Journal of Algebraic Combinatorics (2022) 55:393–411
https://doi.org/10.1007/s10801-021-01055-5

The sum of the Betti numbers of smooth Hilbert schemes

Joseph Donato[1] · Monica Lewis[1] · Tim Ryan[1] · Faustas Udrenas[1] · Zijian Zhang[1]

Received: 23 March 2021 / Accepted: 8 August 2021 / Published online: 28 September 2021

Abstract
Recently, Skjelnes and Smith classified which Hilbert schemes on projective space are smooth in terms of integer partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $r = 0$, $\lambda = (n+1)$, or $n \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 1$. In particular, they found there to be seven families of smooth Hilbert schemes: one with $r = 0$ or $\lambda = (n+1)$, one with Hilbert schemes on the projective line or plane, 4 families with $\lambda_r = 1$, and one with $\lambda_r \geq 2$. In this paper, we compute the sum of the Betti numbers for all of these families of smooth Hilbert schemes over projective space except the case $\lambda_r \geq 2$.

Keywords Hilbert scheme · Cohomology · Homology

1 Introduction

Hilbert schemes are one of the classic families of varieties. In particular, Hilbert schemes of points on surfaces have been extensively studied; so extensively studied that any reasonable list of example literature would take several pages, see [6,14,15] for introductions to the area. This study was at least in part due to these being one of the only sets of Hilbert schemes which were known to be smooth. Recent work [17] has characterized exactly which Hilbert schemes on projective spaces are smooth

✉ Tim Ryan
rtimothy@umich.edu

Joseph Donato
jsdonato@umich.edu

Monica Lewis
malewi@umich.edu

Faustas Udrenas
fudrenas@umich.edu

Zijian Zhang
zzjharry@umich.edu

1 Department of Mathematics, University of Michigan, Ann Arbor, MI, USA

by giving seven "families" of smooth Hilbert schemes in terms of integer partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ satisfying $\lambda = (n+1)$, $r = 0$, or $n \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 1$. Note, a Hilbert scheme on projective space is nonempty if and only if the corresponding Hilbert polynomial can be written as $p(t) = \sum_{i=1}^{r} \binom{t+\lambda_i-i}{\lambda_i-1}$ for some integer partition λ.

One of the most fundamental topological properties of an algebraic variety is its homology. The homology of Hilbert schemes of points has been extensively studied, e.g., [2,3,7,8,11,12]. It is an immediate consequence of [1] that the smooth Hilbert schemes over $\mathbb{C}$ have freely generated even homology groups and zero odd homology groups. This was used to compute the Betti numbers of Hilbert schemes of points on the plane in [2]. A natural follow-up question then is what are the ranks of the homology groups for all of the smooth Hilbert schemes? In this paper, we compute the sum of the Betti numbers for six of the seven families of smooth Hilbert schemes. Since these Hilbert schemes are smooth, this is equivalent to computing the dimension of the cohomology ring as a vector space over $\mathbb{C}$. Note, in this case, the cohomology and the Chow rings are isomorphic.

In order to state the theorem, recall that Macaulay proved that the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial p, denoted $\mathbb{P}^{n[p]}$, is nonempty if and only if p can be written in the form $p(t) = \sum_{i=1}^{r} \binom{t+\lambda_i-i}{\lambda_i-1}$ for some integer partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers satisfying $\lambda = (n+1)$, $r = 0$, or $n \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 1$. Recently Skjelnes and Smith extended this work by classifying which of these Hilbert schemes were smooth

Theorem 1.1 [17] *Let p be a polynomial in a single variable with some sequence $\lambda = (\lambda_1, ..., \lambda_r)$ with $n \geq \lambda_1 \geq ... \geq \lambda_r \geq 1$ or with $\lambda = (n+1)$. Then, the Hilbert scheme $\mathbb{P}^{n[p]}$ on projective space is smooth if and only if:*

1. $n \leq 2$,
2. $\lambda_r \geq 2$,
3. $\lambda = (1)$ *or* $\lambda = (n^{r-2}, \lambda_{r-1}^1, 1^1)$ *for all* $r \geq 2$,
4. $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 2^0, 1^1)$ *for all* $r \geq 2$,
5. $\lambda = (n^{r-s-5}, 2^{s+4}, 1^1)$ *for all* $0 \leq s \leq r-5$ *and all* $r \geq 5$,
6. $\lambda = (n^{r-3}, 1^3)$ *for all* $r \geq 3$, *or*
7. $r = 0$ *or* $(n+1)$

Our main theorem computes the sum of the Betti numbers for six of those seven families.

Theorem 1.2 *Let $H_{n,\lambda}$ be the sum of the Betti numbers for $\mathbb{P}^{n[p_\lambda]}$ where p_λ corresponds to the integer partition $\lambda = (\lambda_1, \ldots, \lambda_r)$. Explicit formulas for $H_{n,\lambda}$ in terms of only n and λ for families 1, 3, 4, 5, 6, and 7 are given in Propositions (3.2 & 4.4), 5.5, 5.7, 5.8, 5.9, and 5.1, respectively.*

The proofs of these results work by translating the computation of the ranks of the homology groups into counting saturated monomial ideals and then translating that into counting choices of orthants in an $(n+1)$-dimensional lattice.

There are countless future directions for work on the now classified smooth Hilbert schemes. In particular, building on this work, ongoing work aims to complete the count in the remaining case and to refine the computation to compute the Betti numbers for the smooth Hilbert schemes. Going further, one would hope to understand the geometry of the cycles such as the stable base locus decomposition on smooth Hilbert schemes in terms of the integer partition and the geometry of the parametrized varieties.

The organization of the paper is as follows. In Sect. 2, the necessary background is given. In Sect. 3, the case of Hilbert schemes over the projective line is worked out as an example case. In Sect. 4, the case of Hilbert schemes over the projective plane is worked out, which proves finishes the first part of Theorem 1.2. Finally, in Sect. 5, we prove cases 3–7 of Theorem 1.2.

The authors would like to thank Harry Bray, who directed the Laboratory of Geometry at Michigan when this work was carried out there, and Zhan Jiang, for many enlightening conversations. The authors would also like to thank the referees; in particular, one of whom pointed out that Theorem 2.4 could be used to streamline many of the proofs of later theorems.

2 Background

In this section, we review the necessary background material.

Let $I \subset \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal. As the quotient ring R/I is a graded ring, it comes equipped with a Hilbert function, $h_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, which sends d to the dimension of the degree d graded piece. By Hilbert [10], this function agrees with a polynomial H_I for $d >> 0$. This is the *Hilbert polynomial* of I; note, this is more properly the Hilbert polynomial of R/I, but no confusion will arise by this usage.

The degree of this polynomial is the dimension of $V(I)$, and the other coefficients include other geometric information such as the degree. As the polynomial captures a lot of the geometry of the subvariety cut out by I, a natural definition of equivalence on algebraic subvarieties of $\mathbb{P}^n$ is those with the same Hilbert polynomial. By Grothendieck [9], the set of subvarieties of $\mathbb{P}^n$ with the same Hilbert polynomial p forms an algebraic scheme called the *Hilbert scheme*, denoted $\mathrm{Hilb}^p(\mathbb{P}^n)$ or $\mathbb{P}^{n[p]}$. The first question one can ask about these Hilbert schemes is when are they nonempty. This was answered by Macaulay.

