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# Embedding the Picard group inside the class group: the case of $\mathbb{Q}$-factorial complete toric varieties

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## Abstract

Let $X$ be a $\mathbb{Q}$-factorial complete toric variety over an algebraic closed field of characteristic 0. There is a canonical injection of the Picard group $\mathrm{Pic}(X)$ in the group $\mathrm{Cl}(X)$ of classes of Weil divisors. These two groups are finitely generated abelian groups; while the first one is a free group, the second one may have torsion. We investigate algebraic and geometrical conditions under which the image of $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$ is contained in a free part of the latter group.


**Keywords** $\mathbb{Q}$-factorial complete toric varieties · Cartier and Weil divisors · Pure modules · Free and torsion subgroups · Localization · Completion of fans

**Mathematics Subject Classification** 14M25 · 20K15 · 20K10

## 1 Introduction

Let $X$ be an irreducible and normal algebraic variety over an algebraic closed field $k$ of characteristic 0. Then, the group $H^0(X, \mathcal{K}^*/\mathcal{O}^*)$ of Cartier divisors of $X$ can be represented as the subgroup of locally principal divisors of the group $\mathrm{Div}(X)$ of Weil divisors [5, Rem. II.6.11.2]. Quotienting both these groups by their subgroup of principal divisors one realizes the group $\mathrm{CaCl}(X)$ of classes of Cartier divisors as a subgroup of the group $\mathrm{Cl}(X)$ of classes of Weil divisors. In addition it turns out a canonical isomorphism between $\mathrm{CaCl}(X)$ and the Picard group $\mathrm{Pic}(X)$ of classes

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of isomorphic line bundles on $X$ [5, Propositions II.6.13,15], so giving a canonical injection

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)\ . \tag{1}$$

For this reason, in this paper we will not distinguish between linear equivalence classes of Cartier divisors and isomorphism classes of line bundles, so identifying $\mathrm{CaCl}(X) = \mathrm{Pic}(X)$.

Assume now that $\mathrm{Cl}(X)$ is finitely generated. It is well known that a finitely generated abelian group decomposes (noncanonically) in a direct sum of a free part and its torsion subgroup. Both $\mathrm{Pic}(X)$ and $\mathrm{Cl}(X)$ may have nontrivial torsion, and clearly, (1) induces an injection $\mathrm{Tors}(\mathrm{Pic}(X)) \hookrightarrow \mathrm{Tors}(\mathrm{Cl}(X))$. Then, the following natural question arises:

(*) *under which conditions on $X$ there exist free parts $F_C$ and $F_W$ of* $\mathrm{Pic}(X)$ *and* $\mathrm{Cl}(X)$, *respectively* (*see Definition* 1), *such that the injection* (1) *induces an injection*

$$F_C \hookrightarrow F_W\ ? \tag{2}$$

One should expect some geometric condition on $X$ answering to question (*), but we could not find anything, in the current literature. Motivated by algebraic considerations (see Proposition 2) we call *pure* a normal, irreducible algebraic variety $X$ such that $\mathrm{Cl}(X)$ is finitely generated and there exist free parts $F_C$ and $F_W$ positively answering problem (*) (see Definition 3). On the contrary if for each choice of free parts $F_C$ and $F_W$ the injection (1) does not induce any injection (2), then $X$ is called *impure*. Obvious examples of pure varieties are given by those varieties $X$ whose class group $\mathrm{Cl}(X)$ is finitely generated and free, and by smooth varieties admitting a finitely generated class group. Examples of impure varieties are in general more involved: Some of them are given in Sect. 3.3.

In the present paper we will consider the easier case of a $\mathbb{Q}$-factorial complete toric variety $X$, essentially for three reasons:

(a) $\mathrm{Cl}(X)$ is a finitely generated abelian group (see, e.g., [2, Thm. 4.1.3])
(b) $\mathrm{Pic}(X)$ is free, i.e., $\mathrm{Tors}(\mathrm{Pic}(X)) = 0$ (see, e.g., [2, Prop. 4.2.5])
(c) locally principal divisors can be easily described by means of principal divisors on affine open subsets of $X(\Sigma)$ associated with maximal cones of the fan $\Sigma$.

