Breaking up simplicial homology and subadditivity of syzygies

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Abstract

We consider the following question: If a simplicial complex Γ has d-homology, then does the corresponding d-cycle always induce cycles of smaller dimension that are not boundaries? We provide an answer to this question in a fixed dimension. We use the breaking of homology to show the subadditivity property for the maximal degrees of syzygies of monomial ideals in a fixed homological degree.

Keywords Monomial ideals · Subadditivity · Simplicial homology

1 Introduction

The motivation for this paper is the subadditivity property for the maximal degrees of syzygies of monomial ideals in polynomial rings. Let I be a homogeneous ideal in the polynomials ring $S = k[x_1, \ldots, x_n]$ over a field k. Let t_a denoted the maximum value of j such that the graded Betti number $\beta_{a,j}(S/I) \neq 0$. The ideal I satisfies the subadditivity property on the maximal degrees of its syzygies if

$$t_{a+b} \leq t_a + t_b \tag{1}$$

where $a + b$ is not more than the projective dimension of the ideal.

The inequality in (1) arises most naturally in the context of (Castelnuovo–Mumford) regularity, which, for the ideal I, can be described as the maximum value of $t_a - a$,

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for all positive integers a. It has been shown to fail in general by Avramov, Conca and Iyengar [2], even if one restricts to Cohen–Macaulay or even Gorenstein settings (see [22] for examples and for a general survey on the topic). However, many special cases are known: certain algebras with codimension ≤ 1 (Eisenbud, Huneke and Ulrich [11]), certain classes of Koszul rings (Avramov, Conca and Iyengar [2]), certain homological degrees for Gorenstein algebras (El Khoury and Srinivasan [12]), among others.

Avramov, Conca and Iyengar [2] conjectured that the subadditivity property holds for Kozul rings and for all monomial ideals (it is also open for toric ideals [22]). In the case of monomial ideals, there are special cases for which (1) has been verified: when $a = 1$ (Herzog and Srinivasan [19]), when $a = 1, 2, 3$ and I is generated in degree 2 (Fernández-Ramos and Gimenez [15], Abedelfatah and Nevo [3]), Cohen—Macaulay ideals generated by monomials of degree 2 when the base field has characteristic 0 [2], facet ideals of simplicial forests (Faridi [13]), ideals whose Betti diagram has a special "shape" (Bigdeli and Herzog [5]), several classes of edge ideals of graphs and path ideals of rooted trees (Jayanthan and Kumar [20]), and for a where the Stanley–Reisner complex of I has dimension bounded by $t_a - a$ (Abedelfatah [1]).

In the case of monomial ideals, the syzygies can be characterized as dimensions of homology modules of topological objects. This is one of the central themes of Stanley–Reisner Theory, connecting Commutative Algebra to Discrete Geometry and Topology. We refer the reader to the books [6,24] for more details on these rich connections.

By viewing the subadditivity property as a geometric one, the inequality in (1) can be shown to follow from the following general type of question:

Does a topological object with d-homology break into sub-objects that have a-homology and b-homology, where a and b are related to d?

This approach was taken by the first author in [13], where the topological objects were atomic lattices (lcm lattices of monomial ideals); see Question 2.1 and Question 2.2 below. In this paper, using Hochster's formula (Equation (2)), we examine this problem from the point of view of the Stanley–Reisner complex, and we can provide a positive answer to the general question above for a fixed value of d. As a result, we show that subadditivity holds in a fixed homological degree for all monomial ideals. The last section interprets the square-free results of the paper for general monomial ideals.

2 Setup

2.1 The subadditivity property

Throughout the paper, let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. If I is a graded ideal of S with minimal free resolution

$$0 \to \oplus_{j \in \mathbb{N}} S(-j)^{\beta_{p,j}} \to \oplus_{j \in \mathbb{N}} S(-j)^{\beta_{p-1,j}} \to \cdots \to \oplus_{j \in \mathbb{N}} S(-j)^{\beta_{1,j}} \to S,$$

then for each i and j, the rank $\beta_{i,j}(S/I)$ of the free S-modules appearing above are called the **graded Betti numbers** of the S-module S/I.

If we let

$$t_a = \max\{j \ : \ \beta_{a,j}(S/I) \neq 0\},$$

a question is whether the t_a satisfy the **subadditivity property**: $t_{a+b} \leq t_a + t_b$?

The answer is known to be negative for a general homogeneous ideal [2], and unknown in the case of monomial ideals. For the case of monomial ideals, there are special cases that are known [1,3,5,13,15,19].

In the case of monomial ideals, Betti numbers can be interpreted as the homology of objects in discrete topology: simplicial complexes, order complexes of lattices, etc.; see, for example, [24] for a survey of this approach. As a result, the subadditivity question can be viewed as a question of breaking up homology in these objects. This idea was explored in [13] by the first author, where the subadditivity problem was solved for facet ideals of simplicial forests using homology of lattices.

By a method called *polarization* [16] (see Section. 5 for the definition), one can reduce questions regarding Betti numbers of monomial ideals to the class of **square-free** monomial ideals.

If $u \subset [n] = \{1, \ldots, n\}$, then we define

$$\mathbf{m}_u = \Pi_{i \in u} x_i$$

to be the **square-free monomial with support** u.

For our purposes it is useful to consider a finer grading of the Betti numbers by indexing the Betti numbers with monomials of the polynomial ring S. A *multigraded* Betti number of S/I is of the form $\beta_{i,\mathbf{m}}(S/I)$ where $\mathbf{m}$ is a monomial in S and

$$\beta_{i,j}(S/I) = \sum_{u \subseteq [n] \text{ and } |u|=j} \beta_{i,\mathbf{m}_u}(S/I).$$

2.2 Simplicial complexes

A **simplicial complex** Γ on a set W is a set of subsets of W with the property that if $F \in \Gamma$ then for every subset $G \subseteq F$ we have $G \in \Gamma$. Every element of Γ is called a **face**, the maximal faces under inclusion are called **facets**, and a simplicial complex contained in Γ is called a **subcomplex** of Γ. The set of all $v \in W$ such that $\{v\} \in \Gamma$ is called the **vertex set** of Γ, and is denoted by $V(\Gamma)$. The set of facets of Γ is denoted by $\text{Facets}(\Gamma)$. If $\text{Facets}(\Gamma) = \{F_1, \ldots, F_q\}$, then we denote Γ by

$$\Gamma = \langle F_1, \ldots, F_q \rangle.$$

If $A \subset V(\Gamma)$, then the **induced subcomplex** Γ_A is defined as

$$\Gamma_A = \{F \in \Gamma \ : \ F \subseteq A\}.$$

The **Alexander dual** $\Gamma^\vee$ of Γ, if we set $F^c = V(\Gamma) \setminus F$, is defined as

$$\Gamma^\vee = \{F \subseteq V(\Gamma) \ : \ F^c \notin \Gamma\} = \{V(\Gamma) \setminus F \ : \ F \notin \Gamma\}.$$

