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On quantum adjacency algebras of Doob graphs and their irreducible modules

John Vincent S. Morales[1] · Tessie M. Palma[2]

Abstract

For fixed integers $n \geq 1$ and $m \geq 0$, we consider the Doob graph $D = D(n, m)$ formed by taking direct product of n copies of Shrikhande graph and m copies of complete graph K_4. Fix a vertex x of D, and let $T = T(x)$ denote the Terwilliger algebra of D with respect to x. Let A denote the adjacency matrix of D. There exists a decomposition of A into a sum $A = L + F + R$ of elements in T where L, F, and R are the lowering, flat, and raising matrices, respectively. We call $A = L + F + R$ the quantum decomposition of A. Hora and Obata (Quantum Probability and Spectral Analysis of Graphs. Theoretical and Mathematical Physics, Springer, Berlin, 2007) introduced a semi-simple matrix algebra based on the quantum decomposition of the adjacency matrix. This algebra is generated by the quantum components of the decomposition and is called the quantum adjacency algebra of the graph. Let $Q = Q(x)$ denote the quantum adjacency algebra of D with respect to x. In this paper, we display an action of the special orthogonal Lie algebra $\mathfrak{so}_4$ on the standard module for D. We also prove Q is generated by the center and the homomorphic image of the universal enveloping algebra $U(\mathfrak{so}_4)$. To do these, we exploit the work of Tanabe (JAC 6: 173–195, 1997) on irreducible T-modules of D.

Keywords Terwilliger algebra · Quantum adjacency algebra · Doob graphs · Q-polynomial distance-regular graph · Special orthogonal Lie algebra

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✉ John Vincent S. Morales
   john.vincent.morales@dlsu.edu.ph

   Tessie M. Palma
   tessie.palma@dlsl.edu.ph

[1] Mathematics and Statistics Department, De La Salle University, 2401 Taft Avenue, Manila, Philippines

[2] Mathematics Learning Area, De La Salle Lipa, 1962 J.P. Laurel National Highway, Lipa City, Batangas, Philippines

1 Introduction and background

The Terwilliger algebra or subconstituent algebra is a finite-dimensional semi-simple matrix $\mathbb{C}$-algebra that is noncommutative in general. Since its introduction (see [21], [22], [23]), the Terwilliger algebra has become a rich area of research in the study of combinatorial objects such as graphs (e.g., [6], [7], [8]) and association schemes (e.g., [3], [14], [17], [19]). Independent of the notion of subconstituent algebra, Hora and Obata [10] introduced the quantum adjacency algebra of the graph based on a certain partition of the edge set. This partition is called the quantum decomposition of the adjacency matrix, and the quantum adjacency algebra is generated by the quantum components of the decomposition. They used this algebra to explore limiting spectral distributions of infinite sequences of "growing" graphs. The quantum components turned out to be elements of the Terwilliger algebra of the graph with respect to a fixed vertex.

To be able to describe our results, we recall some preliminary concepts (see [1], [2], [16], [21] for more thorough discussion).

Let X denote a nonempty finite set. Denote by $\mathrm{Mat}_X(\mathbb{C})$ the $\mathbb{C}$-algebra of $|X| \times |X|$ matrices with complex entries whose rows and columns are indexed by X. The $\mathbb{C}$-vector space of column vectors whose coordinates are indexed by X is denoted by $V = \mathbb{C}^X$. Observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. The vector space V is called the *standard module*. For all $v, u \in V$, endow V with Hermitian inner product $\langle v, u \rangle = v^t \bar{u}$ where v^t denotes the transpose of v and $\bar{u}$ denotes the complex conjugate of u.

Let $\Gamma = (X, R)$ be a finite, undirected, simple connected graph with vertex set X and edge set R. The *distance* $\partial(x, y)$ from x to y is the length of a shortest path from x to y. By the *diameter* of Γ, we mean the scalar $D = \max\{\partial(x, y) : x, y \in X\}$. If, for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h := \left| \left\{ z \in X : \partial(x, z) = i, \partial(y, z) = j \right\} \right| \tag{2}$$

is independent of x and y, then Γ is said to be *distance-regular*. The scalars (2) are called the *intersection numbers* of Γ. We abbreviate

$$b_i := p_{1\,i+1}^i \quad (0 \leq i \leq D-1),$$
$$c_i := p_{1\,i-1}^i \quad (1 \leq i \leq D).$$

For convenience, $c_0 := 0$ and $b_D := 0$. From here on, we assume that Γ is a distance-regular graph with diameter $D \geq 1$.

We recall the Bose–Mesner algebra of Γ. For each integer i $(0 \leq i \leq D)$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy-entry given by

$$(A_i)_{xy} = \begin{cases} 1 \text{ if } \partial(x, y) = i \\ 0 \text{ if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

The matrix A_i is called the *ith distance matrix* of Γ. We abbreviate $A := A_1$ and refer to this as the *adjacency matrix* of Γ. We observe

$$\sum_{i=0}^{D} A_i = J,$$
$$A_0 = I,$$
$$A_i^t = A_i \qquad (0 \le i \le D),$$
$$\overline{A_i} = A_i \qquad (0 \le i \le D),$$
$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad (0 \le i, j \le D),$$

where I and J are the identity and the all-ones matrices in $\mathrm{Mat}_X(\mathbb{C})$, respectively. Since $p_{ij}^h = p_{ji}^h$, it follows that $A_i A_j = A_j A_i$. Note that $\{A_i\}_{i=0}^{D}$ forms a basis for the commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ known as the *Bose–Mesner algebra* of Γ. The matrix A generates M by [1, p. 190]. Moreover, by [1, pp. 59, 64], M has a second basis $\{E_i\}_{i=0}^{D}$ called *primitive idempotents* of Γ such that

$$\sum_{i=0}^{D} E_i = I,$$
$$E_0 = |X|^{-1} J,$$
$$E_i^t = E_i \qquad (0 \le i \le D),$$
$$\overline{E_i} = E_i \qquad (0 \le i \le D),$$
$$E_i E_j = \delta_{ij} E_i \qquad (0 \le i, j \le D).$$

Since $\{E_i\}_{i=0}^{D}$ forms a basis for M, there are scalars $\theta_0, \ldots, \theta_D$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Observe $A E_i = E_i A = \theta_i E_i$ for each integer i ($0 \le i \le D$). By [1, p. 97], the scalars $\{\theta_i\}_{i=0}^{D}$ are real. Since A generates M, the scalars $\{\theta_i\}_{i=0}^{D}$ are pairwise distinct. For each integer i ($0 \le i \le D$), θ_i is the *eigenvalue* of Γ associated with E_i. Then, V decomposes into

$$V = E_0 V + E_1 V + \cdots + E_D V \text{ (orthogonal direct sum).}$$

For each integer i ($0 \le i \le D$), $E_i V$ is the *eigenspace* of A associated to eigenvalue θ_i.

