ℓ^2 -Betti numbers arising from the lamplighter group

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Abstract

We apply a construction developed in a previous paper by the authors in order to obtain a formula which enables us to compute ℓ^2 -Betti numbers coming from a family of group algebras representable as crossed product algebras. As an application, we obtain a whole family of irrational ℓ^2 -Betti numbers arising from the lamplighter group algebra $\mathbb{Q}[\mathbb{Z}_2 \wr \mathbb{Z}]$. This procedure is constructive, in the sense that one has an explicit description of the elements realizing such irrational numbers. This extends the work made by Grabowski, who first computed irrational ℓ^2 -Betti numbers from the algebras $\mathbb{Q}[\mathbb{Z}_n \wr \mathbb{Z}]$, where $n \geq 2$ is a natural number. We also apply the techniques developed to the generalized odometer algebra $\mathcal{O}(\bar{n})$, where $\bar{n}$ is a supernatural number. We compute its $*$ -regular closure, and this allows us to fully characterize the set of $\mathcal{O}(\bar{n})$ -Betti numbers.

Keywords ℓ^2 -Betti number · Atiyah conjecture · Rank function · Lamplighter · Odometer

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1 Introduction and the Atiyah problem

In [5], Atiyah (in collaboration with Singer) extended the Atiyah–Singer index theorem [6] to the non-compact setting and more concretely to the arena of non-compact manifolds M equipped with a free cocompact action of a discrete group $G \curvearrowright M$ (i.e., with M/G compact). In this situation, the space of solutions of the equation $Df = 0$, where D is an elliptic differential operator, becomes an infinite-dimensional Hilbert space $\mathcal{H}_D$, and the concept of analytic index is extended by using the theory of von Neumann algebras. This theory provides a well-defined notion of dimension for $\mathcal{H}_D$, called the *von Neumann dimension* of $\mathcal{H}_D$. This new framework led Atiyah to define the so-called ℓ^2 -Betti numbers as von Neumann dimensions of ℓ^2 -cohomology Hilbert modules [33].

In his paper [5], Atiyah observed that, in case G is a finite group, the ℓ^2 -Betti numbers coincide, *modulo rescaling by $|G|$* , with the previously known Betti numbers. Thus, in this situation, they are in fact *rational* numbers, and indeed Atiyah posed the following question, which is nowadays commonly known as the *Atiyah Conjecture*.

Question (Atiyah) Is it possible to obtain irrational values of ℓ^2 -Betti numbers?

Although they were defined analytically, ℓ^2 -Betti numbers can be defined in an algebraic setting, purely in terms of the group G and without explicit mention of the manifold M (see Sect. 2.2 for a detailed explanation). In this line, a real positive number r is said to be an ℓ^2 -Betti number arising from G , with coefficients in a fixed subfield $K \subseteq \mathbb{C}$ closed under complex conjugation, whenever there exists a matrix operator $T \in M_n(KG)$ such that the von Neumann dimension of $\ker T$ is equal to r . Following this algebraic point of view, stronger questions—in relation to the original Atiyah question—were formulated over the years. One of its strongest versions is the so-called *strong Atiyah conjecture*, considered by Schick in [38, Definition 1].

Strong Atiyah conjecture (SAC) The set of ℓ^2 -Betti numbers arising from G with coefficients in K is contained in the subgroup $\sum_H \frac{1}{|H|} \mathbb{Z}$ of $\mathbb{Q}$, where H ranges over the finite subgroups of G .

The lamplighter is precisely the first counterexample to the SAC, as proved by R. I. Grigorchuk and A. Żuk [19,20], see also [13] for a more elementary proof. More recently, the original Atiyah’s question has been solved in the negative, and some authors including Austin [7], Grabowski [17,18] and Pichot, Schick and Żuk [36] have found examples of groups having irrational values of ℓ^2 -Betti numbers. In particular, Grabowski shows in [18] that there are *transcendental* numbers that appear as ℓ^2 -Betti numbers of the lamplighter group. Nevertheless, the following version of the SAC is still open.

Strong Atiyah conjecture, bounded case (BSAC) Suppose that there exists an upper bound for the orders of the finite subgroups of G . Then the SAC holds for G .

In particular, it is open for torsion-free groups, which gives a generalization of the famous Kaplansky’s zero divisor conjecture. For an extensive study of Atiyah’s original question and strong versions of it, see [14,21–23,27–34].

During the last 40 years, the importance of both the SAC and BSAC has increased due to the wide variety of their consequences in several branches of mathematics. For instance, in differential geometry and topology the SAC has connections with the Hopf conjecture on the possible sign of the Euler characteristic of a Riemannian manifold ([12], see also [33, Chapters 10 and 11]). In group theory, the SAC has implications relating group-theoretic properties of a group and its homological dimension. In particular, it is known that if a group has homological dimension one and satisfies the SAC, then it must be locally free [25].

One of the main problems (following these lines) is to actually compute the whole set $\mathcal{C}(G, K)$ of ℓ^2 -Betti numbers arising from G with coefficients in K . In this paper, we uncover a portion of this set for the lamplighter group Γ with coefficients in $\mathbb{Q}$. The group Γ is defined to be the semidirect product of $\mathbb{Z}$ copies of the finite group $\mathbb{Z}_2$ by $\mathbb{Z}$, i.e.,

$$\Gamma = \left(\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2 \right) \rtimes_{\rho} \mathbb{Z},$$

whose automorphism ρ implementing the semidirect product is the well-known Bernoulli shift. A precursor of our work here can be found in the paper [4] by Ara and Goodearl. In that article, the authors attack this problem algebraically, by trying to uncover the structure of what is called the **-regular closure* $\mathcal{R}_{K\Gamma}$ of the lamplighter group algebra $K\Gamma$ inside $\mathcal{U}(\Gamma)$, the algebra of unbounded operators affiliated to the group von Neumann algebra $\mathcal{N}(\Gamma)$ or, more algebraically, the classical ring of quotients of $\mathcal{N}(\Gamma)$. The precise connection between the **-regular closure* $\mathcal{R}_{K\Gamma}$ and the set $\mathcal{C}(\Gamma, K)$ has been provided recently by a result of Jaikin-Zapirain [22], which states that the rank function on $\mathcal{R}_{K\Gamma}$, obtained by restricting the canonical rank function on $\mathcal{U}(\Gamma)$, is completely determined by its values on matrices over $K\Gamma$. Since $\mathcal{R}_{K\Gamma}$ is **-regular*, this can be rephrased in the form

$$\phi(K_0(\mathcal{R}_{K\Gamma})) = \mathcal{G}(\Gamma, K),$$

where ϕ is the state on $K_0(\mathcal{R}_{K\Gamma})$ induced by the restriction of the rank function on $K\Gamma$, and $\mathcal{G}(\Gamma, K)$ is the subgroup of $\mathbb{R}$ generated by $\mathcal{C}(\Gamma, K)$ [3, Proposition 4.1] (see, e.g., [37] for the definition of the K_0 -group of a ring). The algebra $\mathcal{R}_{K\Gamma}$, together with tight connections with $\mathcal{C}(\Gamma, K)$, has been recently studied by the authors in [3].

A key observation in light of the development of the work presented in this paper is to realize the lamplighter group algebra as a $\mathbb{Z}$ -crossed product **-algebra*

$$K\Gamma \cong C_K(X) \rtimes_T \mathbb{Z}$$

through the Fourier transform. This method was first (somewhat implicitly) used in [13], and very explicitly in [7]. Here $X = \{0, 1\}^{\mathbb{Z}}$ is the Pontryagin dual of the group $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2$, topologically identified with the Cantor set, $C_K(X)$ is the set of locally constant functions $f : X \rightarrow K$, and $T : X \rightarrow X$ is the homeomorphism of X implemented by the Bernoulli shift. There is a natural measure μ on X , namely the usual product measure, having taken the $(\frac{1}{2}, \frac{1}{2})$ -measure on each component $\{0, 1\}$.

This measure is ergodic, full and T -invariant, so we can apply the techniques developed in [2] to study the $\mathbb{Z}$ -crossed product algebra $\mathcal{A} := C_K(X) \rtimes_T \mathbb{Z}$ by giving ‘ μ -approximations’ of the space X , which at the level of the algebra $\mathcal{A}$ correspond to certain ‘approximating’ $*$ -subalgebras $\mathcal{A}_n \subseteq \mathcal{A}$ (see [2, Section 4.1], also [3, Section 6]). By using [2, Theorem 4.7 and Proposition 4.8], we obtain a canonical faithful Sylvester matrix rank function $\text{rk}_{\mathcal{A}}$ on $\mathcal{A}$ which coincides, in case K is a subfield of $\mathbb{C}$ closed under complex conjugation, with the rank function $\text{rk}_{K\Gamma}$ on the group algebra $K\Gamma$ naturally inherited from the canonical rank function in the $*$ -regular ring $\mathcal{U}(\Gamma)$ [3, Proposition 5.10]. In light of this, one can define ‘generalized’ ℓ^2 -Betti numbers in this more general setting, that is, arising from the $\mathbb{Z}$ -crossed product algebra $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$, for K an arbitrary field and T an arbitrary homeomorphism on a Cantor set X .

Turning back to the lamplighter group algebra $K\Gamma$, Graboski has shown in a recent paper [18] the existence of irrational (in fact transcendental) ℓ^2 -Betti numbers arising from Γ , exhibiting a concrete example in [18, Theorem 2]. With this result, the lamplighter group has become the simplest known example which gives rise to irrational ℓ^2 -Betti numbers. Very roughly, his idea is to compute ℓ^2 -Betti numbers by means of decomposing them as an infinite sum of (normalized) dimensions of kernels of finite-dimensional operators (i.e., matrices). He then realizes these matrices as adjacency-labeled matrices of certain graphs in order to determine the global behavior of the dimensions of their kernels. We use these ideas in this work, but applied to our construction. Our main result is the following:

Theorem 1.1 (Theorem 4.5) *Let Γ be the lamplighter group. Then for any family of natural numbers $\{d_i \mid 1 \leq i \leq n\}$, and any family of polynomials $\{p_i(x) \mid 0 \leq i \leq n\}$ of positive degrees and nonnegative integer coefficients, such that the linear coefficient of $p_0(x)$ is nonzero, there exists a matrix operator A over $\mathbb{Q}\Gamma$ such that*

$$b^{(2)}(A) = q_0 + q_1 \sum_{k \geq 1} \frac{1}{2^{p_0(k) + p_1(k)d_1^k + \dots + p_n(k)d_n^k}}$$

for some nonzero rational numbers q_0, q_1 . If moreover the constant coefficient of $p_0(x)$ is 0, then

$$\sum_{k \geq 1} \frac{1}{2^{p_0(k) + p_1(k)d_1^k + \dots + p_n(k)d_n^k}} \in \mathcal{G}(\Gamma, \mathbb{Q}).$$

Here $b^{(2)}(A)$ stands for the ℓ^2 -Betti number of A (see Definition 2.2). Thus, we obtain a whole family of irrational and even transcendental ℓ^2 -Betti numbers arising from Γ . Another source of irrational ℓ^2 -Betti numbers arising from Γ comes from the fact that the $*$ -regular closure $\mathcal{R}_{K\Gamma}$ contains a copy of the algebra of non-commutative rational power series $K_{\text{rat}}\langle \mathcal{X} \rangle$ in infinitely many indeterminates, see [3, Subsection 6.2] and Sect. 4.3. Using this, we show that $\mathcal{G}(\Gamma, \mathbb{Q})$ contains for instance the irrational algebraic number $\frac{1}{4}\sqrt{\frac{2}{7}}$ (Theorem 4.13).

We also apply our machinery to study a particular crossed product algebra known as the odometer algebra. It is defined as the $\mathbb{Z}$ -crossed product algebra $\mathcal{O} := C_K(X) \rtimes_T \mathbb{Z}$ where $X = \{0, 1\}^{\mathbb{N}}$ is the one-sided shift space, and the automorphism $T : X \rightarrow X$ implementing the crossed product is given by addition of the element $(1, 0, 0, \dots)$ with carry over. Although it is not possible to realize the odometer algebra as a group algebra, this example is interesting in its own right because we are able to fully determine the structure of its $*$ -regular closure $\mathcal{R}_{\mathcal{O}}$ (see Theorem 5.4), thus giving a complete description of the set of $\mathcal{O}$ -Betti numbers (Theorem 5.6). The algebra $\mathcal{O}$ has also been studied by Elek in [15], although the author does not exactly compute the $*$ -regular closure $\mathcal{R}_{\mathcal{O}}$; instead, the author computes its rank completion, showing that it must be isomorphic to the von Neumann continuous factor $\mathcal{M}_K$, which is by definition the rank completion of $\varinjlim_i M_{2^i}(K)$ with respect to its unique rank metric (cf. [3, Proposition 4.2]). In fact, we study a general version of the classical odometer algebra, namely the dynamical system generated by ‘addition of 1’ for arbitrary profinite completions of $\mathbb{Z}$.

This work is structured as follows. In Sect. 2, we provide essential background and preliminary concepts about Sylvester matrix rank functions and ℓ^2 -Betti numbers for general group algebras. We summarize, in Sect. 3, a general approximation construction for $\mathbb{Z}$ -crossed product algebras using measure-theoretic tools, which enables us to construct a canonical Sylvester matrix rank function over the crossed product algebra. Using this construction, we derive a formula for computing generalized ℓ^2 -Betti numbers (see Definition 3.3) over this large class of algebras (Formula (3.4)). In Sect. 4, we focus on the particular case of the lamplighter group algebra. We explicitly realize it as a $\mathbb{Z}$ -crossed product algebra, thus enabling us to apply the whole theory developed in [2, 3]. Our main result (Theorem 4.5) provides a vast family of irrational, and even transcendental, ℓ^2 -Betti numbers arising from the lamplighter group.

We also explore another method to find ℓ^2 -Betti numbers arising from the lamplighter group, using the algebra of non-commutative rational series. In this direction, we obtain in Theorem 4.13 that the irrational algebraic number $\frac{1}{4}\sqrt{\frac{2}{7}}$ belongs to $\mathcal{G}(\Gamma, \mathbb{Q})$.

Finally we study, in Sect. 5, the generalized odometer algebra $\mathcal{O}(\bar{n})$ in great detail, and we are able to completely determine the algebraic structure of its $*$ -regular closure (Theorem 5.4). We use this characterization in Theorem 5.6, where we explicitly compute the whole set of $\mathcal{O}(\bar{n})$ -Betti numbers.

2 Preliminaries

2.1 $*$ -Regular rings and rank functions

A $*$ -regular ring is a regular ring R endowed with a *proper* involution $*$, that is, $x^*x = 0$ if and only if $x = 0$. In a $*$ -regular ring R , for every $x \in R$ there exist unique projections $e, f \in R$ such that $xR = eR$ and $Rx = Rf$. It is common to denote them by $e = \text{LP}(x)$ and $f = \text{RP}(x)$, and are termed the *left* and *right projections* of x , respectively. We refer the reader to [1, 8] for further information on $*$ -regular rings.

For any subset $S \subseteq R$ of a unital $*$ -regular ring, there exists a smallest unital $*$ -regular subring of R containing S ([4, Proposition 6.2], see also [32, Proposition 3.1] and [22, Proposition 3.4]). This $*$ -regular ring is denoted by $\mathcal{R}(S, R)$, and called the *$*$ -regular closure* of S in R .

Let us denote by $M(R)$ the set of finite matrices over R of arbitrary size, i.e., $\bigcup_{n \geq 1} M_n(R)$.

Definition 2.1 A *Sylvester matrix rank function* on a unital ring R is a map $\text{rk} : M(R) \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- (a) $\text{rk}(0) = 0$ and $\text{rk}(1) = 1$;
- (b) $\text{rk}(M_1 M_2) \leq \min\{\text{rk}(M_1), \text{rk}(M_2)\}$ for any matrices M_1 and M_2 of appropriate sizes;
- (c) $\text{rk} \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \text{rk}(M_1) + \text{rk}(M_2)$ for any matrices M_1 and M_2 ;
- (d) $\text{rk} \begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix} \geq \text{rk}(M_1) + \text{rk}(M_2)$ for any matrices M_1, M_2 and M_3 of appropriate sizes.

The notion of Sylvester matrix rank function was first introduced by Malcolmson in [35] on a question of characterizing homomorphisms from a fixed ring to division rings. Equivalent definitions exist, introduced by Malcolmson itself and Schofield [39]. For more theory and properties about Sylvester matrix rank functions, we refer the reader to [22] and [39, Part I, Chapter 7].

In a regular ring R , any Sylvester matrix rank function rk is uniquely determined by its values on elements of R , see, e.g., [16, Corollary 16.10]. This is no longer true if R is not regular.

Any Sylvester matrix rank function rk on a unital ring R defines a pseudo-metric by the rule $d(x, y) = \text{rk}(x - y)$. The rank function is called *faithful* if the only element with zero rank is the zero element. In this case, d becomes a metric on R . The ring operations are continuous with respect to d , and rk extends uniquely to a Sylvester matrix rank function $\overline{\text{rk}}$ on the completion $\overline{R}$ of R with respect to d .

For a $*$ -subring S of a $*$ -regular ring R , there are tight connections between the structure of projections of the $*$ -regular closure $\mathcal{R}(S, R)$ and possible values of a Sylvester matrix rank function defined on R , see for instance [3, 22].

2.2 ℓ^2 -Betti numbers for group algebras

In this subsection, we define ℓ^2 -Betti numbers arising from a group G with coefficients in a subfield $K \subseteq \mathbb{C}$ closed under complex conjugation.