The **link** of a face F of Γ is

$$\text{lk}_\Gamma(F) = \{G \in \Gamma \ : \ F \cap G = \emptyset \text{ and } F \cup G \in \Gamma\}.$$

If I is a square-free monomial ideal in S, it corresponds uniquely to a simplicial complex

$$\mathcal{N}(I) = \{u \subset [n] \ : \ m_u \notin I\}$$

called the **Stanley–Reisner complex** of I. Conversely, if Γ is a simplicial complex whose vertices are labelled with $x_1, \ldots, x_n$, then one can associate to it its unique **Stanley–Reisner ideal**

$$\mathcal{N}(\Gamma) = \{m_u \ : \ u \subset [n] \text{ and } u \notin \Gamma\}.$$

The uniqueness of the Stanley–Reisner correspondence implies that

$$\mathcal{N}(\Gamma) = I \iff \mathcal{N}(I) = \Gamma.$$

2.3 The LCM lattice

A *lattice* is a partially ordered set where every two elements have a greatest lower bound called their *meet* and a lowest upper bound called their *join*. A bounded lattice has an upper and a lower bound denoted by $\hat{1}$ and $\hat{0}$, respectively.

If L is a lattice with r elements, then the **order complex** of L is the simplicial complex on r vertices, where the elements of each chain in L form a face.

If I is a monomial ideal, then the **lcm lattice** of I, denoted by $\text{LCM}(I)$, is a bounded lattice ordered by divisibility, whose elements are the generators of I and their least common multiples, and the meet of two elements is their least common multiple.

Two elements of a lattice are called **complements** if their join is $\hat{1}$ and their meet is $\hat{0}$. If the lattice is $\text{LCM}(I)$, then it was shown in [13] that two monomials in $\text{LCM}(I)$ are complements if their gcd is not in I and their lcm is the lcm of all the generators of I.

Gasharov, Peeva and Welker [17,24] showed that multigraded Betti numbers of S/I can be calculated from the homology of (the order complex of) the lattice $\text{LCM}(I)$: If $\mathbf{m}$ is a monomial in $L = \text{LCM}(I)$, then

$$\beta_{i,\mathbf{m}}(S/I) = \dim_k \tilde{H}_{i-2}\left((1, \mathbf{m})_L; k\right)$$

where $(1, \mathbf{m})_L$ refers to the subcomplex of the order complex consisting of all non-trivial monomials in L strictly dividing $\mathbf{m}$.

On the other hand, in a 1977 paper, Baclawski [4] showed that if L is a finite lattice whose proper part has nonzero homology, then every element of L has a complement.

The following question was raised in [13] as a potential way to answer the subadditivity question.

Question 2.1 If I is a square-free monomial ideal in variables $x_1, \ldots, x_n$, and $\beta_{i,n}(S/I) \neq 0$, $a, b > 0$ and $i = a + b$, are there complements $\mathbf{m}$ and $\mathbf{m}'$ in LCM(I) with $\beta_{a,\mathbf{m}}(S/I) \neq 0$ and $\beta_{b,\mathbf{m}'}(S/I) \neq 0$?

Considering that it is enough to study the "top degree" Betti numbers (those of degree n, in this case) [9,13], a positive answer to Question 2.1 will establish the subadditivity property for all monomial ideals, since

$$t_a + t_b \geq \deg(\mathbf{m}) + \deg(\mathbf{m}') \geq n = t_i.$$

Question 2.1 can be written more generally as a question about the homology of the lcm lattice, or in fact, any finite lattice.

Question 2.2 If $L = \text{LCM}(I)$ and $\tilde{H}_{i-2}((1, x_1 \cdots x_n)_L; k) \neq 0$, $a, b > 0$ and $i = a + b$, are there complements $\mathbf{m}$ and $\mathbf{m}'$ in LCM(I) with $\tilde{H}_{a-2}((1, \mathbf{m})_L; k) \neq 0$ and $\tilde{H}_{b-2}((1, \mathbf{m}')_L; k) \neq 0$?

With the same idea, one could translate Question 2.1 into breaking up simplicial homology using Hochster's formula.

2.3.1 Hochster's formula

Let $I = (\mathbf{m}_1, \ldots, \mathbf{m}_q)$ be a square-free monomial ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$. Hochster's formula (see for example [18, Cor. 8.1.4 and Prop. 5.1.8]) states that if $I = \mathcal{N}(\Gamma)$ and $\mathbf{m}_u$ a monomial, then

$$\beta_{i,\mathbf{m}_u}(S/I) = \dim_k \tilde{H}_{i-2}(\text{lk }_{\Gamma^\vee}(u^c), k) = \dim_k \tilde{H}_{|u|-i-1}(\Gamma_u, k) \qquad (2)$$

where $u^c = [n] \setminus u$ is the set complement of u. We would now like to reinterpret Question 2.1 in the language of Hochster's formula. To begin with, since we are dealing with square-free monomials, we can consider a monomial $\mathbf{m}_u$ equivalent to the set u and use intersections for gcd, unions for lcm, and $\mathbf{m}_u^c$ for u^c.

Suppose

$$\beta_{i,x_1 \ldots x_n}(S/I) = \dim_k \tilde{H}_{i-2}(\text{lk }_{\Gamma^\vee}(\emptyset), k) = \dim_k \tilde{H}_{i-2}(\Gamma^\vee, k) \neq 0$$

and $i = a + b$ where $a, b > 0$. We would like to know if there are complements $\mathbf{m}, \mathbf{m}' \in \text{LCM}(I)$ such that

$$\beta_{a,\mathbf{m}}(S/I) \neq 0 \text{ and } \beta_{b,\mathbf{m}'}(S/I) \neq 0.$$

First observe that, $\Gamma^\vee = \langle \mathbf{m}_1^c, \ldots, \mathbf{m}_q^c \rangle$ (e.g. [18] or [14, Prop. 2.4]).