Theorem 2.1 [13] *Given $R = \mathbb{C}[x_0, \dots, x_n]$ and nonzero polynomial $p(d)$ in one variable, there exists ideals in R with Hilbert polynomial $p(d)$ if and only if $p(d)$ can be written in the form $\Sigma_{i=1}^m \binom{d+\lambda_i-i}{\lambda_i-1}$ for some integer partition $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 1$. Further, the zero polynomial corresponds to the integer partition $(n+1)$.*

This theorem means any λ-sequence defines a Hilbert polynomial, and we can also get a λ-sequence from any Hilbert polynomial written in the form shown in Theorem 3.2. For example, if $\lambda = (3, 2)$, then the corresponding Hilbert polynomial is $\binom{d+3-1}{3-1} + \binom{d+2-2}{2-1} = \binom{d+2}{2} + \binom{d}{1} = \frac{(d+2)(d+1)}{2!} + d = \frac{1}{2}d^2 + \frac{5}{2}d + 1$. We will abuse notation and refer interchangeably to λ and p_λ.
It is a natural question to ask about the homology of a Hilbert scheme, but this question is most interesting for the smooth Hilbert schemes where Poincare duality holds, which makes the homology dual to the cohomology. That naturally leads one to ask which

Hilbert schemes are smooth. This was recently answered by Skjelnes and Smith [17]. Note, the $\lambda's$ in the following theorem are exactly the same as the $\lambda's$ mentioned in the previous theorem.

Theorem 2.2 [17] *Let $p = p_\lambda$ be a polynomial in a single variable corresponding to the integer partition $\lambda = (\lambda_1, ..., \lambda_r)$ with $n \geq \lambda_1 \geq ... \geq \lambda_r \geq 1$ or $\lambda = (n+1)$. Then, the Hilbert scheme $\mathbb{P}^{n[p]}$ on projective space is smooth if and only if:*

1. $n \leq 2$,
2. $\lambda_r \geq 2$,
3. $\lambda = (1)$ *or* $\lambda = (n^{r-2}, \lambda_{r-1}^1, 1^1)$ *for all* $r \geq 2$,
4. $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 2^0, 1^1)$ *for all* $r \geq 2$,
5. $\lambda = (n^{r-s-5}, 2^{s+4}, 1^1)$ *for all* $0 \leq s \leq r-5$ *and all* $r \geq 5$,
6. $\lambda = (n^{r-3}, 1^3)$ *for all* $r \geq 3$, *or*
7. $r = 0$ *or* $(n+1)$.

Using this classification is simplified by a special case of Theorem 1.4 of [4], which we restate in our notation, that allows one to treat the n^k part of the partition separately.

Theorem 2.3 [4] *Let $p = p_\lambda$ with $\lambda = (n^k, \lambda') = (n^k, \lambda_{k+1}, ..., \lambda_{k+r})$ with $n-1 \geq \lambda_{k+1} \geq \cdots \geq \lambda_{k+r} \geq 1$. Then, the Hilbert scheme $\mathbb{P}^{n[p]}$ splits into the product*

$$\mathbb{P}^{n[p]} = \mathbb{P}^{n[q]} \times \mathbb{P}^{n[r]}$$

where q and r correspond to the partitions n^k and λ', respectively.

The Hilbert schemes over $\mathbb{P}^n$ inherit the $\mathrm{PGL}(n+1)$ action from projective space itself. In particular, this restricts to a $\mathbb{C}^*$-action; actually it restricts to many $\mathbb{C}^*$-actions, any of which suffices for what follows. The fixed points of this action are points corresponding to the finitely many saturated monomial ideals with that Hilbert polynomial. That lets us apply the following theorem of Bialynicki-Birula to smooth Hilbert schemes.

Theorem 2.4 [1] *Let X be a smooth projective variety with an action of $\mathbb{C}^*$. Suppose that the fixpoint set $\{p_1, \ldots, p_m\}$ is finite, and let $X_i = \{x \in X : \lim_{t\to 0} tx = p_i\}$. Then, X has a cellular decomposition with cells X_i.*

Pairing this theorem with the following result of Fulton shows that a smooth Hilbert scheme has freely generated even cohomology groups and no odd cohomology groups.

Theorem 2.5 [5] *Let X be a scheme with a cellular decomposition. Then for $0 \leq i \leq \dim(X)$,*

(1) $H_{2i+1}(X) = 0$,
(2) $H_{2i}(X)$ *is a $\mathbb{Z}$-module freely generated by the classes of the closures of the i-dimensional cells, and*
(3) *The cycle map* $\mathrm{cl} : A_*(X) \to H_*(X)$ *is an isomorphism.*

Thus, in order to count the sum of the Betti numbers of a smooth Hilbert scheme it suffices to count the number of saturated monomial ideals with that Hilbert polynomial.

3 The projective line

We first consider the case where $n = 1$; equivalently, this is the case where the polynomial ring is $R = \mathbb{C}[x_0, x_1]$. In this case, the only possible partitions are $\lambda = (1^m)$ or (2) which are equivalent to the constant Hilbert polynomial m or $t+1$. It is well known that $\mathbb{P}^{1[m]} = \mathbb{P}^m$ and $\mathbb{P}^{1[t+1]}$ is a reduced point so Theorem 1.2 is immediate in these cases. However, for completeness and clarity, we will give a basic argument in the case of $\mathbb{P}^{1[m]} = \mathbb{P}^m$ which will illuminate the argument which is somewhat obscured by indexing in the later sections.

3.1 Translation for the two variable case

Monomials in the variables x_0 and x_1 are of the form $x_0^a x_1^b$ where $a, b \in \mathbb{Z}_{\geq 0}$. Monomials in two variables are equivalent to points in the lattice $\mathbb{Z}^2_{\geq 0}$ by pairing the point (a, b) with the monomial $x_0^a x_1^b$. By a *ray* or *1-orthant* in this lattice, we will mean the points corresponding to the monomials in a set of the form (Fig. 1)

$$P_a^{i_0} = \{x_{i_0}^a x_{i_1}^b \mid b \in \mathbb{Z}_{\geq 0}\} \text{ for some fixed } a \in \mathbb{Z}_{\geq 0}.$$

We want to see how a set of rays/1-orthants corresponds to (the complement of) a monomial ideal. Since all saturated ideals in two variables are principal, consider the monomial ideal $I = (x_0^a x_1^b)$. Every monomial $\overline{x} = x_0^c x_1^d \notin I$ either has that $c < a$ or that $d < b$. That can be rephrased as

$$\overline{x} \notin I \text{ if and only if } \overline{x} \in \left(\bigcup_{i=0}^{b-1} P_i^1\right) \bigcup \left(\bigcup_{j=0}^{a-1} P_j^0\right).$$

This shows that the Hilbert polynomial of $I = (x_0^a x_1^b)$ is $a + b$, which gives the following lemma.

Lemma 3.1 *The number of saturated monomial ideals in two variables with Hilbert polynomial d, which corresponds to the partition (1^d), is the same as the number of ways to choose d rays/1-orthants in the lattice $\mathbb{Z}^2_{\geq 0}$ in stacks along the two axes.*

The next proposition uses this lemma to count the number of saturated monomial ideals.