Conditions (a) and (b) translate problem (*) in the following

(**) *under which conditions on $X$ there exists a free part $F$ of* $\mathrm{Cl}(X)$ *such that* (1) *induces an injection* $\mathrm{Pic}(X) \hookrightarrow F$ ?

The main result of the present paper is a sufficient condition for a $\mathbb{Q}$-factorial complete toric variety to be a pure variety. This is given by Theorem 2 and can be geometrically summarized as follows:

**Theorem 1** *[see Theorem 2 and Remark 2] Let $X(\Sigma)$ be a $\mathbb{Q}$-factorial complete toric variety of dimension $n$. Then, it admits a canonical covering $Y(\widehat{\Sigma}) \twoheadrightarrow X(\Sigma)$, unramified in codimension 1, such that the class group $\mathrm{Cl}(Y)$ is free (this follows by* [8, Thm. 2.2]*). Both $X$ and $Y$ are orbifolds* [2, Thm. 3.1.19 (b)]*; let*

$$\{U_{\widehat{\sigma}}\}_{\widehat{\sigma}\in\widehat{\Sigma}(n)}$$

*be the collection of affine charts given by the maximal cones and covering $Y$. Calling* $\mathrm{mult}(\widehat{\sigma})$ *the maximum order of a quotient singularity in the affine chart $U_{\widehat{\sigma}}$ (i.e., the* multiplicity *of the cone $\widehat{\sigma} \in \widehat{\Sigma}(n)$, see Definition 4), $X$ is a pure variety if $m_\Sigma := \gcd\{\mathrm{mult}(\widehat{\sigma}) \mid \widehat{\sigma} \in \widehat{\Sigma}(n)\}$ is coprime with the order of the Galois group of the covering $Y \to X$.*

This is not a necessary condition: Example 2 gives a counterexample.

Section 3.3 is entirely devoted to give nontrivial examples of pure and impure varieties. A big class of nontrivial examples of pure varieties is exhibited in Sect. 3.4: Namely, it is given by

– *all $\mathbb{Q}$-factorial complete toric varieties whose small $\mathbb{Q}$-factorial modifications* (s$\mathbb{Q}$m) *are actually isomorphisms.*

Here s$\mathbb{Q}$m *of $X$* means a birational map $f : X \dashrightarrow Y$ such that $f$ is an isomorphism in codimension 1 and $Y$ is still a complete $\mathbb{Q}$-factorial toric variety. By the combinatorial point of view the previous geometric property translates in requiring that *there is a unique simplicial and complete fan $\Sigma$ admitting 1-skeleton given by* $\Sigma(1)$. Our proof that those varieties are pure (see Proposition 5) passes through showing that every maximal simplicial cone generated by rays in $\Sigma(1)$ and not containing any further element of $\Sigma(1)$ other than its generators (we call *minimal* such a maximal simplicial cone) is actually a cone of a complete and simplicial fan. This fact produces a completion procedure of fans looking to be of some interest by itself (see Lemma 4 and Sect. 3.4.1), when compared with standard completion procedures [3, Thm. III.2.8], [4,6].

Further results of the present paper are given by:

– algebraic considerations given in Sect. 2; apart from the definition of a pure submodule given in Definition 2 and some consequences appearing in Proposition 2, the rest of this section consists of original considerations, as far as we know;
– a characterization of $\mathrm{Pic}(X)$ as a subgroup of $\mathrm{Cl}(X)$, when $X$ is a pure, $\mathbb{Q}$-factorial, complete, toric variety: In [7, Thm. 2.9.2] we gave a similar characterization in the case of a poly weighted space (PWS: see Notation 3.1), that is, when $\mathrm{Cl}(X)$ is free; a first generalization was given in [9, Thm. 3.2 (3)] which is here improved in Sect. 4 and in particular by Theorem 3;
– an example of a four-dimensional simplicial fan whose completions necessarily require the addition of some new rays (see Example 3): Ewald, in his book [3], already announced the existence of examples of this kind (see the Appendix to Chapter III in [3]), but we were not able to recover it. We then believe that Example 3 may fill up a lack in the literature on these topics.