We recall the dual Bose–Mesner algebra of Γ. Fix a vertex $x \in X$ and call it *base vertex*. For each integer i ($0 \le i \le D$), let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy-entry given by

$$(E_i^*)_{yy} = \begin{cases} 1 \text{ if } \partial(x, y) = i \\ 0 \text{ if } \partial(x, y) \ne i \end{cases} \quad (y \in X).$$

We call E_i^* the *ith dual primitive idempotent* of Γ with respect to the base vertex x. For convenience, we define $E_i^* = 0$ whenever $i < 0$ or $i > D$. Observe that

$$\sum_{i=0}^{D} E_i^* = I,$$
$$E_i^{*t} = E_i^* \qquad (0 \le i \le D),$$
$$\overline{E_i^*} = E_i^* \qquad (0 \le i \le D),$$
$$E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \le i, j \le D).$$

Note that $\left\{ E_i^* \right\}_{i=0}^{D}$ is linearly independent and forms a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$ known as the *dual Bose–Mesner algebra of Γ with respect to x.*

We recall Terwilliger algebra of Γ and its irreducible modules. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *Terwilliger algebra of Γ with respect to x.* Since M is generated by A and M^* is generated by $\left\{ E_i^* \right\}_{i=0}^{D}$, it follows that T is generated by A and $\{E_i^*\}_{i=0}^{D}$. Suppose for a moment W is a subspace of V. For each $B \in \mathrm{Mat}_X(\mathbb{C})$, we define

$$BW = \{ Bw \ : \ w \in W \} \subseteq V.$$

We say W is *B-invariant* whenever $BW \subseteq W$. If W is B-invariant for all $B \in T$, then W is called a *T-module*. A T-module W is said to be *irreducible* if $W \ne 0$ and W contains no other T-modules other than 0 and W. If W is a T-module, then its *orthogonal complement* $W^{\perp} := \{ v \in V \ : \ \langle v, w \rangle = 0 \ \forall w \in W \}$ is also a T-module. In fact, if W is a T-module containing another T-module W', then $W'^{\perp} \cap W$ is also a T-module and $W = W' \oplus \left(W'^{\perp} \cap W \right)$. Consequently, any nonzero T-module (e.g., the standard module V) is an orthogonal direct sum of irreducible T-modules. Now, let W denote an irreducible T-module. Define $W_s := \{ i \ : \ 0 \le i \le D, \ E_i^* W \ne 0 \}$. We call $|W_s| - 1$ and $\min(W_s)$ the *diameter* and *endpoint* of W, respectively. On the other hand, define $W_{s'} := \{ i \ : \ 0 \le i \le D, \ E_i W \ne 0 \}$. We call $|W_{s'}| - 1$ and $\min(W_{s'})$ the *dual-diameter* and *dual-endpoint* of W, respectively. We say W is *thin* (resp. *dual-thin*) whenever $\dim(E_i^* W) \le 1$ (resp. $\dim(E_i W) \le 1$) for all integers i $(0 \le i \le D)$. Let W and W' denote T-modules. By a *T-module isomorphism from W to W'*, we mean a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. If such an isomorphism exists, then W and W' are said to be *isomorphic T-modules.*

Now, we recall the quantum adjacency algebra of Γ. To describe this algebra, we define the matrices $L = L(x)$, $F = F(x)$, and $R = R(x)$ by

$$L = \sum_{i=0}^{D} E_{i-1}^* A E_i^*, \quad F = \sum_{i=0}^{D} E_i^* A E_i^*, \quad R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*.$$

We call L, F, and R the *lowering matrix*, *flat matrix*, and *raising matrix*, respectively. Observe that $L, F, R \in T$ since A and $\{E_i^*\}_{i=0}^{D}$ are generators of T. Let $Q = Q(x)$ denote the subalgebra of T generated by L, F, and R. We call Q the *quantum adjacency algebra* of Γ with respect to x. Since $E_j^* A E_k^* = 0$ if $|j - k| > 1$, we have

$$
\begin{aligned}
A &= \left(\sum_{i=0}^{D} E_i^*\right) A \left(\sum_{j=0}^{D} E_j^*\right) \\
&= \sum_{i=0}^{D} E_{i-1}^* A E_i^* + \sum_{i=0}^{D} E_i^* A E_i^* + \sum_{i=0}^{D} E_{i+1}^* A E_i^* \\
&= L + F + R.
\end{aligned}
\tag{3}
$$

We call (3) the *quantum decomposition of the adjacency matrix A with respect to x*. Observe that

$$
\overline{L} = L, \quad \overline{F} = F, \quad \overline{R} = R, \quad F^t = F, \quad R^t = L. \tag{4}
$$

Hence, Q is closed under the conjugate-transpose map and is semi-simple. Moreover,

$$
L E_i^* V \subseteq E_{i-1}^* V, \quad F E_i^* V \subseteq E_i^* V, \text{ and } R E_i^* V \subseteq E_{i+1}^* V. \tag{5}
$$

We define *Q-modules*, *irreducible Q-modules*, and *Q-module isomorphism* analogous to that of T-modules, irreducible T-modules, and T-module isomorphism, respectively. Observe that every T-module turns into a Q-module by restricting the action of T to Q.

We compare our results with previous works on Terwilliger algebras of particular distance-regular graphs. The theory of Terwilliger algebra has been most useful when it is applied to Q-polynomial distance-regular graphs (see [2, p. 135] for definition). Hamming graphs and Doob graphs are common examples. Go [8] described the irreducible modules of the Terwilliger algebras of binary Hamming graphs. She showed implicitly that the Terwilliger algebras in this case are homomorphic images of the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the complex Lie algebra $\mathfrak{sl}_2$. Terwilliger [20] considered general Hamming graphs and showed that Terwilliger algebras of Hamming graphs also contain homomorphic images of $U(\mathfrak{sl}_2)$ (cf. [14]). Generalizing these results, Morales and Pascasio [18] considered a Lie algebra in terms of generators and relations that contains subalgebras that are isomorphic to $\mathfrak{sl}_2$. This Lie algebra is known as the *tetrahedron algebra* $\boxtimes$. They showed the Terwilliger algebra of any Hamming graph or Doob graph is generated by the center and the homomorphic image of the universal enveloping algebra $U(\boxtimes)$. In this paper, we consider the special orthogonal Lie algebra $\mathfrak{so}_4$ and show that the quantum adjacency algebra of any Doob graph is generated by the center and homomorphic image of the universal enveloping algebra $U(\mathfrak{so}_4)$. To do this, we exploit the work of Tanabe [19] on irreducible modules of the Terwilliger algebras of Doob graphs.

The paper is organized as follows: In Sect. 2, we review important properties of Doob graphs and their Terwilliger algebras along with Tanabe's description of irreducible modules. In Sect. 3, we prove several relations among lowering, raising, and

flat matrices of Doob graphs. We obtain these relations by restricting quantum components to an arbitrary irreducible module. In Sect. 4, we recall the classical Lie algebra $\mathfrak{so}_4$. We also review the representation theory of finite-dimensional $\mathfrak{so}_4$-modules from the point of view of highest weight theory. In Sect. 5, we display an action of $\mathfrak{so}_4$ on the standard module for Doob graphs and prove other main results.

2 Doob graphs and their Terwilliger algebras

Let $\Gamma = (X, R)$ and $\Gamma' = (X', R')$ be finite, undirected, simple connected graphs. The *direct product* $\Gamma \square \Gamma'$ is the graph on vertex set $X \times X'$ such that (a, a') and (b, b') are adjacent if and only if either $ab \in R$ and $a' = b'$ or $a = b$ and $a'b' \in R'$. Let d denote a positive integer. We write $\Gamma^{\square d}$ instead of $\Gamma \square \cdots \square \Gamma$ (d copies). A graph is *complete* if every pair of distinct vertices are adjacent. For each integer $q \geq 3$, let K_q denote the complete graph on q vertices. By *Hamming graph $H(d, q)$*, we mean the graph $K_q^{\square d}$ which is distance-regular with diameter d and has intersection numbers

$$b_i = (q - 1)(d - i),$$
$$c_i = i,$$

for integers i ($0 \leq i \leq d$) (see [2, p. 261]). The eigenvalues of $H(d, q)$ are $\{q(d - i) - d \mid$ for $0 \leq i \leq d\}$. On the other hand, the *Shrikhande graph S* has vertex set consisting of all cyclic permutations of the codes $000000, 110000, 010111,$ and 011011 such that two vertices are adjacent if and only if they differ in exactly two coordinates. The graph S is distance-regular and has the same intersection numbers as $H(2, 4)$.

