted lines correspond to color 2. The cycles representing $(Z_8)^+$ (on the left, in color 2), and $(Z_{12})^+$ (on the right, in color 1) are drawn in a way grouping vertices corresponding to cosets; this is to make the picture more readable.

To prove that $\mathrm{Aut}(\Gamma) \subseteq B \oplus C$, it remains to show that O is a fixed block for $\mathrm{Aut}(\Gamma)$. This may be achieved by suitable rearrangement of colors of edges in Γ_1, so that the quadruple of color degrees of vertices in O is different than that in Q (note that because $\mathrm{Aut}(\Gamma)$ is transitive on its orbits, this quadruple is the same for all vertices in the given orbit). Such a rearrangement is impossible only in one case, when both

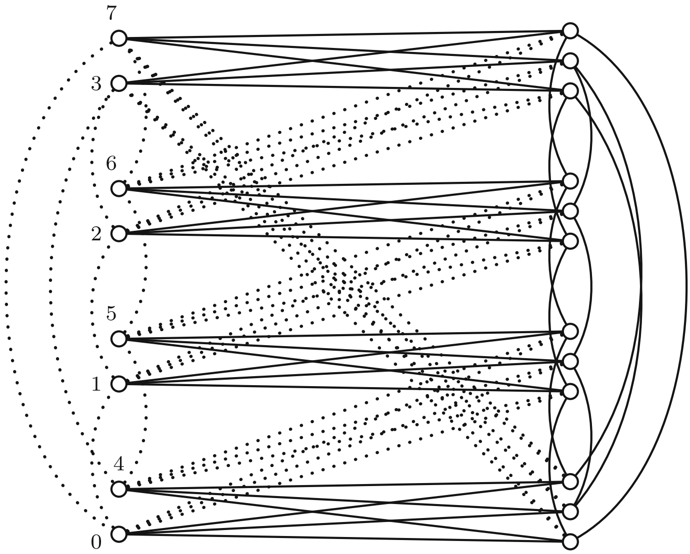

Fig. 3 Graph Γ with $\mathrm{Aut}(\Gamma) = Z_8[Z_2] \oplus_\phi Z_{12}[Z_3]$, consisting of two cycles representing Z_8 and Z_{12}, and edges joining the cycles.

graphs Γ_i are isomorphic, all 4 colors are used, and degree in each color is the same. It follows that, according to Lemma 4, this happens only in the case when B and C are isomorphic with Z_3^2. Moreover, it follows that, in such a case, A is a group of the form $Z_3^2[H] \oplus_\phi Z_3^2[H]$, and since the orbits are adjacent, $H = Z_3$ or $H = I_9$. Yet, by Lemma 5, $A \in GR(4)$ in such a case, which completes the proof for the adjacent orbits.

Now, assume that O and Q are not adjacent. Then, by condition (2), both B and C are elementary abelian 2-groups of the form Z_2^m. Moreover, since A is nontrivial, at least for one of these group $m > 0$. In this case, the involutions are trivial, so we do not need any special construction to prevent them. Whence, in this case, all the edges between X and O other than the edges guaranteeing parallel action between the cosets are colored 0. Since no group of the form Z_2^m, $m > 0$, has $4n + 1$ elements for any n, as in the previous case, the colors of edges in Γ_i can be rearranged so that to ensure that O is fixed block for $\mathrm{Aut}(\Gamma)$. Then the result follows as before, completing the proof. □

Remark 1 For future reference note that the edges between the orbits are colored in at most three colors 0, 1, 2. This includes the case covered by Lemma 5.

Remark 2 The assumption is that A is nontrivial is only to exclude the exceptional case of the trivial permutation group acting on exactly 2 elements, which (because of lack of room) is not representable by any 2-element graph.

Now we prove the "if" part.

Lemma 11 *If A is a nontrivial abelian permutation group satisfying condition* (1) *and* (2) *of the theorem, then $A \in GR(4)$.*

Proof The proof is by induction on the number of orbits r of A. If $r = 1$, A is transitive, and condition (2) means that A is an elementary abelian 2-group, since in this case $A_i^* = A$. By Lemma 3 $A \in GR(4)$. If A has 2 orbits, then the result holds by Lemma 10.

Now, suppose that A has $r > 2$ orbits, and the result holds for all groups with the number of orbits less than r.

Consider an arbitrary orbit X_i and the decomposition of A with regard to this orbit, that is, let $A = A_i[A_i^*] \oplus_\phi B[B']$, where B is the restriction of A to $X \setminus X_i$. Since A is nontrivial, we may assume in addition that B is nontrivial (A_i may happen to be a fixed point). Let $\bar{B}$ be the 2-orbit closure of B. By Lemma 6, $\bar{B}$ is abelian, and it has $r - 1$ orbits. Moreover, it satisfies condition (2), providing all isolated orbits in $\bar{B}$ are those isolated in A. Let us continue under this additional assumption.