This paper is structured as follows: The first section gives all needed algebraic ingredients. Section 3 is devoted to state and prove the main result, given by Theorem 2, and produce examples of pure and impure varieties in the toric setup (see Sects. 3.3 and 3.4). Section 4 gives the above-mentioned characterization of $\mathrm{Pic}(X)$ as a subgroup of $\mathrm{Cl}(X)$ when $X$ is a pure, $\mathbb{Q}$-factorial, complete, toric variety.

## 2 Algebraic considerations

Let $A$ be a PID, $\mathcal{M}$ be a finitely generated $A$-module of rank $r$, and $T = \mathrm{Tors}_A(\mathcal{M})$.

**Definition 1** A *free part* of $\mathcal{M}$ is a free submodule $L \subseteq \mathcal{M}$ such that $\mathcal{M} = L \oplus T$.

It is well known that a free part of $\mathcal{M}$ always exists.
Let $H \subseteq \mathcal{M}$ be a free submodule of rank $h$. In general it is not true that $H$ is contained in a free part of $\mathcal{M}$. For example, let $A = \mathbb{Z}$, $\mathcal{M} = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and let $H = \langle(2, 1)\rangle$.

**Proposition 1** *There exist elements $a_1, \ldots, a_h \in A$ such that $a_1|a_2|\ldots|a_h$ satisfying the following property: every free part $L$ of $\mathcal{M}$ has a basis $\mathbf{f}_1, \ldots, \mathbf{f}_r$ such that $a_1\mathbf{f}_1 + t_1, \ldots, a_h\mathbf{f}_h + t_h$ is a basis of $H$ for suitable $t_1, \ldots, t_h \in T$.*

***Proof*** Let $\pi : \mathcal{M} \to \mathcal{M}/T$ the quotient map. The group $\mathcal{M}/T$ is free of rank $r$. The restriction of $\pi$ to $H$ is injective, since $H$ is free and $\ker(\pi) = T$. Therefore, $\pi(H)$ is a subgroup of rank $h$ of the free group $\mathcal{M}/T$. By the elementary divisor theorem, there exist a basis $\tilde{\mathbf{f}}_1, \ldots, \tilde{\mathbf{f}}_r$ and element $a_1, \ldots, a_h \in A$ such that $a_1|a_2|\ldots|a_h$ and $a_1\tilde{\mathbf{f}}_1, \ldots, a_h\tilde{\mathbf{f}}_h$ is a basis of $\pi(H)$. Let $\mathbf{h}_1, \ldots, \mathbf{h}_h$ be the basis of $H$ such that $\pi(\mathbf{h}_i) = a_i\tilde{\mathbf{f}}_i$.
Now let $L$ be a free part of $\mathcal{M}$. The decomposition $\mathcal{M} = L \oplus T$ gives rise to a section $s : \mathcal{M}/T \to L$, i.e., $\pi \circ s = id$. By putting $\mathbf{f}_i = s(\tilde{\mathbf{f}}_i)$ we get a basis $\mathbf{f}_1, \ldots, \mathbf{f}_r$ of $L$ such that $a_1\mathbf{f}_1, \ldots, a_h\mathbf{f}_h$ is a basis of $s(\pi(H))$. For $i = 1, \ldots, h$ put $t_i = \mathbf{h}_i - a_i\mathbf{f}_i$. Then, $t_i \in \ker(\pi) = T$, so that the claim is proved. □