Let G be a discrete, countable group. For any subring $R \subseteq \mathbb{C}$ closed under complex conjugation, let RG denote the *group $*$ -algebra* of G with coefficients in R , consisting of formal finite sums $\sum_{\gamma \in G} a_\gamma \gamma$ with $a_\gamma \in R$. The sum operation is defined pointwise, the product is induced by the group product and the $*$ -operation is defined by linearity and according to the rule $(a_\gamma \gamma)^* = \overline{a_\gamma} \gamma^{-1}$, for $a_\gamma \in R$ and $\gamma \in G$. Let also $\ell^2(G)$ denote the Hilbert space of all square-summable functions $f : G \rightarrow \mathbb{C}$ with obvious

addition and scalar multiplication, and inner product defined by

$$\langle f, g \rangle_{\ell^2(G)} = \sum_{\gamma \in G} f(\gamma) \overline{g(\gamma)} \quad \text{for } f, g \in \ell^2(G).$$

The space $\ell^2(G)$ has an orthonormal basis, naturally identified with G , consisting of indicator functions $\xi_\gamma \in \ell^2(G)$ for each $\gamma \in G$. Here ξ_γ is defined to be 1 over the element γ and 0 otherwise.

Observe that G acts faithfully on $\ell^2(G)$ by left (resp. right) multiplication $\lambda : G \rightarrow \mathcal{B}(\ell^2(G))$ (resp. $\rho : G \rightarrow \mathcal{B}(\ell^2(G))$), defined by

$$\lambda_\gamma(f)(\delta) = f(\gamma^{-1}\delta) \quad \text{and} \quad \rho_\gamma(f)(\delta) = f(\delta\gamma)$$

for $f \in \ell^2(G)$ and $\gamma, \delta \in G$. Either λ or ρ extend R -linearly to actions $RG \curvearrowright \ell^2(G)$ by bounded operators, preserving the $*$ -operation. We will identify RG with the image of λ inside $\mathcal{B}(\ell^2(G))$.

We denote by $\mathcal{N}(G)$ the weak completion of $\mathbb{C}G \subseteq \mathcal{B}(\ell^2(G))$, which is commonly known as the *group von Neumann algebra of G* . An equivalent algebraic definition can be given: It consists exactly of those bounded operators $T : \ell^2(G) \rightarrow \ell^2(G)$ that are G -equivariant, i.e., the relation $\rho_\gamma \circ T = T \circ \rho_\gamma$ is satisfied for every $\gamma \in G$. The algebra $\mathcal{N}(G)$ is endowed with a normal, positive and faithful trace, defined as

$$\mathrm{tr}_{\mathcal{N}(G)}(T) := \langle T(\xi_e), \xi_e \rangle_{\ell^2(G)} \quad \text{for } T \in \mathcal{N}(G).$$

Note that for an element $T = \sum_{\gamma \in G} a_\gamma \gamma \in \mathbb{C}G$, its trace is simply the coefficient a_e .

All the above constructions can be extended to $k \times k$ matrices: The ring $M_k(RG)$ acts faithfully on $\ell^2(G)^k$ by left (resp. right) multiplication. We denote the extended actions by λ_k and ρ_k , respectively. We identify $M_k(RG)$ with its image $M_k(RG) \subseteq \mathcal{B}(\ell^2(G)^k)$ under λ_k . We denote by $\mathcal{N}_k(G)$ the weak completion of $M_k(\mathbb{C}G)$ inside $\mathcal{B}(\ell^2(G)^k)$, which is easily seen to be equal to $M_k(\mathcal{N}(G))$. The previous trace can be extended to an unnormalized trace over $M_k(\mathcal{N}(G))$ by setting, for a matrix $T = (T_{ij}) \in M_k(\mathcal{N}(G))$,

$$\mathrm{Tr}_{\mathcal{N}_k(G)}(T) := \sum_{i=1}^k \mathrm{tr}_{\mathcal{N}(G)}(T_{ii}).$$

A *finitely generated Hilbert (right) G -module* is any closed subspace V of $\ell^2(G)^k$, invariant with respect to the *right* action $\rho^{\oplus k} := \rho \oplus \dots \oplus \rho$. For $V \leq \ell^2(G)^k$ a finitely generated Hilbert G -module, the corresponding orthogonal projection operator $p_V : \ell^2(G)^k \rightarrow \ell^2(G)^k$ onto V belongs to $\mathcal{N}_k(G)$. One then defines the *von Neumann dimension of V* as the trace of p_V :

$$\dim_{\mathcal{N}(G)}(V) := \mathrm{Tr}_{\mathcal{N}_k(G)}(p_V).$$

In the particular case of matrix group rings $M_k(KG)$, being $K \subseteq \mathbb{C}$ a subfield closed under complex conjugation, every matrix operator $A \in M_k(KG)$ gives rise to an ℓ^2 -

Betti number, in the following way. Consider A as an operator $A : \ell^2(G)^k \rightarrow \ell^2(G)^k$ acting on the left, and take $p_A \in \mathcal{N}_k(G)$ to be the projection onto $\ker A$, which is a finitely generated Hilbert G -module. One can then consider the von Neumann dimension of $\ker A$, which is simply the trace of the projection p_A .

Definition 2.2 Let A be a matrix operator in $M_k(KG)$ for some integer $k \geq 1$. We define the ℓ^2 -Betti number of A by

$$b^{(2)}(A) := \dim_{\mathcal{N}(G)}(\ker A) = \mathrm{Tr}_{\mathcal{N}_k(G)}(p_A).$$

The set of all ℓ^2 -Betti numbers of operators $A \in M_k(KG)$ will be denoted by $\mathcal{C}(G, K)$, and will be referred to as the set of all ℓ^2 -Betti numbers arising from G with coefficients in K . It should be noted that this set is always a subsemigroup of $(\mathbb{R}^+, +)$. We also write $\mathcal{G}(G, K)$ for the subgroup of $(\mathbb{R}, +)$ generated by $\mathcal{C}(G, K)$.

It is also possible to define the von Neumann dimension by means of a *Sylvester matrix rank function*, as follows. Let $\mathcal{U}(G)$ be the algebra of unbounded operators affiliated to $\mathcal{N}(G)$; equivalently, the classical ring of quotients of $\mathcal{N}(G)$. It is a $*$ -regular ring possessing a Sylvester matrix rank function $\mathrm{rk}_{\mathcal{U}(G)}$ defined by

$$\mathrm{rk}_{\mathcal{U}(G)}(U) := \mathrm{Tr}_{\mathcal{N}_k(G)}(\mathrm{LP}(U)) = \mathrm{Tr}_{\mathcal{N}_k(G)}(\mathrm{RP}(U))$$

for any matrix $U \in M_k(\mathcal{U}(G))$, where $\mathrm{LP}(U)$ and $\mathrm{RP}(U)$ are the left and right projections of U inside the $*$ -regular algebra $M_k(\mathcal{U}(G))$, respectively. Notice that these projections actually belong to $\mathcal{N}_k(G)$. In particular, we obtain by restriction a Sylvester matrix rank function rk_{KG} over KG . So for a matrix operator $A \in M_k(KG)$ we have $p_A = 1_k - \mathrm{RP}(A)$ and we get the equality

$$b^{(2)}(A) = k - \mathrm{rk}_{KG}(A). \quad (2.1)$$

Here 1_k stands for the identity matrix in k dimensions.

3 Approximating crossed product algebras through a dynamical perspective

We recall the general construction used in [2] on approximating $\mathbb{Z}$ -crossed product algebras.

Let $T : X \rightarrow X$ be a homeomorphism of a totally disconnected, compact metrizable space X , which we also assume to be infinite (e.g., one can take X to be the Cantor space). Let also K be an arbitrary field endowed with a positive definite involution, that is, an involution $*$ such that for all $n \geq 1$ and $a_1, \dots, a_n \in K$, we have $\sum_{i=1}^n a_i^* a_i = 0 \implies a_i = 0$ for each $i = 1, \dots, n$.

The algebras of interest are $\mathbb{Z}$ -crossed product algebras of the form

$$\mathcal{A} := C_K(X) \rtimes_T \mathbb{Z},$$

where $C_K(X)$ denotes the algebra of locally constant functions $f : X \rightarrow K$; equivalently, the algebra of continuous functions $f : X \rightarrow K$ when K is endowed with the discrete topology. For the approximation process, we choose a T -invariant, ergodic and full probability measure μ on X . We refer the reader to [2, Section 3] for a detailed exposition of the construction.

For any clopen subset $\emptyset \neq E \subseteq X$ and any (finite) partition $\mathcal{P}$ of the complement $X \setminus E$ into clopen subsets, let $\mathcal{B}$ be the unital $*$ -subalgebra of $\mathcal{A}$ generated by the partial isometries $\{\chi_Z t \mid Z \in \mathcal{P}\}$. Here t denotes the generator of the copy of $\mathbb{Z}$ inside $\mathcal{A}$, and χ_A denotes the characteristic function of the set A . There exists a quasi-partition of X (i.e., a countable family of non-empty, pairwise disjoint clopen subsets whose union has full measure) given by the T -translates of clopen subsets W of the form

$$W = E \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E) \quad (3.1)$$

for $k \geq 1$ and $Z_i \in \mathcal{P}$, whenever these are non-empty. In fact, if we write $|W| := k$ (the length of W) and $\mathbb{V} := \{W \neq \emptyset \text{ as above}\}$, then for a fixed $W \in \mathbb{V}$ and $0 \leq i < |W|$ the element $\chi_{T^i(W)}$ belongs to $\mathcal{B}$, and moreover, the set of elements

$$e_{ij}(W) = (\chi_{X \setminus E} t)^i \chi_W (t^{-1} \chi_{X \setminus E})^j, \quad 0 \leq i, j < |W|,$$

forms a set of $|W| \times |W|$ matrix units in $\mathcal{B}$ (that is, they satisfy $e_{ik}(W)e_{lj}(W) = \delta_{k,l}e_{ij}(W)$ for all indices $0 \leq i, j, k, l < |W|$). In addition, by [2, Proposition 3.11] the element $h_W := e_{00}(W) + \cdots + e_{|W|-1, |W|-1}(W)$ is central in $\mathcal{B}$ and we have a $*$ -isomorphism

$$h_W \mathcal{B} \cong M_{|W|}(K).$$

In this way, one obtains an injective $*$ -representation $\pi : \mathcal{B} \hookrightarrow \prod_{W \in \mathbb{V}} M_{|W|}(K) =: \mathfrak{R}_{\mathcal{B}}$ defined by $\pi(a) = (h_W \cdot a)_W$ [2, Proposition 3.13].

From now on, the $*$ -algebra $\mathcal{B}$ corresponding to $(E, \mathcal{P})$ as above will be denoted by $\mathcal{A}(E, \mathcal{P})$.

Take now $\{E_n\}_{n \geq 1}$ to be a decreasing sequence of clopen sets of X together with a family $\{\mathcal{P}_n\}_{n \geq 1}$ consisting of (finite) partitions into clopen sets of the corresponding complements $X \setminus E_n$, satisfying:

- (a) the intersection of all the E_n consists of a single point $y \in X$;
- (b) $\mathcal{P}_{n+1} \cup \{E_{n+1}\}$ is a partition of X finer than $\mathcal{P}_n \cup \{E_n\}$;
- (c) $\bigcup_{n \geq 1} (\mathcal{P}_n \cup \{E_n\})$ generates the topology of X .

By writing $\mathbb{V}_n$ for the set of all the non-empty subsets W of the form (3.1) corresponding to the pair $(E_n, \mathcal{P}_n)$, and setting $\mathcal{A}_n := \mathcal{A}(E_n, \mathcal{P}_n)$ and $\mathfrak{R}_n := \prod_{W \in \mathbb{V}_n} M_{|W|}(K)$,

we get injective $*$ -representations $\pi_n : \mathcal{A}_n \hookrightarrow \mathfrak{R}_n$, in such a way that the diagrams

$$\begin{array}{ccccccc} \mathcal{A}_n & \xhookrightarrow{\iota_n} & \mathcal{A}_{n+1} & \xhookrightarrow{\iota_{n+1}} & \mathcal{A}_{n+2} & \hookrightarrow \cdots \hookrightarrow & \mathcal{A}_\infty \\ \downarrow \pi_n & & \downarrow \pi_{n+1} & & \downarrow \pi_{n+2} & & \downarrow \pi_\infty \\ \mathfrak{R}_n & \xhookrightarrow{j_n} & \mathfrak{R}_{n+1} & \xhookrightarrow{j_{n+1}} & \mathfrak{R}_{n+2} & \hookrightarrow \cdots \hookrightarrow & \mathfrak{R}_\infty \end{array}$$

commute. Here ι_n is the natural embedding $\iota_n(\chi_Z t) = \sum_{Z'} \chi_{Z'} t$ where the sum is taken with respect to all the $Z' \in \mathcal{P}_{n+1}$ satisfying $Z' \subseteq Z$, the maps $j_n : \mathfrak{R}_n \hookrightarrow \mathfrak{R}_{n+1}$ are the embeddings given in [2, Proposition 4.2], and $\mathcal{A}_\infty, \mathfrak{R}_\infty$ are the inductive limits of the direct systems $(\mathcal{A}_n, \iota_n), (\mathfrak{R}_n, j_n)$, respectively.

Remark 3.1 The algebra $\mathcal{A}_\infty$ can be explicitly described in terms of the crossed product, as follows. For $U \subseteq X$ an open set, denote by $C_{c,K}(U)$ the ideal of $C_K(X)$ generated by the characteristic functions χ_V , where V ranges over the clopen subsets $V \subseteq X$ contained in U . By [2, Lemma 4.3], $\mathcal{A}_\infty$ coincides with the $*$ -subalgebra of $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ generated by $C_K(X)$ and $C_{c,K}(X \setminus \{y\})t$. (Recall from a) above that $\{y\} = \bigcap_{n \geq 1} E_n$.)

One can define a Sylvester matrix rank function on each $\mathfrak{R}_n$ by the rule

$$\mathrm{rk}_n(M) = \sum_{W \in \mathbb{V}_n} \mu(W) \mathrm{Rk}(M_W) \quad \text{for } M = (M_W)_W \in \mathfrak{R}_n, \quad (3.2)$$

being Rk the usual rank of matrices. These Sylvester matrix rank functions are compatible with respect to the embeddings j_n , so they give rise to a well-defined Sylvester matrix rank function rk_∞ on $\mathfrak{R}_\infty$.

Theorem 3.2 [2, Theorem 4.7 and Proposition 4.8] *Following the above notation, if $\mathfrak{R}_{\mathrm{rk}} := \overline{\mathfrak{R}_\infty}$ denotes the rank completion of $\mathfrak{R}_\infty$ with respect to its Sylvester matrix rank function rk_∞ (see Sect. 2.1), then there exists an injective $*$ -homomorphism $\pi_{\mathrm{rk}} : \mathcal{A} \rightarrow \mathfrak{R}_{\mathrm{rk}}$ making the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\pi_{\mathrm{rk}}} & \mathfrak{R}_{\mathrm{rk}} \\ \text{IU} & & \text{IU} \\ \mathcal{A}_\infty & \xrightarrow{\pi_\infty} & \mathfrak{R}_\infty \end{array}$$

commutative, and sending the element t to $\pi_{\mathrm{rk}}(t) = \lim_n \pi_n(\chi_{X \setminus E_n} t)$. Moreover, we obtain a Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A}$ by restriction of rk_∞ (the extension of rk_∞ to $\mathfrak{R}_{\mathrm{rk}}$) on $\mathcal{A}$, which is extremal and unique with respect to the following property:

$$\mathrm{rk}_{\mathcal{A}}(\chi_U) = \mu(U) \quad \text{for every clopen subset } U \subseteq X. \quad (3.3)$$

Finally, the rank completion of $\mathcal{A}$ with respect to $\mathrm{rk}_{\mathcal{A}}$ gives back $\mathfrak{R}_{\mathrm{rk}}$, that is $\overline{\mathcal{A}} = \mathfrak{R}_{\mathrm{rk}}$.

In particular, since $\mathfrak{R}_{\text{rk}}$ has the structure of a $*$ -regular algebra, we can consider the $*$ -regular closure of $\mathcal{A}$ inside $\mathfrak{R}_{\text{rk}}$, which we denote by $\mathcal{R}_{\mathcal{A}} := \mathcal{R}(\mathcal{A}, \mathfrak{R}_{\text{rk}})$, see [3] for more details. The $*$ -regular closure $\mathcal{R}_{\mathcal{A}}$ is extensively studied in [3]. For our purposes, it will become important when computing $\mathcal{O}(\bar{n})$ -Betti numbers arising from the generalized odometer algebra $\mathcal{O}(\bar{n})$ in Sect. 5, and also in Sect. 4.3.

3.1 A formula for computing $\mathcal{A}$ -Betti numbers inside $\mathcal{B}$

In this subsection, we give a formula for computing $\mathcal{A}$ -Betti numbers of elements from matrix algebras over the approximation algebra $\mathcal{B} = \mathcal{A}(E, \mathcal{P})$. We will use ideas from [18], although applied to our construction.

By Theorem 3.2, the algebra $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ possesses a ‘canonical’ Sylvester matrix rank function $\text{rk}_{\mathcal{A}}$, unique with respect to Property (3.3). It is then natural to ask which is the set of positive real numbers reached by such a Sylvester matrix rank function.

Definition 3.3 Let $A \in M_k(\mathcal{A})$. We define the $\mathcal{A}$ -Betti number of A by (cf. Definition 2.2, see also Eq. (2.1))

$$b^{\mathcal{A}}(A) := k - \text{rk}_{\mathcal{A}}(A).$$

The set consisting of all $\mathcal{A}$ -Betti numbers of elements $A \in M_k(\mathcal{A})$ will be denoted by $\mathcal{C}(\mathcal{A})$. This set has the structure of a semigroup of $(\mathbb{R}^+, +)$ (cf. Definition 2.2). The subgroup of $(\mathbb{R}, +)$ generated by $\mathcal{C}(\mathcal{A})$ will be denoted by $\mathcal{G}(\mathcal{A})$. It is shown in Sect. 4 that the definition of $\mathcal{A}$ -Betti number coincides with Definition (2.2) in case $\mathcal{A}$ is the lamplighter group algebra Γ , and thus, $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\Gamma, K)$.