Let $n \geq 1$ and $m \geq 0$ denote integers. By *Doob graph $D(n, m)$*, we mean the direct product of n copies of S and m copies of K_4. This graph was first introduced by Doob in 1972 (see [4]). The graph $D(n, m)$ is distance-regular and has the same intersection numbers as $H(2n + m, 4)$ (see [5]). We note that $H(n, q)$ and $D(n, m)$ have integral eigenvalues. In this section, objects in reference to the Doob graph $D(n, m)$ are labeled with (n, m). For example, $A_i(n, m)$ refers to the ith distance matrix of $D(n, m)$ for each integer i ($0 \leq i \leq 2n + m$). Fix an integer i ($0 \leq i \leq 2n + m$) and note that

$$A_i(n, m) = \sum A_{i_1}(1, 0) \otimes \cdots \otimes A_{i_n}(1, 0) \otimes A_{j_1}(0, 1) \otimes \cdots \otimes A_{j_m}(0, 1)$$

where the sum ranges to all $i_1, i_2, \ldots, i_n \in \{0, 1, 2\}$ and $j_1, j_2, \ldots, j_m \in \{0, 1\}$ such that $i_1 + \cdots + i_n + j_1 + \cdots + j_m = i$. The primitive idempotent $E_i(n, m)$ is obtained similarly. Choose the base vertex $x(n, m)$ of $D(n, m)$ such that

$$x(n, m) = (\underbrace{x(1, 0), x(1, 0), \ldots, x(1, 0)}_{n \text{ copies}}, \underbrace{x(0, 1), x(0, 1), \ldots, x(0, 1)}_{m \text{ copies}}).$$

Then, the ith dual-primitive idempotent $E_i^*(n, m)$ with respect to $x(n, m)$ is given by

$$E_i^*(n, m) = \sum E_{i_1}^*(1, 0) \otimes \cdots \otimes E_{i_n}^*(1, 0) \otimes E_{j_1}^*(0, 1) \otimes \cdots \otimes E_{j_m}^*(0, 1)$$

where the sum ranges to all $i_1, i_2, \ldots, i_n \in \{0, 1, 2\}$ and $j_1, j_2, \ldots, j_m \in \{0, 1\}$ such that $i_1 + \cdots + i_n + j_1 + \cdots + j_m = i$. Let $T(n, m)$ denote the Terwilliger algebra of $D(n, m)$ with respect to $x(n, m)$ and let $V(n, m)$ denote the standard module. Since $T(n, m)$ is semi-simple, $V(n, m)$ decomposes as a direct sum of irreducible $T(n, m)$-modules. We end this section with Tanabe's description of irreducible $T(n, m)$-modules on the standard module $V(n, m)$ (see [19]).

Proposition 2.1 [19, Proposition 3]
Let $n, m, v, d, p, t \in \mathbb{Z}$ such that $n - 1, m, v, d, p \geq 0$. Let $W := W(n, m; v, d, p, t)$ denote a $T(n, m)$-module on $V(n, m)$ with endpoint v, diameter $d + p$, dimension $(d + 1)(p + 1)$, and basis

$$\{w_{ij} \in E^*_{v+i+j}(n, m)W : 0 \leq i \leq d \text{ and } 0 \leq j \leq p\}$$

satisfying

$$A_1(n, m)w_{ij} = 3(d - i + 1)w_{i-1,j} + (p - j + 1)w_{i,j-1} + (t + 2(i - j))w_{ij}$$
$$+ 3(j + 1)w_{i,j+1} + (i + 1)w_{i+1,j}, \tag{6}$$

where $w_{ij} := 0$ if $i \notin \{0, \ldots, d\}$ or $j \notin \{0, \ldots, p\}$. Then, each of the following holds:

(i) *W is an irreducible $T(n, m)$-module.*
(ii)

$$\dim E^*_{v+k}(n, m)W = \begin{cases} k + 1 & \text{if } 0 \leq k \leq \min\{d, p\}, \\ \min\{d, p\} + 1 & \text{if } \min\{d, p\} < k \leq \max\{d, p\}, \\ d + p + 1 - k & \text{if } \max\{d, p\} < k \leq d + p. \end{cases}$$

(iii) *W is thin if and only if $dp = 0$.*
(iv) *If μ is the dual-endpoint of W, then*

$$\mu = \frac{3(2n + m) - t - 3d - p}{4},$$
$$\dim E_{\mu+k}(n, m)W = \dim E^*_{v+k}(n, m)W.$$

Moreover, the diameter of W and the dual-diameter of W are equal.
(v) *W and $W' := W(n, m; v', d', p', t')$ are isomorphic $T(n, m)$-modules if and only if $(v, d, p, t) = (v', d', p', t')$.*

Proposition 2.2 *[19, Proposition 1, Lemma 2, and Proposition 4] With reference to above notations, we have the following:*

(i) *The set $\{U_0, U_1, U_2, U_3, U_4, U_5\}$ forms a complete set of pairwise nonisomorphic irreducible $T(1, 0)$-modules on standard module $V(1, 0)$ of $D(1, 0)$ where*

$$U_0 \cong W(1, 0; 0, 2, 0, 0), \quad U_1 \cong W(1, 0; 1, 1, 0, -1), \quad U_2 \cong W(1, 0; 1, 0, 1, 1),$$
$$U_3 \cong W(1, 0; 1, 0, 0, -2), \quad U_4 \cong W(1, 0; 2, 0, 0, 2), \quad U_5 \cong W(1, 0; 2, 0, 0, -2).$$

(ii) *The set* $\{V_0, V_1\}$ *forms a complete set of pairwise nonisomorphic irreducible* $T(0, 1)$-*modules on standard module* $V(0, 1)$ *of* $D(0, 1)$ *where*

$$V_0 \cong W(0, 1; 0, 1, 0, 0), \quad V_1 \cong W(0, 1; 1, 0, 0, -1).$$

(iii) $V(n, m)$ *is isomorphic to a direct sum of spaces*

$$\left(U_0^{\otimes N_0} \otimes U_1^{\otimes N_1} \otimes U_2^{\otimes N_2} \otimes U_3^{\otimes N_3} \otimes U_4^{\otimes N_4} \otimes U_5^{\otimes N_5}\right) \otimes \left(V_0^{M_0} \otimes V_1^{M_1}\right) \quad (7)$$

where $N_0, \ldots, N_5, M_0, M_1$ *are nonnegative integers satisfying* $N_0 + \cdots + N_5 = n$, *and* $M_0 + M_1 = m$. *The space* (7) *decomposes into irreducible* $T(n, m)$-*modules isomorphic to* $W(n, m; v, d, p, t)$ *such that*

$$v = r + s - 2N_0 - N_1 - N_2 - N_3 - M_0 + 2n + m,$$
$$d = 2N_0 + N_1 + M_0 - 2r,$$
$$p = N_2 - 2s,$$
$$t = 2r - 2s + 2N_0 + N_1 + 3N_2 + 4N_4 + M_0 - 2n - m.$$

where

$$\begin{cases} r = 0, & \text{if } N_0 = 1 \text{ and } N_1 = M_0 = 0, \\ r = 0, 1, \ldots, N_0 + \lfloor \frac{N_1 + M_0}{2} \rfloor, & \text{otherwise,} \end{cases}$$

and $s = 0, 1, \ldots, \lfloor \frac{N_2}{2} \rfloor$.