Proposition 3.2 *Given the ring $R = \mathbb{C}[x_0, x_1]$ and the partition $\lambda = (1^m)$, there are exactly $m + 1$ saturated monomial ideals of R with Hilbert polynomial $p_\lambda = m$.*

Proof By the previous lemma, the problem of finding the number of saturated monomial ideals boils down to finding the number of ways of choosing m rays/1-orthants along either the x_0-axis or x_1-axis, which is clearly equal to $m + 1$. □

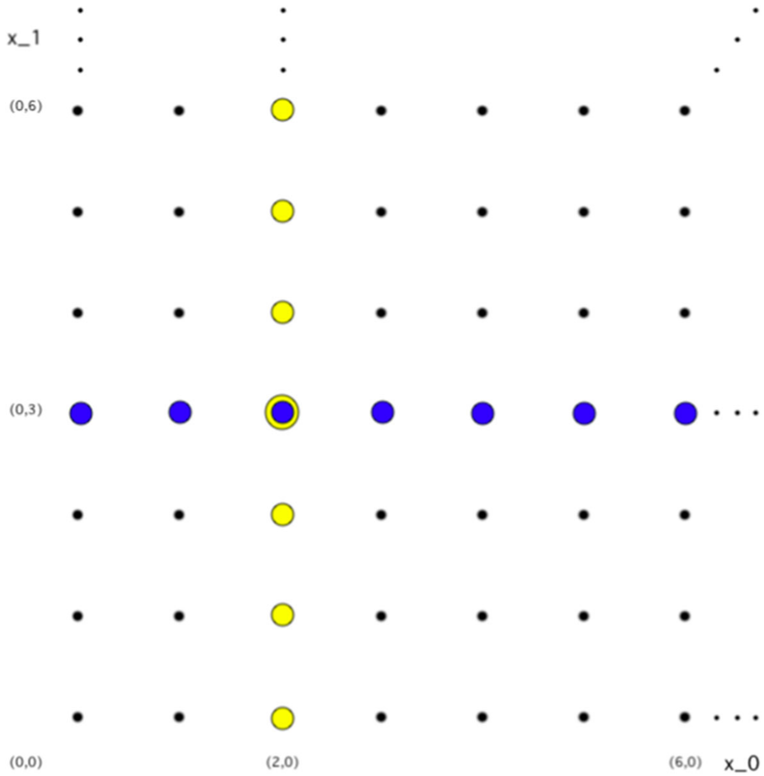

Fig. 1 A visualization of the sets P_3^1 (in blue) and P_2^0 (in maize) in the $\mathbb{Z}^2_{\geq 0}$ (x_0,x_1) lattice

4 The projective plane

In this section, we prove the remainder of the first case of Theorem 1.2, which is the case of Hilbert schemes on the projective plane. We note that this is known, at least implicitly by [2]. We include this case at least in part to exemplify some of the more difficult counting arguments which did not appear in the case of Hilbert schemes on the projective line. To prove this case, we first show the correspondence of saturated monomial ideals in 3 variables and the choice of rays and quadrants in $\mathbb{Z}^3_{\geq 0}$ similarly to the correspondence in Sect. 3.

The three-variable case In the three-dimensional lattice, we have *rays* of the form

$$P_{a,b}^{i_0,i_1} = \{x_{i_0}^a x_{i_1}^b x_{i_2}^c | c \in \mathbb{Z}_{\geq 0}\} \text{ for some fixed } a, b \in \mathbb{Z}_{\geq 0}.$$

On the other hand, we have *quadrants* of the form

$$P_a^{i_0} = \{x_{i_0}^a x_{i_1}^b x_{i_2}^c | b, c \in \mathbb{Z}_{\geq 0}\} \text{ for some fixed } a \in \mathbb{Z}_{\geq 0}.$$

We want to see how the complement of a monomial ideal in this case is the union of quadrants and rays; that is the content of the following lemma.

Lemma 4.1 *Given the ring $R = \mathbb{C}[x_0, x_1, x_2]$ and a saturated monomial ideal $I = (x_0^{\alpha_0^1} x_1^{\alpha_1^1} x_2^{\alpha_2^1}, ..., x_0^{\alpha_0^m} x_1^{\alpha_1^m} x_2^{\alpha_2^m})$, then the set of monomials not in I is of the form $\left(\bigcup_{i=0}^k P_{p_1^i, p_2^i}^{c_1^i, c_2^i}\right) \bigcup \left(\bigcup_{j=0}^\ell P_{p_1^j}^{c_1^j}\right)$ where*

$$P_{a,b}^{i_0,i_1} = \{x_{i_0}^a x_{i_1}^b x_{i_2}^c \mid c \in \mathbb{Z}_{\geq 0}\} \textit{ for some fixed } a, b \in \mathbb{Z}_{\geq 0} \textit{ and}$$
$$P_a^{i_0} = \{x_{i_0}^a x_{i_1}^b x_{i_2}^c \mid b, c \in \mathbb{Z}_{\geq 0}\} \textit{ for some fixed } a \in \mathbb{Z}_{\geq 0}.$$

These are equivalent to quadrants/2-orthants and rays/1-orthants, respectively, in the $\mathbb{Z}_{\geq 0}^3$ lattice where each point in our lattice (y_0, y_1, y_2) corresponds to the monomial $x_0^{y_0} x_1^{y_1} x_2^{y_2}$.

Proof First consider the monomial ideal generated by a single monomial $I = (x_0^\alpha x_1^\beta x_2^\gamma)$. A monomial $\overline{x} = x_0^a x_1^b x_2^c$ is not in I if and only if $a < \alpha$, $b < \beta$, or $c < \gamma$. In other words,

$$\overline{x} \notin I \text{ if and only if } \overline{x} \in \left(\bigcup_{i=0}^{\alpha-1} P_i^0\right) \cup \left(\bigcup_{j=0}^{\beta-1} P_j^1\right) \cup \left(\bigcup_{k=0}^{\gamma-1} P_k^2\right).$$

Now let us consider a general monomial ideal, which we denote $I = (x_0^{\alpha_1} x_1^{\beta_1} x_2^{\gamma_1}, ..., x_0^{\alpha_m} x_1^{\beta_m} x_2^{\gamma_m})$ where $\alpha_\ell, \beta_\ell, \gamma_\ell \in \mathbb{Z}_{\geq 0}$, $\forall \ell \in \{1, ..., m\}$. Defining $I_\ell = (x_0^{\alpha_\ell} x_1^{\beta_\ell} x_2^{\gamma_\ell})$ for $\ell \in \{1, .., m\}$, we notice that $I = \bigcup_{\ell=1}^m I_\ell$. By De Morgan's Law, we see that

$$\overline{x} \notin I \text{ if and only if } \overline{x} \in \bigcap_{\ell=1}^m \left[\left(\bigcup_{i=0}^{\alpha_\ell-1} P_i^0\right) \cup \left(\bigcup_{j=0}^{\beta_\ell-1} P_j^1\right) \cup \left(\bigcup_{k=0}^{\gamma_\ell-1} P_k^2\right)\right].$$

To make set manipulation simpler, we define

$$A_\ell = \bigcup_{i=0}^{\alpha_\ell-1} P_i^0,\ B_\ell = \bigcup_{j=0}^{\beta_\ell-1} P_j^1, \text{ and } C_\ell = \bigcup_{k=0}^{\gamma_\ell-1} P_k^2.$$

Using this, we can rewrite the compliment of I as

$$\overline{x} \notin I \text{ if and only if } \overline{x} \in \bigcap_{\ell=1}^m (A_\ell \cup B_\ell \cup C_\ell).$$

Recall that this intersection is equivalent to a union of 3^m intersections of size m by the distributive property of set intersections over unions.

In order to formally write this, we need some notation. Recall the symmetric group on m letters, denoted S_m, is the set of permutations of the set $\{1, \ldots, m\}$. Define the subset $G_{2,m}$ of S_m as the set of permutations σ of $\{1, \ldots, m\}$ such that $\sigma_1 < \cdots < \sigma_k$, $\sigma_{k+1} < \cdots < \sigma_\ell$, and $\sigma_{\ell+1} < \cdots < \sigma_m$ for some $1 \leq k \leq \ell \leq m$, i.e., permutations which decrease at most twice. With this notation, we can write the complement of I as

$$\overline{x} \notin I \text{ if and only if } \overline{x} \in \bigcup_{g \in G_{2,m}} \left(\left(\bigcap_{j=1}^{k} A_{g(j)} \right) \bigcap \left(\bigcap_{j=k+1}^{\ell} B_{g(j)} \right) \bigcap \left(\bigcap_{j=\ell+1}^{m} C_{g(j)} \right) \right).$$

We must consider 3 possible cases for our m sized intersections depending on the values of k and ℓ.