We now focus on the approximation algebra $\mathcal{B} = \mathcal{A}(E, \mathcal{P})$, where recall that E is any non-empty clopen subset of X , and $\mathcal{P}$ a (finite) partition of the complement $X \setminus E$ into clopen subsets. Let $\pi : \mathcal{B} \rightarrow \mathfrak{R}_{\mathcal{B}}$ be the faithful $*$ -representation on $\mathfrak{R}_{\mathcal{B}} = \prod_{W \in \mathbb{V}} M_{|W|}(K)$ given by $\pi(a) = (h_W \cdot a)_W$. We extend π to a faithful $*$ -representation, also denoted by π , over matrix algebras

$$\pi : M_k(\mathcal{B}) \rightarrow M_k(\mathfrak{R}_{\mathcal{B}}) \cong \prod_{W \in \mathbb{V}} M_k(K) \otimes M_{|W|}(K)$$

in a canonical way. Hence, $\text{rk}_{\mathcal{A}}$ can be computed over elements $A \in M_k(\mathcal{B})$ by means of the formula (recall (3.2))

$$\text{rk}_{\mathcal{A}}(A) = \sum_{W \in \mathbb{V}} \mu(W) \text{Rk}(\pi(A)_W),$$

where Rk is the usual rank of matrices. Note that $\pi(A)_W \in M_k(K) \otimes M_{|W|}(K) = M_{k|W|}(K)$. We thus obtain the following proposition.

Proposition 3.4 *With the above notation, for a given element $A \in M_k(\mathcal{B})$ we have the formula*

$$b^A(A) = \sum_{W \in \mathbb{V}} \mu(W) \dim_K(\ker \pi(A)_W).$$

Here $\dim_K(\cdot)$ denotes the usual K -dimension of finite-dimensional K -vector spaces.

Proof It is just a matter of computation, using that $\text{Rk}(M) = m - \dim_K(\ker M)$ for any matrix $M \in M_m(K)$ and that $\mathbb{V}$ forms a quasi-partition of X , so $\sum_{W \in \mathbb{V}} \mu(W)|W| = 1$. $\square$

Fix now $\{e_{ij} \mid 0 \leq i, j < k\}$ a complete system of matrix units for $M_k(K)$. For a fixed $W \in \mathbb{V}$, the family $\{e_{ij} \otimes e_{i'j'}(W) \mid 0 \leq i, j < k; 0 \leq i', j' < |W|\}$ is a complete system of matrix units for $M_k(K) \otimes M_{|W|}(K)$, where the $e_{i'j'}(W)$ are the matrix units for $h_W \mathcal{B} = M_{|W|}(K)$ introduced in Sect. 3. In particular, $\pi(e_{ij} \otimes 1)_W = e_{ij} \otimes h_W$. Let now $M_k(K) \otimes M_{|W|}(K)$ act on the K -vector space $K^k \otimes K^{|W|}$, with K -basis $\mathcal{B} = \{e_i \otimes e_{i'}(W) \mid 0 \leq i < k; 0 \leq i' < |W|\}$, by left matrix multiplication:

$$(e_{ij} \otimes e_{i'j'}(W)) \cdot (e_a \otimes e_{a'}(W)) = \delta_{j,a} \delta_{j',a'} e_i \otimes e_{i'}(W).$$

Let $\langle \cdot, \cdot \rangle$ be the scalar product on $K^k \otimes K^{|W|}$ rendering the basis $\mathcal{B}$ orthonormal. Then for an element $A \in M_k(\mathcal{B})$, the entry of the matrix $\pi(A)_W$ corresponding to the $e_{ij} \otimes e_{i'j'}(W)$ component is given by

$$\langle \pi(A)_W \cdot (e_j \otimes e_{j'}(W)), e_i \otimes e_{i'}(W) \rangle.$$

One can think of the matrix $\pi(A)_W$ as the *adjacency-labeled* matrix of an edge-labeled graph $E_A(W)$ defined as follows:

- (a) The vertices are the pairs $(e_i, e_{i'}(W))$, for $0 \leq i < k, 0 \leq i' < |W|$.
- (b) We have an arrow from $(e_j, e_{j'}(W))$ to $(e_i, e_{i'}(W))$ if and only if

$$\langle \pi(A)_W \cdot (e_j \otimes e_{j'}(W)), e_i \otimes e_{i'}(W) \rangle \neq 0.$$

In this case, we label the arrow as $d_{(j,j'),(i,i')} := \langle \pi(A)_W \cdot (e_j \otimes e_{j'}(W)), e_i \otimes e_{i'}(W) \rangle$.

Here *adjacency-labeled* matrix means that the coefficient of $\pi(A)_W$ corresponding to the $e_{ij} \otimes e_{i'j'}(W)$ component is exactly $d_{(j,j'),(i,i')}$. There is an example of such a graph in Fig. 1, corresponding to the matrix

$$\begin{aligned} \pi(A)_W &= d_1 \cdot e_{11} \otimes e_{21}(W) + d_2 \cdot e_{k-2,k-2} \otimes e_{21}(W) \\ &\quad + d_3 \cdot e_{k-1,k-2} \otimes e_{22}(W) + d_4 \cdot e_{11} \otimes e_{|W|-2,|W|-2}(W). \end{aligned}$$

If we denote by $\mathcal{G}_A(W)$ the set consisting of the connected components of the graph $E_A(W)$ and by A_C , for $C \in \mathcal{G}_A(W)$, the adjacency-labeled matrix of the corresponding connected component C , then the matrix $\pi(A)_W$ is similar to the block-diagonal

Fig. 1 An example of a graph $E_A(W)$

matrix $\text{diag}(A_{C_1}, A_{C_2}, \dots, A_{C_r})$, being $\mathcal{G}r_A(W) = \{C_1, \dots, C_r\}$. As a consequence, and by using Proposition 3.4, we obtain the formula

$$\dim_{\mathcal{A}}(\ker A) = \sum_{W \in \mathbb{V}} \sum_{C \in \mathcal{G}r_A(W)} \mu(W) \dim_K(\ker A_C), \quad (3.4)$$

which will be used in Sect. 4 for explicit computations of ℓ^2 -Betti numbers.

The following lemma, whose proof is trivial, is an adaptation of [18, Lemma 20] in our notation, and provides a graphical way for computing dimensions of kernels of finite matrices.

Lemma 3.5 (Flow lemma at $(e_i, e_{i'}(W))$) *An element $\alpha = \sum_{j,j'} \lambda_{(j,j')} e_j \otimes e_{j'}(W) \in K^k \otimes K^{|W|}$ belongs to the kernel of the matrix $\pi(A)_W$ if and only if, for every vertex $(e_i, e_{i'}(W))$,*

$$\sum_{j,j'} \lambda_{(j,j')} d_{(j,j'),(i,i')} = 0.$$

One can think of this system of equations in a graphical way: If we think of the variable $\lambda_{(i,i')}$ as the label of the vertex $(e_i, e_{i'}(W))$, then the flow lemma at this vertex is telling us that

$$\sum_{\substack{\text{all arrows having} \\ \text{range } (i, i')}} (\text{label of the arrow}) \cdot (\text{label of the source of the arrow}) = 0.$$

To see how exactly the flow lemma works explicitly, we refer the reader to the appendix given in [18], where some applications of it in concrete examples are described.

4 The lamplighter group algebra

In this section, we apply the construction given in Sect. 3 to the lamplighter group algebra. This algebra is of great relevance because, among other things, it gave the first counterexample to the strong Atiyah conjecture (SAC), see for example [13, 19]. We refer the reader to [3, Section 5], where a further application of the construction from Sect. 3 to a wider class of group algebras is given.

Definition 4.1 The lamplighter group Γ is defined to be the wreath product of the finite group of two elements, $\mathbb{Z}_2$, by $\mathbb{Z}$. In other words,

$$\Gamma = \mathbb{Z}_2 \wr \mathbb{Z} = \left(\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2 \right) \rtimes_{\sigma} \mathbb{Z},$$

where the action implementing the semidirect product is the well-known Bernoulli shift σ defined by

$$\sigma_n(x)_i = x_{i+n} \quad \text{for } x = (x_i) \in \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2.$$

If we denote by t the generator corresponding to $\mathbb{Z}$ and by a_i the generator corresponding to the i^{th} copy of $\mathbb{Z}_2$, we have the following presentation for Γ :

$$\Gamma = \langle t, \{a_i\}_{i \in \mathbb{Z}} \mid a_i^2, a_i a_j a_i a_j, t a_i t^{-1} a_{i-1} \text{ for } i, j \in \mathbb{Z} \rangle.$$

Let now $K \subseteq \mathbb{C}$ be a subfield of $\mathbb{C}$ closed under complex conjugation, which will be the involution on K . As in [3, Section 6], the Fourier transform $\mathcal{F}$ gives a $*$ -isomorphism $K\Gamma \cong C_K(X) \rtimes_T \mathbb{Z} =: \mathcal{A}$ through the identifications

$$1 \mapsto \chi_X, \quad e_i := \frac{1 + a_i}{2} \mapsto \chi_{U_i}, \quad t \mapsto t,$$

where $X = \{0, 1\}^{\mathbb{Z}}$ is the Cantor set, T the shift map defined by $T(x)_i = x_{i+1}$ for $x = (x_i) \in X$, and U_i is the clopen set consisting of all points $x \in X$ having a 0 at the i^{th} component.

Given $\epsilon_{-k}, \dots, \epsilon_l \in \{0, 1\}$, the cylinder set $\{x = (x_i) \in X \mid x_{-k} = \epsilon_{-k}, \dots, x_l = \epsilon_l\}$ will be throughout denoted by $[\epsilon_{-k} \cdots \epsilon_0 \cdots \epsilon_l]$. The collection of all the cylinder sets constitutes a basis for the topology of X .

We take μ to be the product measure on X , where the $(\frac{1}{2}, \frac{1}{2})$ -measure on each component $\{0, 1\}$ is considered. It is well known (see, e.g., [24, Example 3.1]) that μ is an ergodic, full and shift-invariant probability measure on X . Therefore, by Theorem 3.2 we can construct a Sylvester matrix rank function $\text{rk}_{\mathcal{A}}$ on $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$, as exposed in Sect. 3.

Proposition 4.2 *As in Sect. 2.2, let $\text{rk}_{K\Gamma}$ be the Sylvester matrix rank function on $K\Gamma$ inherited from $\mathcal{U}(\Gamma)$. Then $\text{rk}_{K\Gamma}$ and $\text{rk}_{\mathcal{A}}$ coincide through the Fourier transform $\mathcal{F} : K\Gamma \cong \mathcal{A}$. As a consequence, the ℓ^2 -Betti number of a matrix operator $A \in M_k(K\Gamma)$ equals its $\mathcal{A}$ -Betti number, that is,*

$$b^{(2)}(A) = b^{\mathcal{A}}(\mathcal{F}(A)). \quad (4.1)$$

Hence, $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\Gamma, K)$ and $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\Gamma, K)$.

Proof Both $\text{rk}_{K\Gamma}$ and $\text{rk}_{\mathcal{A}}$ coincide by Theorem 3.2, as observed in [3, Subsection 6.1]. The result follows from Formula (2.1). $\square$

4.1 The approximating algebras $\mathcal{A}_n$ for the lamplighter group algebra

In this subsection, we summarize the contents of [3, Subsection 6.1] in relation to the approximating algebras $\mathcal{A}_n$ in the particular case of the lamplighter group algebra. We identify $K\Gamma = C_K(X) \rtimes_T \mathbb{Z}$ under the Fourier transform.

We take $E_n = [1 \cdots \underline{1} \cdots 1]$ (with $2n + 1$ one's) for the sequence of clopen sets, whose intersection gives the point $y = (\dots, 1, \underline{1}, 1, \dots) \in X$. We also take the partitions $\mathcal{P}_n$ of the complements $X \setminus E_n$ to be

$$\mathcal{P}_n = \{[00 \cdots \underline{0} \cdots 00], [00 \cdots \underline{0} \cdots 01], \dots, [01 \cdots \underline{1} \cdots 11]\}.$$

Let $\mathcal{A}_n := \mathcal{A}(E_n, \mathcal{P}_n)$ be the unital $*$ -subalgebra of $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ generated by the partial isometries $\{\chi_{Zt} \mid Z \in \mathcal{P}_n\}$. The set $\mathbb{V}_n$ consists of the non-empty W of the following types:

- (a) $W_0 = [11 \cdots \underline{1} \cdots 111]$ of length 1 (there are $2n + 2$ ones);
- (b) $W_1 = [11 \cdots \underline{1} \cdots 11011 \cdots 1 \cdots 11]$ of length $2n + 2$ (there are $4n + 2$ ones, and a zero);
- (c) $W(\epsilon_1, \dots, \epsilon_l) = [11 \cdots \underline{1} \cdots 110\epsilon_1 \cdots \epsilon_l 011 \cdots 1 \cdots 11]$ of length $(2n + 3) + l$,

where in the last type $l \geq 0$, and each $\epsilon_i \in \{0, 1\}$, but with at most $2n$ consecutive ones in the sequence $(\epsilon_1, \dots, \epsilon_l)$. We understand $l = 0$ as having no ϵ_i , so there are only 2 zeros in the middle of the corresponding W .

We now recall the definition of the m -step Fibonacci sequence, as in [3, Definition 6.2].

Definition 4.3 Fix an integer $m \geq 2$. For $k \in \mathbb{Z}^+$, we define the k^{th} m -acci number, denoted by $\text{Fib}_m(k)$, recursively by setting

$$\text{Fib}_m(0) = 0, \quad \text{Fib}_m(1) = 1 \quad \text{and} \quad \text{Fib}_m(l) = 2^{l-2} \text{ for } 2 \leq l \leq m - 1,$$

and for $r \in \mathbb{Z}^+$,

$$\text{Fib}_m(r+m) = \text{Fib}_m(r+m-1) + \cdots + \text{Fib}_m(r).$$

Lemma 4.4 (Lemma 6.3 of [3]) *For integers $k, m \geq 2$, $\text{Fib}_m(k)$ is exactly the number of possible sequences $(\epsilon_1, \dots, \epsilon_l)$ of length $l = k - 2$ that one can construct with zeroes and ones, but having at most $m - 1$ consecutive one's.*

We thus have injective $*$ -representations $\pi_n : \mathcal{A}_n \hookrightarrow \mathfrak{R}_n$ defined by the rule $\pi_n(x) = (h_W \cdot x)_W$, where we have the concrete realization $\mathfrak{R}_n = K \times \prod_{k \geq 1} M_{2n+1+k}(K)^{\text{Fib}_{2n+1}(k)}$. We will use the sequence $\{\mathcal{A}_n\}_{n \geq 0}$ in the next subsection for computing rational values of ℓ^2 -Betti numbers arising from Γ .

4.2 Some computations of ℓ^2 -Betti numbers for the lamplighter group algebra

Although it does not appear in our approximating sequence, we now introduce the algebra $\mathcal{A}_{1/2}$, which gives the first non-trivial approximating algebra exhibiting irrational behavior of ℓ^2 -Betti numbers. Throughout this subsection, K will be any subfield of $\mathbb{C}$ closed under complex conjugation, which will be the involution on K .

We take as a non-empty clopen subset $E_{1/2} = [1\bar{1}]$, and the partition $\mathcal{P}_{1/2} = \{[00], [01], [10]\}$ of the complement $X \setminus E_{1/2}$. The algebra $\mathcal{A}_{1/2} := \mathcal{A}(E_{1/2}, \mathcal{P}_{1/2})$ is then the unital $*$ -subalgebra of $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ generated by the partial isometries $\{\chi_{[00]}t, \chi_{[01]}t, \chi_{[10]}t\}$; equivalently, it is the unital $*$ -subalgebra generated by $s_{-1} = e_{-1}t$ and $s_0 = e_0t$.

The set $\mathbb{V}_{1/2}$ consists of the non-empty W of the form:

- (a) either $W = [1\bar{1}\bar{1}]$ of length 1, or
- (b) $W = [1\bar{1}0^{k_1}10^{k_2}1 \cdots 0^{k_r}1\bar{1}]$ of length $k = k_1 + \cdots + k_r + (r+1)$, with $r, k_i \geq 1$,

where 0^{k_i} denotes $0 \cdots 0$. It is easily checked that

$$\begin{aligned} \sum_{W \in \mathbb{V}_{1/2}} |W| \mu(W) &= \frac{1}{2^3} + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{k_1 + \cdots + k_r + (r+1)}{2^{k_1 + \cdots + k_r + (r+3)}} \\ &= \frac{1}{2^3} + \sum_{r \geq 1} \frac{3r+1}{2^{r+3}} = 1, \end{aligned}$$

so indeed the T -translates of the $W \in \mathbb{V}_{1/2}$ form a quasi-partition of X .

We are now ready to give examples of matrix operators with coefficients in $\mathbb{Q}$, i.e., $A \in M(\mathbb{Q}\Gamma)$, with irrational ℓ^2 -Betti number by using part (i) of Proposition 4.2, together with Formula (3.4). Our main result is the following.

Theorem 4.5 *Fix n a nonnegative integer. For $0 \leq l \leq n$, take*

$$p_l(x) = a_{0,l} + a_{1,l}x + \cdots + a_{m_l,l}x^{m_l}$$

polynomials of degrees $m_l \geq 1$ with nonnegative integer coefficients, and we require that $a_{1,0} \geq 1$. Take also $d_1, \dots, d_n \geq 2$ natural numbers.

There exists then an element A inside some matrix algebra over $\mathcal{A}_{1/2}$ with rational coefficients such that

$$b^{(2)}(A) = q_0 + q_1 \sum_{k \geq 1} \frac{1}{2^{p_0(k) + p_1(k)d_1^k + \dots + p_n(k)d_n^k}} \in \mathcal{C}(\Gamma, \mathbb{Q}), \quad (4.2)$$

where q_0, q_1 are nonzero rational numbers which can be explicitly computed. Therefore, we get a collection of irrational and even transcendental ℓ^2 -Betti numbers arising from the lamplighter group Γ . If moreover $a_{0,0} = 0$, then

$$\sum_{k \geq 1} \frac{1}{2^{p_0(k) + p_1(k)d_1^k + \dots + p_n(k)d_n^k}} \in \mathcal{G}(\Gamma, \mathbb{Q}).$$

Proof Since the proof of the theorem is quite technical, we will start by giving some examples of elements whose ℓ^2 -Betti number behave as stated, first by just considering a single polynomial $p(k)$ and then by considering $p(k)d^k$. After that, we will present an explicit element having ℓ^2 -Betti number as in (4.2).