3 Quantum adjacency algebras of Doob graphs

As mentioned in Sect. 1, the quantum adjacency algebra of the graph (with respect to a base vertex) is the algebra generated by the components of the quantum decomposition of the adjacency matrix. In the case of Doob graphs, the components of our quantum decompositions are the lowering, flat, and raising matrices. In this section, we describe the quantum adjacency algebras of Doob graphs as well as the relations of the quantum components. We shall adopt the following assumption:

Assumption 3.1 Fix the integers $n \geq 1$ and $m \geq 0$ and consider the Doob graph $D = D(n, m)$. Let A be the adjacency matrix of D. Choose a base vertex x of D and let $E_0^*, E_1^*, \ldots, E_{2n+m}^*$ denote the dual primitive idempotents with respect to x. Let T (resp. Q) denote the Terwilliger algebra (resp. quantum adjacency algebra) of D with respect to x. Let L, F, and R denote lowering, flat, and raising matrices with respect to x, respectively. Define the mapping $[\ ,\ ] : T \times T \to T$ such that $[Y, Z] = YZ - ZY$ for all $Y, Z \in T$. Finally, let V denote the standard module for D.

Lemma 3.2 *With reference to Assumption 3.1, let* $W = W(n, m; v, d, p, t)$ *denote an irreducible* T-*module with basis* $\{w_{ij}\}$ *as described in Proposition 2.1. Then,*

(i) $L w_{ij} = 3(d - i + 1) w_{i-1,j} + (p - j + 1) w_{i,j-1}$

(ii) $F w_{ij} = (t + 2(i - j)) w_{ij}$,

(iii) $R w_{ij} = 3(j + 1) w_{i,j+1} + (i + 1) w_{i+1,j}$,

for all integers i $(0 \leq i \leq d)$ and for all integers j $(0 \leq j \leq p)$.

Proof Follows from Proposition 2.1 and definitions of matrices L, F, and R. □

Lemma 3.3 *With reference to Assumption 3.1, let $W = W(n, m; v, d, p, t)$ denote an irreducible T-module with basis $\{w_{ij}\}$ as described in Proposition 2.1. Then,*

(i) $[L, R] w_{ij} = 3(p + d - 2i - 2j) w_{ij}$,

(ii) $[L, F] w_{ij} = 6(d - i + 1) w_{i-1,j} - 2(p - j + 1) w_{i,j-1}$,

(iii) $[R, F] w_{ij} = 6(j + 1) w_{i,j+1} - 2(i + 1) w_{i+1,j}$,

(iv) $[R, [F, L]] w_{ij} = 6(d - p - 2i + 2j) w_{ij}$,

for all integers i $(0 \leq i \leq d)$ and for all integers j $(0 \leq j \leq p)$.

Proof Follows from Lemma 3.2. □

Lemma 3.4 *With reference to Assumption 3.1, the matrices F, $[L, R]$, and $[R, [F, L]]$ mutually commute on V.*

Proof Let $W = W(n, m; v, d, p, t)$ denote an irreducible T-module with basis $\{w_{ij}\}$ as described in Proposition 2.1. From Lemma 3.2(ii), Lemma 3.3(i), and Lemma 3.3(iv), these matrices mutually commute on W. Since W is arbitrary and V is a direct sum of irreducible T-modules, these matrices mutually commute on V. □

Lemma 3.5 *With reference to Assumption 3.1, we have*

i) $[L, [L, F]] = 0$,

ii) $[R, [R, F]] = 0$,

iii) $[F, [F, R]] = 4R$,

iv) $[F, [F, L]] = 4L$,

v) $[R, [R, L]] = -6R$,

vi) $[L, [L, R]] = -6L$,

vii) $[L, [F, R]] = [R, [F, L]]$.

Proof Let $W = W(n, m; v, d, p, t)$ denote an irreducible T-module with basis $\{w_{ij}\}$ as described in Proposition 2.1. By Lemmas 3.2 and 3.3, the equations are true on W. Since W is arbitrary and V is a direct sum of irreducible T-modules, these equations hold on V. □

Lemma 3.6 *With reference to Assumption 3.1, we have*

(i) $[R, [R, [F, L]]] = -[[R, F], [L, R]] = -6[R, F]$,

(ii) $[L, [R, [F, L]]] = [[L, F], [L, R]] = -6[L, F]$,

(iii) $[[L, F], [R, F]] = -4[L, R]$,

(iv) $[[R, F], [R, [F, L]]] = -24R$,

(v) $[[L, F], [R, [F, L]]] = -24L$.

Proof Similar to the proof of Lemma 3.5. □

Lemma 3.7 *With reference to Assumption 3.1, let* $W = W(n, m; v, d, p, t)$ *denote an irreducible T-module with basis* $\{w_{ij}\}$ *as described in Proposition 2.1. Then, the matrix* $F + \frac{1}{6}[R, [F, L]]$ *acts as the scalar* $t - p + d$ *on* W.

Proof Follows immediately from Lemma 3.2(ii) and Lemma 3.3(iv). □

Lemma 3.8 *With reference to Assumption 3.1, the set*

$$\{L, R, [L, R], [L, F], [R, F], [R, [F, L]]\} \tag{8}$$

is linearly independent in Q.

Proof Let W denote an irreducible T-module with basis $\{w_{ij}\}$ as in Proposition 2.1. Assume there exists a matrix in (8) that can be expressed as a linear combination of the remaining matrices. Among the matrices in (8), only $[L, R]$ and $[R, [F, L]]$ leave invariant $\mathbb{C}w_{ij}$ for all $i, j \in \mathbb{Z}$. But $[L, R]$ and $[R, [F, L]]$ cannot be linearly dependent by Lemma 3.3. On the other hand, the matrices L and $[L, F]$ (resp. R and $[R, F]$) shift the space $\mathbb{C}w_{ij}$ to the space $\mathbb{C}w_{i,j-1} \oplus \mathbb{C}w_{i-1,j}$ (resp. $\mathbb{C}w_{i,j+1} \oplus \mathbb{C}w_{i+1,j}$) for all $i, j \in \mathbb{Z}$. By Lemma 3.2 and Lemma 3.3, the pair cannot be linearly dependent. This shows that any matrix cannot be expressed as a linear combination of the others. Statement holds. □

Lemma 3.9 *With reference to Assumption 3.1, the subspace of Q spanned by the matrices in (8) is closed under* [,]. *In particular, we have the relations*

[,]	H_1	H_2	X_1	X_2	Y_1	Y_2
H_1	0	0	X_1	X_2	$-Y_1$	$-Y_2$
H_2	0	0	X_1	$-X_2$	$-Y_1$	Y_2
X_1	$-X_1$	$-X_1$	0	0	$H_1 + H_2$	0
X_2	$-X_2$	X_2	0	0	0	$H_1 - H_2$
Y_1	Y_1	Y_1	$-H_1 - H_2$	0	0	0
Y_2	Y_2	$-Y_2$	0	$-H_1 + H_2$	0	0

where $H_1 = \frac{1}{6}[L, R]$, $H_2 = \frac{1}{12}[R, [F, L]]$, $X_1 = \frac{1}{12}(2L + [L, F])$, $X_2 = \frac{1}{4}(2L - [L, F])$, $Y_1 = \frac{1}{4}(2R - [R, F])$, *and* $Y_2 = \frac{1}{12}(2R + [R, F])$.