Case I: Consider the case where $0 < k < \ell < m$. An intersection of such a form will contain finitely many monomials. Since we are interested only in saturated ideals, we can ignore this case since the Hilbert polynomial $H_I(d)$ does not consider the elements in this intersection.

Case II: Next, consider any m-intersection of the form $0 = k < \ell < m$, $0 < k = \ell < m$, or $0 < k < \ell = m$. If $0 < k < \ell = m$, the intersection is of the following form:

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} \cap B_{i_{k+1}} \cap \cdots \cap B_{i_{m-1}} \cap B_{i_m}.$$

Let $\theta_{g,0} = \min\{\alpha_{i_1}, ..., \alpha_{i_k}\}$ and $\theta_{g,1} = \min\{\beta_{i_{k+1}}, ..., \beta_{i_m}\}$. The above intersection is now equivalent to

$$\left(\bigcup_{i=0}^{\theta_{g,0}-1} P_i^0 \right) \cap \left(\bigcup_{j=0}^{\theta_{g,1}-1} P_j^1 \right) = \bigcup_{0 \leq i \leq \theta_{g,0}-1 \, , \, 0 \leq j \leq \theta_{g,1}-1} P_{i,j}^{0,1}.$$

Similarly, if $0 < k = \ell < m$ and $\theta_{g,0} = \min\{\alpha_{i_1}, ..., \alpha_{i_k}\}$ and $\theta_{g,2} = \min\{\gamma_{i_{k+1}}, ..., \gamma_{i_m}\}$, then the intersection above is now equivalent to

$$\left(\bigcup_{i=0}^{\theta_{g,0}-1} P_i^0 \right) \cap \left(\bigcup_{j=0}^{\theta_{g,2}-1} P_j^2 \right) = \bigcup_{0 \leq i \leq \theta_{g,0}-1 \, , \, 0 \leq j \leq \theta_{g,2}-1} P_{i,j}^{0,2}.$$

Finally, if $0 = k < \ell < m$ and $\theta_{g,1} = \min\{\beta_{i_1}, ..., \beta_{i_k}\}$ and $\theta_{g,2} = \min\{\gamma_{i_{k+1}}, ..., \gamma_{i_m}\}$, then the intersection above is now equivalent to

$$\left(\bigcup_{i=0}^{\theta_{g,1}-1} P_i^1 \right) \cap \left(\bigcup_{j=0}^{\theta_{g,2}-1} P_j^2 \right) = \bigcup_{0 \leq i \leq \theta_{g,1}-1 \, , \, 0 \leq j \leq \theta_{g,2}-1} P_{i,j}^{1,2}.$$

Case III: Lastly, consider any m-intersection with $0 < k = \ell = m$, $0 = k < \ell = m$, or $0 = k = \ell < m$. If $0 < k = \ell = m$, we have the intersection

$$A_1 \cap \cdots \cap A_m$$

Now, let $\theta_0 = \min\{\alpha_1, \ldots, \alpha_n\}$, and note that

$$A_1 \cap \cdots \cap A_m = \bigcup_{i=0}^{\theta_0 - 1} P_i^0.$$

Similarly, if $0 = k < \ell = m$ and $\theta_1 = \min\{\beta_1, \ldots, \beta_n\}$, then

$$B_1 \cap \cdots \cap B_m = \bigcup_{i=0}^{\theta_1 - 1} P_i^1.$$

Finally, if $0 = k = \ell < m$ and $\theta_2 = \min\{\gamma_1, \ldots, \gamma_n\}$, then

$$C_1 \cap \cdots \cap C_m = \bigcup_{i=0}^{\theta_2 - 1} P_i^2.$$

Since every intersection is one of those three cases, the result follows. □

This lemma means that counting saturated monomial ideals on the plane with fixed Hilbert polynomial is equivalent to counting quadrants and rays whose complement is the monomials of an ideal with that Hilbert polynomial. We want to bound the exact number of quadrants and rays that will give the correct Hilbert polynomial, but first we need to know some more information about Hilbert polynomials when $n = 2$.

Proposition 4.2 *Suppose $p(d)$ is a polynomial in d such that $p(d) = Md - r$ for some $M, r \in \mathbb{R}$, then $p(d)$ is a Hilbert polynomial if and only if $M \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}$, and*

$$r \leq \frac{M^2 - 3M}{2} \text{ and therefore } \lambda = (2^M, 1^{\frac{M^2-3M}{2} - r}).$$

Proof We first prove the forward direction. Assume $p(d)$ is a Hilbert polynomial. By Macaulay [13], this implies that there exists a lambda sequence, say $\lambda = (2^A, 1^B)$, where $A, B \in \mathbb{Z}_{\geq 0}$ such that

$$p(d) = \sum_{i=1}^{A} \binom{d + 2 - i}{1} + \sum_{i=A+1}^{A+B} \binom{d + 1 - i}{0}$$

Simplifying the expression for $p(d)$, we get

$$p(d) = Ad - \left(\frac{A^2 - 3A}{2} - B\right) = Md - r$$

Thus, we see here that $M = A$ and $r = \frac{A^2-3A}{2} - B$. Since $A, B \in \mathbb{Z}_{\geq 0}$ and $p(d)$ is a Hilbert polynomial, we have that $M \in \mathbb{Z}_{\geq 0}$ and $r = B - \frac{A(A-3)}{2} \in \mathbb{Z}$. Since $B \geq 0$, we get that $\frac{M^2-3M}{2} \geq r$.

Now, we proceed with the reverse direction. Let $p(d) = Md - r$, $M \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}$, and $r \leq \frac{M^2-3M}{2}$. Then, $p(d)$ is a Hilbert polynomial because we can choose $\lambda = (2^M, 1^{\frac{M^2-3M}{2}-r})$. With that choice, we get that

$$
\begin{aligned}
H_\lambda(d) \ = & \sum_{i=1}^{\frac{M^2-3M}{2}-r+M} \binom{d+\lambda_i - i}{\lambda_i - 1} = \sum_{i=1}^{M}(d+2-i) \\
& + \sum_{i=M+1}^{\frac{M^2-3M}{2}-r+M} (1) = Md - r = p(d).
\end{aligned}
$$

□

Next, we know from Lemma 4.1 that the number of saturated monomial ideals in three variables for a given Hilbert polynomial $p(d)$ is equivalent to the number of ways of choosing some number of quadrants and rays in the $\mathbb{Z}^3_{\geq 0}$ lattice. So, now we establish the connection between a given Hilbert polynomial and the exact number of quadrants and rays that will be chosen in the $\mathbb{Z}^3_{\geq 0}$ lattice.

Lemma 4.3 *In $R = \mathbb{C}[x_0, x_1, x_2]$, given a Hilbert polynomial p_λ with associated lambda partition $\lambda = (2^A, 1^B)$, the number of saturated monomial ideals in R with associated Hilbert polynomial p_λ is equivalent to the number of ways of choosing A quadrants and B rays in $\mathbb{Z}^3_{\geq 0}$ (none of which are contained in another).*

Proof Consider a Hilbert polynomial p_λ. By Theorem 4.2, we note that this Hilbert polynomial must take the form $p_\lambda(d) = Md - r$ where $A = M \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}$, and $B = \frac{M^2-3M}{2} - r \geq 0$.