We have

$$\mathbf{m} \in \mathrm{LCM}(I)$$

$$\Longleftrightarrow \mathbf{m} = \mathbf{m}_{i_1} \cup \mathbf{m}_{i_2} \cup \cdots \cup \mathbf{m}_{i_s} \text{ for some } 1 \le i_1 < i_2 < \cdots < i_s \le q$$

$$\Longleftrightarrow \mathbf{m}^c = \mathbf{m}_{i_1}^c \cap \mathbf{m}_{i_2}^c \cap \cdots \cap \mathbf{m}_{i_s}^c \text{ for some } 1 \le i_1 < i_2 < \cdots < i_s \le q$$

$$\Longleftrightarrow \mathbf{m}^c \text{ is the intersection of some facets of } \Gamma^\vee.$$

Moreover, if $\mathbf{m}, \mathbf{m}' \in \mathrm{LCM}(I)$, then

$$\mathbf{m}, \mathbf{m}' \text{are complements} \Longleftrightarrow \mathbf{m} \cup \mathbf{m}' = [n] \text{ and } \mathbf{m} \cap \mathbf{m}' \notin I$$

$$\Longleftrightarrow \mathbf{m}^c \cap \mathbf{m}'^c = \emptyset \text{ and } \mathbf{m} \cap \mathbf{m}' \in \Gamma$$

$$\Longleftrightarrow \mathbf{m}^c \cap \mathbf{m}'^c = \emptyset \text{ and } (\mathbf{m} \cap \mathbf{m}')^c \notin \Gamma^\vee$$

$$\Longleftrightarrow \mathbf{m}^c \cap \mathbf{m}'^c = \emptyset \text{ and } \mathbf{m}^c \cup \mathbf{m}'^c \notin \Gamma^\vee.$$

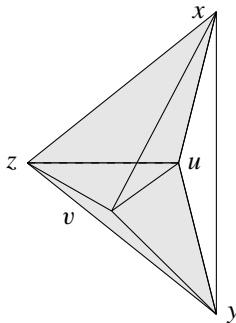So we are looking for subsets $A, B \subseteq [q]$ such that

1. $\mathbf{m}^c = \bigcap_{j \in A} \mathbf{m}_j^c$ and $\mathbf{m}'^c = \bigcap_{j \in B} \mathbf{m}_j^c$
2. $\mathbf{m}^c \cap \mathbf{m}'^c = \emptyset$
3. $\mathbf{m}^c \cup \mathbf{m}'^c \notin \Gamma^\vee$
4. $\tilde{H}_{a-2}(\mathrm{lk}_{\Gamma^\vee}(\mathbf{m}^c), k) \ne 0$ and $\tilde{H}_{b-2}(\mathrm{lk}_{\Gamma^\vee}(\mathbf{m}'^c), k) \ne 0$.

Now we can state Question 2.1 in the following form.

Question 2.3 If $\Gamma = \langle F_1, \dots, F_q \rangle$ is a simplicial complex with $\tilde{H}_{i-2}(\Gamma, k) \ne 0$ and $i = a + b$ where $a, b > 0$, can we find subsets $A, B \subseteq [q]$ such that

1. $F = \bigcap_{j \in A} F_j$ and $G = \bigcap_{j \in B} F_j$
2. $F \cap G = \emptyset$
3. $F \cup G \notin \Gamma$
4. $\tilde{H}_{a-2}(\mathrm{lk}_{\Gamma}(F), k) \ne 0$ and $\tilde{H}_{b-2}(\mathrm{lk}_{\Gamma}(G), k) \ne 0$?

Example 2.4 If $\mathcal{N}(I)^\vee = \Gamma = \langle xzu, xzv, xuv, yzu, yzv, yuv, xy \rangle$,

then $I = (xz, yz, xu, yu, xv, yv, zuv)$ has Betti table

$$
\begin{array}{llllll}
& 0 & 1 & 2 & 3 & 4 \\
\text{total}: & 1 & 7 & 11 & 6 & 1 \\
0: & 1 & . & . & . & . \\
1: & . & 6 & 9 & 5 & 1 \\
2: & . & 1 & 2 & 1 & .
\end{array}
$$

So $\beta_{i,xyzuv} \neq 0$ when $i = 3, 4$, which corresponds to nonvanishing of homology of links of faces of Γ in dimensions 1, 2. We consider each case separately:

1. $i = 3, a = 1, b = 2$. Then $\tilde{H}_1(\Gamma, k) \neq 0$. Let $F = xy$ and $G = xuv \cap yuv = uv$, then $F \cap G = \emptyset$, $F \cup G = xyuv \notin \Gamma$, and

$$
\tilde{H}_{a-2}(\text{lk }_\Gamma(F), k) = \tilde{H}_{-1}(\langle\emptyset\rangle, k) \neq 0
$$
$$
\text{and } \tilde{H}_{b-2}(\text{lk }_\Gamma(G), k) = \tilde{H}_0(\langle x, y\rangle, k) \neq 0.
$$

2. $i = 4, a = 1, b = 3$. Then $\tilde{H}_2(\Gamma, k) \neq 0$. Let $F = yzu$ and $G = xzu \cap xuv \cap xzv \cap xy = x$, then $F \cap G = \emptyset$, $F \cup G = xyzu \notin \Gamma$, and

$$
\tilde{H}_{a-2}(\text{lk }_\Gamma(F), k) = \tilde{H}_{-1}(\langle\emptyset\rangle, k) \neq 0
$$
$$
\text{and } \tilde{H}_{b-2}(\text{lk }_\Gamma(G), k) = \tilde{H}_1(\langle zu, uv, zv, y\rangle, k) \neq 0.
$$

3. $i = 4, a = 2, b = 2$. Then $\tilde{H}_2(\Gamma, k) \neq 0$. Let $F = yzu \cap yuv = yu$ and $G = xzu \cap xzv = xz$, then $F \cap G = \emptyset$, $F \cup G = xyzu \notin \Gamma$, and

$$
\tilde{H}_{a-2}(\text{lk }_\Gamma(F), k) = \tilde{H}_0(\langle z, v\rangle, k) \neq 0
$$
$$
\text{and } \tilde{H}_{b-2}(\text{lk }_\Gamma(G), k) = \tilde{H}_0(\langle u, v\rangle, k) \neq 0.
$$

A dual version of Question 2.3 can be stated as follows (see Corollary 3.6 for the justification).

Question 2.5 If Γ is a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$, and $\tilde{H}_{i-2}(\Gamma, k) \neq 0$, and $n - i + 1 = a + b$, where a and b are positive integers, are there nonempty subsets $C, D \subseteq \{x_1, \ldots, x_n\}$ such that

1. $C \cup D = \{x_1, \ldots, x_n\}$
2. $C \cap D \in \Gamma$
3. $\tilde{H}_{|C|-a-1}(\Gamma_C, k) \neq 0$ and $\tilde{H}_{|D|-b-1}(\Gamma_D, k) \neq 0$?

Example 2.6 Let $\mathcal{N}(I) = \Gamma = \langle zwx, vwx, uvx, zux, zuy, uvy, vwy, zwy\rangle$.