Proof Immediate from Lemmas 3.4, 3.5, and 3.6. □

4 The special orthogonal Lie algebra $\mathfrak{so}_4$

By a *complex Lie algebra*, we mean a vector space $\mathfrak{g}$ over $\mathbb{C}$ together with a bracket operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following conditions:

(i) $[,]$ is bilinear,

(ii) $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$, and

(iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Let $\mathfrak{g}$ be a complex Lie algebra. A *subalgebra* of $\mathfrak{g}$ is a subspace that is closed under the operation $[\ ,\ ]$. On the other hand, an *ideal* $\mathfrak{i}$ of $\mathfrak{g}$ is a subspace such that $[x, y] \in \mathfrak{i}$ for all $x \in \mathfrak{g}$ and for all $y \in \mathfrak{i}$. We say that $\mathfrak{g}$ is *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. We say $\mathfrak{g}$ is *simple* if it is a non-abelian complex Lie algebra whose only ideals are 0 and $\mathfrak{g}$ itself. We say that $\mathfrak{g}$ is *semi-simple* if it is a direct sum of simple complex Lie algebras.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be complex Lie algebras. A linear map $\pi : \mathfrak{g} \to \mathfrak{h}$ is called a *Lie algebra homomorphism* if $\pi([x, y]) = [\pi(x), \pi(y)]$ for all $x, y \in \mathfrak{g}$. Let V be an n-dimensional vector space over $\mathbb{C}$. Let $\mathfrak{gl}(V)$ denote the complex Lie algebra of all linear transformations on V together with the bracket operation $[x, y] = x \circ y - y \circ x$ for all $x, y \in \mathfrak{gl}(V)$ where $\circ$ means composition. Fixing an ordered basis for V, we view $\mathfrak{gl}(V)$ as the complex Lie algebra $\mathfrak{gl}_n$ of square matrices of order n with the bracket operation $[x, y] = xy - yx$ for all $x, y \in \mathfrak{gl}_n$. We say that V is a $\mathfrak{g}$-*module* if there exists a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. In this case, the element $x \in \mathfrak{g}$ acts on V as the image $\pi(x)$. We say that a $\mathfrak{g}$-module V is *irreducible* if V contains no other $\mathfrak{g}$-modules aside from 0 and V. Two spaces V and V' are *isomorphic* $\mathfrak{g}$-*modules* if there exists a vector space isomorphism $\rho : V \to V'$ such that x has the same action on v and $\rho(v)$ for all $x \in \mathfrak{g}$ and for all $v \in V$.

Let

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and consider the subalgebra $\mathfrak{so}_4 = \{ P \in \mathfrak{gl}_4 : PX + XP^t = 0 \}$ of $\mathfrak{gl}_4$. Note that $\mathfrak{so}_4$ is a six-dimensional complex Lie algebra with a basis consisting of

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that the Lie bracket relations of basis matrices of $\mathfrak{so}_4$ coincide with the Lie bracket relations in Lemma 3.9. In this section, we focus on the special orthogonal complex Lie algebra $\mathfrak{so}_4$ which is semi-simple and is one of the classical Lie algebras. We recall the representation theory of finite-dimensional $\mathfrak{so}_4$-modules based on a theorem of highest weight. This states that every irreducible $\mathfrak{so}_4$-module has a highest weight and two irreducible $\mathfrak{so}_4$-modules with the same highest weight are isomorphic.

The discussion on the highest weight theory for classical Lie algebras are discussed in many reference books (e.g., [9] and [11]).

For the rest of the section, suppose V is a finite-dimensional vector space over $\mathbb{C}$ and $\pi : \mathfrak{so}_4 \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism (i.e., V is an $\mathfrak{so}_4$-module).

Definition 4.1 An ordered pair $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ is called a *weight* on V if there exists a nonzero vector $v \in V$ such that

$$\pi(H_1)v = \lambda_1 v,$$
$$\pi(H_2)v = \lambda_2 v.$$

In this case, we call v a *weight vector corresponding to the weight* λ. If λ is a weight, then the set of all corresponding weight vectors forms a subspace of V is called the *weight space corresponding to the weight* λ. The *multiplicity* of the weight is the dimension of the corresponding weight space.

Proposition 4.2 *If V is a nonzero $\mathfrak{so}_4$-module, then V has at least one weight.*

Proof Since $\mathbb{C}$ is algebraically closed, $\pi(H_1)$ has at least one eigenvalue $\lambda_1 \in \mathbb{C}$. Let $W \subseteq V$ be the eigenspace for $\pi(H_1)$ with eigenvalue λ_1. Since $[H_1, H_2] = 0$, $\pi(H_1)$ commutes with $\pi(H_2)$ and so W is $\pi(H_2)$-invariant. The restriction of $\pi(H_2)$ to W must have at least one eigenvector w with eigenvalue $\lambda_2 \in \mathbb{C}$. Therefore, w is a simultaneous eigenvector for $\pi(H_1)$ and $\pi(H_2)$ with eigenvalues λ_1 and λ_2. □

Definition 4.3 An ordered pair $\alpha = (a_1, a_2) \in \mathbb{C}^2$ is called a *root* if $(a_1, a_2) \neq (0, 0)$ and there exists a $Z_\alpha \in \mathfrak{so}_4$ such that

$$[H_1, Z_\alpha] = a_1 Z_\alpha,$$
$$[H_2, Z_\alpha] = a_2 Z_\alpha.$$

The element Z_α is called a *root vector corresponding to the root* α.

Lemma 4.4 *Let $\alpha = (a_1, a_2)$ denote a root with corresponding root vector Z_α. Suppose $\lambda = (\lambda_1, \lambda_2)$ is a weight on V with corresponding weight vector v. Then, we have*

$$\pi(H_1)\pi(Z_\alpha)v = (\lambda_1 + a_1)\pi(Z_\alpha)v, \tag{9}$$
$$\pi(H_2)\pi(Z_\alpha)v = (\lambda_2 + a_2)\pi(Z_\alpha)v. \tag{10}$$

Hence, either $\pi(Z_\alpha)v = 0$ or $\pi(Z_\alpha)v$ is a weight vector corresponding to the weight $\lambda + \alpha = (\lambda_1 + a_1, \lambda_2 + a_2)$.

Proof Since π is a Lie algebra homomorphism, we have

$$\pi(H_1)\pi(Z_\alpha)v = [\pi(H_1), \pi(Z_\alpha)]v + \pi(Z_\alpha)\pi(H_1)v$$
$$= \pi([H_1, Z_\alpha])v + \pi(Z_\alpha)\pi(H_1)v$$
$$= a_1\pi(Z_\alpha)v + \lambda_1\pi(Z_\alpha)v$$

$$= (\lambda_1 + a_1)\pi(Z_\alpha)v.$$

This proves (9). We prove (10) analogously. If $\pi(Z_\alpha)v \neq 0$, then $\pi(Z_\alpha)v$ is a weight vector corresponding to the weight $\lambda + \alpha$. □

Note that the set of all roots is $R = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. Let E be the vector space $\mathbb{R}^2$ with standard inner product $\langle \ , \ \rangle_E$. It can be checked that (E, R) forms a *root system* (see [9, Definition 8.1]) which is denoted by $A_1 \times A_1$. We shall fix the *base* $\Delta = \{(1, 1), (1, -1)\}$ for E. Observe that each root is expressed as a linear combination of the vectors in Δ with either all nonnegative coefficients or all nonpositive coefficients. We say that a root is *positive* (resp. *negative*) if these coefficients are all nonnegative (resp. nonpositive). The table below summarizes the roots and corresponding root vectors.

Root	Root vector	Type
$(1, 1)$	X_1	Positive
$(1, -1)$	X_2	Positive
$(-1, -1)$	Y_1	Negative
$(-1, 1)$	Y_2	Negative

Definition 4.5 Let λ and λ' denote weights. We say λ *is higher than* λ' (with respect to the base Δ) and we write $\lambda \succeq \lambda'$ if there exists nonnegative real numbers c_1 and c_2 such that

$$\lambda - \lambda' = c_1(1, 1) + c_2(1, -1).$$

A weight λ on V is said to be a *highest weight* if $\lambda \succeq \lambda'$ for all weights λ' on V.