Any quadrant in $\mathbb{Z}^3_{\geq 0}$, say $P_a^{i_0}$, represents all monomials of the form $x_{i_0}^a x_{i_1}^\rho x_{i_2}^\eta$ where a is fixed and $\rho, \eta \in \mathbb{Z}_{\geq 0}$ are variable. We choose the quadrants in order, subject to listing $P_a^{i_j}$ before $P_{a+1}^{i_j}$ for all a and j. Thus, the number of monomials of degree d in which x_{i_0} has degree a is $\binom{d-a+1}{1} = d - a + 1$. However, if $P_a^{i_j}$ is the i-th quadrant we list, then it has $i - a$ monomials of degree d in common with previously listed quadrants (one for each quadrant which is not parallel to it). Putting this together, the i-th listed quadrant contributes $d - a - (i - 1 - a) + 1 = d + 2 - i$ to the Hilbert polynomial; by the *contribution* of a (k-)orthant we mean the number of monomials of degree d for $d >> 0$ in that (k-)orthant not in a previously listed (k-)orthant. This implies that in order for the linear term of the Hilbert polynomial $p_\lambda(x)$ to have a coefficient of M, there must be M distinct quadrants in the complement of the ideal as the rays do not contribute to the linear term. Thus, the contribution of all of the quadrants to the Hilbert polynomial is $\sum_{i=1}^{M} \binom{t+2-i}{2-1}$.

Then, the rays must contribute the remaining part of the constant term which is easily seen to be B. Since rays in $\mathbb{Z}^3_{\geq 0}$ contain only one point in the $\mathbb{Z}^3_{\geq 0}$ lattice for each $d > 0$, they each contribute 1 to the Hilbert polynomial. Thus, the monomials not in the ideal consist of exactly A quadrants and B rays. □

Thus, given a Hilbert polynomial p_λ with associated lambda partition $\lambda = (2^A, 1^B)$, $A, B \in \mathbb{Z}_{\geq 0}$ we note that the number of saturated monomial ideals for this Hilbert polynomial in $R = \mathbb{C}[x_0, x_1, x_2]$ is equivalent to number of ways of choosing A quadrants and B rays in $\mathbb{Z}^3_{\geq 0}$. Finally, we can establish the following proposition, which is part of Theorem 1.2.

Proposition 4.4 *Given the ring $R = \mathbb{C}[x_0, x_1, x_2]$ and the partition $\lambda = (2^m, 1^r)$, the number of saturated monomial ideals of R with Hilbert polynomial p_λ is exactly*

$$\binom{m+2}{2} \cdot \sum_{c_1+c_2+c_3=r} \Big[f_1(c_1) \cdot f_1(c_2) \cdot f_1(c_3) \Big]$$

where $c_1, c_2, c_3 \in \mathbb{Z}_{\geq 0}$ and $f_1 : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{>0}$ is the function which maps an integer c to the number of integer partitions of c.

Proof First, consider the case where $r = 0$. In that case, the lambda partition of p_λ is of the form $\lambda = (2^m)$. By the previous lemma, we know that the number of saturated ideals $I \in R = \mathbb{C}[x_0, x_1, x_2]$ is equivalent to the number of ways to choose m quadrants in $\mathbb{Z}^3_{\geq 0}$, or in other words, the number of unique sets

$$\left(\bigcup_{i=0}^{\alpha-1} P_i^0 \right) \cup \left(\bigcup_{j=0}^{\beta-1} P_j^0 \right) \cup \left(\bigcup_{k=0}^{\gamma-1} P_k^0 \right)$$

where $\alpha + \beta + \gamma = m$.

Clearly, the uniqueness of each set is determined by the assignment of α, β and γ. Thus, for the case where $r = 0$, we get that the number of saturated monomial ideals for p_λ for $n = 2$ is $\binom{m+2}{2}$.

Next, consider when $m = 0$, i.e., when $\lambda = (1^r)$. By the previous lemma, the number of saturated monomial ideals with Hilbert polynomial p_λ for $n = 2$ is equivalent to the number of ways to choose r rays in $\mathbb{Z}^3_{\geq 0}$. Much like for the case where $r = 0$, we note that there are only 3 forms of which these rays can be; for a fixed $a, b \in \mathbb{Z}_{\geq 0}$ these are

$$\begin{aligned} P_{a,b}^{i_0,i_1} &= \{x_{i_0}^a x_{i_1}^b x_{i_2}^c | c \in \mathbb{Z}_{\geq 0}\},\ P_{a,b}^{i_0,i_2} = \{x_{i_0}^a x_{i_1}^c x_{i_2}^b | c \in \mathbb{Z}_{\geq 0}\}, \text{ or } P_{a,b}^{i_1,i_2} \\ &= \{x_{i_0}^c x_{i_1}^a x_{i_2}^b | c \in \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

Visually, these r rays can only extend in one of 3 directions in the $\mathbb{Z}^3_{\geq 0}$ lattice: the x_{i_0}, the x_{i_1}, or the x_{i_2} direction. In order to count the number of ways to pick r rays, we first count the ways to divide the rays into one of the 3 directions, and then count the number of ways to orient the rays in each direction. These two counts are independent of each other. It is easy to see that there are $\binom{r+2}{2}$ unique distributions of these r rays into the x_{i_0}, x_{i_1} and x_{i_2} directions.

If $P^{i_j,i_k}_{a,b}$ is in the complement of a saturated ideal then so are $P^{i_j,i_k}_{a-1,b}$ or $P^{i_j}_{a-1}$ unless $a = 0$. Similarly, if $P^{i_j,i_k}_{a,b}$ is in the complement of a saturated ideal then so are $P^{i_j,i_k}_{a,b-1}$ or $P^{i_k}_{b-1}$ unless $b = 0$. Given the previous facts and that we have already picked which quadrants are included, any valid choice of a set of rays in one direction is such that the lattice points of fixed degree in the rays form (the centers of the squares of) a Young diagram. In other words, the number of valid arrangement of k rays in one direction is the number of integer partitions of k.

If we do this for each distinct distribution of rays then the number of ways to choose r rays in $\mathbb{Z}^3_{\geq 0}$ is given by

$$\sum_{c_0+c_1+c_2=r} \left[f_1(c_0) \cdot f_1(c_1) \cdot f_1(c_2) \right]$$

Lastly, when m and r are both nonzero, we find that the choice of the m quadrants will not affect the number of choices possible for the r rays. To see this, recall that a given choice of the m quadrants in $\mathbb{Z}^3_{\geq 0}$ is of the form

$$\left(\bigcup_{i=0}^{\alpha-1} P^0_i \right) \cup \left(\bigcup_{j=0}^{\beta-1} P^0_j \right) \cup \left(\bigcup_{k=0}^{\gamma-1} P^0_k \right)$$

where $\alpha + \beta + \gamma = m$.

This choice "shifts" the region in which the remaining r rays can be chosen. We can think about choosing the remaining r rays in the $\mathbb{Z}^3_{\geq 0}$ lattice in which the coordinate axes are defined by $x_0 - \alpha$, $x_1 - \beta$ and $x_2 - \gamma$. Thus, the number of ways to choose the r rays in this region is exactly the same as choosing r rays when $m = 0$ so the choice of the m quadrants is independent of the choice of the r rays. Thus, combining the earlier two cases, we have that there are

$$\binom{m+2}{2} \cdot \sum_{c_1+c_2+c_3=r} \left[f_1(c_1) \cdot f_1(c_2) \cdot f_1(c_3) \right]$$

many ways to choose m quadrants and r rays in $\mathbb{Z}^3_{\geq 0}$ where $c_1, c_2, c_3 \in \mathbb{Z}_{\geq 0}$. Along with Lemma 4.3, this concludes the proof. □

5 General case

In this section, we prove the remaining cases of Theorem 1.2.

We first note that one case of Theorem 1.2 is immediate as the relevant Hilbert schemes are each a single reduced point.

Proposition 5.1 *If $r = 0$ or $\lambda = (n + 1)$, then $H_{n,\lambda} = 1$.*

For the rest of the cases, we must first show that a saturated monomial ideal corresponds to a union of 1-orthants, 2-orthants, 3-orthants, etc., in the lattice $\mathbb{Z}^{n+1}_{\geq 0}$.