***Remark 1*** When $A = \mathbb{Z}$, the objects whose existence is established by Proposition 1 are effectively computable. In fact, assume that $\mathcal{M} = \mathbb{Z}^r \oplus T$, where $T$ is a finite group, and that $\mathbf{g}_1 + s_1, \ldots, \mathbf{g}_h + s_h$ is a basis of $H$, with $\mathbf{g}_1, \ldots, \mathbf{g}_h \in \mathbb{Z}^r$ and $s_1, \ldots, s_h \in T$. Let $G$ be the $h \times r$ matrix having rows $\mathbf{g}_1, \ldots, \mathbf{g}_h$. It is possible to compute the Smith normal form $S$ of $G$ and matrices $U \in \mathrm{GL}_h(\mathbb{Z})$, $V \in \mathrm{GL}_r(\mathbb{Z})$ such that $UGV = S$. Then, the rows of $V^{-1}$ give the basis $\mathbf{f}_1, \ldots, \mathbf{f}_r$, the diagonal entries of $S$ give $a_1, \ldots, a_h \in \mathbb{Z}$. Moreover, we recover the elements $t_1, \ldots, t_h$ by putting (with the obvious notation)

$$\begin{pmatrix} t_1 \\ \vdots \\ t_h \end{pmatrix} = U \begin{pmatrix} s_1 \\ \vdots \\ s_h \end{pmatrix}.$$

The following definition is standard (see, for example, [10, Ex. B-3.6]):

**Definition 2** Let $\mathcal{M}$ be an $A$-module. A submodule $\mathcal{M}' \subseteq \mathcal{M}$ is said *pure* if the following property is satisfied:

$$\text{if } am \in \mathcal{M}' \text{ for some } a \in A, m \in \mathcal{M}, \text{ then there is } m' \in \mathcal{M}' \text{ such that } am' = am.$$

**Proposition 2** *The following are equivalent:*

(a) *$H$ is contained in a free part of $\mathcal{M}$.*
(b) *The image of $T$ in $\mathcal{M}/H$ is a free summand.*
(c) *The image of $T$ in $\mathcal{M}/H$ is a pure submodule.*
(d) *Let $L$ be a free part of $\mathcal{M}$ and $\mathbf{f}_1, \dots, \mathbf{f}_r$ be a basis of $L$ as in Proposition 1; then, for $i = 1, \dots, h$ the element $t_i$ is divisible by $a_i$ in $T$, that is, there exists $u_i \in T$ such that $t_i = a_i u_i$;*

***Proof*** a) $\Rightarrow$ b): Let $L$ be a free part of $\mathcal{M}$ such that $H \subseteq L$. Then, $\mathcal{M}/H = (L \oplus T)/H \cong (L/H) \oplus T$.
The equivalence of $(b)$ and $(c)$ is a the well-known fact that, for modules finitely generated over a PID, pure submodules and direct summands coincide (see, for example, [10, Ex. B-3.7 (ii)]).
c) $\Rightarrow$ d): Since $a_i\mathbf{f}_i + t_i \in H$, the image of $a_i\mathbf{f}_i$ in $\mathcal{M}/H$ belongs to the image of $T$, for $i = 1, \dots, r$. By purity, there exists $u_i \in T$ such that the images of $a_i\mathbf{f}_i$ and $-a_i u_i$ coincide in $\mathcal{M}/H$, that is $a_i\mathbf{f}_i + a_i u_i$ in $H$. But then $t_i - a_i u_i \in H \cap T = \{0\}$, because $H$ is free.
c) $\Rightarrow$ d): Let $L'$ be the submodule of $\mathcal{M}$ generated by $\mathbf{f}_1 + u_1, \dots, \mathbf{f}_h + u_h, \mathbf{f}_{h+1}, \dots, \mathbf{f}_r$. Then, $L'$ is a free part of $\mathcal{M}$ containing $H$. □

Notice that since $H$ is free, $H \cap T = \{0\}$, so that the image of $T$ in $\mathcal{M}/H$ is isomorphic to $T$.

For every prime element $p$ of $A$, we denote by $A_{(p)}$ the localization of $A$ at the prime ideal $(p)$. If $\mathcal{M}$ is an $A$-module, $\mathcal{M}_{(p)}$ is the localized $A_{(p)}$-module.