Note that the relation $\succeq$ depends on the chosen base Δ and it forms a partial order on the set of all weights on V.

Proposition 4.6 *If V is an irreducible $\mathfrak{so}_4$-module, then V is a direct sum of its weight spaces.*

Proof Let V' denote the sum of the weight spaces of V. By Proposition 4.2, $V' \neq 0$. Since V' is the sum of all weight spaces, we may view V' as the span of all simultaneous eigenvectors of $\pi(H_1)$ and $\pi(H_2)$. It follows that V' is invariant under the actions of H_1 and H_2. Now, take a root α with corresponding root vector Z_α. By Lemma 4.4, we see that V' is invariant under $\pi(Z_\alpha)$. By irreducibility of V, we have $V = V'$. The sum is direct since weight vectors with different weights are linearly independent. □

Proposition 4.7 *If $\lambda = (\lambda_1, \lambda_2)$ is a weight on V, then $\lambda_1 + \lambda_2$ and $\lambda_1 - \lambda_2$ are integers. Hence, either λ_1 and λ_2 are both integers or both half-integers.*

Proof Since (λ_1, λ_2) is a weight on V, there exists a simultaneous eigenvector v such that $\pi(H_1)v = \lambda_1 v$ and $\pi(H_2)v = \lambda_2 v$. By applying $\pi(X_1)$ repeatedly on v, we have

$$\pi(H_1)\pi(X_1)^k v = (\lambda_1 + k)\pi(X_1)^k v,$$

$$\pi(H_2)\pi(X_1)^k v = (\lambda_2 + k)\pi(X_1)^k v.$$

for all integer $k \geq 0$. Since V is finite-dimensional, there is a nonnegative integer N such that $\pi(X_1)^N v \neq 0$ but $\pi(X_1)^{N+1} v = 0$. By repeatedly applying $\pi(Y_1)$ on $\pi(X_1)^N v$, we have

$$\pi(H_1)\pi(Y_1)^\ell \pi(X_1)^N v = (\lambda_1 + N - \ell)\pi(Y_1)^\ell \pi(X_1)^N v,$$
$$\pi(H_2)\pi(Y_1)^\ell \pi(X_1)^N v = (\lambda_2 + N - \ell)\pi(Y_1)^\ell \pi(X_1)^N v.$$

for all integer $\ell \geq 0$. Since V is finite-dimensional, there is a nonnegative integer M such that $\pi(Y_1)^M \pi(X_1)^N v \neq 0$ but $\pi(Y_1)^{M+1}\pi(X_1)^N v = 0$. So, we have

$$
\begin{aligned}
0 &= \pi(X_1)\pi(Y_1)^{M+1}\pi(X_1)^N v \\
&= [\pi(X_1), \pi(Y_1)]\pi(Y_1)^M \pi(X_1)^N v + \pi(Y_1)[\pi(X_1), \pi(Y_1)]\pi(Y_1)^{M-1}\pi(X_1)^N v \\
&\quad + \cdots + \pi(Y_1)^{M-1}[\pi(X_1), \pi(Y_1)]\pi(Y_1)\pi(X_1)^N v + \pi(Y_1)^M[\pi(X_1), \pi(Y_1)]\pi(X_1)^N v \\
&= \pi(H_1 + H_2)\pi(Y_1)^M \pi(X_1)^N v + \pi(Y_1)\pi(H_1 + H_2)\pi(Y_1)^{M-1}\pi(X_1)^N v \\
&\quad + \cdots + \pi(Y_1)^{M-1}\pi(H_1 + H_2)\pi(Y_1)\pi(X_1)^N v + \pi(Y_1)^M \pi(H_1 + H_2)\pi(X_1)^N v \\
&= \left(\sum_{i=N-M}^{N} (\lambda_1 + \lambda_2 + 2i)\right)\pi(Y_1)^M \pi(X_1)^N v.
\end{aligned}
$$

Since $\pi(Y_1)^M \pi(X_1)^N v \neq 0$, $\left(\sum_{i=N-M}^{N}(\lambda_1 + \lambda_2 + 2i)\right) = 0$. Thus, $\lambda_1 + \lambda_2 = M - 2N$. Similarly, there exist nonnegative integers P and Q such that $\lambda_1 - \lambda_2 = Q - 2P$. To prove this, replace $\pi(X_1)$ by $\pi(X_2)$ and replace $\pi(Y_1)$ by $\pi(Y_2)$ above. □

Proposition 4.8 *If V is an irreducible $\mathfrak{so}_4$-module, then V has a unique highest weight.*

Proof By Proposition 4.6, V is a direct sum of its weight spaces. Since $\dim(V)$ is finite, there are only a finite number of weights on V. By Lemma 4.4 and since there are only finitely many weights on V, there exists a weight λ with weight vector v such that $\pi(Z_\alpha)v = 0$ for each root $\alpha \in \Delta$ with corresponding root vector Z_α. Let V' denote the smallest $\mathfrak{so}_4$-invariant subspace of V that contains v. Then, V' is the span of vectors of the form

$$\pi(Z_{-\alpha_1})\pi(Z_{-\alpha_2})\cdots\pi(Z_{-\alpha_k})v \qquad (11)$$

where $\alpha_1, \ldots, \alpha_k \in \Delta$. If the vector (11) is nonzero, then it is a weight vector corresponding to the weight $\lambda - \alpha_1 - \cdots - \alpha_k$ by Lemma 4.4. Observe that $\lambda \succeq \lambda - \alpha_1 - \cdots - \alpha_k$ and so λ is higher than all other weights on V'. Thus, V' is an $\mathfrak{so}_4$-module with a unique highest weight λ. Observe that $V = V'$ by irreducibility of V. □

We end the section with a highest weight theorem for irreducible $\mathfrak{so}_4$-modules.

Theorem 4.9 *Two irreducible $\mathfrak{so}_4$-modules have the same highest weight if and only if they are isomorphic as $\mathfrak{so}_4$-modules.*

5 An action of $\mathfrak{so}_4$ on the standard module for Doob graphs

In this section, we establish a Lie algebra homomorphism $\pi : \mathfrak{so}_4 \to Q$ and show that Q is generated by the center and $\pi(\mathfrak{so}_4)$. In addition, we prove a necessary and sufficient condition for irreducible T-modules to be isomorphic irrducible Q-modules.

Theorem 5.1 *With reference to Assumption 3.1, there exists a Lie algebra homomorphism $\pi : \mathfrak{so}_4 \to Q$ such that*

the element of $\mathfrak{so}_4$	acts as
H_1	$\frac{1}{6}[L, R]$
H_2	$\frac{1}{12}[R, [F, L]]$
X_1	$\frac{1}{12}(2L + [L, F])$
X_2	$\frac{1}{4}(2L - [L, F])$
Y_1	$\frac{1}{4}(2R - [R, F])$
Y_2	$\frac{1}{12}(2R + [R, F])$

$$(12)$$

on the standard module V.