Proposition 5.2 *Given the ring $R = \mathbb{C}[x_0, x_1, \ldots, x_n]$ and a saturated monomial ideal*
$I = (x_0^{\alpha_0^1} \ldots x_n^{\alpha_n^1}, \ldots, x_0^{\alpha_0^m} \ldots x_n^{\alpha_n^m})$, then the set of monomials not in I is of the form $\bigcup_{i=1}^{M} P_{p_1^i,\ldots,p_{r_i}^i}^{c_1^i,\ldots,c_{r_i}^i}$ for some positive integer M and for some fixed $p_1, \ldots, p_{n+1-k} \in \mathbb{Z}_{\geq 0}$ where

$$P_{p_1,\ldots,p_{n+1-k}}^{c_1,\ldots,c_{n+1-k}} = \{x_{c_1}^{p_1} \ldots x_{c_{n+1-k}}^{p_{n+1-k}} x_{c_{n+2-k}}^{p_{n+2-k}} \ldots x_{c_n}^{p_n} \mid p_{n+2-k}, \ldots, p_n \in \mathbb{Z}_{\geq 0}\}$$

is a k-orthant in a $n+1$-dimensional lattice where each point in our lattice $(p_0, \ldots, p_n)$ corresponds to the monomial $x_0^{p_0} \ldots x_n^{p_n}$.

Proof Utilizing the same approach as before gives us that a k-orthant in the $n+1$-dimensional lattice would be of the form

$$P_{p_1,\ldots,p_{n+1-k}}^{c_1,\ldots,c_{n+1-k}} = \{x_{c_1}^{p_1} \ldots x_{c_{n+1-k}}^{p_{n+1-k}} x_{c_{n+2-k}}^{p_{n+2-k}} \ldots x_{c_n}^{p_n} \mid p_{n+2-k}, .., p_n \in \mathbb{Z}_{\geq 0}\}$$
$$\text{for some fixed } p_1, \ldots, p_{n+1-k} \in \mathbb{Z}_{\geq 0}$$

As with the previous approach, let us begin by considering a monomial ideal generated by a single element $(x_0^{\alpha_0} \ldots x_n^{\alpha_n})$. Then, the set of monomials in the compliment of I is

$$\left(\bigcup_{i_0=1}^{\alpha_0-1} P_{i_0}^0\right) \cup \cdots \cup \left(\bigcup_{i_n=1}^{\alpha_n-1} P_{i_n}^n\right).$$

Now, let us consider a monomial ideal with m generators $I = (x_0^{\alpha_0^1} \ldots x_n^{\alpha_n^1}, \ldots, x_0^{\alpha_0^m} \ldots x_n^{\alpha_n^m})$ and let $I_\ell = (x_0^{\alpha_0^\ell} \ldots x_n^{\alpha_n^\ell})$ which then gives us that $I = \bigcup_{\ell=1}^{m} I_\ell$. By De Morgan's laws, we have that

$$\overline{x} \notin I \text{ if and only if } \bar{x} \in \bigcap_{\ell=1}^{m} \left[\left(\bigcup_{i_0=1}^{\alpha_0^\ell-1} P_{i_0}^0\right) \cup \cdots \cup \left(\bigcup_{i_n=1}^{\alpha_n^\ell-1} P_{i_n}^n\right)\right]$$

From here, let us simplify our notation by setting

$$A_\ell^j = \bigcup_{i_j=1}^{\alpha_j^\ell-1} P_{i_j}^j.$$

Using this, we can rewrite the compliment of I as

$$\overline{x} \notin I \text{ if and only if } \overline{x} \in \bigcap_{\ell=1}^{m} \left(\bigcup_{j=1}^{n} A_\ell^j\right).$$

The intersection above then simplifies into a $(n+1)^m$ sized union of m sized intersections.

In order to formally write this, we need some notation. Recall the symmetric group on m letters, denoted S_m, is the set of permutations of the set $\{1, \ldots, m\}$. Define the subset $G_{n,m}$ of S_m as the set of permutations σ of $\{1, \ldots, m\}$ such that $\sigma_1 < \cdots < \sigma_{k_1}$, $\ldots$, and $\sigma_{k_{n-1}+1} < \cdots < \sigma_n$ for some $1 \leq k_1 \leq \cdots \leq k_{n-1} \leq n$, i.e., permutations which decrease at most n times. With this notation, we can write the complement of I as

$$\overline{x} \notin I \text{ if and only if } \overline{x} \in \bigcup_{g \in G_{n,m}} \left(\left(\bigcap_{j=1}^{k_1} A^1_{g(j)} \right) \bigcap \cdots \bigcap \left(\bigcap_{j=k_{n-1}+1}^{n} A^n_{g(j)} \right) \right).$$

Now as before, we observe that if a single one of these intersections has all possible A^i, then it contains finitely many monomials. Hence, we can ignore this case since for $d >> 0$ the Hilbert polynomial $H_I(d)$ would not count these. Next, consider an intersection of the form

$$(A^0_1 \cap \cdots \cap A^0_{j_1}) \cap (A^1_{j_1+1} \cap \cdots \cap A^1_{j_2}) \cap \ldots \cdots \cap (A^h_{j_h+1} \cap \cdots \cap A^h_{j_{h+1}})$$

with $h < n$. If we let $\theta_0 = \min\{\alpha^1_0, \ldots, \alpha^{j_1}_0\}$, $\theta_1 = \min\{\alpha^{j_1+1}_1, \ldots, \alpha^{j_2}_1\}, \ldots, \theta_h = \min\{\alpha^{j_h+1}_h, \ldots, \alpha^{j_{h+1}}_h\}$ then our intersection above is equivalent to

$$\left(\bigcup_{l_0=0}^{\theta_0-1} P^{j_0}_{l_0} \right) \cap \cdots \cap \left(\bigcup_{l_h=0}^{\theta_h-1} P^{j_h}_{l_h} \right) = \bigcup_{0 \leq l_0 \leq \theta_0-1, \ldots, 0 \leq l_h \leq \theta_h-1} P^{j_0,\ldots,j_h}_{l_0,\ldots,l_h}.$$

Thus, the intersections of this form correspond to an $n-h$-orthant. Since any intersection in the union has this form for some $1 \leq h \leq n-1$ (or is irrelevant to the saturation), the result follows. □

Based on the case of the line, one may naively expect that any arrangement of (k-)orthants giving the Hilbert polynomial p_λ where $\lambda = (n^{k_n}, \ldots, 1^{k_1})$ consists of k_i many i-orthants, but this is incorrect.

Example 5.1 Consider the Hilbert polynomial $p(d) = 2d + 1$ with lambda sequence $\lambda = (2^2)$ and $n \geq 3$. If the naive correspondence holds true for all $n \geq 3$, then we should expect that every choice of two quadrants in $\mathbb{Z}^{n+1}_{\geq 0}$ will result in a Hilbert function $h(d)$ such that $h(d) = p(d)$ for some $d >> 0$. However, consider the following two quadrants in $\mathbb{Z}^{n+1}_{\geq 0}$

$$P_1 = P^{2,3,4,\ldots,n}_{0,0,0,\ldots,0} \text{ and } P_2 = P^{0,1,4,\ldots,n}_{0,0,0,\ldots,0}$$

Note that P_1 and P_2 do not intersect other than at the origin since recall that the elements of P_1 and P_2 are

$$P_1 = \{1, x_0, x_1, x_0x_1, ..., x_0^a x_1^b, ...\} \text{ for some } \quad a, b \in \mathbb{Z}_{\geq 0}$$
$$P_2 = \{1, x_2, x_3, x_2x_3, ..., x_2^c x_3^d, ...\} \text{ for some } \quad c, d \in \mathbb{Z}_{\geq 0}$$

Thus, P_1 and P_2 each contribute $d + 1$ to the Hilbert polynomial for this specific monomial ideal $h(d)$. Thus, the Hilbert polynomial of the corresponding ideal must be $h(d) = (d + 1) + (d + 1) = 2d + 2$. Thus, the naive correspondence fails.