Then, by the induction hypothesis, $\bar{B} \in GR(4)$, and there exists a 4-colored graph Γ_2 on the set of vertices $X \setminus X_i$ representing $\bar{B}$. We construct a graph Γ on X representing A. Let Γ_1 be a 4-colored graph on X_i representing $(A_i)^+$ (by Lemma 4). We may assume that both the graphs Γ_1 and Γ_2 are connected in colors 2 and 3 (meaning every two vertices in Γ_i are connected by a path using only edges of color 2 or 3; this may be achieved by a suitable change of colors). For each orbit X_j of $\bar{B}$ (that are exactly the orbits of A other than X_i), we put the edges colored 1 between X_i and X_j joining corresponding cosets in parallel manner as in the proof of Lemma 10. The remaining edges joining the vertices of X_i and $X \setminus X_i$ are colored 0. Obviously, $\mathrm{Aut}(\Gamma) \supseteq A$, and since $\bar{B}$ is intransitive, each automorphism of Γ preserves the orbit X_i. Thus, $\mathrm{Aut}(\Gamma) \subseteq (A_i)^+ \oplus \bar{B}$.

We prove that no nontrivial involution on X_i is admitted, that is, $\mathrm{Aut}(\Gamma) \subseteq A_i \oplus \bar{B}$). Indeed, if A_i/A_i^* is an elementary abelian 2-group, then by (2), so is A_i, and the involution is trivial. Then $(A_i)^+ = A_i$, and the claim is obvious. So, we may assume that A_i/A_i^* is not an elementary abelian 2-group. Then by Lemma 8, X_i is not isolated, which means that there is an orbit X_j, $j \neq i$, adjacent to X_i. In particular, A_i/A_i^j is not an elementary abelian 2-group. Consider the restriction of A to $X_i \cup X_j$, which can be presented in the form $A_i[A_i^j] \oplus_\phi A_j[A_j^i]$. Similarly as in Lemma 10, we infer that there is $x \in A_i$ such that x and x^{-1} lie in different cosets of A_i/A_i^j. Now, A_j/A_j^i is regular, as it is transitive and abelian, so if an automorphism of Γ fixes A_j^i, it fixes all the cosets of A_j/A_j^i. Because of the edges between X_i and X_j guaranteeing a parallel action on cosets, we infer that if an automorphism of Γ fixes A_i^j, then it fixes all the cosets in A_i/A_i^j. Consequently, there is no automorphism of Γ whose restriction to X_i would be the involution. This proves our claim.

The construction ensures that for any two orbits X_j and X_k of A, the restriction of $\mathrm{Aut}(\Gamma)$ to $X_j \cup X_k$ is the same as the restriction of A to $X_j \cup X_k$. By Lemma 7, $\mathrm{Aut}(\Gamma) = A$, as required.

Thus, we have proved that $A \in GR(4)$, under conditions $(**)$ *that all isolated orbits in* $\bar{B}$ are those isolated in A, and that B has a nontrivial orbit. Consider now the general situation. If there is a trivial orbit (fixed point) in A, we may take this orbit as X_i above, and the result follows (because conditions $(**)$ are satisfied). Otherwise, if there is any isolated orbit in A, then we may take it as X_i, and again conditions $(**)$

are satisfied, and the result follows. The result also follows in any case when there is an orbit X_i such that (∗∗) are satisfied. All that remains to consider is the situation when A has an even number r of orbits, all nontrivial, and paired in such a way that for every orbit X_i there is a unique orbit X_j in A such that X_i and X_j are adjacent.

If $r = 2$, then the result follows by Lemma 10. If $r \geq 4$, we take a pair of adjacent orbits X_i and X_j, put $Y = X_i \cup X_j$, and $Z = X \setminus Y$, and decompose A with regard to Y and Z: $A = C[C'] \oplus_\phi B[B']$. Now, the proof is the same as in the case of decomposing with regard to a chosen orbit X_i, with the natural modification for C consisting of two orbits, and using Lemma 10 rather than Lemma 4. This makes the proof simpler, since we may omit the part concerning involutions. An additional case is created for $n = 4$, since then we need to use a more sophisticated coloring of edges to prevent transposing sets Y and Z. In this case C and B consist each of two (adjacent) orbits, and the problem arises when Γ_1 and Γ_2, representing C and B, respectively, are isomorphic as colored graphs. Then we make use of Remark 1 following the proof of Lemma 10. According to this remark we may assume that in the graph Γ_1 representing C the edges between the orbits are in colors 0, 1, 2, while for the graph Γ_2 representing B the edges between the orbits are in colors 1, 2, 3. Then Γ_1 and Γ_2 are no longer isomorphic, and the construction works also in this case. This completes the proof of the lemma. □

Now, Theorem 2 follows immediately by Lemmas 9 and 11. In fact, we have proved something more, namely, that in case of abelian permutation groups "four colors suffices."

Corollary 1 *An abelian group A is the automorphism group of a colored graph if and only if it is the automorphism group of a complete graph whose edges are colored in at most 4 colors.*

We note that, by the remark following Lemma 3, the bound 4 above is sharp. In fact, using, e.g., the construction of the direct sum with the summand Z_2^3 one may obtain an infinite family of abelian permutation groups requiring 4 colors to be representable by a colored graph.