The localization of $T$ at $(p)$ coincide with the $p$-torsion of $T$, and $T = \bigoplus_p T_{(p)}$.

If $L$ is a free part of $\mathcal{M}$ and $\mathcal{M} = L \oplus T$ is the corresponding decomposition, then $L_{(p)}$ is a free part of $\mathcal{M}_{(p)}$, that is, there is a decomposition $\mathcal{M}_{(p)} = L_{(p)} \oplus T_{(p)}$. The natural map $\mathcal{M} \to \mathcal{M}_{(p)}$ is the sum of the injection $L \to L_{(p)}$ and the surjection $T \to T_{(p)}$.

**Proposition 3** *$H$ is contained in a free part of $\mathcal{M}$ if and only if $H_{(p)}$ is contained in a free part of $\mathcal{M}_{(p)}$ for every prime element $p \in A$.*

The proof of Proposition 3 is based on the next two lemmas:

**Lemma 1** *Let $A$ be a commutative ring with unity, $\mathcal{M}$ be an $A$-module and $N \subseteq \mathcal{M}$ be a submodule. Then, $N$ is a direct summand of $\mathcal{M}$ if and only if there exists a map $\varphi : \mathcal{M} \to N$ such that $\varphi|_N = id_N$.*

***Proof*** Assume that a map $\varphi : \mathcal{M} \to N$ as in the statement of the Lemma exists, and set $K = \ker(\varphi)$. Then, $K \cap N = \{0\}$, so that the map $\theta : N \oplus K \to \mathcal{M}$ given by the sum of the inclusions is injective. If $m \in \mathcal{M}$, put $n = \varphi(m) = \varphi(n)$; then, $m - n \in K$ and $m = n + (m - n)$, and this shows that $\theta$ is surjective. The converse is obvious. □

**Lemma 2** *Let $A$ be a commutative ring with unity, $\mathcal{M}$ be an $A$-module and $N, K \subseteq \mathcal{M}$ be direct summands of $\mathcal{M}$. Assume that the two ideals $Ann_A(N)$, $Ann_A(K)$ are coprime. Then, $N \oplus K$ is a direct summand of $\mathcal{M}$.*

***Proof*** Firstly notice that by hypothesis there exist $a \in Ann_A(N)$, $b \in Ann_A(K)$ such that $a + b = 1$. This shows that $N \cap K = \{0\}$: If $x \in N \cap K$, then $x = ax + bx = 0$. Write $\mathcal{M} = N \oplus N'$; let $k \in K$ and write $k = k_1 + k_2$, with $k_1 \in N$ and $k_2 \in N'$. Then, from $bk = 0$ we deduce

$$0 = bk_1 = (1 - a)k_1 = k_1,$$

so that $K \subseteq N'$. The composition

$$N' \hookrightarrow K \oplus K' \twoheadrightarrow K$$

is the identity when restricted to $K$ so that $K$ is a direct summand of $N'$. □

*Proof of Proposition* 3. Since localization is an exact functor, we have

$$\mathcal{M}_{(p)}/H_{(p)} \simeq (\mathcal{M}/H)_{(p)} .$$

Let $T_{(p)}$ be the $p$-torsion of $T$; it coincides with the localization at $(p)$ of $T$ and it is a direct summand of $T$. Moreover, the natural maps from $T_{(p)}$ in $\mathcal{M}/H$ and in $\mathcal{M}_{(p)}/H_{(p)}$ are injective, so that we can regard $T_{(p)}$ as a submodule of $\mathcal{M}/H$ and of $\mathcal{M}_{(p)}/H_{(p)}$.