Proof This follows from the fact that the Lie bracket relations of the basis matrices of $\mathfrak{so}_4$ coincide with the Lie bracket relations in Lemma 3.9. $\qquad\square$

Corollary 5.2 *With reference to Assumption 3.1, let π denote the Lie algebra homomorphism in Theorem 5.1. Then, Q is generated by $\pi(\mathfrak{so}_4)$ and $F + \frac{1}{6}[R, [F, L]]$. Consequently, Q is generated by the homomorphic image of the universal enveloping algebra $U(\mathfrak{so}_4)$ and the center.*

Proof Let Q' be the subalgebra of Q generated by $\pi(\mathfrak{so}_4)$ and the matrix $F + \frac{1}{6}[R, [F, L]]$. By (12), the matrices L, F, and R are in Q'. Since Q is generated by L, F, and R, we have $Q' = Q$. $\qquad\square$

Lemma 5.3 *With reference to Assumption 3.1, let W denote an irreducible T-module on the standard module V. Then, W is an irreducible Q-module.*

Proof See [24, Proposition 6.3]. $\qquad\square$

Lemma 5.4 *With reference to Assumption 3.1, let π denote the Lie algebra homomorphism in Theorem 5.1 and let W be an irreducible Q-module on V. If $F + \frac{1}{6}[R, [F, L]]$ acts a scalar on W, then W is an irreducible $\pi(\mathfrak{so}_4)$-module.*

Proof Let W denote an irreducible Q-module. By (12), W is a $\pi(\mathfrak{so}_4)$-module since W is invariant under the actions of L, F, and R. Suppose W is not an irreducible $\pi(\mathfrak{so}_4)$-module. Since $\mathfrak{so}_4$ is semi-simple, there exists a nonzero $\pi(\mathfrak{so}_4)$-module W' that is properly contained in W. Observe that W' is invariant under the actions of $\pi(\mathfrak{so}_4)$ and $F + \frac{1}{6}[R, [F, L]]$. By Corollary 5.2, it follows that W' is a Q-module. Since W is an irreducible Q-module, $W = W'$. Hence, W is an irreducible $\pi(\mathfrak{so}_4)$-module. $\qquad\square$

Theorem 5.5 *With reference to Assumption 3.1, let π denote the Lie algebra homomorphism in Theorem 5.1. Let W denote a subspace of V and suppose $F + \frac{1}{6}[R, [F, L]]$ acts as a scalar on W. Then, the following are equivalent:*

  i) *W is an irreducible T-module,*
 ii) *W is an irreducible Q-module,*
iii) *W is an irreducible $\pi(\mathfrak{so}_4)$-module.*

Proof Assume W is an irreducible $\pi(\mathfrak{so}_4)$-module. We show that W is an irreducible T-module. To do this, we construct a basis $\{w_{ij}\}$ of W and show that this basis satisfies the conditions in Proposition 2.1. Since W is finite-dimensional, there exists a scalar $v := \min\{j : E_j^* W \neq 0\}$. By (5) and (12), $E_v^* W = E_v^* V \cap W$ is invariant under the actions of $\pi(H_1)$ and $\pi(H_2)$. Since $\mathbb{C}$ is algebraically closed, there exists a weight vector $w \in E_v^* W$ such that

$$\pi(H_1)\, w = \lambda_1 w, \tag{13}$$
$$\pi(H_2)\, w = \lambda_2 w, \tag{14}$$

where $\lambda_1, \lambda_2 \in \frac{1}{2}\mathbb{Z}$. By (5) and (12) and since $E_{v-1}^* V \cap W = E_{v-1}^* W = 0$, we have

$$\pi(X_1)\, w = 0, \tag{15}$$
$$\pi(X_2)\, w = 0. \tag{16}$$

Define the vectors

$$w_{ij} = \frac{\pi(Y_1)^i\, \pi(Y_2)^j\, w}{i!\, j!} \tag{17}$$

for all integers $i, j \geq 0$. Observe that $w_{ij} \in E_{v+i+j}^* W$. For convenience, define $w_{\ell k} = 0$ if $\ell < 0$ or $k < 0$. Since $[Y_1, Y_2] = 0$, $\pi(Y_1)$ and $\pi(Y_2)$ commute and one easily verifies that

$$\pi(Y_1)\, w_{ij} = (i+1)\, w_{i+1,j}, \tag{18}$$
$$\pi(Y_2)\, w_{ij} = (j+1)\, w_{i,j+1}, \tag{19}$$

for all integers $i, j \geq 0$. Now, we claim

$$\pi(H_1)\, w_{ij} = (\lambda_1 - i - j)\, w_{ij}, \tag{20}$$
$$\pi(H_2)\, w_{ij} = (\lambda_2 - i + j)\, w_{ij}, \tag{21}$$

for all integers $i, j \geq 0$. We prove this by induction on $i + j$. By (13), (14), and (17), the claim holds for $i + j = 0$. Assume the claim holds for $i + j = \ell + k - 1$ for some integers $\ell, k \geq 0$ such that $\ell + k \geq 1$. By (18) and (19) and by induction hypothesis, we have

$$\pi(H_1)\, w_{\ell k} = \frac{1}{\ell}\, \pi(H_1) \pi(Y_1)\, w_{\ell-1,k}$$

$$
\begin{aligned}
&= \frac{1}{\ell} \left[\pi\left(H_1\right), \pi\left(Y_1\right) \right] w_{\ell-1,k} + \frac{1}{\ell} \pi\left(Y_1\right) \pi\left(H_1\right) w_{\ell-1,k} \\
&= -\frac{1}{\ell} \pi\left(Y_1\right) w_{\ell-1,k} + \frac{1}{\ell} \left(\lambda_1 - \ell + 1 - k\right) \pi\left(Y_1\right) w_{\ell-1,k} \\
&= \left(\lambda_1 - \ell - k\right) w_{\ell k}
\end{aligned}
$$

and

$$
\begin{aligned}
\pi\left(H_2\right) w_{\ell k} &= \frac{1}{k} \pi\left(H_2\right) \pi\left(Y_2\right) w_{\ell,k-1} \\
&= \frac{1}{k} \left[\pi\left(H_2\right), \pi\left(Y_2\right) \right] w_{\ell,k-1} + \frac{1}{k} \pi\left(Y_2\right) \pi\left(H_2\right) w_{\ell,k-1} \\
&= \frac{1}{k} \pi\left(Y_2\right) w_{\ell,k-1} + \frac{1}{k} \left(\lambda_2 - \ell + k - 1\right) \pi\left(Y_2\right) w_{\ell,k-1} \\
&= \left(\lambda_2 - \ell + k\right) w_{\ell k}.
\end{aligned}
$$

Thus, (20) and (21) hold when $i + j = \ell + k$. Similarly, we have

$$
\pi\left(X_1\right) w_{ij} = \left(\lambda_1 + \lambda_2 - i + 1\right) w_{i-1,j}, \tag{22}
$$

$$
\pi\left(X_2\right) w_{ij} = \left(\lambda_1 - \lambda_2 - j + 1\right) w_{i,j-1}, \tag{23}
$$

for all integers $i, j \geq 0$ by induction on $i + j$. Now, we find a basis for W. Since W is finite-dimensional, we may define the scalars

$$
\begin{aligned}
d &= \max\{i : w_{i0} \neq 0\}, \\
p &= \max\{j : w_{0j} \neq 0\}.
\end{aligned}
$$

We claim that $w_{ij} \neq 0$ for integers i, j ($0 \leq i \leq d$, $0 \leq j \leq p$). By (22) and (23),

$$
\begin{aligned}
0 &= w_{d+1,0} = \pi\left(X_1\right) w_{d+1,0} = \left(\lambda_1 + \lambda_2 - d\right) w_{d,0} \\
0 &= w_{0,p+1} = \pi\left(X_2\right) w_{0,p+1} = \left(\lambda_1 - \lambda_2 - p\right) w_{0,p}.
\end{aligned}
$$

Since $w_{d,0} \neq 0$ and $w_{0,p} \neq 0$, it follows that

$$
d = \lambda_1 + \lambda_2 \quad \text{and} \quad p = \lambda_1 - \lambda_2. \tag{24}
$$