We start by giving a concrete example, $p(x) = 2 + x + x^2$. Consider the element from $M_{11}(\mathcal{A}_{1/2})$ given by

$$\begin{aligned} A = & -\chi_{[00]}t^{-1} \cdot e_{1,1} - \chi_{[0]} \cdot e_{2,1} - \chi_{[00]}t^{-1} \cdot e_{2,2} \\ & - \chi_{[10]} \cdot e_{3,2} - \chi_{[00]}t \cdot e_{3,3} - \chi_{[01]} \cdot e_{4,3} \\ & - \chi_{[00]}t^{-1} \cdot e_{4,4} - \chi_{[0]} \cdot e_{5,4} - \chi_{[00]}t^{-1} \cdot e_{5,5} \\ & - \chi_{[10]} \cdot e_{6,5} - \chi_{[00]}t \cdot e_{6,6} - \chi_{[01]} \cdot e_{0,6} \\ & - \chi_{[010]}t^2 \cdot e_{8,1} - \chi_{[00]}t \cdot e_{8,8} + \chi_{[0]} \cdot e_{9,8} \\ & - \chi_{[00]}t \cdot e_{9,9} - \chi_{[01]} \cdot e_{10,9} \\ & - \chi_{[010]}t \cdot e_{9,7} + \chi_{[010]}t^{-1} \cdot e_{0,7} \\ & + \chi_{[0]} \cdot (\text{Id}_{11} - e_{0,0} - e_{7,7} - e_{10,10}) - \chi_{[01]} \cdot e_{1,1} \in M_{11}(\mathbb{Q}\Gamma), \end{aligned}$$

being the family $\{e_{i,j} \mid 0 \leq i, j < 11\}$ a complete system of matrix units for $M_{11}(\mathbb{Q})$.¹ If we take $W = [1\bar{1}1]$, we observe that $\pi(A)_W$ equals the zero 11×11 matrix, so its kernel has dimension 11. Figure 2 gives the prototypical graph $E_A(W)$ that appears in case one takes a W of the second form $[1\bar{1}0^{k_1}10^{k_2}1 \dots 0^{k_r}11]$ with length $k = k_1 + \dots + k_r + (r+1)$. By having in mind Formula (3.4), we study the different types of connected components of the graph $E_A(W)$. From now on, in the diagrams each straight line should be labeled with a $+1$, and each dotted line with a -1 . Nevertheless, we will explicitly write down some of the labels when necessary in the diagrams, and we will use straight lines in such cases.

In this concrete example, we have four different types of connected components C for the graph, namely:

¹ It should be mentioned here that this example is similar to the one given in [18, Section 4], although some differences are present: Apart from the fact that the element is not the same, our construction realizes the ‘levels’ e_i as a result of allowing matrix algebras over $\mathcal{A}_{1/2}$.

Fig. 2 Graph $E_A(W)$ for a $w = [110^k 1 10^k 1 \dots 0^k 1]$ of length k . Each straight line should be labeled with a $+1$, and each dotted line with a -1

- (a) C_1 , given by the graphs with only one vertex $\bullet$. Note that in this case $\dim_{\mathbb{Q}}(\ker A_{C_1}) = 1$, and we have $12+8r+3(k_1+\dots+k_r)$ connected components of this kind.
- (b) C_2 , given by the graph

Let us apply the flow lemma to compute $\dim_{\mathbb{Q}}(\ker A_{C_2})$. We write $\lambda_{(i,j)}$ to denote the labels of the vertices of the i^{th} row, for $1 \leq i \leq 3$ and $1 \leq j \leq k_1$. Here k_1 corresponds to the number of columns of the graph. We analyze the case where $k_1 \geq 2$, being the case $k_1 = 1$ completely analogous, and giving the same final result. Note that for $i = 3$ we only have the vertex corresponding to $j = k_1$. The flow lemma gives the following set of equations for the vertices:

$$\begin{cases} \lambda_{(1,1)} = 0, \\ \lambda_{(2,1)} + \lambda_{(1,1)} = 0, \\ \text{for } 1 \leq j \leq k_1 - 1, \\ \begin{cases} \lambda_{(1,j+1)} - \lambda_{(1,j)} = 0, \\ \lambda_{(2,j+1)} + \lambda_{(1,j+1)} - \lambda_{(2,j)} = 0, \end{cases} \\ -\lambda_{(2,k_1)} = 0. \end{cases}$$

The solution of this system of equations is $\lambda_{(1,j)} = \lambda_{(2,j)} = 0$ for all $1 \leq j \leq k_1$, giving one degree of freedom, namely $\lambda_{(3,k_1)}$. Therefore, $\dim_{\mathbb{Q}}(\ker A_{C_2}) = 1$. We only have one connected component of this kind.

- (c) C_3 , given by the graph

Let us apply again the flow lemma to compute $\dim_{\mathbb{Q}}(\ker A_{C_3})$. The labels of the vertices of the i^{th} row will again be denoted by $\lambda_{(i,j)}$, for $1 \leq i \leq 7$ and

$1 \leq j \leq k_r$. We restrict ourselves to the case where $k_r \geq 2$ (the case $k_r = 1$ gives the same result for $\dim_{\mathbb{Q}}(\ker A_{C_3})$). Note that for $i = 1$ we only have the vertex corresponding to $j = k_r$. The flow lemma gives the following set of equations for the vertices:

$$\left\{ \begin{array}{l} \text{for } 1 \leq j \leq k_r - 1, \\ \lambda_{(2,j)} - \lambda_{(2,j+1)} = 0, \\ \lambda_{(4,j+1)} - \lambda_{(4,j)} = 0, \\ \lambda_{(5,j)} - \lambda_{(5,j+1)} = 0, \\ \lambda_{(7,j+1)} - \lambda_{(7,j)} = 0, \end{array} \right. \quad \left\{ \begin{array}{l} \lambda_{(3,k_r)} - \lambda_{(2,k_r)} = 0, \\ \lambda_{(4,1)} - \lambda_{(3,1)} = 0, \\ \lambda_{(5,k_r)} - \lambda_{(4,k_r)} = 0, \\ \lambda_{(6,k_r)} - \lambda_{(5,k_r)} = 0, \\ \lambda_{(7,1)} - \lambda_{(6,1)} = 0, \\ -\lambda_{(7,k_r)} = 0, \end{array} \right. \quad \left\{ \begin{array}{l} \text{for } 1 \leq j \leq k_r - 1, \\ \lambda_{(3,j)} - \lambda_{(2,j)} - \lambda_{(3,j+1)} = 0, \\ \lambda_{(6,j)} - \lambda_{(5,j)} - \lambda_{(6,j+1)} = 0, \end{array} \right.$$

The first set of equations gives $\lambda_{(2,j)} = \lambda_{(2,1)}$, $\lambda_{(4,j)} = \lambda_{(4,1)}$, $\lambda_{(5,j)} = \lambda_{(5,1)}$, and $\lambda_{(7,j)} = \lambda_{(7,1)}$ for all $1 \leq j \leq k_r$. Analogously, the third set of equations gives $\lambda_{(3,j)} = \lambda_{(3,1)} - (j-1)\lambda_{(2,1)}$, and $\lambda_{(6,j)} = \lambda_{(6,1)} - (j-1)\lambda_{(5,1)}$ for all $2 \leq j \leq k_r$. For now, we are left with $\lambda_{(1,k_r)}$, and $\lambda_{(i,1)}$ for all $2 \leq i \leq 7$ as possible degrees of freedom. But the second set of equations then implies $\lambda_{(2,1)} = \lambda_{(3,1)} - (k_r-1)\lambda_{(2,1)}$, $\lambda_{(3,1)} = \lambda_{(4,1)} = \lambda_{(5,1)}$, $\lambda_{(5,1)} = -(k_r-1)\lambda_{(5,1)}$ and $\lambda_{(6,1)} = \lambda_{(7,1)} = 0$. Solving this gives $\lambda_{(i,1)} = 0$ for all $2 \leq i \leq 7$.

The conclusion is that we have only one degree of freedom, namely $\lambda_{(1,k_r)}$, and so $\dim_{\mathbb{Q}}(\ker A_{C_3}) = 1$. We only have one connected component of this kind.

(d) Finally C_4 , given by the graphs

These connected components are the essential part of the whole graph since they give rise to the irrationality of $b^{(2)}(A)$. Note that we have $r-1$ connected com-

ponents of this kind, and we fix one of them, corresponding to the lengths k_l and k_{l+1} respectively, for $1 \leq l \leq r-1$. Note that k_l coincides with the number of columns of the top left graph, and k_{l+1} coincides with the number of columns of the bottom right graph.

Once more, we apply the flow lemma to compute $\dim_{\mathbb{Q}}(\ker A_{C_4})$. We write $\lambda_{(i,j)}$ to denote the labels of the vertices of the top left graph, where $1 \leq i \leq 7$ are the rows, and $1 \leq j \leq k_l$ are the columns. Note that for $i = 1$ we only have the vertex corresponding to $j = k_l$. Similarly, $\mu_{(i,j)}$ will denote the labels of the vertices of the bottom right graph, where now $1 \leq i \leq 3$ and $1 \leq j \leq k_{l+1}$. As before, note that for $i = 3$ we only have the vertex corresponding to $j = k_{l+1}$. We use ρ to denote the label of the isolated vertex being in the middle of the graph. We have the following set of equations:

$$\left\{ \begin{array}{l} \text{for } 1 \leq j \leq k_l - 1, \\ \lambda_{(2,j)} - \lambda_{(2,j+1)} = 0, \\ \lambda_{(4,j+1)} - \lambda_{(4,j)} = 0, \\ \lambda_{(5,j)} - \lambda_{(5,j+1)} = 0, \\ \lambda_{(7,j+1)} - \lambda_{(7,j)} = 0, \end{array} \right\} \left\{ \begin{array}{l} \lambda_{(3,k_l)} - \lambda_{(2,k_l)} = 0, \\ \lambda_{(4,1)} - \lambda_{(3,1)} = 0, \\ \lambda_{(5,k_l)} - \lambda_{(4,k_l)} = 0, \\ \lambda_{(6,k_l)} - \lambda_{(5,k_l)} = 0, \\ \lambda_{(7,1)} - \lambda_{(6,1)} = 0, \\ \rho - \lambda_{(7,k_l)} = 0, \end{array} \right\} \left\{ \begin{array}{l} \text{for } 1 \leq j \leq k_l - 1, \\ \lambda_{(3,j)} - \lambda_{(2,j)} - \lambda_{(3,j+1)} = 0, \\ \lambda_{(6,j)} - \lambda_{(5,j)} - \lambda_{(6,j+1)} = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} \mu_{(1,1)} - \lambda_{(2,k_l)} = 0, \\ \mu_{(2,1)} + \mu_{(1,1)} - \rho = 0, \\ \text{for } 1 \leq j \leq k_{l+1} - 1, \\ \mu_{(1,j+1)} - \mu_{(1,j)} = 0, \\ \mu_{(2,j+1)} + \mu_{(1,j+1)} - \mu_{(2,j)} = 0, \\ -\mu_{(2,k_{l+1})} = 0. \end{array} \right.$$

The first set of equations gives $\lambda_{(2,j)} = \lambda_{(2,1)}$, $\lambda_{(4,j)} = \lambda_{(4,1)}$, $\lambda_{(5,j)} = \lambda_{(5,1)}$, and $\lambda_{(7,j)} = \lambda_{(7,1)}$ for all $1 \leq j \leq k_l$. Analogously, the third set of equations gives $\lambda_{(3,j)} = \lambda_{(3,1)} - (j-1)\lambda_{(2,1)}$, and $\lambda_{(6,j)} = \lambda_{(6,1)} - (j-1)\lambda_{(5,1)}$ for all $2 \leq j \leq k_l$. Now the fourth set of equations gives $\mu_{(1,j)} = \mu_{(1,1)}$, and $\mu_{(2,j)} = \mu_{(2,1)} - (j-1)\mu_{(1,1)}$ for all $2 \leq j \leq k_{l+1}$, together with $\mu_{(1,1)} = \lambda_{(2,k_l)} = \lambda_{(2,1)}$, $\mu_{(2,1)} + \mu_{(1,1)} = \rho$ and $\mu_{(2,k_{l+1})} = 0$.

For now, the possible degrees of freedom reduces to $\lambda_{(1,k_l)}$, $\lambda_{(i,1)}$ for all $2 \leq i \leq 7$, ρ , and $\mu_{(3,k_{l+1})}$. But imposing the restrictions coming from the second set of equations gives $\rho = \lambda_{(7,1)} = \lambda_{(6,1)} = k_l \lambda_{(5,1)} = k_l \lambda_{(4,1)} = k_l \lambda_{(3,1)} = k_l^2 \lambda_{(2,1)}$, with the condition $(k_l^2 - k_{l+1})\lambda_{(2,1)} = 0$. This implies that the possible degrees of freedom are now $\lambda_{(1,k_l)}$, $\lambda_{(2,1)}$, and $\mu_{(3,k_{l+1})}$, together with the restriction $(k_l^2 - k_{l+1})\lambda_{(2,1)} = 0$.

The conclusion is that, depending on whether $k_l^2 = k_{l+1}$ or not, we have either three degrees of freedom or two, respectively. Thus, we get

$$\dim_{\mathbb{Q}}(\ker A_{C_4}) = \begin{cases} 3 & \text{if } k_{l+1} = k_l^2 \\ 2 & \text{otherwise} \end{cases} = 2 + \delta_{k_{l+1}, k_l^2}.$$

As commented, we have $r-1$ connected components of this kind.

We can now compute the ℓ^2 -Betti number of A using Proposition 3.4 and Formula (3.4):

$$\begin{aligned}
 b^{(2)}(A) &= \sum_{W \in \mathbb{V}_{1/2}} \mu(W) \dim_{\mathbb{Q}}(\ker \pi(A)_W) \\
 &= \mu([1\bar{1}\bar{1}]) \dim_{\mathbb{Q}}(\ker \pi(A)_{[1\bar{1}\bar{1}]}) \\
 &\quad + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \mu([1\bar{1}0^{k_1} 10^{k_2} \dots 10^{k_r} 1\bar{1}]) \left(\sum_{i=1}^4 \dim_{\mathbb{Q}}(\ker A_{C_i}) \right) \\
 &= \frac{11}{2^3} + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{1}{2^{k_1 + \dots + k_r + (r+3)}} \left[(12 + 8r + 3(k_1 + \dots + k_r)) \right. \\
 &\quad \left. + 1 + 1 + \sum_{l=1}^{r-1} (2 + \delta_{k_{l+1}, k_l^2}) \right] \\
 &= \frac{11}{8} + \frac{1}{8} \sum_{r \geq 1} \frac{1}{2^r} \left[\sum_{k_1, \dots, k_r \geq 1} \frac{12 + 10r}{2^{k_1} \dots 2^{k_r}} \right. \\
 &\quad \left. + 3 \sum_{k_1, \dots, k_r \geq 1} \frac{k_1 + \dots + k_r}{2^{k_1} \dots 2^{k_r}} + \sum_{k_1, \dots, k_r \geq 1} \sum_{l=1}^{r-1} \frac{\delta_{k_{l+1}, k_l^2}}{2^{k_1} \dots 2^{k_r}} \right] \\
 &= \frac{11}{8} + \frac{1}{8} \sum_{r \geq 1} \frac{1}{2^r} \left[(12 + 10r) \left(\sum_{k \geq 1} \frac{1}{2^k} \right)^r \right. \\
 &\quad \left. + 3r \left(\sum_{k \geq 1} \frac{k}{2^k} \right) \left(\sum_{k \geq 1} \frac{1}{2^k} \right)^{r-1} \right. \\
 &\quad \left. + (r-1) \left(\sum_{k \geq 1} \frac{1}{2^{k^2+k}} \right) \left(\sum_{k \geq 1} \frac{1}{2^k} \right)^{r-2} \right] \\
 &= \frac{11}{8} + \frac{1}{8} \sum_{r \geq 1} \frac{12 + 16r}{2^r} + \frac{1}{8} \left(\sum_{r \geq 1} \frac{r-1}{2^r} \right) \left(\sum_{k \geq 1} \frac{1}{2^{k^2+k}} \right) \\
 &= \frac{55}{8} + \frac{1}{2} \sum_{k \geq 1} \frac{1}{2^{k^2+k+2}} \approx 6.9082337619 \dots
 \end{aligned}$$

which is an irrational number. (Its binary expansion is clearly non-periodic.)