Γ Γ_C Γ_D
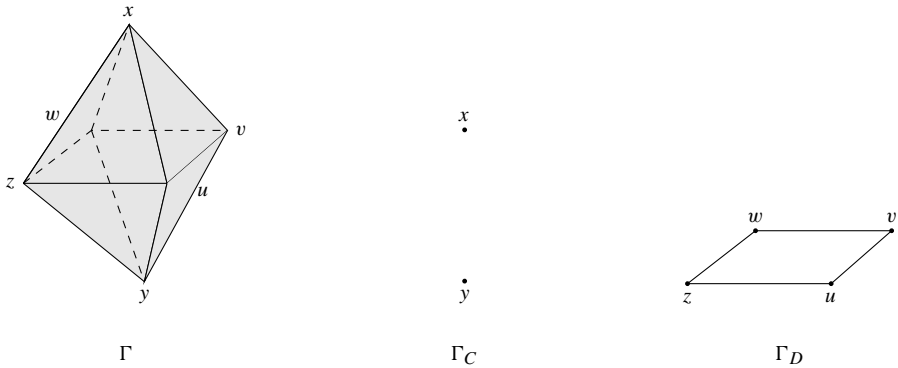
Then $I = (xy, zv, uw)$ has Betti table

```
              0 1 2 3
     total : 1 3 3 1
         0 : 1 . . .
         1 : . 3 . .
         2 : . . 3 .
         3 : . . . 1
```

So $\beta_{3,xyzuvw}(S/I) \neq 0$ which corresponds to nonvanishing homology of Γ in dimension 2 (i.e., $\tilde{H}_2(\Gamma, k) \neq 0$). Let $a = 1$ and $b = 2$. Choose $C = \{x, y\}$ and $D = \{z, u, v, w\}$. Then $C \cup D = \{x, y, z, u, v, w\}$, $C \cap D = \emptyset \in \Gamma$ and

$$\tilde{H}_{|C|-a-1}(\Gamma_C, k) = \tilde{H}_0(\langle x, y \rangle, k) \neq 0 \text{ and}$$
$$\tilde{H}_{|D|-b-1}(\Gamma_D, k) = \tilde{H}_1(\langle zu, uv, vw, zw \rangle, k) \neq 0.$$

A positive answer to either Questions 2.3 or 2.5 would settle the subadditivity question for syzygies.

3 Main results

The following lemma is an easy exercise.

Lemma 3.1 Γ *simplicial complex and* $A \in \Gamma$ *and* $B \in \mathrm{lk}_\Gamma(A)$, *then*

$$\mathrm{lk}_{\mathrm{lk}_\Gamma(A)}(B) = \mathrm{lk}_\Gamma(A \cup B).$$

In a simplicial complex Γ we say a d-cycle Σ is **supported on** faces $F_1, \ldots, F_q$ if $\Sigma = a_1 F_1 + \cdots + a_q F_q$ for nonzero scalars $a_1, \ldots, a_q \in k$. We say that Σ is **a face-minimal cycle** or **minimally supported on** $F_1, \ldots, F_q$ if additionally no proper subset of $F_1, \ldots, F_q$ is the support of a d-cycle. If Σ is supported on $F_1 \ldots, F_q$, we call the simplicial complex $\langle F_1, \ldots, F_q \rangle$ the **support complex of** Σ.

Example 3.3 can guide the reader through the statement of the theorem below, a variation of which appears as Theorem 4.2 of Theorem [25].

Theorem 3.2 *Let k be a field, Γ a d-dimensional simplicial complex, and*

$$\Sigma = a_1 F_1 + \cdots + a_q F_q, \ a_1, \ldots, a_q \in k$$

a d-cycle in Γ supported on $F_1, \ldots, F_q$, which is not a boundary so that $\tilde{H}_d(\Gamma, k) \neq 0$. Suppose A is a face of the support complex of Σ such that for some $s \leq q$ we have

$$A \subseteq F_1 \cap \ldots \cap F_s, \ and \ A \nsubseteq F_j \ if \ j > s$$

and $0 \leq |A| \leq d + 1$. Then

  1. *there are $\epsilon_i \in \{\pm 1\}$ for $i = 1, \ldots, s$ such that*

$$\Sigma_A = \epsilon_1 a_1 (F_1 \setminus A) + \cdots + \epsilon_s a_s (F_s \setminus A)$$

   is a $(d - |A|)$-cycle in $\operatorname{lk}_\Gamma(A)$ that is not a boundary in $\operatorname{lk}_\Gamma(A)$;
  2. $\tilde{H}_{d-|A|}(\operatorname{lk}_\Gamma(A), k) \neq 0$;
  3. $A = F_1 \cap \ldots \cap F_s$.

Proof The case $|A| = d + 1$ will result in $\operatorname{lk}_\Gamma(A) = \{\emptyset\}$ which has (-1)-homology. So we can assume that $0 \leq |A| \leq d$. To prove Statement 1, we will proceed using induction on $a = |A|$. If $a = 0$, then $\operatorname{lk}_\Gamma(A) = \Gamma$, $\Sigma_A = \Sigma$ and there is nothing to prove.

Suppose $a > 0$, $A = \{v_1, \ldots, v_a\}$, $A' = \{v_1, \ldots, v_{a-1}\}$ (or $A' = \emptyset$ when $a = 1$) and $\Gamma' = \operatorname{lk}_\Gamma(A')$, and suppose without loss of generality

$$A' \subseteq F_1 \cap \ldots \cap F_t \text{ and } A' \nsubseteq F_j \text{ for } j > t \geq s.$$

By the induction hypothesis, for some $\epsilon_i' \in \{\pm 1\}$ there is a $(d - (a - 1))$-cycle

$$\Sigma_{A'} = a_1 \epsilon_1' (F_1 \setminus A') + \cdots + a_t \epsilon_t' (F_t \setminus A')$$

in Γ' that is not a boundary in Γ' and $\tilde{H}_{d-(a-1)}(\Gamma', k) \neq 0$. In particular, we must have $t \neq s$ as otherwise the support complex of $\Sigma_{A'}$ would be a cone with every facet containing v_a, a contradiction.

We know that $v_a \in (F_i \setminus A')$ if and only if $i \leq s$. Depending on the orientation of the faces of the complex Γ', for some $\epsilon_i'' \in \{\pm 1\}$, we can write

$$0 = \partial(\Sigma_{A'})$$

$$= \epsilon_1' a_1 \partial(F_1 \setminus A') + \cdots + \epsilon_t' a_t \partial(F_t \setminus A')$$

$$= \epsilon_1'' \epsilon_1' a_1 (F_1 \setminus A) + \cdots + \epsilon_s'' \epsilon_s' a_s (F_s \setminus A) + \mathcal{U}$$

$$+ \partial(\epsilon_{s+1}' a_{s+1} F_{s+1} \setminus A' + \cdots + \epsilon_t' a_t F_t \setminus A')$$

where $\mathcal{U}$ consists of all the summands above which contain the vertex v_a, and hence

$$\mathcal{U} = \sum_{j=1}^{s} \epsilon'_j a_j \left(\partial(F_j \setminus A') - \epsilon''_j F_j \setminus A \right) = 0.$$

If we set $\epsilon_i = \epsilon''_i \epsilon'_i$ and $\Sigma_A = \epsilon_1 a_1 (F_1 \setminus A) + \cdots + \epsilon_s a_s (F_s \setminus A)$ it follows that

$$\Sigma_A = -\partial(\epsilon'_{s+1} a_{s+1} (F_{s+1} \setminus A') + \cdots + \epsilon'_t a_t (F_t \setminus A'))$$

and

$$\partial(\Sigma_A) = -\partial^2(\epsilon'_{s+1} a_{s+1} (F_{s+1} \setminus A') + \cdots + \epsilon'_t a_t (F_t \setminus A')) = 0.$$

So Σ_A is a $(d-a)$-cycle in lk $_{\Gamma'}(v_a) = $ lk $_\Gamma(A)$ by Lemma 3.1 (and since $v_a \in \Gamma'$). Since $\dim(\text{lk}_\Gamma(A)) = d - |A|$, the $(d - |A|)$-cycle Σ_A is not a boundary in lk $_\Gamma(A)$. Therefore, $\tilde{H}_{d-|A|}(\Gamma, k) \neq 0$, proving Statement 2.