Consider integers $\ell, k \geq 0$ such that $w_{\ell,k} \neq 0$ but $w_{\ell+1,k} = 0$ and $w_{\ell,k+1} = 0$. By (22) and (23),

$$
\begin{aligned}
0 &= w_{\ell+1,k} = \pi\left(X_1\right) w_{\ell+1,k} = \left(\lambda_1 + \lambda_2 - \ell\right) w_{\ell,k}, \\
0 &= w_{\ell,k+1} = \pi\left(X_2\right) w_{\ell,k+1} = \left(\lambda_1 - \lambda_2 - k\right) w_{\ell,k}.
\end{aligned}
$$

Since $w_{\ell,k} \neq 0$, we have $\ell = d$ and $k = p$. This proves the claim. Now, let

$$
W' = \operatorname{span}\{w_{ij} : 0 \leq i \leq d, 0 \leq j \leq p\}.
$$

Note that W' is $\pi(\mathfrak{so}_4)$-invariant by (18)–(23). Since W is an irreducible $\pi(\mathfrak{so}_4)$-module, $W = W'$. By (20) and (21) and since $w_{ij} \neq 0$, w_{ij} is a weight vector corresponding to the weight $(\lambda_1 - i - j, \lambda_2 - i + j)$. Hence, w_{ij}'s belong to different weight spaces and are linearly independent. This proves $\{w_{ij}\}$ is a basis for W. Finally, we show the action of A on w_{ij}. Since $F + \frac{1}{6}[R, [F, L]]$ acts as a scalar on W, we may define

$$\left(F + \frac{1}{6}[R, [F, L]]\right) w_{ij} = (t + d - p)\, w_{ij} \tag{25}$$

for all integers i, j ($0 \leq i \leq d, 0 \leq j \leq p$). By (12), (21), and (24), we have

$$[R, [F, L]]\, w_{ij} = 6(d - p - 2i + 2j) w_{ij} \tag{26}$$

for all integers i, j ($0 \leq i \leq d, 0 \leq j \leq p$). By (25)–(26), we have

$$F w_{ij} = (t + 2\,(i - j))\, w_{ij}. \tag{27}$$

By (12), (22)–(23), and (18)–(19), we have

$$L w_{ij} = 3\,(d - i + 1)\, w_{i-1,j} + (p - j + 1)\, w_{i,j-1}, \tag{28}$$
$$R w_{ij} = 3\,(j + 1)\, w_{i,j+1} + (i + 1)\, w_{i+1,j}. \tag{29}$$

for all integers i, j ($0 \leq i \leq d, 0 \leq j \leq p$). We obtain

$$A w_{ij} = 3\,(d - i + 1)\, w_{i-1,j} + (p - j + 1)\, w_{i,j-1} + (t + 2\,(i - j))\, w_{ij}$$
$$+ 3\,(j + 1)\, w_{i,j+1} + (i + 1)\, w_{i+1,j}$$

by (3) and (27)–(29). Note that $\sum w_{ij}$ is an eigenvector for A with eigenvalue $3d + p + t$. Since A has integral eigenvalues, t is an integer. Thus, W is an irreducible T-module by Proposition 2.1. The remaining assertions of the theorem follow from Lemma 5.3 and Lemma 5.4. $\qquad\square$

Corollary 5.6 *With reference to Assumption 3.1, let π denote the Lie algebra homomorphism in Theorem 5.1. If $W = W(n, m; v, d, p, t)$ is an irreducible T-module, then W is an irreducible $\pi(\mathfrak{so}_4)$-module with highest weight $\left(\frac{1}{2}(d + p), \frac{1}{2}(d - p)\right)$.*

Proof Let $\{w_{ij}\}$ denote the basis of W described in Proposition 2.1. By (12) and Lemma 3.3, we see that

$$\left\{\left(\frac{d + p}{2} - i - j, \frac{d - p}{2} - i + j\right) : 0 \leq i \leq d,\ 0 \leq j \leq p\right\}$$

is the set of all weights on W. Observe that $\left(\frac{1}{2}(d + p), \frac{1}{2}(d - p)\right)$ is the highest weight in the set. $\qquad\square$

Corollary 5.7 *With reference to Assumption 3.1, let π denote the Lie algebra homomorphism in Theorem 5.1. Let $W = W(n, m; v, d, p, t)$ and $W' = W(n, m; v', d', p', t')$ denote irreducible T-modules. Then, W and W' are isomorphic Q-modules if and only if $(d, p, t) = (d', p', t')$.*

Proof Let W and W' denote irreducible T-modules. By Lemma 3.7, $F + \frac{1}{6}[R, [F, L]]$ acts on W (resp. W') as the scalar $t - d + p$ (resp. $t' - d' + p'$). Moreover, W and W' are irreducible Q-modules and irreducible $\pi(\mathfrak{so}_4)$-modules by Theorem 5.5. By Corollary 5.6, the highest weight of W (resp. W') is $\left(\frac{1}{2}(d + p), \frac{1}{2}(d - p)\right)$ (resp. $\left(\frac{1}{2}(d' + p'), \frac{1}{2}(d' - p')\right)$).

Note that $(d, p, t) = (d', p', t')$ if and only if the spaces W and W' are isomorphic $\pi(\mathfrak{so}_4)$-modules and $F + \frac{1}{6}[R, [F, L]]$ has the same action on them. Since Q is generated by $\pi(\mathfrak{so}_4)$ and $F + \frac{1}{6}[R, [F, L]]$ by Corollary 5.2, statement holds. $\square$

Remark 5.8 In [24, Theorem 9.1], Terwilliger and Žitnik gave equivalent conditions for $T \neq Q$ that work for general distance-regular graphs. Among these conditions is the existence of a pair of quasi-isomorphic irreducible T-modules with unequal endpoints. In Doob graph $D(n, m)$, one can prove that $W(n, m; v, d, p, t)$ and $W(n, m; v', d', p', t')$ is a pair of quasi-isomorphic irreducible T-modules with unequal endpoints if and only if $v \neq v'$ and $(d, p, t) = (d', p', t')$. This establishes Corollary 5.7 from the context of quasi-isomorphism.

Remark 5.9 The complex Lie algebra $\mathfrak{sl}_2$ is a simple Lie algebra with basis $\{e, f, h\}$ satisfying the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \text{ and } [e, f] = h.$$

Let $\mathfrak{h}_1$ (resp. $\mathfrak{h}_2$) denote the subalgebra of $\mathfrak{so}_4$ spanned by X_1, Y_1, and $H_1 + H_2$ (resp. X_2, Y_2, and $H_1 - H_2$). One checks that $\mathfrak{sl}_2$, $\mathfrak{h}_1$, and $\mathfrak{h}_2$ are pairwise isomorphic and in particular, $\mathfrak{so}_4$ is a direct sum of $\mathfrak{h}_1$ and $\mathfrak{h}_2$. With reference to Assumption 3.1, let π denote the homomorphism in Theorem 5.1. Let $W = W(n, m; v, d, p, t)$ denote an irreducible T-module with basis $\{w_{ij}\}$ as described in Proposition 2.1. Now, consider the spaces

$$\text{span}\left\{w_{ij} \mid 0 \le i \le d\right\} \text{ for a fixed } j \ (0 \le j \le p), \tag{30}$$
$$\text{span}\left\{w_{ij} \mid 0 \le j \le p\right\} \text{ for a fixed } i \ (0 \le i \le d). \tag{31}$$

By (18), (20)–(21), and (22), the spaces (30) are irreducible $\pi(\mathfrak{h}_1)$-modules. On the other hand, the spaces (31) are irreducible $\pi(\mathfrak{h}_2)$-modules by (19), (20)–(21), and (23). Therefore, the spaces above are irreducible $\mathfrak{sl}_2$-modules.

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