With this example in mind, we now construct an element whose ℓ^2 -Betti number equals

$$\frac{12n + 31}{8} + \frac{2^{a_0}}{8} \sum_{k \geq 1} \frac{1}{2^{p(k)}},$$

being $p(x) = a_0 + a_1x + \dots + a_nx^n$ a polynomial of degree $n \geq 2$ with nonnegative integer coefficients, whose linear coefficient satisfies $a_1 \geq 1$. We consider the element from $M_{3n+5}(\mathcal{A}_{1/2})$ given by

$$\begin{aligned}
A = & -\chi_{[00]}t^{-1} \cdot e_{1,1} - \chi_{[0]} \cdot e_{2,1} - \chi_{[00]}t^{-1} \cdot e_{2,2} \\
& - \chi_{[10]} \cdot e_{3,2} - \chi_{[00]}t \cdot e_{3,3} - \chi_{[01]} \cdot e_{4,3} - (a_1 - 1)\chi_{[01]} \cdot e_{0,3} \\
& + \sum_{i=1}^{n-2} \left(-\chi_{[00]}t^{-1} \cdot e_{3i+1,3i+1} - \chi_{[0]} \cdot e_{3i+2,3i+1} - \chi_{[00]}t^{-1} \cdot e_{3i+2,3i+2} \right. \\
& \left. - \chi_{[10]} \cdot e_{3i+3,3i+2} - \chi_{[00]}t \cdot e_{3i+3,3i+3} - \chi_{[01]} \cdot e_{3i+4,3i+3} - a_{i+1}\chi_{[01]} \cdot e_{0,3i+3} \right) \\
& - \chi_{[00]}t^{-1} \cdot e_{3n-2,3n-2} - \chi_{[0]} \cdot e_{3n-1,3n-2} - \chi_{[00]}t^{-1} \cdot e_{3n-1,3n-1} \\
& - \chi_{[10]} \cdot e_{3n,3n-1} - \chi_{[00]}t \cdot e_{3n,3n} - a_n\chi_{[01]} \cdot e_{0,3n} \\
& - \chi_{[010]}t^2 \cdot e_{3n+2,1} - \chi_{[00]}t \cdot e_{3n+2,3n+2} + \chi_{[0]} \cdot e_{3n+3,3n+2} \\
& - \chi_{[00]}t \cdot e_{3n+3,3n+3} - \chi_{[01]} \cdot e_{3n+4,3n+3} - \chi_{[010]}t \cdot e_{3n+3,3n+1} + \chi_{[010]}t^{-1} \cdot e_{0,3n+1} \\
& + \chi_{[0]} \cdot (\text{Id}_{3n+5} - e_{0,0} - e_{3n+1,3n+1} - e_{3n+4,3n+4}) - \chi_{[01]} \cdot e_{1,1} \in M_{3n+5}(\mathbb{Q}\Gamma).
\end{aligned}$$

Again, $\pi(A)_W$ gives the zero $(3n+5) \times (3n+5)$ matrix for $W = [1\underline{1}1]$, so its kernel has dimension $3n+5$. For a $W = [1\underline{1}0^{k_1}10^{k_2}1 \cdots 0^{k_r}11]$ of length $k = k_1 + \cdots + k_r + (r+1)$, its graph $E_A(W)$ has four different types of connected components C , namely:

- (a) C_1 , given by the graphs with only one vertex $\bullet$. Here $\dim_{\mathbb{Q}}(\ker A_{C_1}) = 1$, and we have $(3n+6) + (3n+2)r + 3(k_1 + \cdots + k_r)$ connected components of this kind.
- (b) C_2 , given by the graph

The flow lemma gives $\dim_{\mathbb{Q}}(\ker A_{C_2}) = 1$. We only have one connected component of this kind.

(c) C_3 , given by the graph

We get $\dim_{\mathbb{Q}}(\ker A_{C_3}) = 1$. We only have one connected component of this type.

(d) Finally C_4 , given by the graphs

Here

$$\dim_{\mathbb{Q}}(\ker A_{C_4}) = \begin{cases} 3 & \text{if } k_{i+1} = p(k_i) - k_i - a_0 \\ 2 & \text{otherwise} \end{cases} = 2 + \delta_{k_{i+1}, p(k_i) - k_i - a_0}.$$

We have $r - 1$ connected components of this kind.

Putting everything together, we compute

$$\begin{aligned} b^{(2)}(A) &= \mu([1\bar{1}1]) \dim_{\mathbb{Q}}(\ker \pi(A)_{[1\bar{1}1]}) \\ &\quad + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \mu([1\bar{1}10^{k_1}10^{k_2} \dots 10^{k_r}11]) \left(\sum_{i=1}^4 \dim_{\mathbb{Q}}(\ker A_{C_i}) \right) \\ &= \frac{3n+5}{2^3} + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{1}{2^{k_1 + \dots + k_r + (r+3)}} \left[(3n+6) \right. \\ &\quad \left. + (3n+4)r + 3(k_1 + \dots + k_r) \right] \\ &\quad + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{1}{2^{k_1 + \dots + k_r + (r+3)}} \left[\sum_{i=1}^{r-1} \delta_{k_{i+1}, p(k_i) - k_i - a_0} \right] \\ &= \frac{3n+5}{8} + \frac{1}{8} \sum_{r \geq 1} \frac{(3n+6) + (3n+10)r}{2^r} \end{aligned}$$

$$\begin{aligned}
& + \frac{2^{a_0}}{8} \left(\sum_{r \geq 1} \frac{r-1}{2^r} \right) \left(\sum_{k \geq 1} \frac{1}{2^{p(k)}} \right) \\
& = \frac{12n+31}{8} + \frac{2^{a_0}}{8} \sum_{k \geq 1} \frac{1}{2^{p(k)}},
\end{aligned}$$

which is an irrational number since the degree of the polynomial is at least 2. Note that, in the special case $a_0 = 0$, we obtain a number of the form $q_0 + \frac{1}{2^m}\alpha$, being $\alpha = \sum_{k \geq 1} \frac{1}{2^{p(k)}}$.

Let us do a step further: we now construct an element with ℓ^2 -Betti number exactly

$$\frac{12n+35}{8} + \frac{1}{8} \sum_{k \geq 1} \frac{1}{2^{k+p(k)d^k}},$$

with $p(x)$ a polynomial of degree at least 1 with nonnegative integer coefficients, and $d \geq 2$ a natural number. We consider the element from $M_{3n+6}(\mathcal{A}_{1/2})$ given by

$$\begin{aligned}
A = & -d\chi_{[10]} \cdot e_{1,2} - \chi_{[00]}t \cdot e_{1,1} - a_0\chi_{[01]} \cdot e_{0,1} \\
& -d\chi_{[00]}t^{-1} \cdot e_{2,2} - \chi_{[0]} \cdot e_{3,2} - d\chi_{[00]}t^{-1} \cdot e_{3,3} \\
& -d\chi_{[10]} \cdot e_{4,3} - \chi_{[00]}t \cdot e_{4,4} - \chi_{[01]} \cdot e_{5,4} - a_1\chi_{[01]} \cdot e_{0,4} \\
& + \sum_{i=1}^{n-2} \left(-\chi_{[00]}t^{-1} \cdot e_{3i+2,3i+2} - \chi_{[0]} \cdot e_{3i+3,3i+2} - \chi_{[00]}t^{-1} \cdot e_{3i+3,3i+3} \right. \\
& \left. - \chi_{[10]} \cdot e_{3i+4,3i+3} - \chi_{[00]}t \cdot e_{3i+4,3i+4} - \chi_{[01]} \cdot e_{3i+5,3i+4} - a_{i+1}\chi_{[01]} \cdot e_{0,3i+4} \right) \\
& - \chi_{[00]}t^{-1} \cdot e_{3n-1,3n-1} - \chi_{[0]} \cdot e_{3n,3n-1} - \chi_{[00]}t^{-1} \cdot e_{3n,3n} \\
& - \chi_{[10]} \cdot e_{3n+1,3n} - \chi_{[00]}t \cdot e_{3n+1,3n+1} - a_n\chi_{[01]} \cdot e_{0,3n+1} \\
& - \chi_{[010]}t^2 \cdot e_{3n+3,2} - \chi_{[00]}t \cdot e_{3n+3,3n+3} + \chi_{[0]} \cdot e_{3n+4,3n+3} \\
& - \chi_{[00]}t \cdot e_{3n+4,3n+4} - \chi_{[01]} \cdot e_{3n+5,3n+4} - \chi_{[010]}t \cdot e_{3n+4,3n+2} + \chi_{[010]}t^{-1} \cdot e_{0,3n+2} \\
& + \chi_{[0]} \cdot (\text{Id}_{3n+6} - e_{0,0} - e_{3n+2,3n+2} - e_{3n+5,3n+5}) - \chi_{[01]} \cdot e_{2,2} \in M_{3n+6}(\mathbb{Q}\Gamma).
\end{aligned}$$

For $W = [111]$, $\pi(A)_W$ gives the zero $(3n+6) \times (3n+6)$ matrix, so its kernel has dimension $3n+6$. For a $W = [110^{k_1}10^{k_2}1 \dots 0^{k_r}11]$ of length $k = k_1 + \dots + k_r + (r+1)$, its graph $E_A(W)$ has four different types of connected components C , namely:

- C_1 , given by the graphs with only one vertex $\bullet$. Here $\dim_{\mathbb{Q}}(\ker A_{C_1}) = 1$, and we have $(3n+7) + (3n+3)r + 3(k_1 + \dots + k_r)$ connected components of this type.
- C_2 , given by the graph

We compute $\dim_{\mathbb{Q}}(\ker A_{C_2}) = 1$. We only have one connected component of this type.

(c) C_3 , given by the graph

We get $\dim_{\mathbb{Q}}(\ker A_{C_3}) = 1$. We only have one connected component of this type.

(d) Finally C_4 , given by the graphs

Here

$$\dim_{\mathbb{Q}}(\ker A_{C_4}) = \begin{cases} 3 & \text{if } k_{i+1} = p(k_i)d^{k_i} \\ 2 & \text{otherwise} \end{cases} = 2 + \delta_{k_{i+1}, p(k_i)d^{k_i}}.$$

We have $r - 1$ connected components of this kind.

In this case, we compute

$$\begin{aligned} b^{(2)}(A) &= \mu([1\bar{1}1])\dim_{\mathbb{Q}}(\ker \pi(A)_{[1\bar{1}1]}) \\ &+ \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \mu([1\bar{1}0^{k_1}10^{k_2} \dots 10^{k_r}1]) \left(\sum_{i=1}^4 \dim_{\mathbb{Q}}(\ker A_{C_i}) \right) \\ &= \frac{3n+6}{2^3} + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{1}{2^{k_1 + \dots + k_r + (r+3)}} \left[(3n+7) \right. \\ &\quad \left. + (3n+5)r + 3(k_1 + \dots + k_r) \right] \\ &+ \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{1}{2^{k_1 + \dots + k_r + (r+3)}} \left[\sum_{i=1}^{r-1} \delta_{k_{i+1}, p(k_i)d^{k_i}} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{3n+6}{8} + \frac{1}{8} \sum_{r \geq 1} \frac{(3n+7) + (3n+11)r}{2^r} \\
&\quad + \frac{1}{8} \left(\sum_{r \geq 1} \frac{r-1}{2^r} \right) \left(\sum_{k \geq 1} \frac{1}{2^{k+p(k)d^k}} \right) \\
&= \frac{12n+35}{8} + \frac{1}{8} \sum_{k \geq 1} \frac{1}{2^{k+p(k)d^k}}
\end{aligned}$$

which is again an irrational number, and even transcendental (see, e.g., [40]). Just as before, the rational number accompanying $\alpha = \sum_{k \geq 1} \frac{1}{2^{k+p(k)d^k}}$ is of the form $\frac{1}{2^m}$.

After these examples, one can derive the pattern in order to obtain an exponent of the form $p_0(k) + p_1(k)d_1^k + \cdots + p_n(k)d_n^k$ by simply adding more levels, i.e., by considering matrices of higher dimension, and gluing the corresponding graphs in an appropriate way. We write down the corresponding element that gives rise to such a pattern. If we let $N = m_0 + \cdots + m_n$ to be the sum of the degrees of the polynomials $p_0, \dots, p_n$ respectively, with $p_i(x) = \sum_{j=0}^{m_i} a_{j,i}x^j$ for $0 \leq i \leq n$, then the element realizing the preceding pattern belongs to $M_{3N+n+5}(\mathcal{A}_{1/2})$, and is given explicitly by

$$\begin{aligned}
A = & -\chi_{[\underline{00}]}t^{-1} \cdot e_{1,1} - \chi_{[\underline{0}]} \cdot e_{2,1} - \chi_{[\underline{00}]}t^{-1} \cdot e_{2,2} \\
& - \chi_{[\underline{10}]} \cdot e_{3,2} - \chi_{[\underline{00}]}t \cdot e_{3,3} - \chi_{[\underline{01}]} \cdot e_{4,3} - (a_{1,0} - 1)\chi_{[\underline{01}]} \cdot e_{0,3} \\
& + \sum_{j=1}^{m_0-2} \left(-\chi_{[\underline{00}]}t^{-1} \cdot e_{3j+1,3j+1} - \chi_{[\underline{0}]} \cdot e_{3j+2,3j+1} - \chi_{[\underline{00}]}t^{-1} \cdot e_{3j+2,3j+2} \right. \\
& \quad - \chi_{[\underline{10}]} \cdot e_{3j+3,3j+2} - \chi_{[\underline{00}]}t \cdot e_{3j+3,3j+3} \\
& \quad \left. - \chi_{[\underline{01}]} \cdot e_{3j+4,3j+3} - a_{j+1,0}\chi_{[\underline{01}]} \cdot e_{0,3j+3} \right) \\
& - \chi_{[\underline{00}]}t^{-1} \cdot e_{3m_0-2,3m_0-2} - \chi_{[\underline{0}]} \cdot e_{3m_0-1,3m_0-2} - \chi_{[\underline{00}]}t^{-1} \cdot e_{3m_0-1,3m_0-1} \\
& - \chi_{[\underline{10}]} \cdot e_{3m_0,3m_0-1} - \chi_{[\underline{00}]}t \cdot e_{3m_0,3m_0} - a_{m_0,0}\chi_{[\underline{01}]} \cdot e_{0,3m_0} \\
& - \chi_{[\underline{01}]} \cdot e_{3m_0+2,1} \\
& - d_1\chi_{[\underline{10}]} \cdot e_{3m_0+1,3m_0+2} - \chi_{[\underline{00}]}t \cdot e_{3m_0+1,3m_0+1} - a_{0,1}\chi_{[\underline{01}]} \cdot e_{0,3m_0+1} \\
& - d_1\chi_{[\underline{00}]}t^{-1} \cdot e_{3m_0+2,3m_0+2} - \chi_{[\underline{0}]} \cdot e_{3m_0+3,3m_0+2} \\
& - d_1\chi_{[\underline{00}]}t^{-1} \cdot e_{3m_0+3,3m_0+3} - d_1\chi_{[\underline{10}]} \cdot e_{3m_0+4,3m_0+3} \\
& - \chi_{[\underline{00}]}t \cdot e_{3m_0+4,3m_0+4} - \chi_{[\underline{01}]} \cdot e_{3m_0+5,3m_0+4} - a_{1,1}\chi_{[\underline{01}]} \cdot e_{0,3m_0+4} \\
& + \sum_{j=1}^{m_1-2} \left(-\chi_{[\underline{00}]}t^{-1} \cdot e_{3m_0+3j+2,3m_0+3j+2} - \chi_{[\underline{0}]} \cdot e_{3m_0+3j+3,3m_0+3j+2} \right. \\
& \quad - \chi_{[\underline{00}]}t^{-1} \cdot e_{3m_0+3j+3,3m_0+3j+3} - \chi_{[\underline{10}]} \cdot e_{3m_0+3j+4,3m_0+3j+3} \\
& \quad - \chi_{[\underline{00}]}t \cdot e_{3m_0+3j+4,3m_0+3j+4} - \chi_{[\underline{01}]} \cdot e_{3m_0+3j+5,3m_0+3j+4} \\
& \quad \left. - a_{j+1,1}\chi_{[\underline{01}]} \cdot e_{0,3m_0+3j+4} \right) \\
& - \chi_{[\underline{00}]}t^{-1} \cdot e_{3(m_0+m_1)-1,3(m_0+m_1)-1} - \chi_{[\underline{0}]} \cdot e_{3(m_0+m_1),3(m_0+m_1)-1} \\
& - \chi_{[\underline{00}]}t^{-1} \cdot e_{3(m_0+m_1),3(m_0+m_1)} - \chi_{[\underline{10}]} \cdot e_{3(m_0+m_1)+1,3(m_0+m_1)} \\
& - \chi_{[\underline{00}]}t \cdot e_{3(m_0+m_1)+1,3(m_0+m_1)+1} - a_{m_1,1}\chi_{[\underline{01}]} \cdot e_{0,3(m_0+m_1)+1} \\
& - \chi_{[\underline{01}]} \cdot e_{3(m_0+m_1)+3,1} \\
& \quad \vdots \\
& - \chi_{[\underline{01}]} \cdot e_{3(N-m_n)+n+1,1} \\
& - d_n\chi_{[\underline{10}]} \cdot e_{3(N-m_n)+n,3(N-m_n)+n+1} - \chi_{[\underline{00}]}t \cdot e_{3(N-m_n)+n,3(N-m_n)+n} \\
& - a_{0,n}\chi_{[\underline{01}]} \cdot e_{0,3(N-m_n)+n} - d_n\chi_{[\underline{00}]}t^{-1} \cdot e_{3(N-m_n)+n+1,3(N-m_n)+n+1} \\
& - \chi_{[\underline{0}]} \cdot e_{3(N-m_n)+n+2,3(N-m_n)+n+1} - d_n\chi_{[\underline{00}]}t^{-1} \cdot e_{3(N-m_n)+n+2,3(N-m_n)+n+2} \\
& - d_n\chi_{[\underline{10}]} \cdot e_{3(N-m_n)+n+3,3(N-m_n)+n+2} - \chi_{[\underline{00}]}t \cdot e_{3(N-m_n)+n+3,3(N-m_n)+n+3} \\
& - \chi_{[\underline{01}]} \cdot e_{3(N-m_n)+n+4,3(N-m_n)+n+3} - a_{1,n}\chi_{[\underline{01}]} \cdot e_{0,3(N-m_n)+n+3} \\
& + \sum_{j=1}^{m_n-2} \left(-\chi_{[\underline{00}]}t^{-1} \cdot e_{3(N-m_n)+3j+n+1,3(N-m_n)+3j+n+1} \right. \\
& \quad - \chi_{[\underline{0}]} \cdot e_{3(N-m_n)+3j+n+2,3(N-m_n)+3j+n+1} \\
& \quad - \chi_{[\underline{00}]}t^{-1} \cdot e_{3(N-m_n)+3j+n+2,3(N-m_n)+3j+n+2} \\
& \quad - \chi_{[\underline{10}]} \cdot e_{3(N-m_n)+3j+n+3,3(N-m_n)+3j+n+2} \\
& \quad - \chi_{[\underline{00}]}t \cdot e_{3(N-m_n)+3j+n+3,3(N-m_n)+3j+n+3} \\
& \quad - \chi_{[\underline{01}]} \cdot e_{3(N-m_n)+3j+n+4,3(N-m_n)+3j+n+3} \\
& \quad \left. - a_{j+1,n}\chi_{[\underline{01}]} \cdot e_{0,3(N-m_n)+3j+n+3} \right) \\
& - \chi_{[\underline{00}]}t^{-1} \cdot e_{3N+n-2,3N+n-2} - \chi_{[\underline{0}]} \cdot e_{3N+n-1,3N+n-2} \\
& - \chi_{[\underline{00}]}t^{-1} \cdot e_{3N+n-1,3N+n-1} - \chi_{[\underline{10}]} \cdot e_{3N+n,3N+n-1} \\
& - \chi_{[\underline{00}]}t \cdot e_{3N+n,3N+n} - a_{m_n,n}\chi_{[\underline{01}]} \cdot e_{0,3N+n} \\
& \quad - \chi_{[\underline{010}]}t^2 \cdot e_{3N+n+2,1} \\
& \quad + \chi_{[\underline{010}]}t^{-1} \cdot e_{0,3N+n+1} \\
& \quad - \chi_{[\underline{010}]}t \cdot e_{3N+n+3,3N+n+1} \\
& - \chi_{[\underline{00}]}t \cdot e_{3N+n+2,3N+n+2} + \chi_{[\underline{0}]} \cdot e_{3N+n+3,3N+n+2} \\
& - \chi_{[\underline{00}]}t \cdot e_{3N+n+3,3N+n+3} - \chi_{[\underline{01}]} \cdot e_{3N+n+4,3N+n+3} \\
& + \chi_{[\underline{0}]} \cdot (\text{Id}_{3N+n+5} - e_{0,0} - e_{3N+n+1,3N+n+1} - e_{3N+n+4,3N+n+4}) - \chi_{[\underline{01}]} \cdot e_{1,1} \in M_{3N+n+5}(\mathbb{Q}\Gamma).
\end{aligned}$$

$\left. \begin{array}{l} \text{1st polynomial} \\ \text{2nd polynomial} \\ \text{nth polynomial} \\ \text{nth polynomial} \\ \text{last graph} \end{array} \right\}$

The elements in between connect the different polynomials p_i , and the contributions (monomials) of the polynomials (that is, the sum $p_0(k) + p_1(k)d_1^k + \cdots + p_n(k)d_n^k$) are accumulated in the $\chi_{[0,1]} \cdot e_{0,0}$ component. A simplified schematic of a prototypical graph appearing here is as follows.