To see Statement 3, note that if $F_1 \ldots, F_s$ all contain a vertex outside A, then the support complex of Σ_A would be a cone contradicting Statement 2. $\qquad \square$

Example 3.3 Let $\Gamma = \langle xy, zu, zv, uv \rangle$, which is the Alexander dual of the simplicial complex Γ in Example 2.4.
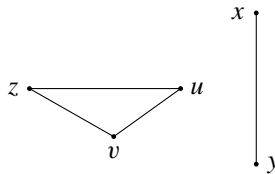
As stated in Theorem 3.2, Γ is a 1-dimensional simplicial complex and has $\Sigma = uz + zv + vu$ as a 1-cycle so that $\tilde{H}_1(\Gamma, k) \neq 0$. Taking $A = \{z\}$, then $\Sigma_A = u - v$ is a 0-cycle in lk $_\Gamma(A) = \langle u, v \rangle$ with $\tilde{H}_0(\text{lk}_\Gamma(A), k) \neq 0$.

Corollary 3.4 *Let k be a field, Γ a d-dimensional simplicial complex with $\tilde{H}_d(\Gamma, k) \neq 0$, and let Σ be a d-cycle in Γ which is not a boundary. Let A be a face of the support complex of Σ, and suppose $F_1 \ldots, F_q$ are the facets of Γ that contain A. Then*

$$A = \bigcap_{j=1}^{q} F_j.$$

Proof Since lk $_\Gamma(A) = \langle F_1 \setminus A, \ldots, F_q \setminus A \rangle$, if there is a vertex of $\bigcap_{j=1}^{q} F_j$ which is not in A, then lk $_\Gamma(A)$ would be a cone and would therefore have no homology, contradicting Theorem 3.2. $\qquad \square$

Theorem 3.5 below is a formal statement on breaking homological cycles. We refer the reader to parts (2) and (3) of Example 2.4 where we demonstrated the theorem's statement. Note also that the case in part (1) of Example 2.4 follows the same pattern, though a proof is not known yet.

Theorem 3.5 *(Breaking up cycles on links). Let k be a field and $\Gamma = \langle F_1, \ldots, F_r \rangle$ be a d-dimensional simplicial complex such that*

$$\tilde{H}_d(\Gamma, k) \neq 0 \text{ and } d + 2 = a + b \text{ for some } a, b > 0.$$

Suppose Γ contains a d-dimensional cycle

$$\Sigma = \sum_{j=1}^{q} a_j F_j$$

supported on the facets $F_1, \ldots, F_q$ of Γ, and Σ is not a boundary in Γ. Then there are subsets $A, B \subseteq [q] \subseteq [r]$ with

$$F = \bigcap_{j \in A} F_j \text{ and } G = \bigcap_{j \in B} F_j$$

such that

1. *$F \cap G = \emptyset$;*
2. *$F \cup G \notin \Gamma$;*
3. *$\tilde{H}_{a-2}(\text{lk }_\Gamma(F), k) \neq 0$ and $\tilde{H}_{b-2}(\text{lk }_\Gamma(G), k) \neq 0$.*

Moreover, if $a, b > 1$, F and G and $\epsilon_j, \delta_j \in \{\pm 1\}$ could be chosen to additionally satisfy:

4. *$|F| = b$ and $|G| = a$;*
5. *$\Sigma_F = \sum_{j \in A} \epsilon_j a_j (F_j \setminus F)$ is an $(a-2)$-cycle in $\text{lk }_\Gamma(F)$ which is not a boundary ;*
6. *$\Sigma_G = \sum_{j \in B} \delta_j a_j (F_j \setminus G)$ is a $(b-2)$-cycle in $\text{lk }_\Gamma(G)$ which is not a boundary.*

Proof Set $i = d + 2$. We first consider the case $b = 1$ and $a = i - 1$. If $a = 1$, then $d = 0$ and Γ is disconnected. Let F and G be two facets each belonging to a distinct connected component of Γ. Then we clearly have $F \cap G = \emptyset$ and $F \cup G \notin \Gamma$. Moreover, $\text{lk }_\Gamma(F) = \text{lk }_\Gamma(G) = \{\emptyset\}$ and so

$$\tilde{H}_{a-2}(\text{lk }_\Gamma(F), k) = \tilde{H}_{b-2}(\text{lk }_\Gamma(G), k) = \tilde{H}_{-1}(\{\emptyset\}, k) \neq 0$$

as desired.

If $b = 1$ and $a = i - 1 > 1$, then $d = a + b - 2 > 0$. By Theorem 3.2, if we take a vertex v in the support complex of Σ, then $\tilde{H}_{i-3}(\text{lk }_\Gamma(v), k) \neq 0$.

Since Σ is a cycle, not all of $F_1, \ldots, F_q$ contain v. Let G be one of the facets $F_1, \ldots, F_q$ that does not contain v. Then $F \cap G = \emptyset$ and $F \cup G \notin \Gamma$ (as G is a facet), and moreover

$$\tilde{H}_{a-2}(\mathrm{lk}\ _\Gamma(F), k) = \tilde{H}_{i-3}(\mathrm{lk}\ _\Gamma(v), k) \neq 0$$
$$\text{and}\ \ \tilde{H}_{b-2}(\mathrm{lk}\ _\Gamma(G), k) = \tilde{H}_{-1}(\{\emptyset\}, k) \neq 0.$$

Now suppose $a, b \geq 2$ and $a = i - b$. Suppose $F_1 = \{w_1, v_1, \ldots, v_{i-2}\}$. Then since F_1 is in the support of the $(i-2)$-cycle Σ, $\{w_1, v_2, \ldots, v_{i-2}\}$ must appear in another one of the F_j in the support of Σ, say F_2. Suppose $F_2 = \{w_1, w_2, v_2, \ldots, v_{i-2}\}$. Considering that $a = i - b \leq i - 2$, let

$$G = \{v_1, \ldots, v_a\} \text{ and } F = \{v_{a+1}, \ldots, v_{i-2}, w_1, w_2\}.$$

Then $|G| = a$ and $|F| = i - 2 + 2 - a = b$. Moreover $F \cap G = \emptyset$ by construction, and if $i - 2 = d$, then $F \cup G \notin \Gamma$ since $|F \cup G| = d + 2$ which is larger than the size of any face of Γ.