Now assume that $H$ is contained in a free part of $\mathcal{M}$. Then, $T$ is a direct summand of $\mathcal{M}/H$, by Proposition 2, so that there is a map $\mathcal{M}/H \to T$ which is the identity over $T$. Let $p$ be a prime element in $A$; by localizing at $(p)$ we find a map $\mathcal{M}_{(p)}/H_{(p)} \to T_{(p)}$ which is the identity over $T_{(p)}$, so that $T_{(p)}$ results to be a direct summand of $\mathcal{M}_{(p)}/H_{(p)}$; therefore, $H_{(p)}$ is contained in a free part of $\mathcal{M}_{(p)}$.

Conversely, assume that $H_{(p)}$ is contained in a free part of $\mathcal{M}_{(p)}$, for every prime element $p$ of $A$. Then, $T_{(p)}$ is a direct summand of $\mathcal{M}_{(p)}/H_{(p)}$, so that there is a map $(\mathcal{M}/H)_{(p)} \simeq M_{(p)}/H_{(p)} \to T_{(p)}$ which is the identity when restricted to $T_{(p)}$. If we compose with the natural map $\mathcal{M}/H \to (\mathcal{M}/H)_{(p)}$ we get a map $\mathcal{M}/H \to T_{(p)}$ which is the identity over $T_{(p)}$. Then, $T_{(p)}$ is a direct summand of $\mathcal{M}/H$ for every prime $p$. By Lemma 2, $T = \prod_p T_{(p)}$ is a direct summand of $\mathcal{M}/H$, so that $H$ is contained in a free part of $\mathcal{M}$, again by Proposition 2. (Notice that the product above is in fact a finite product since by hypothesis $\mathcal{M}$ and hence $T$, are finitely generated modules.)

## 3 Application to toric varieties

As already mentioned in the Introduction, we put the following

**Definition 3** Let $X$ be an irreducible and normal algebraic variety such that $\mathrm{Cl}(X)$ is finitely generated. Then, $X$ is called *pure* if there exist free parts $F_C$ and $F_W$ of $\mathrm{Pic}(X)$ and $\mathrm{Cl}(X)$, respectively, such that the canonical injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$ descends to give the following commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(X) & \hookrightarrow & \mathrm{Cl}(X) \\ \uparrow & & \uparrow \\ F_C & \hookrightarrow & F_W \end{array}$$

In particular, if $X = X(\Sigma)$ is a $n$-dimensional toric variety whose $n$-skeleton $\Sigma(n)$ is not empty (see the following Sect. 3.1 for notation), then $\mathrm{Pic}(X)$ is free (see, e.g., [2, Prop. 4.2.5]), meaning that $X$ is pure if and only if $\mathrm{Pic}(X)$ is contained in a free part of $\mathrm{Cl}(X)$.
If $X$ is not pure, it is called *impure*.

Of course, if $\mathrm{Cl}(X)$ is free, then $X$ is pure; moreover if $X$ is smooth and $\mathrm{Cl}(X)$ is finitely generated, then it is pure, because the injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$ is an isomorphism.
Conversely, producing examples of impure varieties is definitely more complicated (see Example 1).

### 3.1 Notation on toric varieties

Let $X = X(\Sigma)$ be a $n$-dimensional toric variety associated with a fan $\Sigma$. Calling $\mathbb{T} \cong (k^*)^n$ the torus acting on $X$, we use the standard notation $M$, for the characters group of $\mathbb{T}$, and $N := \mathrm{Hom}(M, \mathbb{Z})$. Then, $\Sigma$ is a collection of cones in $N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^n$. $\Sigma(i)$ denotes the $i$-skeleton of $\Sigma$, that is, the collection of $i$-dimensional cones in the fan $\Sigma$. We shall use the notation $\tau \preceq \sigma$ to indicate that the cone $\tau$ is a face of $\sigma$.