Using the same procedure as before, a straightforward (but quite tedious) computation allows us to conclude the proof. We only write down the contribution corresponding to the components C_4 of the graph $E_A(W)$ where W is of the second type, which are the ones that contribute to the irrationality of the ℓ^2 -Betti number. We get

$$\dim_{\mathbb{Q}}(\ker A_{C_4}) = 2 + \delta_{k_{i+1}, p_0(k_i) - k_i - a_{0,0} + p_1(k_i)d_1^{k_i} + \cdots + p_n(k_i)d_n^{k_i}}.$$

We have $r - 1$ connected components of this kind. Thus, the contribution to $b^{(2)}(A)$ coming from these graphs is

$$\begin{aligned} & \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{1}{2^{k_1 + \cdots + k_r + (r+3)}} \left[2(r-1) + \sum_{i=1}^{r-1} \delta_{k_{i+1}, p_0(k_i) - k_i - a_{0,0} + p_1(k_i)d_1^{k_i} + \cdots + p_n(k_i)d_n^{k_i}} \right] \\ &= \frac{2}{8} \sum_{r \geq 1} \frac{r-1}{2^r} + \frac{2^{a_{0,0}}}{8} \left(\sum_{r \geq 1} \frac{r-1}{2^r} \right) \left(\sum_{k \geq 1} \frac{1}{2^{p_0(k) + p_1(k)d_1^k + \cdots + p_n(k)d_n^k}} \right) \\ &= \frac{1}{4} + \frac{2^{a_{0,0}}}{8} \sum_{k \geq 1} \frac{1}{2^{p_0(k) + p_1(k)d_1^k + \cdots + p_n(k)d_n^k}}. \end{aligned}$$

We leave the rest of the details to the reader. Note, again, that in the special case $a_{0,0} = 0$ the irrational number $\alpha = \sum_{k \geq 1} \frac{1}{2^{p_0(k)+p_1(k)d_1^k+\dots+p_n(k)d_n^k}}$ is accompanied by a rational number of the form $\frac{1}{2^m}$.

For the second part of the theorem, note that under the assumption $a_{0,0} = 0$ we have always obtained irrational numbers of the form $q_0 + q_1\alpha$, being α the irrational number

$$\alpha = \sum_{k \geq 1} \frac{1}{2^{p_0(k)+p_1(k)d_1^k+\dots+p_n(k)d_n^k}},$$

and q_0, q_1 nonzero rational numbers with q_1 of the form $\frac{1}{2^m}$ for some $m \geq 1$. In particular, $q_0 + q_1\alpha$ belongs to $\mathcal{G}(\Gamma, \mathbb{Q})$. By a result of Ara and Goodearl [4, Corollary 6.14], the group $\mathcal{G}(\Gamma, \mathbb{Q})$ contains the rational numbers, and hence, the element $q_1\alpha$ belongs to $\mathcal{G}(\Gamma, \mathbb{Q})$ too. Since $q_1 = \frac{1}{2^m}$, we get

$$\alpha = q_1\alpha + \dots + q_1\alpha \in \mathcal{G}(\Gamma, \mathbb{Q}).$$

This completes the proof of the theorem. $\square$

Remark 4.6 By making use of the ideas and techniques developed in Theorem 4.5, one can construct elements whose associated ℓ^2 -Betti number has a binary expansion following different kinds of exotic patterns. For example, by gluing in an appropriate way two graphs corresponding to different polynomials $p_1(x)$ and $p_2(x)$, that is, by constructing graphs of the form C_3 but substituting the bottom right graph (the one contributing to k_{i+1}) by another graph corresponding to the polynomial $p_2(x)$, we can obtain terms in the computation of the associated ℓ^2 -Betti number of the form

$$\sum_{k_1, k_2 \geq 1} \frac{\delta_{p_1(k_1), p_2(k_2)}}{2^{k_1+k_2}},$$

and more generally of the form

$$\sum_{k_1, k_2 \geq 1} \frac{\delta_{p_0(k_1)+p_1(k_1)d_1^{k_1}+\dots+p_n(k_1)d_n^{k_1}, q_0(k_2)+q_1(k_2)l_1^{k_2}+\dots+q_m(k_2)l_m^{k_2}}}{2^{k_1+k_2}},$$

being $p_i(x), q_j(x)$ polynomials satisfying the hypotheses of the theorem, and $d_i, l_j \geq 2$ integers.

To conclude this section, it may be instructive to compute some rational values of ℓ^2 -Betti numbers. In [19] (cf. [4, 13, 17]) the authors compute the ℓ^2 -Betti number of the element $a_0 = e_0t + t^{-1}e_0 = \chi_{X \setminus E_0}t + t^{-1}\chi_{X \setminus E_0}$, which belongs to the first of our approximating algebras, $\mathcal{A}_0$. We will compute, in general, the ℓ^2 -Betti number of the element $a_n = \chi_{X \setminus E_n}t + t^{-1}\chi_{X \setminus E_n} \in \mathbb{Q}\Gamma$ belonging to the $*$ -subalgebra $\mathcal{A}_n$. Under

π_n —and recalling that $\mathfrak{R}_n = \mathbb{Q} \times \prod_{k \geq 1} M_{m+k}(\mathbb{Q})^{\text{Fib}_m(k)}$ with $m = 2n + 1$ —it gives

$$\pi_n(a_n) = (0, (t_{m+k} + t_{m+k}^*, \text{Fib}_m(k), t_{m+k} + t_{m+k}^*)_{k \geq 1}),$$

where t_r is the $r \times r$ lower triangular matrix given by

$$t_r = \begin{pmatrix} 0 & & & & \mathbf{0} \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ \mathbf{0} & & & 1 & 0 \end{pmatrix}.$$

It is then straightforward to show that

$$\dim_{\mathbb{Q}}(\ker(t_{m+k} + t_{m+k}^*)) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

so

$$b^{(2)}(a_n) = \frac{1}{2^{m+1}} + \sum_{k \geq 1} \frac{\text{Fib}_m(2k)}{2^{2m+2k}}.$$

This sum can be computed by using the summation rule

$$\sum_{k \geq 1} \frac{\text{Fib}_m(2k)}{2^{2k}} = 2^{m-1} \frac{2^{m+1} - 1}{2^{m+2} + 1},$$

whose proof can be found in [10, Lemma 3.2.11]. We then have

$$b^{(2)}(a_n) = \frac{3}{1 + 2^{2n+3}}.$$

Note that $b^{(2)}(a_0) = \frac{1}{3}$, and we recover the result from [13, 19]. Also, as $n \rightarrow \infty$, this value tends to zero, as expected since $a_n \rightarrow t + t^{-1}$ in rank, which is invertible inside $\mathfrak{R}_{\text{rk}}$.

4.3 Rational series and ℓ^2 -Betti numbers

In this section, we let again $K \subseteq \mathbb{C}$ be any field closed under complex conjugation. A more algebraic perspective to attack the problem of computing values from $\mathcal{G}(\mathcal{A}_{1/2}) \subseteq \mathcal{G}(\Gamma, K)$ is through the determination of the $*$ -regular closure $\mathcal{R}_{1/2}$ of the algebra $\mathcal{A}_{1/2}$ seen inside $\mathfrak{R}_{1/2} = K \times \prod_{k \geq 1} M_{k+2}(K)^{\text{Fib}_2(k)}$ through the embedding $\pi_{1/2} : \mathcal{A}_{1/2} \hookrightarrow$

$\mathfrak{R}_{1/2}$, and taking advantage of [3, Proposition 4.1] which states that the values of ranks of elements from $\mathcal{R}_{1/2}$ are all included in $\mathcal{G}(\Gamma, K)$.

In order to clarify the exact relationship of this approach with Atiyah's problem, it is convenient to introduce the following definitions for a general unital ring.

Definition 4.7 Let R be a unital ring and let rk be a Sylvester matrix rank function on R . We denote by $\mathcal{C}_{\text{rk}}(R)$ the set of real numbers of the form $k - \text{rk}(A)$ for $A \in M_k(R)$, $k \geq 1$. We denote by $\mathcal{C}'_{\text{rk}}(R)$ the set of real numbers of the form $\text{rk}(A)$ for $A \in M_k(R)$, $k \geq 1$. Both $\mathcal{C}_{\text{rk}}(R)$ and $\mathcal{C}'_{\text{rk}}(R)$ are subsemigroups of $(\mathbb{R}^+, +)$. We denote by $\mathcal{G}_{\text{rk}}(R)$ the subgroup of $\mathbb{R}$ generated by $\mathcal{C}_{\text{rk}}(R)$. Clearly

$$\mathcal{G}_{\text{rk}}(R) = \mathcal{C}_{\text{rk}}(R) - \mathcal{C}_{\text{rk}}(R) = \mathcal{C}'_{\text{rk}}(R) - \mathcal{C}'_{\text{rk}}(R).$$

Remark 4.8 With this notation, [22, Corollary 6.2] says that, if $\mathcal{U}$ is a $*$ -regular ring and R is a $*$ -subring of $\mathcal{U}$ such that $\mathcal{U}$ is the $*$ -regular closure of R in $\mathcal{U}$, then we have

$$\mathcal{G}_{\text{rk}}(R) = \mathcal{G}_{\text{rk}}(\mathcal{U})$$

for every Sylvester matrix rank function rk on $\mathcal{U}$. Thus, the main object of study within this approach is the group $\mathcal{G}_{\text{rk}}(R)$. Of course, $\mathcal{C}(G, K) = \mathcal{C}_{\text{rk}}(KG)$ for any discrete group G , where rk is the canonical rank. It would be interesting to know conditions under which $\mathcal{C}_{\text{rk}}(R) = \mathcal{C}'_{\text{rk}}(R)$ and under which $\mathcal{C}_{\text{rk}}(R) = \mathcal{G}_{\text{rk}}(R) \cap \mathbb{R}^+$, or $\mathcal{C}'_{\text{rk}}(R) = \mathcal{G}_{\text{rk}}(R) \cap \mathbb{R}^+$. Note that

$$\forall g \in \mathcal{G}_{\text{rk}}(R) \exists t \in \mathbb{Z}^+ : g + t \in \mathcal{C}_{\text{rk}}(R) \iff \forall g \in \mathcal{G}_{\text{rk}}(R) \exists t \in \mathbb{Z}^+ : g + t \in \mathcal{C}'_{\text{rk}}(R).$$

In case R is von Neumann regular, we have that $\mathcal{C}_{\text{rk}}(R) = \mathcal{C}'_{\text{rk}}(R)$ and the above equivalent conditions hold automatically, for every Sylvester matrix rank function rk .

We briefly remind some concepts from [3]. Write $\mathcal{A}_{1/2} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_{1/2, i} t^i$, where $\mathcal{A}_{1/2, 0} = \mathcal{A}_{1/2} \cap C_K(X)$ and

$$\begin{aligned} \mathcal{A}_{1/2, i} &= \chi_{X \setminus (E_{1/2} \cup T(E_{1/2}) \cup \dots \cup T^{i-1}(E_{1/2}))} \mathcal{A}_{1/2, 0} \quad \text{and} \\ \mathcal{A}_{1/2, -i} &= \chi_{X \setminus (T^{-1}(E_{1/2}) \cup \dots \cup T^{-i}(E_{1/2}))} \mathcal{A}_{1/2, 0} \end{aligned}$$

for $i > 0$. Following [3, Subsection 4.2], we consider a skew partial power series ring $\mathcal{A}_{1/2, 0}[[t; T]]$ by taking infinite formal sums

$$\sum_{i \geq 0} b_i (\chi_{X \setminus E_{1/2}} t)^i = \sum_{i \geq 0} b_i t^i, \quad \text{where } b_i \in \mathcal{A}_{1/2, i} \text{ for all } i \geq 0.$$

Specializing [3, Definition 4.17] to our situation, we see that a *special term* of positive degree i is a monomial $b_i t^i$ in $\mathcal{A}_{1/2, i}$, where $b_i = \chi_S$ with

$$S = [10^{i_1} 10^{i_2} 1 \dots 10^{i_r} \underline{1}], \quad \text{where } r \geq 1, i_1, \dots, i_r \geq 1 \text{ and } i = i_1 + \dots + i_r + r.$$

Moreover, since $E_{1/2} \cap T^{-1}(E_{1/2}) \neq \emptyset$ and $S_0 = T^{-1}(S_1) = [1]$, we get from [3, Definition 4.7(c)] that the unique special term of degree 0 is $\chi_{[1]}$.

Next, consider the subset $S_{1/2}[[t; T]]$ of $\mathcal{A}_{1/2,0}[[t; T]]$ consisting of those elements

$$\sum_{i \geq 0} b_i (\chi_{X \setminus E_{1/2}} t)^i = \sum_{i \geq 0} b_i t^i$$

such that each $b_i t^i \in \mathcal{A}_{1/2,i} t^i$ belongs to the K -linear span of the special terms of degree i . We denote by $S_{1/2}[t; T]$ the subspace of $S_{1/2}[[t; T]]$ consisting of elements with finite support.

Let $\mathcal{X} = \{x_1, x_2, \dots\}$ be an infinite countable set, and consider the algebras $K\langle \mathcal{X} \rangle$ and $K\langle\langle \mathcal{X} \rangle\rangle$ of non-commutative polynomials and non-commutative formal power series, respectively, with coefficients in K . We consider the degree in $K\langle \mathcal{X} \rangle$ given by $d(x_i) = i + 1$ for all $i \geq 1$ and $d(1) = 0$. With this degree function, $K\langle \mathcal{X} \rangle$ is a graded K -algebra, where K has the trivial degree.

The following result is shown in [3, Lemma 6.5 and Proposition 6.6]. Note that [3, Propositions 6.6 and 6.7] are stated only for $n \geq 1$, but they hold, with the same proof, for $n = 1/2$.

Proposition 4.9 *The linear subspaces $S_{1/2}[t; T]$ and $S_{1/2}[[t; T]]$ are indeed subalgebras of $\mathcal{A}_{1/2}[[t; T]]$. There is an isomorphism of unital graded algebras $K\langle \mathcal{X} \rangle \cong S_{1/2}[t; T]$ sending the element $x_{i_r} \cdots x_{i_1}$ to $\chi_{[10^{i_1} 10^{i_2} 1 \dots 10^{i_r} 1]} t^i$, where $i = i_1 + \cdots + i_r + r$, and 1 to $\chi_{[1]}$. This isomorphism extends naturally to an isomorphism between $K\langle\langle \mathcal{X} \rangle\rangle$ and $S_{1/2}[[t; T]]$.*

The special terms of the form $\chi_{[10^i 1]} t^{i+1}$, for $i \geq 1$, which are the algebra generators of $S_{1/2}[t; T]$, are called *pure special terms*, see the proof of [3, Proposition 6.6].

Let $\mathcal{X}^*$ denote the free monoid on $\mathcal{X}$. We now recall the concept of the *Hadamard product* $\odot$ on $K\langle\langle \mathcal{X} \rangle\rangle$, see [9] for more details. If $a = \sum_{w \in \mathcal{X}^*} a_w w$ and $b = \sum_{w \in \mathcal{X}^*} b_w w$ are elements in $K\langle\langle \mathcal{X} \rangle\rangle$, set

$$a \odot b = \left(\sum_{w \in \mathcal{X}^*} a_w w \right) \odot \left(\sum_{w \in \mathcal{X}^*} b_w w \right) = \sum_{w \in \mathcal{X}^*} (a_w b_w) w \in K\langle\langle \mathcal{X} \rangle\rangle.$$

We will denote by $K\langle\langle \mathcal{X} \rangle\rangle^\circ$ the algebra of non-commutative formal power series endowed with the Hadamard product. Note that $K\langle\langle \mathcal{X} \rangle\rangle^\circ$ is a $*$ -regular ring with respect to the involution given by $\sum_{w \in \mathcal{X}^*} a_w w = \sum_{w \in \mathcal{X}^*} \overline{a_w} \overline{w}$. The algebra of *non-commutative rational series*, denoted by $K_{\text{rat}}\langle \mathcal{X} \rangle$ is defined as the division closure of $K\langle \mathcal{X} \rangle$ in $K\langle\langle \mathcal{X} \rangle\rangle$, that is, $K_{\text{rat}}\langle \mathcal{X} \rangle$ is the smallest subalgebra of $K\langle\langle \mathcal{X} \rangle\rangle$ containing $K\langle \mathcal{X} \rangle$ and closed under inversion, see [9] for details. It is well known that $K_{\text{rat}}\langle \mathcal{X} \rangle$ is closed under the Hadamard product [9, Theorems 1.5.5 and 1.7.1].