By Theorem 3.2, and noting that $i - 2 - |G| = b - 2$ and $i - 2 - |F| = a - 2$, we have

$$\tilde{H}_{a-2}(\mathrm{lk}\ _\Gamma(F), k) \neq 0 \text{ and } \tilde{H}_{b-2}(\mathrm{lk}\ _\Gamma(G), k) \neq 0,$$

conditions 5 and 6 are satisfied, and if

$$A = \{j \in [q]\ :\ F \subset F_j\} \text{ and } B = \{j \in [q]\ :\ G \subset F_j\}$$

then

$$F = \bigcap_{j \in A} F_j \text{ and } G = \bigcap_{j \in B} F_j.$$

$\square$

Another version of Theorem 3.5 below is one which gives lower-dimensional cycles in induced subcomplexes.

Corollary 3.6 *(Breaking up cycles). Let Γ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$, and suppose $\tilde{H}_{d-2}(\Gamma, k) \neq 0$, where d is the smallest possible size of a nonface of Γ. Suppose $n - d + 1 = a + b$, where a and b are positive integers. Then there are nonempty subsets $C, D \subseteq \{x_1, \ldots, x_n\}$ such that*

1. *$C \cup D = \{x_1, \ldots, x_n\}$;*
2. *$C \cap D \in \Gamma$;*
3. *$\tilde{H}_{|C|-a-1}(\Gamma_C, k) \neq 0$ and $\tilde{H}_{|D|-b-1}(\Gamma_D, k) \neq 0$.*

Proof By Alexander duality—see Prop. 5.1.10 and the discussion preceding Prop. 5.1.8 in [18]—we have that $\tilde{H}_{n-d-1}(\Gamma^\vee, k) \neq 0$. Now d is the smallest possible size of a nonface of Γ, so by the definition of Alexander duals, $\dim(\Gamma^\vee) = n - d - 1$.

Suppose $\Gamma^\vee = \langle F_1, \ldots, F_r \rangle$. If $n - d + 1 = a + b$, then, by Theorem 3.5, there are subsets A and B of $[r]$ such that

$$F = \bigcap_{j \in A} F_j \text{ and } G = \bigcap_{j \in B} F_j$$

and

(i) $F \cap G = \emptyset$;
(ii) $F \cup G \notin \Gamma^\vee$;
(iii) $\tilde{H}_{a-2}(\text{lk }_{\Gamma^\vee}(F), k) \neq 0$ and $\tilde{H}_{b-2}(\text{lk }_{\Gamma^\vee}(G), k) \neq 0$.

Now let

$$C = F^c = \bigcup_{j \in A} F_j^c \text{ and } D = G^c = \bigcup_{j \in B} F_j^c.$$

Then by (i), $C \cup D = (F \cap G)^c = \{x_1, \ldots, x_n\}$. By (ii), $(C \cap D)^c = F \cup G \notin \Gamma^\vee$ so $C \cap D \in \Gamma$. Finally by (iii) and Equation (2), $\tilde{H}_{|C|-a-1}(\Gamma_C, k) \neq 0$ and $\tilde{H}_{|D|-b-1}(\Gamma_D, k) \neq 0$. □

Theorem 3.7 (*Subadditivity of syzygies of square-free monomial ideals*). *If I is a square-free monomial ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$ where k is a field, and d is the smallest possible degree of a generator of I. Suppose $i = n - d + 1$, $\beta_{i,n}(S/I) \neq 0$ and $i = a + b$, for some positive integers a and b. Then $t_i \leq t_a + t_b$.*

Proof By Hochster's formula (Equation (2)), if $\Gamma = \mathcal{N}(I)$, then

$$\beta_{n-d+1,n}(S/I) = \beta_{n-d+1, x_1 \cdots x_n}(S/I) = \dim_k \tilde{H}_{d-2}(\Gamma, k) \neq 0.$$

If $n - d + 1 = a + b$, then by Corollary 3.6, there are nonempty subsets $C, D \subseteq \{x_1, \ldots, x_n\}$ such that

$$C \cup D = \{x_1, \ldots, x_n\} \text{ and } C \cap D \in \Gamma,$$

and

$$\tilde{H}_{|C|-a-1}(\Gamma_C, k) \neq 0 \text{ and } \tilde{H}_{|D|-b-1}(\Gamma_D, k) \neq 0.$$

By Equation (2), this means that

$$\beta_{a,|C|}(S/I) \neq 0 \text{ and } \beta_{b,|D|}(S/I) \neq 0,$$

so that $t_a \geq |C|$ and $t_b \geq |D|$. Putting this all together, we get

$$t_a + t_b \geq |C| + |D| \geq n = t_i,$$

which settles our claim. □

Discussion 3.8 Given a square-free monomial ideal I if we are looking for top degree Betti numbers, by Hochster's formula (Equation (2))

$$\beta_{n-i-1,n}(S/I) = \dim_k \tilde{H}_i(\Gamma, k).$$

Now if d is the smallest possible degree of a generator of I, then all monomials of degree $\leq d - 1$ are not in I, which means all possible faces of dimension $\leq d - 2$ are in $\Gamma = \mathcal{N}(I)$. This means that the smallest index i with $\tilde{H}_i(\Gamma, k) \neq 0$ is $d - 2$, that is

$$\tilde{H}_i(\Gamma, k) = 0 \text{ for } i < d - 2$$

and hence

$$\beta_{j,n}(S/I) = 0 \text{ for } j = n - i - 1 > n - d + 1.$$

So $n - d + 1$ is the maximum homological degree where we could have a nonvanishing top degree Betti number. We do not have an example of our setting where $n - d + 1$ is not the projective dimension. After comparing with bounds on the projective dimension of S/I given by Dao and Schweig [8, Theorem 3.2, Remark 3.4] in terms of dominance parameters of clutters, we concluded that $n - d + 1$ is often either the projective dimension of S/I or very close to it, though we were not able to determine how close.