Given a toric variety $X(\Sigma)$ we will denote by $\mathcal{W}_T(X) \subseteq \mathrm{Div}(X)$ the subgroup of *torus invariant* Weil divisors and by $\mathcal{C}_T(X) \subseteq \mathcal{W}_T(X)$ the subgroup of Cartier torus invariant divisors. It is well known that

$$\mathcal{W}_T(X) = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} \cdot D_\rho \quad \text{where} \quad D_\rho := \overline{\mathbb{T} \cdot x_\rho}$$

the latter being the closure of the torus orbit of the *distinguished point* $x_\rho$ of the ray $\rho$ [2, § 3.2, § 4.1]. In particular the homomorphism $D \mapsto [D]$, sending a Weil divisor to its linear equivalence class, when restricted to torus invariant divisors still gives a epimorphism $d_X : \mathcal{W}_T(X) \twoheadrightarrow \mathrm{Cl}(X)$ [2, Thm. 4.1.3].

In [7, Def. 2.7] we introduced the notion of a *poly weighted space* (PWS), which is a $\mathbb{Q}$-factorial complete toric variety $Y$ whose class group $\mathrm{Cl}(Y)$ is free. This is equivalent to say that $Y$ is connected in codimension 1 (1-connected); when $k = \mathbb{C}$ this means that the regular locus $Y_{\mathrm{reg}}$ of $Y$ is simply connected, as $Y$ is a normal variety: Recall that $\pi_1(Y_{\mathrm{reg}}) \cong \mathrm{Tors}(\mathrm{Cl}(Y)) = 0$ [8, Cor. 1.8, Thm. 2.1]. As proved in [8, Thm. 2.2], every $\mathbb{Q}$-factorial complete toric variety $X(\Sigma)$ is a finite quotient of a unique PWS $Y(\widehat{\Sigma})$, which is its universal *covering unramified in codimension 1* (1-covering). The Galois group of the torus equivariant covering $Y \twoheadrightarrow X$ is precisely the dual group $\mu(X) = \mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), k^*)$. At lattice level, the equivariant surjection $Y \twoheadrightarrow X$ induces an injective automorphism $\beta : N \hookrightarrow N$ whose $\mathbb{R}$-linear extension

$\beta_{\mathbb{R}} : N_{\mathbb{R}} \hookrightarrow N_{\mathbb{R}}$ identifies the associated fans, that is $\beta_{\mathbb{R}}(\widehat{\Sigma}) = \Sigma$. Recall that one has the following commutative diagram (see diagram (5) in [9])

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & \ker(\overline{\alpha}) = T & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow[V^T]{div_X} & \mathcal{W}_T(X) = \mathbb{Z}^{|\Sigma(1)|} & \xrightarrow[Q\oplus\Gamma]{d_X} & \mathrm{Cl}(X) \cong \mathbb{Z}^r \oplus T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta^T} & & {\scriptstyle \mathbf{I}_{n+r}}\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \overline{\alpha}} & & \\
0 & \longrightarrow & M & \xrightarrow[\widehat{V}^T]{div_Y} & \mathcal{W}_T(Y) = \mathbb{Z}^{|\widehat{\Sigma}(1)|} & \xrightarrow[Q]{d_Y} & \mathrm{Cl}(Y) \cong \mathbb{Z}^r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}(\beta^T) \cong T & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\tag{3}
$$

where

- $T = \mathrm{Tors}(\mathrm{Cl}(X))$ ;
- $\mathrm{div}_X, \mathrm{div}_Y$ are the morphisms sending a character in $M$ to the associated principal divisor in $\mathcal{W}_T(X), \mathcal{W}_T(Y)$, respectively;
- $d_X, d_Y$ are the morphisms sending a torus invariant divisor in $\mathcal{W}_T(X), \mathcal{W}_T(Y)$, respectively, to its class in $\mathrm{Cl}(X)$, $\mathrm{Cl}(Y)$, respectively;
- $\alpha$ is the identification $\mathcal{W}_T(X) \cong \mathcal{W}_T(Y)$ induced by inverse images of rays by $\beta_{\mathbb{R}}$, that is,

$$
\alpha\left(\sum_{\rho\in\Sigma(1)} a_\rho D_\rho\right) = \sum_{\beta_{\mathbb{R}}^{-1}(\rho)\in\widehat{\Sigma}(1)} a_\rho D_{\beta_{\mathbb{R}}^{-1}(\rho)} ;
$$