6 Characterization of 2-closed abelian permutation groups

We show that the notions "2-closed" and "2-orbit-closed" coincide. The result is an essential characterization, since although checking 2-orbit-closure is not easy, checking 2-closure is computationally much harder. It is enough to note that the most elementary definition of 2-closure [5] involves orbitals of a permutation group (and hence, the induced action on the pairs of points), while the definition of 2-orbit-closure refers merely to the basic notion of orbit.

Theorem 3 *Let A be an abelian permutation group. Then A is the automorphism group of a colored directed graph if and only if A is 2-orbit-closed.*

Proof To prove this result we follow the approach in the proof of Theorem 2. In fact, the proof is simpler because we can start the induction from $r = 1$ orbits, and there

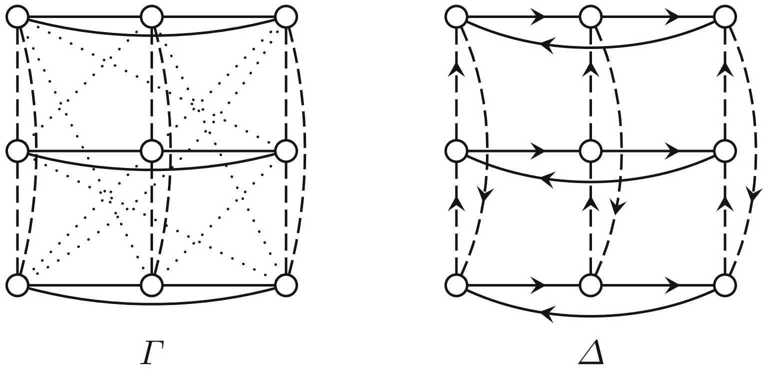

Fig. 4 Simple and directed graphs: $\mathrm{Aut}(\Gamma) = (Z_3^2)^+$ and $\mathrm{Aut}(\Delta) = Z_3^2$

is no problem of involution and special cases, as in the previous proof. In addition, we may omit the assumption about nontriviality of A, since the result holds for trivial groups, as well (in particular, $I_n \in DGR(2)$ can be represented by the directed graph consisting of one directed path of length $n-1$). As the arguments are similar, we give here only a sketch, referring the reader for details to the previous proof.

The "only if part" is immediate by the remark before Lemma 7. We prove the "if part" by induction on the number of orbits r of A. If $r = 1$, A is transitive, and hence regular, and by the result of Babai [4], every nontrivial regular abelian group $A \in DGR(2)$, except for $A = Z_2^n$ with $n = 2, 3, 4$, and $A = Z_3^2$. In the first case, by Lemma 1, $Z_2^n \in DGR(4)$ (and again Z_2^3 requires 4 colors; here we have $x = x^{-1}$ and the cases of directed and undirected graphs are the same). In the second case, $Z_3^2 \in DGR(4)$ by Lemma 4. In fact, it can be easily seen that $Z_3^2 \in DGR(3)$ (see the right-hand side of Fig. 4).

Now, suppose that A has $r > 1$ orbits, and the result holds for all groups with the number of orbits less than r.

Consider an arbitrary orbit X_i and the decomposition of A with regard to this orbit, that is, let $A = A_i[A_i^*] \oplus_\phi B[B']$, where B is the restriction of A to $X \setminus X_i$. Let $\bar{B}$ be the 2-orbit closure of B. By Lemma 6, $\bar{B}$ is abelian, and it has $r-1$ orbits.

It follows, by the induction hypothesis, that $\bar{B} \in DGR(4)$, and there exists a 4-colored digraph Γ_2 on the set of vertices $X \setminus X_i$ representing $\bar{B}$. We construct a digraph Γ on X representing A. Let Γ_1 be a 4-colored digraph on X_i representing A_i (which exists by the proof for the case $r = 1$). We may assume that both the graphs Γ_1 and Γ_2 are connected (as undirected graphs) in colors 2 and 3 (this may be achieved by suitable change of colors). For each orbit X_j of $\bar{B}$, we put the edges colored 1 from X_i to X_j joining corresponding cosets in parallel manner as in the proof of Lemma 10. We assume that these edges are directed from X_i to X_j. The remaining edges joining the vertices of X_i and $X \setminus X_i$ are colored 0.

Then, obviously, $\mathrm{Aut}(\Gamma) \supseteq A$, Moreover, since Γ_1 and Γ_2 are connected in colors 2 and 3, and edges in color 1 between Γ_1 and Γ_2 are directed from Γ_1 to Γ_2, each automorphism of Γ preserves the orbit X_i. Thus, $\mathrm{Aut}(\Gamma) \subseteq A_i \oplus \bar{B}$.

The construction ensures that for any two orbits X_j and X_k of A, the restriction of Aut(Γ) to $X_j \cup X_k$ is the same as the restriction of A to $X_j \cup X_k$. Hence, by Lemma 7, Aut(Γ) $= A$, as required. This completes the proof. □

Again, what we have proved in addition is that "four colors suffices," and that the bound 4 below is sharp.

Corollary 2 *An abelian group A is the automorphism group of a colored directed graph if and only if it is the automorphism group of a complete directed graph* (*without loops*) *whose edges are colored in at most* 4 *colors.*

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