Example 3.9 Let $I = (xyz, xzv, xuv, yzu, yuv)$ be an ideal of $S = k[x, y, z, u, v]$ in 5 variables. Here the smallest degree of a generator of I is $d = 3$, so $n - d + 1 = 3$, so we pick $a = 1$ and $b = 2$. According to Macaulay2 [23], the Betti table of S/I is

```
            0 1 2 3
   total : 1 5 5 1
       0 : 1 . . .
       1 : . . . .
       2 : . 5 5 1
```

which verifies that

$$t_3 = 5, t_2 = 4, t_1 = 3 \implies t_3 < t_1 + t_2 = 7.$$

Example 3.10 In Example 2.4, $I = (xz, yz, xu, yu, xv, yv, zuv)$ is a square-free monomial ideal in 5 variables where $d = 2$ and $n - d + 1 = 4$. According to the Betti table of I, $t_4 = 5$, $t_3 = 5$, $t_2 = 4$ and $t_1 = 3$. Here $t_4 < t_1 + t_3 = 8$ and $t_4 < 2t_2 = 8$. Note that we also have $\beta_{3,5}(S/I) \neq 0$ where $3 < 4 = n - d + 1$ while still we have $t_3 < t_1 + t_2 = 7$.
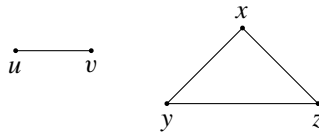
4 Special cases of breaking up simplicial homology

In this section, we consider breaking up special classes of cycles, where we can provide a combinatorial description for the lower-dimensional cycles.

4.1 The case of a disconnected simplicial complex

We begin with an example.

Example 4.1 Let $\mathcal{N}(I) = \Gamma = \langle uv, xy, yz, xz \rangle$ be a simplicial complex on $n = 5$ vertices.

Here $\tilde{H}_0(\Gamma, k) \neq 0$ and hence $\beta_{4,uvxyz}(S/I) \neq 0$. If $4 = a + b$, then using Corollary 3.6 we have the following two cases to consider.

1. $a = 1$ and $b = 3$. Let $C = \{u, x\}$ and $D = \{u, v, y, z\}$. Then $C \cup D = \{u, v, x, y, z\}$, $C \cap D = \{u\} \in \Gamma$ and

$$\tilde{H}_{|C|-a-1}(\Gamma_C, k) = \tilde{H}_0(\langle u, x \rangle, k) \neq 0$$
$$\text{and } \tilde{H}_{|D|-b-1}(\Gamma_D, k) = \tilde{H}_0(\langle uv, yz \rangle, k) \neq 0.$$

2. $a = b = 2$. Let $C = \{u, x, v\}$ and $D = \{u, y, z\}$. Then $C \cup D = \{u, v, x, y, z\}$, $C \cap D = \{u\} \in \Gamma$ and

$$\tilde{H}_{|C|-a-1}(\Gamma_C, k) = \tilde{H}_0(\langle uv, x \rangle, k) \neq 0$$
$$\text{and } \tilde{H}_{|D|-b-1}(\Gamma_D, k) = \tilde{H}_0(\langle u, yz \rangle, k) \neq 0.$$

In general, if Γ is a disconnected complex on n vertices with Stanley–Reisner ideal I, then $\beta_{n-1,n}(S/I) \neq 0$, and if $n - 1 = a + b$ for some $a, b > 0$, then we can always find disconnected induced subcomplexes Γ_C and Γ_D where $C = a+1$ and $D = b+1$, as in the example above. Below we demonstrate how this can be done.

If Γ is disconnected, then it has the form

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_t$$

where $\Gamma_1, \ldots, \Gamma_t$ are connected components and $t > 1$. In this case, $|V(\Gamma_i)| \geq 1$ for all $1 \leq i \leq t$, $V(\Gamma) = V(\Gamma_1) \cup \cdots \cup V(\Gamma_t)$ and $V(\Gamma_k) \cap V(\Gamma_l) = \emptyset$ for all $1 \leq k < l \leq t$.

Without loss of generality and up to renaming the variables, we can assume the following:

- $|V(\Gamma_1)| \leq |V(\Gamma_2)| \leq \cdots \leq |V(\Gamma_t)|$,
- $x_k \in V(\Gamma_k)$ for $1 \leq k \leq t$,
- $V(\Gamma_1) = \{x_1, x_{t+1}, \ldots, x_{t+|V(\Gamma_1)|-1}\}$
- $V(\Gamma_k) = \{x_k, x_{(t+|V(\Gamma_1)|+\cdots+|V(\Gamma_{k-1})|-k+2)}, \ldots, x_{(t+|V(\Gamma_1)|+\cdots+|V(\Gamma_k)|-k)}\}$ for each $1 < k \leq t$.

Example 4.2 The simplicial complex Γ in Example 4.1 can be relabeled and written as $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \langle x_1 x_3 \rangle$ and $\Gamma_2 = \langle x_2 x_4, x_4 x_5, x_2 x_5 \rangle$.
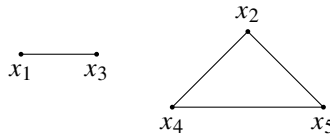
For each $1 \leq a < n - 1$, define

$$C = \{x_1, x_2, \ldots, x_{a+1}\} \text{ and } D = \{x_1, x_{a+2}, \ldots, x_n\}.$$

Clearly $C \cup D = \{x_1, \ldots, x_n\}$, $|C| = a + 1$, $|D| = n - a$ and $C \cap D = \{x_1\} \in \Gamma$. Moreover, it is easy to see that both Γ_C and Γ_D are disconnected induced subcomplexes of Γ on the subsets $\{x_1, x_2, \ldots, x_{a+1}\}$ and $\{x_1, x_{a+2}, \ldots, x_n\}$, respectively. Therefore, if $b = n - a - 1$

$$\tilde{H}_{|C|-a-1}(\Gamma_C, k) = \tilde{H}_0(\Gamma_C, k) \neq 0 \text{ and } \tilde{H}_{|D|-b-1}(\Gamma_D, k) = \tilde{H}_0(\Gamma_D, k) \neq 0.$$

4.2 The case of a graph cycle

Recall that a *cycle* in a graph G is an ordered list of distinct vertices $x_1, \ldots, x_n$ where the edges are $x_{i-1} x_i$ for $2 \leq i \leq n$ and $x_n x_1$. Graph cycles characterize nontrivial 1-homology in simplicial complexes; see, for example, Theorem 3.2 in [7].

Suppose Γ is a simplicial complex on the set $\{x_1, \ldots, x_n\}$ that is the support complex of a face-minimal graph cycle, so that $\tilde{H}_1(\Gamma, k) \neq 0$. This means that $\beta_{n-2,n}(S/I) \neq 0$. Suppose $n - 2 = a + b$ for some $a, b > 0$.