In order to study the case where the naive correspondence does hold, or where we can salvage an adapted version of it, we first need a general remark

Remark 5.3 Consider a k-orthant K in the complement of a saturated monomial ideal I such that every $(k+1)$-orthant containing it intersects the ideal. The number of degree d monomials in K is $\binom{d+k-1}{k-1}$ so K's contribution to the Hilbert polynomial only effects the coefficients of the terms with degree at most $k - 1$ and adds nonnegatively to the degree $k - 1$ coefficient.

Using this remark, we can show that the naive correspondence does hold for the n-orthants in the arrangement and the n^k in the partition, which splits off as a product by 2.3.

Lemma 5.4 *Let p_λ be the Hilbert polynomial corresponding to the partition $\lambda = (n^k)$. Then, the complement of any saturated monomial ideal in $n + 1$ variables with Hilbert polynomial p_λ is exactly k many n-orthants and*

$$H_{n,\lambda} = \binom{n+k}{k}.$$

Proof We first show that the complement of any monomial with Hilbert polynomial p_λ contains exactly k many n-orthants. Given λ, the corresponding Hilbert polynomial $\sum_{i=1}^{m} \binom{t+\lambda_i-i}{\lambda_i-1}$ has degree $n-1$. Since there are no terms of degree greater than $n-1$, the complement of the ideal cannot contain the entire $n+1$-dimensional lattice so only the n-orthants contained in the complement of I contribute to the leading coefficient. Since the n-orthants overlap with other (k-)orthants in lower-dimensional spaces, each n-orthant contributes $\frac{1}{(n-1)!}$ to the leading coefficient. Since the leading coefficient of p_λ is $\frac{k}{(n-1)!}$, the complement of I contains exactly k many n-orthants.

We now exclude the possibility of any other k-orthants being in the complement of the ideal. If we think of choosing those n-orthants in order by the amount which they are shifted away from the axes, consider the i-th chosen n-orthant, K. It is parallel to j previously chosen n-orthants for some $0 \leq j \leq i - 1$. Then, there are $\binom{d+n-j}{n-1}$ monomials of degree d in K. Similarly, it overlaps each of the k-th previously chosen nonparallel n-orthants (of the $i - 1 - j$ previously n-orthants) in $\binom{d+n-1-k}{n-2}$ many monomials of degree d. Thus, the contribution of the i-th chosen n-orthant is $\binom{d+n-j}{n-1} - \sum_{k=1}^{i-j} \binom{d+n-k-j}{n-2} = \binom{d+n-i}{n-1}$. Thus, the contribution of all of the n-orthants to the Hilbert polynomial is $\sum_{i=1}^{k} \binom{d+n-i}{n-1}$ so there are no other k-orthants in the complement of the ideal.

Finally, we count $H_{n,\lambda}$. By the proof so far, a saturated monomial with this Hilbert polynomial is equivalent to the choice of k many n-orthants so there are $\binom{n+k}{k}$ ways to choose k of them. $\square$

These lemmas form the basis for proving the remaining cases of Theorem 1.2, in the following there propositions.

Proposition 5.5 *If* $\lambda = (n^k, 1)$, *then* $H_{n,\lambda} = \binom{n+k}{k}(n+1)$.

If $\lambda = (n^k, 1^2)$, *then* $H_{n,\lambda} = 3\binom{n+k}{k}\binom{n+1}{2}$.

If $\lambda = (n^{r-2}, \lambda_{r-1}, 1)$ *where* $r \geq 2$ *and* $n-1 \geq \lambda_{r-1} \geq 2$, *then* $H_{n,\lambda} = \binom{n+r-2}{r-2}\binom{n+1}{\lambda_{r-1}}(n+1-\lambda_{r-1})(\lambda_{r-1}+1)$, *respectively.*

Proof In the case $\lambda = (1)$, the Hilbert polynomial is $p_\lambda = 1$. Then, the Hilbert scheme is $\mathbb{P}^n$ so the result is immediate. Similarly, the case of $\lambda = (n^k, 1^2)$ is immediate by Lemma 5.4, Theorem 2.3, and Lemma 2.1 of [16].

In the last case, it suffices to show that $H_{n,\lambda} = \binom{n+1}{\lambda_{r-1}}(n+1)$ where λ corresponds to the partition $(\lambda_1, 1)$ with $n-1 \geq \lambda_1 \geq 2$ by Lemma 5.4 and Theorem 2.3. The Hilbert polynomial in this case is $p_\lambda = \binom{d+\lambda_1}{\lambda_1-1} + 1$ which has degree $\lambda_1 - 1$, the complement of the ideal must contain at least 1 λ_1-orthant. The first λ_1-orthant chosen contributes $\binom{d+\lambda_1}{\lambda_1-1}$ to the Hilbert polynomial. After having chosen this, there is only a remaining 1 to be contributed to the Hilbert polynomial, which again must be contributed by a ray. Thus, a saturated monomial ideal with Hilbert polynomial p_λ is equivalent to picking a λ_1-orthant and a ray.

There is $\binom{n+1}{\lambda_1}$ ways to choose the λ_1-orthant, and there are then $n+1-\lambda_1$ ways to choose the ray not parallel to the λ_1 orthant and $\lambda_1(n+1-\lambda_1)$ ways to choose the ray parallel to it given the previous choice. Putting these together gives

$$H_{n,\lambda} = \binom{n+1}{\lambda_1}(n+1-\lambda_1)(\lambda_1+1).$$

$\square$

Lemma 5.6 *If* $\lambda = (n^r, \lambda_{r+1}^s, 1)$ *where* $r, s \geq 0$ *and* $n-1 \geq \lambda_{r+1} \geq 1$ *and* $\mathbb{P}^{n[p_\lambda]}$ *is smooth, then*

$$H_{n,\lambda} = \binom{n+r}{r}\binom{n+1}{\lambda_{r+1}+1}\left(\binom{\lambda_{r+1}+1+s}{s}\right.$$
$$\left.(n-\lambda_{r+1})(\lambda_{r+1}+2) - (\lambda_{r+1}+1)(n-1-2\lambda_{r+1})\right).$$

Proof By Lemma 5.4 and Theorem 2.3, it suffices to show that

$$H_{n,\lambda} = \left(\binom{n+1}{\lambda_{r+1}+1}\left(\binom{\lambda_{r+1}+1+s}{s} - (\lambda_{r+1}+1)\right) + \binom{n+1}{\lambda_{r+1}}\right)(n+1)$$

where λ corresponds to the partition $(\lambda_1^s, 1)$. Arguing analogously to the previous cases using the coefficients of the Hilbert polynomials, we see the complement must also contain exactly s many λ_1-orthants. However, not all choices of s many λ_1-orthants live in the complement some ideal with the correct Hilbert polynomial. In particular, a generalization of Example 5.1 to 6 variables (choosing two 3-orthants such that they all only intersect at the origin) shows that not all choices of s quadrants live in the complement some ideal with the correct Hilbert polynomial. We now characterize those that do.