Let $p_{1/2} := p_{E_{1/2}} = \pi_{1/2}(\chi_{E_{1/2}})$ be the projection in $\mathfrak{R}_{1/2} = \prod_{W \in \mathbb{V}_{1/2}} M_{|W|}(K) = K \times \prod_{k \geq 1} M_{k+2}(K)^{\text{Fib}_2(k)}$ which has a 1 in the left upper corner of each matrix factor $M_{|W|}(K)$, and 0's elsewhere, that is, $(p_{1/2})_W = e_{00}(W)$ for each $W \in \mathbb{V}_{1/2}$. Recall from the beginning of Sect. 4.2 that the elements of $\mathbb{V}_{1/2}$ are all the clopen sets of the form $[110^{i_1} 10^{i_2} 1 \dots 10^{i_r} 11]$, where $r \geq 0$ and $i_1, \dots, i_r \geq 1$. (We interpret this term to be $[111]$ if $r = 0$.)

By [3, Proposition 4.20], there is a $*$ -isomorphism from $(\mathcal{S}_{1/2}[[t; T]], \odot, -)$ onto $p_{1/2}\mathfrak{R}_{1/2}p_{1/2}$. Composing the isomorphism $K\langle\langle\mathcal{X}\rangle\rangle \cong \mathcal{S}_{1/2}[[t; T]]$ from Proposition 4.9 (which respects the respective Hadamard products and involutions) with this isomorphism, we obtain a $*$ -isomorphism

$$\Delta: K\langle\langle\mathcal{X}\rangle\rangle^\circ \xrightarrow{\cong} p_{1/2}\mathfrak{R}_{1/2}p_{1/2}$$

such that

$$\Delta(f)_{[1\underline{1}0^{i_1}1\dots 10^{i_r}11]} = (f, x_{i_r} \cdots x_{i_1})e_{00}([1\underline{1}0^{i_1}1 \cdots 10^{i_r}11]) \quad (f \in K\langle\langle\mathcal{X}\rangle\rangle)$$

for each $i_1, \dots, i_r \geq 1$. Here, $(f, x_{i_r} \cdots x_{i_1})$ is the coefficient of f corresponding to the monomial $x_{i_r} \cdots x_{i_1}$.

For each $n \geq 1$, set $\mathcal{X}_n = \{x_1, \dots, x_n\}$. Let L be a *language* over $\mathcal{X}_n$, i.e., a subset of the free monoid $\mathcal{X}_n^*$ on $\mathcal{X}_n$. We now give a formula for the rank of the element $\Delta(\underline{L})$, where

$$\underline{L} := \sum_{w \in L} w \in K\langle\langle\mathcal{X}_n\rangle\rangle$$

is the *characteristic series* of L ([9, p.8]). Note that, for $W = [1\underline{1}0^{i_1}10^{i_2}1 \cdots 10^{i_r}11] \in \mathbb{V}_{1/2}$, we have

$$\Delta(\underline{L})_W = \begin{cases} e_{00}(W) & \text{if } x_{i_r} \cdots x_{i_2}x_{i_1} \in L \\ 0 & \text{if } x_{i_r} \cdots x_{i_2}x_{i_1} \notin L \end{cases}.$$

Moreover, we have $\mu(W) = 2^{-(i_1+\dots+i_r+r+3)} = 2^{-3}2^{-d(w)}$, where $w = x_{i_r} \cdots x_{i_2}x_{i_1}$. Let $s(L) = \sum_{j \geq 0} \alpha_j x^j$ be the series in $\mathbb{Z}^+[[x]]$ such that α_j is the number of words $w \in L$ such that $d(w) = j$. Then we have

$$\text{rk}(\Delta(\underline{L})) = \sum_{W \in \mathbb{V}_{1/2}} \mu(W) \text{Rk}(\Delta(\underline{L})_W) = \frac{1}{2^3} \sum_{w \in L} \frac{1}{2^{d(w)}} = \frac{1}{8} s(L)\left(\frac{1}{2}\right). \quad (4.3)$$

Here we make use of (3.2) for the first equality, of the above observations for the second, and of the definition of $s(L)$ for the third. We say that $s(L) \in \mathbb{Z}^+[[x]]$ is the *generating function* of L with respect to d , and we denote by α_L the real number $\text{rk}(\Delta(\underline{L}))$ described in (4.3).

We now recall the notion of a rational language (see [9, Chapter 3]).

Definition 4.10 The set of *rational languages* over a finite set F is the smallest set of subsets of the free monoid F^* containing all the finite subsets of F^* and closed under the following operations:

- (a) union $L_1 \cup L_2$;
- (b) product $L_1 L_2 := \{w_1 w_2 \in F^* \mid w_1 \in L_1, w_2 \in L_2\}$;

(c) $\ast$ -product $L^* := \bigcup_{k \geq 0} L^k$.

A characterization of the rational languages is the following ([9, Lemma 3.1.4]): a language over a finite alphabet F is rational if and only if it is the support of some rational series $z \in \mathbb{Z}^+ \langle\langle F \rangle\rangle$. In fact, if L is a rational language over F , its characteristic series $\underline{L} = \sum_{w \in L} w$ belongs to $R_{\text{rat}}(F)^\circ$ for any semiring R ([9, Proposition 3.2.1]).

It turns out that the set of rational languages over F , which we will denote by $\mathfrak{L}(F)$, forms a Boolean subalgebra of $P(F^*)$, the power set of F^* ([9, Corollary 3.1.5]). However, the converse may not be true. Indeed, for a subfield K of $\mathbb{C}$ closed under complex conjugation, given a rational series $z = \sum_{w \in F^*} \lambda_w w \in K_{\text{rat}} \langle\langle F \rangle\rangle$ it may be the case that its support $L = \text{supp}(z)$ is not a rational language. We put

$$\mathfrak{R}(F) := \{\text{supp}(z) \mid z \in K_{\text{rat}} \langle\langle F \rangle\rangle^\circ\} \subseteq P(F^*),$$

and we let $\mathbf{B}_{\text{rat}}^K(F)$ be the Boolean algebra of subsets of F^* generated by $\mathfrak{R}(F)$. Then by [3, Proposition 6.10] the $\ast$ -regular closure of $K_{\text{rat}} \langle\langle F \rangle\rangle^\circ$ in $K \langle\langle F \rangle\rangle^\circ$ is contained in the set of formal power series whose support belongs to $\mathbf{B}_{\text{rat}}^K(F)$, and for each set $L \in \mathbf{B}_{\text{rat}}^K(F)$, the characteristic series $\underline{L}$ belongs to the $\ast$ -regular closure.

We gather some of these facts in the following.

Proposition 4.11 *Let $n \geq 1$ and let $\mathcal{X}_n = \{x_1, \dots, x_n\}$, endowed with the degree function $d(x_i) = i + 1$. With the above notation, we have $\mathfrak{L}(\mathcal{X}_n) \subseteq \mathfrak{R}(\mathcal{X}_n) \subseteq \mathbf{B}_{\text{rat}}^K(\mathcal{X}_n)$. The sets $\mathfrak{L}(\mathcal{X}_n)$ and $\mathbf{B}_{\text{rat}}^K(\mathcal{X}_n)$ are Boolean subalgebras of $P(\mathcal{X}_n^*)$, but this is not the case in general for $\mathfrak{R}(\mathcal{X}_n)$. Moreover, if $L \in \mathbf{B}_{\text{rat}}^K(\mathcal{X}_n)$, then $\alpha_L \in \mathcal{G}(\Gamma, K)$.*

Proof See [9, Chapter 3]. The fact that $\alpha_L \in \mathcal{G}(\Gamma, K)$ for $L \in \mathbf{B}_{\text{rat}}^K(\mathcal{X}_n)$ follows from [3, Propositions 4.1, 4.26, 6.7 and 6.10]. $\square$

We now show that rational languages give rise to rational ℓ^2 -Betti numbers.

Lemma 4.12 *If a series $\sum_{j \geq 0} \alpha_j x^j \in \mathbb{Z}^+[[x]]$ is the generating function of a rational language $L \in \mathfrak{L}(\mathcal{X}_n)$ with respect to d , then it is a rational series with constant term either 0 or 1. Consequently, $\alpha_L \in \mathbb{Q}$.*

Proof Suppose $\alpha_j = \#(L \cap d^{-1}(\{j\}))$, being $L \in \mathfrak{L}(\mathcal{X}_n)$ a rational language. In particular the series $\underline{L} = \sum_{w \in L} w$ is rational. Define a map $\rho : \mathcal{X}_n \rightarrow \mathbb{Z}^+[[x]]$ by setting $\rho(b) = x^{d(b)}$. By [9, Proposition 1.4.2], the map ρ extends uniquely to a morphism $\rho : \mathbb{Z}^+ \langle\langle \mathcal{X}_n \rangle\rangle \rightarrow \mathbb{Z}^+[[x]]$ which induces the identity on $\mathbb{Z}^+$ and, moreover, preserves rationality. Then

$$\rho\left(\sum_{w \in L} w\right) = \sum_{w \in L} x^{d(w)} = \sum_{j \geq 0} \alpha_j x^j$$

is rational in $\mathbb{Z}^+[[x]]$, and clearly its constant term is either 0 or 1.

By [9, Proposition 6.1.1], there exist polynomials $p(x), q(x) \in \mathbb{Z}[x]$ such that the series $\sum_{j \geq 0} \alpha_j x^j$ is the power series expansion of the rational function $\frac{p(x)}{1-xq(x)}$, that

is,

$$s(L) = \sum_{j \geq 0} \alpha_j x^j = \frac{p(x)}{1 - xq(x)} = p(x) \left[1 + \sum_{j \geq 0} (xq(x))^j \right].$$

Consequently, if $L \in \mathcal{L}(\mathcal{X}_n)$, then $\alpha_L = \frac{1}{8}s(L)(\frac{1}{2}) \in \mathbb{Q}$. $\square$

We obtain now a concrete irrational algebraic number of the form α_L for a suitable $L \in \mathbf{B}_{\text{rat}}^{\mathbb{Q}}(\mathcal{X}_2)$.

Theorem 4.13 *There is $L \in \mathbf{B}_{\text{rat}}^{\mathbb{Q}}(\mathcal{X}_2)$ such that $\alpha_L = \frac{1}{4}\sqrt{\frac{2}{7}}$. In particular, $\frac{1}{4}\sqrt{\frac{2}{7}} \in \mathcal{G}(\Gamma, \mathbb{Q})$.*

Proof Let L be the language on $\mathcal{X}_2 = \{x_1, x_2\}$ given by

$$L = \{w \in \mathcal{X}_2^* \mid |w|_{x_1} = |w|_{x_2}\}.$$

Here $|w|_{x_i}$ is the number of appearances of x_i in w , for $i = 1, 2$. Then by [9, Example 3.4.1], $L \in \mathbf{B}_{\text{rat}}^{\mathbb{Q}}(\mathcal{X}_2)$ (although L is *not* a rational language). If $w \in L$, then since $d(x_1) = 2$ and $d(x_2) = 3$, we get that $d(w) = 5l$, where $l = |w|_{x_1} = |w|_{x_2}$. Therefore, we get from Proposition 4.11 that

$$\alpha_L = \frac{1}{8} \sum_{j \geq 0} \frac{\alpha_j}{2^j} = \frac{1}{8} \sum_{l \geq 0} \binom{2l}{l} \frac{1}{2^{5l}} \in \mathcal{G}(\Gamma, \mathbb{Q}).$$

Since

$$\sum_{l \geq 0} \binom{2l}{l} x^l = \frac{1}{\sqrt{1-4x}} \quad \text{for } |x| < \frac{1}{4},$$

we obtain $\alpha_L = \frac{1}{4}\sqrt{\frac{2}{7}} \in \mathcal{G}(\Gamma, \mathbb{Q})$. $\square$

Taking different choices of variables x_r, x_s , with $r \neq s$, we indeed obtain that $\mathcal{G}(\Gamma, \mathbb{Q})$ contains all the algebraic numbers

$$\frac{1}{8} \sqrt{\frac{2^t}{2^t - 1}}, \quad \text{for } t \geq 3.$$

Similar formulas can be obtained to compute the ranks of elements coming from the copy of the algebra $\mathbb{Q}_{\text{rat}}\langle \mathcal{X} \rangle$ in the $*$ -regular closure $\mathcal{R}_n$ of $\mathcal{A}_n$ in $\mathfrak{R}_n$, where $\mathcal{A}_n$ are the approximations of $\mathbb{Q}\Gamma$ described in Sect. 4.1. The only differences are that we have different degree functions on $\mathcal{X}$, depending on n , and that the factor $1/8$ must be substituted by the factor 2^{-2n-2} .

Remark 4.14 By Theorem 4.13, $\frac{1}{4}\sqrt{\frac{2}{7}}$ is a ‘virtual’ ℓ^2 -Betti number arising from the lamplighter group Γ . This means that there are two ℓ^2 -Betti numbers arising from Γ , α_1 and α_2 , such that $\frac{1}{4}\sqrt{\frac{2}{7}} = \alpha_1 - \alpha_2$. It would be interesting to know whether $\frac{1}{4}\sqrt{\frac{2}{7}}$ is itself an ℓ^2 -Betti number, and in that case, to determine a concrete matrix over $\mathbb{Q}\Gamma$ witnessing this fact. In particular, this would solve a question posed by Grabowski in [17, p. 32], see also [18, Question 3]. We do not know any example of a group G for which $\mathcal{C}(G, \mathbb{Q}) \neq \mathcal{G}(G, \mathbb{Q}) \cap \mathbb{R}^+$.

We close this section with the following problem:

Question 4.15 With the notation established before, is α_L always an algebraic number when $L \in \mathbf{B}_{\text{rat}}^{\mathbb{Q}}(\mathcal{X}_n)$?

5 The odometer algebra

In this last section, we focus on computing the whole set of values that the Sylvester matrix rank function constructed from Theorem 3.2 can achieve in the case of the generalized odometer algebra. We first recall its definition.

Fix a sequence of natural numbers $\bar{n} = (n_i)_{i \in \mathbb{N}}$ with $n_i \geq 2$ for all $i \in \mathbb{N}$, and consider X_i to be the finite space $\{0, 1, \dots, n_i - 1\}$ endowed with the discrete topology. From these we form the topological space $X = \prod_{i \in \mathbb{N}} X_i$ endowed with the product topology, which is in fact a Cantor space. Let T be the homeomorphism on X given by the odometer, namely for $x = (x_i) \in X$, T is given by

$$T(x) = \begin{cases} (x_1 + 1, x_2, x_3, \dots) & \text{if } x_1 \neq n_1 - 1, \\ (0, \dots, 0, x_{m+1} + 1, x_{m+2}, \dots) & \text{if } x_j = n_j - 1 \text{ for } 1 \leq j \leq m \text{ and } x_{m+1} \neq n_{m+1} - 1, \\ (0, 0, \dots) & \text{if } x_i = n_i - 1 \text{ for all } i \in \mathbb{N}. \end{cases}$$

Note that the odometer action is just addition of $(1, 0, \dots)$ by carryover.

Let $(K, -)$ be any field with a positive definite involution $-$. The *generalized odometer algebra* is defined as the crossed product $*$ -algebra $\mathcal{O}(\bar{n}) := C_K(X) \rtimes_T \mathbb{Z}$. We obtain a measure μ on X by taking the usual product measure, where we consider the measure on each component X_i which assigns mass $\frac{1}{n_i}$ on each point in X_i . It is well known (e.g., [11, Section VIII.4]) that μ is an ergodic, full and T -invariant probability measure on X , which in turn coincides with the Haar measure $\hat{\mu}$ on X if one considers X as an abelian group with addition by carryover. We denote by $\text{rk}_{\mathcal{O}(\bar{n})}$ the Sylvester matrix rank function on $\mathcal{O}(\bar{n})$ given by Theorem 3.2.

5.1 Characterizing the $*$ -regular closure $\mathcal{R}_{\mathcal{O}(\bar{n})}$

We explicitly compute the $*$ -regular closure $\mathcal{R}_{\mathcal{O}(\bar{n})}$ of the odometer algebra. Recall that it is defined as the $*$ -regular closure of $\mathcal{O}(\bar{n})$ inside $\mathfrak{R}_{\text{rk}}$ through the embedding $\mathcal{O}(\bar{n}) \hookrightarrow \mathfrak{R}_{\text{rk}}$ given by Theorem 3.2. We will use the notation $[a_1 a_2 \dots a_r]$ for the cylinder sets $\{(x_j)_{j \in \mathbb{N}} \mid x_i = a_i, 1 \leq i \leq r\}$, where $0 \leq a_i < n_i, 1 \leq i \leq r$.