Without loss of generality, Γ can be written in the form

$$\Gamma = \langle x_1 x_2, x_2 x_3, \ldots, x_{n-1} x_n, x_n x_1 \rangle.$$

For $1 \leq a < n - 2$, define

$$C = \{x_1, x_3, x_4, \ldots, x_{a+2}\} \text{ and } D = \{x_2, x_{a+3}, \ldots, x_n\}.$$

Clearly, $C \cup D = \{x_1, \ldots, x_n\}$, $|C| = a + 1$, $|D| = n - a - 1$ and $C \cap D = \emptyset \in \Gamma$. Moreover, it is easy to see that both Γ_C and Γ_D are disconnected induced subcomplexes of Γ on the subsets $\{x_1, x_3, x_4, \ldots, x_{a+2}\}$ and $\{x_2, x_{a+3}, \ldots, x_n\}$, respectively. Therefore,

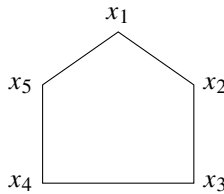$$\tilde{H}_{|C|-a-1}(\Gamma_C, k) = \tilde{H}_0(\Gamma_C, k) \neq 0$$

and

$$\tilde{H}_{|D|-b-1}(\Gamma_D, k) = \tilde{H}_0(\Gamma_D, k) \neq 0$$

where $b = n - a - 2$.

Example 4.3 Let $\mathcal{N}(I) = \Gamma = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5 \rangle$.

Then $\tilde{H}_1(\Gamma, k) \neq 0$ and hence $\beta_{3, x_1 \cdots x_5}(S/I) \neq 0$. Taking $a = 1$ and $b = 2$, set $C = \{x_1, x_3\}$ and $D = \{x_2, x_4, x_5\}$. Then

$$\tilde{H}_{|C|-a-1}(\Gamma_C, k) = \tilde{H}_0(\langle x_1, x_3 \rangle, k) \neq 0 \text{ and}$$
$$\tilde{H}_{|D|-b-1}(\Gamma_D, k) = \tilde{H}_0(\langle x_2, x_4x_5 \rangle, k) \neq 0.$$

5 The case of general monomial ideals

The polarization [16] of a monomial ideal I is a method to transform I to a square-free monomial ideal, by adding new variables to the polynomial ring. The procedure is described below.

Definition 5.1 (Polarization). Let I be minimally generated by monomials $\mathbf{m}_1, \ldots \mathbf{m}_q$ in the polynomial ring $R = k[x_1, \ldots, x_n]$. For $i \in \{1, \ldots, n\}$, let

$$p_i = \begin{cases} 1 & \text{if } x_i \nmid \mathbf{m}_u \text{ for every } u \in [q] \\ \max \left\{ j \ : \ x_i^j \mid \mathbf{m}_u \text{ for some } u \in [q] \right\} & \text{otherwise.} \end{cases}$$

Let S be the polynomial ring in $\mathbf{p} = p_1 + \cdots + p_n$ variables

$$S = k[x_{i,j} \ : \ 1 \le i \le n, \ 1 \le j \le p_i]$$

and let the **polarization of** I be the square-free monomial ideal

$$\mathcal{P}(I) = \big(\mathcal{P}(\mathbf{m}_1), \ldots, \mathcal{P}(\mathbf{m}_q) \big)$$

where, if $\mathbf{m} = x_{a_1}^{b_1} \cdots x_{a_c}^{b_c}$ where the a_i are distinct integers in $\{1, \ldots, n\}$ and $1 \le b_i \le p_i$ for $1 \le i \le c$, then

$$\mathcal{P}(\mathbf{m}) = x_{a_1,1} \cdots x_{a_1,b_1} x_{a_2,1} \cdots x_{a_2,b_2} \cdots x_{a_c,1} \cdots x_{a_c,b_c}.$$

Example 5.2 If $I = (x^2, xy^3z^2) \subseteq k[x, y, z]$, then its polarization is the square-free monomial ideal $\mathcal{P}(I) = (x_1x_2, x_1y_1y_2y_3z_1z_2)$ in the polynomial ring $k[x_1, x_2, y_1, y_2, y_3, z_1, z_2]$.

Corollary 5.3 *(Subadditivity of syzygies of monomial ideals). If I is a monomial ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$ where k is a field, d is the smallest possible degree of a generator of I, and $\mathbf{p}$ is defined as in Definition* 5.1. *Suppose $i = \mathbf{p} - d + 1$, $\beta_{i,\mathbf{p}}(R/I) \neq 0$ and $i = a + b$, for some positive integers a and b. Then $t_i \leq t_a + t_b$.*

Proof Let $I = (\mathbf{m}_1, \ldots, \mathbf{m}_q)$, whose polarization is the square-free monomial ideal $\mathcal{P}(I)$ in the polynomial ring S in $\mathbf{p}$ variables in Definition 5.1. Since $\beta_{i,\mathbf{p}}(R/I) \neq 0$, we must have $\beta_{i,\mathbf{m}}(R/I) \neq 0$ for some $\mathbf{m} \in \mathrm{LCM}(I)$. On the other hand, $\mathbf{p}$ is the largest possible degree for a monomial in $\mathrm{LCM}(I)$, and so $\mathbf{m} = \mathrm{lcm}(\mathbf{m}_1, \ldots, \mathbf{m}_q)$, the top monomial in the lcm lattice of I.

Now the two lcm lattices $\mathrm{LCM}(I)$ and $\mathrm{LCM}(\mathcal{P}(I))$ are isomorphic ([17]), and the degree $\mathbf{p}$ square-free monomial $\mathcal{P}(\mathbf{m})$ sits on top of the lattice $\mathrm{LCM}(\mathcal{P}(I))$, and so $\beta_{i,\mathbf{p}}(S/\mathcal{P}(I)) \neq 0$. Now since $\deg(\mathbf{m}_i) = \deg(\mathcal{P}(\mathbf{m}_i))$ for all $1 \leq i \leq q$, the conditions for Theorem 3.7 hold, and therefore $t_i \leq t_a + t_b$ holds for the ideal $\mathcal{P}(I)$. But as the graded Betti numbers of I and $\mathcal{P}(I)$ are equal, the inequality also holds for I, and we are done. $\qquad\square$

Example 5.4 Let $I = (xy^2, xyz, y^3, y^2z)$ be an ideal of $R = k[x, y, z]$. Here $\mathbf{p} = 5$ and the smallest degree of a generator is $d = 3$, so $\mathbf{p} - d + 1 = 3$. We pick $a = 1$ and $b = 2$. According to Macaulay2 [23] the Betti table of R/I is

$$
\begin{array}{rcccc}
 & 0 & 1 & 2 & 3 \\
\text{total} : & 1 & 4 & 4 & 1 \\
0 : & 1 & . & . & . \\
1 : & . & . & . & . \\
2 : & . & 4 & 4 & 1
\end{array}
$$

which verifies that

$$t_3 = 5, t_2 = 4, t_1 = 3 \implies t_3 < t_1 + t_2 = 7.$$

6 Final remarks

Questions 2.1, 2.2, 2.3, and 2.5 are all equivalent, though their different settings allow the application of different (inductive) tools. All of them are open in their full generality as far as we know, though each can be answered positively for certain classes of ideals or combinatorial objects. A positive answer to either would settle the subadditivity question for monomial ideals in a polynomial ring.

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