If the λ_1-orthants are not all contained in any $(\lambda_1 + 1)$-orthant, then we claim that they do not give the correct Hilbert polynomial. In particular, choose the first two of them to not lie in the same $(\lambda_1 + 1)$-orthant, then they contribute at least

$$\binom{d+\lambda_1-1}{\lambda_1-1} + \binom{d+\lambda_1-2}{\lambda_1-1} + \left(\binom{d+\lambda_1-2}{\lambda_1-2} - \binom{d+\lambda_1-2}{\lambda_1-3}\right)$$

and all of the subsequently chosen ones contribute at least the expected amount. In either case, this would force a negative number of $(\lambda_1 - 1)$-orthants in order to have the correct Hilbert polynomial, which is obviously impossible.

On the other hand, if all of them are chosen within a $(\lambda_1 + 1)$-orthant, then the k-th chosen one contributes $\binom{d+\lambda_1-k}{\lambda_1-1}$ to the Hilbert polynomial as expected.

Finally, given a choice of s many λ_1-orthants in some $(\lambda_1 + 1)$-orthant, there is a remaining one in the Hilbert polynomial which must be contributed by a single additional ray. Thus, a saturated monomial ideal with Hilbert polynomial p_λ is equivalent to the choice of s many λ_1-orthants in some $(\lambda_1 + 1)$-orthant and a single ray.

There at $\binom{n+1}{\lambda_1+1}$ ways to choose the (λ_1+1)-orthant containing the λ_1-orthants. We count the case where all of the λ_1-orthants are parallel separately as the number of choices of ray depends on that difference. Given a chosen $(\lambda_1 + 1)$-orthant, there are $\binom{\lambda_1+1+s}{s} - (\lambda_1 + 1)$ ways to choose the λ_1-orthants such that they are not all parallel and $(\lambda_1 + 1)$ ways to choose them all parallel. If the λ_1-orthants are not all parallel, then there are $(n + 1 - (\lambda_1 + 1))(\lambda_1 + 2) = (n - \lambda_1)(\lambda_1 + 2)$ ways to choose a ray. If the λ_1-orthants are all parallel, then there are $(n + 1 - \lambda_1)(\lambda_1 + 1)$ ways to choose a ray.

Putting this together gives

$$\begin{aligned} H_{n,\lambda} &= \binom{n+1}{\lambda_1+1}\left(\left(\binom{\lambda_1+1+s}{s} - (\lambda_1+1)\right)(n-\lambda_1)(\lambda_1+2)\right. \\ &\qquad \left. +(n+1-\lambda_1)(\lambda_1+1)^2\right) \\ &= \binom{n+1}{\lambda_1+1}\left(\binom{\lambda_1+1+s}{s}(n-\lambda_1)(\lambda_1+2) - (\lambda_1+1)(n-1-2\lambda_1)\right). \end{aligned}$$

□

When $\lambda_{r+1} \geq 3$, replacing r and s with $r-s-3$ and $s+2$, respectively, immediately gives the following corollary.

Proposition 5.7 *If* $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$ *where* $r-3 \geq s \geq 0$ *and* $n-1 \geq \lambda_{r-s-2} \geq 3$*, then*

$$H_{n,\lambda} = \binom{n+r-s-3}{r-s-3}\binom{n+1}{\lambda_{r-s-2}+1}$$
$$* \left(\binom{\lambda_{r-s-2}+s+3}{s+2}(n-\lambda_{r-s-2})(\lambda_{r-s-2}+2) - (\lambda_{r-s-2}+1)(n-1-2\lambda_{r-s-2})\right).$$

Similarly, when $\lambda_{r+1} = 2$, replacing r and s with $r-s-5$ and $s+4$, respectively, immediately gives the following corollary.

Proposition 5.8 *If* $\lambda = (n^{r-s-5}, 2^{s+4}, 1)$ *where* $r-5 \geq s \geq 0$*, then*

$$H_{n,\lambda} = \binom{n+r-s-5}{r-s-5}\binom{n+1}{3}\left(4\binom{s+7}{s+4}(n-2) - 3(n-5)\right).$$

Proposition 5.9 *If* $\lambda = (n^k, 1^3)$*, then* $H_{n,\lambda} = \binom{k+n}{n}\frac{n(n+1)(5n+1)}{3}$.

Proof By Lemma 5.4 and Theorem 2.3, it suffices to show that $H_{n,\lambda} = \frac{n(n+1)(5n+1)}{3}$ where λ corresponds to the partition (1^3). Since rays contribute 1 each and any k-orthant for $k \geq 2$ would change higher order terms, this means that there are 3 rays to be chosen. Thus, a saturated monomial with this Hilbert polynomial is equivalent to the choice of 3 rays.

There are $\binom{n+1}{3}$ ways to choose the rays in distinct directions. There are $2n\binom{n+1}{2}$ ways to choose the rays in only two directions. Finally, there are $(n+1)\left(n+\binom{n}{2}\right)$ ways to choose the three rays in the same direction. In total, there are

$$\left[\binom{n+1}{3} + 2n\binom{n+1}{2} + (n+1)\left(n + \frac{n(n-1)}{2}\right)\right]$$

saturated monomial ideals in $n+1$ variables with Hilbert polynomial p_λ. This count simplifies to

$$H_{n,\lambda} = \left(\frac{n(n+1)(5n+1)}{3}\right).$$

□

References

1. Białynicki-Birula, A.: Some theorems on actions of algebraic groups. Ann. of Math. **2**(98), 480–497 (1973)
2. Ellingsrud, G., Strømme, S.A.: On the homology of the Hilbert scheme of points in the plane. Invent. Math. **87**(2), 343–352 (1987)
3. Evain, L.: The Chow ring of punctual Hilbert schemes on toric surfaces. Transform. Groups **12**(2), 227–249 (2007)
4. Fogarty, J.: Algebraic families on an algebraic surface. Amer. J. Math. **90**, 511–521 (1968)

5. Fulton, W.: Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas, 2nd edn. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Berlin (1998)
6. Göttsche, L.: Hilbert schemes of points on surfaces. In: Proceedings of the International Congress of Mathematicians, vol. II (Beijing, 2002), pp. 483–494. Higher Ed. Press, Beijing (2002)
7. Göttsche, L.: The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann. **286**(1–3), 193–207 (1990)
8. Grojnowski, I.: Instantons and affine algebras. I. The Hilbert scheme and vertex operators. Math. Res. Lett. **3**(2), 275–291 (1996)
9. Grothendieck, A.: Techniques de construction et théorémes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In Séminaire Bourbaki, vol. 6, Exp. No. 221, pp. 249–276. Soc. Math. France, Paris (1995)
10. Hilbert, D.: Gesammelte Abhandlungen. Band II: Algebra, Invariantentheorie, Geometrie. Springer, Berlin (1970)
11. Li, W., Qin, Z., Wang, W.: Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces. Math. Ann. **324**(1), 105–133 (2002)
12. Lehn, M., Sorger, C.: Symmetric groups and the cup product on the cohomology of Hilbert schemes. Duke Math. J. **110**(2), 345–357 (2001)
13. Macaulay, F.S.: Some properties of enumeration in the theory of modular systems. Proc. Lond. Math. Soc. **2**(26), 531–555 (1927)
14. Nakajima, H.: Lectures on Hilbert Schemes of Points on Surfaces. University Lecture Series, vol. 18. American Mathematical Society, Providence (1999)
15. Nakajima, H.: More lectures on Hilbert schemes of points on surfaces. In: Fujino, O., Kondō, S., Moriwaki, A., Saito, M.-H., Yoshioka, K. (eds.) Development of Moduli Theory-Kyoto 2013, volume 69 of Advanced Studies in Pure Mathematics, pp. 173–205. Mathematical Society of Japan, Tokyo (2016)
16. Ryan, T.: The geometry of Hilbert schemes of two points on projective space (2021). arXiv:2103.12674
17. Skjelnes, R., Smith, G.G.: Smooth Hilbert schemes: their classification and geometry (2020). arXiv:2008.08938