Define new integers $p_m := n_1 \cdots n_m$ for $m \in \mathbb{N}$. At each level $m \geq 1$, we take $E_m = [00 \cdots 0]$ (with m zeroes) for the sequence of clopen sets, whose intersection gives the point $y = (0, 0, \dots) \in X$. We take the partition $\mathcal{P}_m$ of the complement $X \setminus E_m$ to be the obvious one, namely

$$\mathcal{P}_m = \{[10 \cdots 0], \dots, [(n_1 - 1)0 \cdots 0], \dots, [(n_1 - 1)(n_2 - 1) \cdots (n_m - 1)]\}.$$

We denote by $\mathcal{O}(\bar{n})_m$ the unital $*$ -subalgebra of $\mathcal{O}(\bar{n})$ generated by the partial isometries $\{\chi_{Zt} \mid Z \in \mathcal{P}_m\}$. The quasi-partition $\bar{\mathcal{P}}_m$ is really simple in this case: write $Z_{m,l} = T^l(E_m)$ for $1 \leq l < p_m$. Note that these clopen sets form exactly the partition $\mathcal{P}_m$, and that $T(Z_{m,p_m-1}) = E_m$. Therefore, there is only one possible $W \in \mathbb{V}_m$, which has length p_m and is given by

$$W = E_m \cap T^{-1}(Z_{m,1}) \cap T^{-2}(Z_{m,2}) \cap \cdots \cap T^{-p_m+1}(Z_{m,p_m-1}) \cap T^{-p_m}(E_m) = E_m.$$

The representations $\pi_m : \mathcal{O}(\bar{n})_m \rightarrow \mathfrak{K}_m, x \mapsto (h_W \cdot x)_W$ become $*$ -isomorphisms, where $\mathfrak{K}_m = M_{p_m}(K)$. Under this identification, the embeddings $\iota_m : \mathcal{O}(\bar{n})_m \hookrightarrow \mathcal{O}(\bar{n})_{m+1}$ become the block-diagonal embeddings

$$M_{p_m}(K) \hookrightarrow M_{p_{m+1}}(K), \quad x \mapsto \text{diag}(x, \overset{n_{m+1}}{\cdot}, x),$$

so that $\mathcal{O}(\bar{n})_\infty := \varinjlim_m \mathcal{O}(\bar{n})_m \cong \varinjlim_m M_{p_m}(K)$. Note that $\mathcal{O}(\bar{n})_\infty$ is already $*$ -regular, but it does not contain $\mathcal{O}(\bar{n})$: in fact, it is contained in $\mathcal{O}(\bar{n})$. Intuitively, in order to get containment of the whole algebra $\mathcal{O}(\bar{n})$ we need to adjoin to $\mathcal{O}(\bar{n})_\infty$ the element t .

Definition 5.1 For every $m \geq 1$, we denote by $\mathcal{O}(\bar{n})_m^t$ the unital $*$ -subalgebra of $\mathcal{O}(\bar{n})$ generated by $\mathcal{O}(\bar{n})_m$ and t .

We can completely characterize these $*$ -subalgebras.

Lemma 5.2 *There exists a $*$ -isomorphism $\mathcal{O}(\bar{n})_m^t \cong M_{p_m}(K[t^{p_m}, t^{-p_m}])$.*

Proof For $0 \leq i, j < p_m$, the elements $e_{ij}^{(m)} := e_{ij}(E_m)$ form a complete system of matrix units inside $\mathcal{O}(\bar{n})_m^t$, so there is an isomorphism $\mathcal{O}(\bar{n})_m^t \cong M_{p_m}(T)$ with T being the centralizer of the family $\{e_{ij}^{(m)} \mid 0 \leq i, j < p_m\}$ in $\mathcal{O}(\bar{n})_m^t$. The isomorphism is given explicitly by

$$a \mapsto \sum_{i,j=0}^{p_m-1} a_{ij} e_{ij}^{(m)}, \quad \text{with } a_{ij} = \sum_{k=0}^{p_m-1} e_{ki}^{(m)} \cdot a \cdot e_{jk}^{(m)} \in T,$$

which is also a $*$ -isomorphism. Since $t^{p_m} e_{ij}^{(m)} t^{-p_m} = e_{ij}^{(m)}$ and

$$t = \sum_{i=0}^{p_m-1} t e_{ii}^{(m)} = \sum_{i=0}^{p_m-2} e_{i+1,i}^{(m)} + t^{p_m} e_{0,p_m-1}^{(m)} \in M_{p_m}(K[t^{p_m}, t^{-p_m}]),$$

we deduce that $T = K[t^{p_m}, t^{-p_m}]$, as desired. $\square$

The inclusion $\mathcal{O}(\bar{n})_m^t \subseteq \mathcal{O}(\bar{n})_{m+1}^t$ translates to an embedding from $M_{p_m}(K[t^{p_m}, t^{-p_m}])$ to $M_{p_{m+1}}(K[t^{p_{m+1}}, t^{-p_{m+1}}])$ that extends the previous one $M_{p_m}(K) \hookrightarrow M_{p_{m+1}}(K)$, and sends the element $t^{p_m} \cdot \text{Id}_{p_m}$ to the element

$$\begin{pmatrix} \mathbf{0}_{p_m} & & & \mathbf{0}_{p_m} & t^{p_{m+1}} \cdot \text{Id}_{p_m} & \\ \text{Id}_{p_m} & \mathbf{0}_{p_m} & & & \mathbf{0}_{p_m} & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \mathbf{0}_{p_m} & \\ & & & \ddots & \text{Id}_{p_m} & \mathbf{0}_{p_m} \\ \mathbf{0}_{p_m} & & & & & \mathbf{0}_{p_m} \end{pmatrix}.$$

Lemma 5.3 $\mathcal{O}(\bar{n})$ is $*$ -isomorphic to the direct limit $\varinjlim_m M_{p_m}(K[t^{p_m}, t^{-p_m}])$ with respect to the previous embeddings.

Proof By Remark 3.1, the algebra $\mathcal{O}(\bar{n})_\infty$ contains $C_K(X)$; hence, the $*$ -subalgebra of $\mathcal{O}(\bar{n})$ generated by t and $\mathcal{O}(\bar{n})_\infty$ is $\mathcal{O}(\bar{n})$ itself. Now each $\mathcal{O}(\bar{n})_m$ sits inside $\mathcal{O}(\bar{n})_m^t$; hence, $\mathcal{O}(\bar{n})_\infty = \varinjlim_m \mathcal{O}(\bar{n})_m \subseteq \varinjlim_m \mathcal{O}(\bar{n})_m^t$. But $t \in \varinjlim_m \mathcal{O}(\bar{n})_m^t$ already, so $\mathcal{O}(\bar{n}) = \varinjlim_m \mathcal{O}(\bar{n})_m^t \cong \varinjlim_m M_{p_m}(K[t^{p_m}, t^{-p_m}])$ by Lemma 5.2. $\square$

We are now ready to compute $\mathcal{R}_{\mathcal{O}(\bar{n})}$.

Theorem 5.4 There is a $*$ -isomorphism $\mathcal{R}_{\mathcal{O}(\bar{n})} \cong \varinjlim_m M_{p_m}(K(t^{p_m}))$, where the maps $M_{p_m}(K(t^{p_m})) \hookrightarrow M_{p_{m+1}}(K(t^{p_{m+1}}))$ are induced by the inclusions of matrices over Laurent polynomial rings.

Proof We have embeddings $\mathcal{O}(\bar{n})_m^t \hookrightarrow \mathcal{O}(\bar{n}) \hookrightarrow \mathcal{R}_{\mathcal{O}(\bar{n})} \hookrightarrow \mathfrak{R}_{\text{rk}}$. By [3, Lemma 4.4], the rational function field $K(t^{p_m})$ sits inside $\mathcal{R}_{\mathcal{O}(\bar{n})}$. Hence, there is, for each $m \geq 1$, a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(\bar{n})_m^t & \hookrightarrow & M_{p_m}(K(t^{p_m})) & \hookrightarrow & \mathcal{R}_{\mathcal{O}(\bar{n})} \\ \downarrow & & & & \parallel \\ \mathcal{O}(\bar{n})_{m+1}^t & \hookrightarrow & M_{p_{m+1}}(K(t^{p_{m+1}})) & \hookrightarrow & \mathcal{R}_{\mathcal{O}(\bar{n})} \end{array}$$

It is straightforward to prove that for any nonzero element $q(t) \in K[t^{p_m}, t^{-p_m}]$, the corresponding matrix in $M_{p_{m+1}}(K[t^{p_{m+1}}, t^{-p_{m+1}}])$ becomes invertible in $M_{p_{m+1}}(K(t^{p_{m+1}}))$. Therefore, the embedding $\mathcal{O}(\bar{n})_m^t \hookrightarrow \mathcal{O}(\bar{n})_{m+1}^t$ extends uniquely to an embedding $M_{p_m}(K(t^{p_m})) \hookrightarrow M_{p_{m+1}}(K(t^{p_{m+1}}))$, and the previous commutative diagrams give embeddings

$$\mathcal{O}(\bar{n}) \cong \varinjlim_m \mathcal{O}(\bar{n})_m^t \hookrightarrow \varinjlim_m M_{p_m}(K(t^{p_m})) \hookrightarrow \mathcal{R}_{\mathcal{O}(\bar{n})}.$$

But $\varinjlim_m M_{p_m}(K(t^{p_m}))$ is already $*$ -regular and contains $\mathcal{O}(\bar{n})$, so necessarily $\mathcal{R}_{\mathcal{O}(\bar{n})} \cong \varinjlim_m M_{p_m}(K(t^{p_m}))$, as required. $\square$

In particular, since $\lim_{\rightarrow m} M_{p_m}(K(r^{p_m}))$ has a unique Sylvester matrix rank function, it coincides with the one given by Theorem 3.2 under the previous $*$ -isomorphism.

5.2 Characterizing the set $\mathcal{C}'(\mathcal{O}(\bar{n}))$

We characterize the set $\mathcal{C}'(\mathcal{O}(\bar{n}))$ of all positive real values that the Sylvester matrix rank function $\text{rk}_{\mathcal{O}(\bar{n})}$ can achieve, and the subgroup $\mathcal{G}(\mathcal{O}(\bar{n}))$ it generates. First, let

$$\mathcal{C}'(\mathcal{R}_{\mathcal{O}(\bar{n})}) := \text{rk}_{\mathcal{R}_{\mathcal{O}(\bar{n})}} \left(\bigcup_{k \geq 1} M_k(\mathcal{R}_{\mathcal{O}(\bar{n})}) \right) \subseteq \mathbb{R}^+$$

and note that $\mathcal{C}'(\mathcal{O}(\bar{n})) \subseteq \mathcal{C}'(\mathcal{R}_{\mathcal{O}(\bar{n})}) = \mathcal{C}(\mathcal{R}_{\mathcal{O}(\bar{n})})$ (see Definition 4.7 and Remark 4.8). The following definition will be essential.

Definition 5.5 For each sequence $\bar{n} = (n_1, n_2, \dots)$ of positive integers $n_i \geq 2$, one may associate with it the *supernatural number*

$$n = \prod_{i \in \mathbb{N}} n_i = \prod_{q \in \mathbb{Q}} q^{\varepsilon_q(n)},$$

where $\mathbb{Q}$ is the set of prime numbers ordered with respect to the natural ordering, and each $\varepsilon_q(n) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. In more detail, if $n_i = \prod_{q \in \mathbb{Q}} q^{\varepsilon_q(n_i)}$ is the prime decomposition of n_i , then, for each $q \in \mathbb{Q}$, $\varepsilon_q(n)$ is defined by $\varepsilon_q(n) = \sum_{i=1}^{\infty} \varepsilon_q(n_i) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, and n is defined as the formal product $\prod_{q \in \mathbb{Q}} q^{\varepsilon_q(n)}$.

As in [26, Definition 7.4.2], from any supernatural number n one can construct an additive subgroup of $\mathbb{Q}$ containing 1, denoted by $\mathbb{Z}(n)$, consisting of those fractions $\frac{a}{b}$ with $a \in \mathbb{Z}$, and $b \in \mathbb{Z} \setminus \{0\}$ being of the form

$$b = \prod_{q \in \mathbb{Q}} q^{\varepsilon_q(b)},$$

where $\varepsilon_q(b) \leq \varepsilon_q(n)$ for all $q \in \mathbb{Q}$, and $\varepsilon_q(b) = 0$ for all but finitely many q 's. If n comes from a sequence $\bar{n} = (n_1, n_2, \dots)$ as above, $\mathbb{Z}(n)$ is exactly the additive subgroup of $\mathbb{Q}$ consisting of those fractions of the form

$$\frac{a}{n_1 \cdots n_r}, \quad \text{with } a \in \mathbb{Z} \text{ and } r \geq 1.$$

Each group $\mathbb{Z}(n)$ gives rise to a subgroup $\mathbb{G}(n) = \mathbb{Z}(n)/\mathbb{Z}$ of the group $\mathbb{Q}/\mathbb{Z}$. Indeed the groups $\mathbb{G}(n)$ describe all the subgroups of $\mathbb{Q}/\mathbb{Z}$. These groups are called *Prüfer groups*.

Theorem 5.6 We have $\mathcal{C}'(\mathcal{O}(\bar{n})) = \mathcal{C}(\mathcal{O}(\bar{n})) = \mathcal{C}'(\mathcal{R}_{\mathcal{O}(\bar{n})}) = \mathbb{Z}(n)^+$. In particular, $\mathcal{G}(\mathcal{O}(\bar{n})) = \mathbb{Z}(n)$.

Proof The argument is similar to the one given in the proof of [3, Proposition 4.1]. Since $\mathcal{R}_{\mathcal{O}(\bar{n})}$ is a $*$ -regular ring with positive definite involution, each matrix algebra $M_k(\mathcal{R}_{\mathcal{O}(\bar{n})})$ is also $*$ -regular. Hence, for each $A \in M_k(\mathcal{R}_{\mathcal{O}(\bar{n})})$ there exists a projection $P \in M_k(\mathcal{R}_{\mathcal{O}(\bar{n})})$ such that $\text{rk}_{\mathcal{R}_{\mathcal{O}(\bar{n})}}(A) = \text{rk}_{\mathcal{R}_{\mathcal{O}(\bar{n})}}(P)$. We conclude that $\mathcal{C}'(\mathcal{R}_{\mathcal{O}(\bar{n})})$ equals the set of positive real numbers of the form $\text{rk}_{\mathcal{R}_{\mathcal{O}(\bar{n})}}(P)$, where P ranges over matrix projections with coefficients in $\mathcal{R}_{\mathcal{O}(\bar{n})}$. Now each such projection P is equivalent to a diagonal projection [16, Proposition 2.10], that is of the form $\text{diag}(p_1, p_2, \dots, p_r)$ for some projections $p_1, \dots, p_r \in \mathcal{R}_{\mathcal{O}(\bar{n})}$, so that $\text{rk}_{\mathcal{R}_{\mathcal{O}(\bar{n})}}(P) = \text{rk}_{\mathcal{R}_{\mathcal{O}(\bar{n})}}(p_1) + \dots + \text{rk}_{\mathcal{R}_{\mathcal{O}(\bar{n})}}(p_r)$. But since $\mathcal{R}_{\mathcal{O}(\bar{n})} \cong \varinjlim_m M_{p_m}(K(t^{p_m}))$ by Theorem 5.4, the set of ranks of elements in $\mathcal{R}_{\mathcal{O}(\bar{n})}$ is contained in $\mathbb{Z}(n)^+ \cap [0, 1]$. Therefore, $\text{rk}_{\mathcal{R}_{\mathcal{O}(\bar{n})}}(P) \in \mathbb{Z}(n)^+$. This proves the inclusion $\mathcal{C}'(\mathcal{R}_{\mathcal{O}(\bar{n})}) \subseteq \mathbb{Z}(n)^+$. The inclusion $\mathbb{Z}(n)^+ \subseteq \mathcal{C}'(\mathcal{O}(\bar{n}))$ is straightforward, since for $0 \leq i < p_m$ we have $\text{rk}_{\mathcal{O}(\bar{n})}(e_{ii}^{(m)}) = \frac{1}{p_m}$. The equality $\mathcal{C}(\mathcal{O}(\bar{n})) = \mathbb{Z}(n)^+$ also follows from the above.

The last part of the theorem is immediate. $\square$

In a sense, Theorem 5.6 confirms the SAC for a class of crossed product algebras, as follows:

Remark 5.7 Let n be a supernatural number and let $n = \prod_{i \in \mathbb{N}} n_i$ a decomposition of n , as above. Consider the presentation of $\mathbb{G}(n)$ given by generators $\{g_m\}_{m \in \mathbb{N}}$ and relations $g_1^{n_1} = 1$ and $g_{m+1}^{n_{m+1}} = g_m$ for all $m \geq 1$. The topological group $X(n) = \prod_{i \in \mathbb{N}} X_i$ is naturally isomorphic with the Pontryagin dual $\widehat{\mathbb{G}(n)}$ of the Prüfer group $\mathbb{G}(n)$. Indeed, let ξ_{p_m} denote the canonical primitive p_m -root of unity in $\mathbb{C}$, where $p_m = n_1 \cdots n_m$, and observe that $\xi_{p_{m+1}}^{m_{n+1}} = \xi_{p_m}$ for each $m \in \mathbb{N}$. The map $\phi: X(n) \rightarrow \widehat{\mathbb{G}(n)}$ defined by

$$\phi_x(g_m) = \xi_{p_m}^{a_1 + a_2 n_1 + \dots + a_m n_{m-1}},$$

where $x = (a_1, a_2, \dots) \in X(n)$, is a group isomorphism and a homeomorphism. Therefore, if K is a subfield of $\mathbb{C}$ closed under complex conjugation containing all p_m^{th} roots of unity, Fourier transform defines an isomorphism

$$\mathcal{F}: K[\mathbb{G}(n)] \longrightarrow C_K(X(n)),$$

and we can pull back the automorphism induced by T on $C_K(X(n))$ to an automorphism ρ on $K[\mathbb{G}(n)]$ (which is *not* induced by an automorphism of $\mathbb{G}(n)$). Note that $\rho(g_m) = \xi_{p_m} g_m \in K \cdot \mathbb{G}(n)$ for all $m \geq 1$. Therefore, we can interpret Theorem 5.6 as giving a positive answer to the SAC for the crossed product $\mathcal{A} = K[\mathbb{G}(n)] \rtimes_{\rho} \mathbb{Z} \cong \mathcal{O}(n)$: the set of $\mathcal{A}$ -Betti numbers is exactly the semigroup $\mathbb{Z}(n)^+$ generated by the inverses of the orders of the elements of $\mathbb{G}(n)$. Note finally that the groups $X(n)$, n a supernatural, give all the profinite completions of $\mathbb{Z}$. With this view, the dynamical system considered on $X(n)$ is generated by addition by 1.

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