- $\overline{\alpha}$ is what induced by $\alpha$ on classes groups;
- $V, \widehat{V}$ are matrices whose transposed represent $div_X, div_Y$, respectively, w.r.t. a chosen a basis of $M$ and standard bases of torus orbits of rays of $\mathcal{W}_T(X)$ and $\mathcal{W}_T(Y)$, respectively; since $|\Sigma(1)| = |\widehat{\Sigma}(1)| = n + r$, where

$$
r = \mathrm{rk}(\mathrm{Cl}(Y)) = \mathrm{rk}(\mathrm{Cl}(X))
$$

both $V$ and $\widehat{V}$ are $n \times (n + r)$ integer matrices called *fan matrices* of $X$ and $Y$, respectively; they turn out to be $F$*-matrices*, in the sense of [7, Def. 3.10], and $\widehat{V}$ is also a CF-matrix; notice that, still calling $\beta$ the representative matrix of the homonymous morphism $\beta : N \hookrightarrow N$ w.r.t. the basis dual to that chosen in $M$, there is the relation

$$V = \beta \cdot \widehat{V}$$

(see [7, Prop. 3.1 (3)] and [8, Rem. 2.4]); concretely, both $V$ and $\widehat{V}$ can be obtained as matrices whose columns represent primitive generators of rays in $\Sigma(1)$ and $\widehat{\Sigma}(1)$, respectively, w.r.t. the dual basis, i.e.,

$$V = \left( \mathbf{v}_1 \cdots \mathbf{v}_{n+r} \right) , \quad \widehat{V} = \left( \widehat{\mathbf{v}}_1 \cdots \widehat{\mathbf{v}}_{n+r} \right)$$

where $\Sigma(1) = \{\langle \mathbf{v}_1 \rangle, \ldots, \langle \mathbf{v}_{n+r} \rangle\}$, $\widehat{\Sigma}(1) = \{\langle \widehat{\mathbf{v}}_1 \rangle, \ldots, \langle \widehat{\mathbf{v}}_{n+r} \rangle\}$, being $\langle \mathbf{v} \rangle$ the ray generated by $\mathbf{v}$ in $\mathbb{R}^n \cong N_{\mathbb{R}}$ ;

- $Q$ is a matrix representing $d_Y$ w.r.t. a chosen basis of $\mathrm{Cl}(Y)$; it is a $r \times (n+r)$ integer matrix which turns out to be a *Gale dual matrix* of both $V$ and $\widehat{V}$, in the sense of [7, § 3.1] and a *W-matrix*, in the sense of [7, Def. 3.9]; it is called a *weight matrix* of both $X$ and $Y$;
- the choice of a basis of $\mathrm{Cl}(Y)$ as above determines a basis of a free part of $\mathrm{Cl}(X)$; complete such a basis with a set of generators of the torsion subgroup $T \subseteq \mathrm{Cl}(X)$; then, $d_X$ decomposes as $d_X = f_X \oplus \tau_X$ where

$$\begin{array}{ccc} & & \mathbb{Z}^r \\ & \overset{f_X}{\nearrow} & \uparrow \pi_1 \\ \mathcal{W}_T(X) & \xrightarrow{d_X} & \mathrm{Cl}(X) \cong \mathbb{Z}^r \oplus T \\ & \underset{\tau_X}{\searrow} & \downarrow \pi_2 \\ & & T \end{array} \qquad ;$$

with respect to these choices, the weight matrix $Q$ turns out to be a representative matrix of $f_X$, too, while morphism $\tau_X$ is represented by a *torsion matrix* $\Gamma$ [9, Thm. 3.2 (6)].

#### 3.1.1 Some further notation

Let $A \in \mathbf{M}(d, m; \mathbb{Z})$ be a $d \times m$ integer matrix, then

$\mathcal{L}_r(A) \subseteq \mathbb{Z}^m$ denotes the sublattice spanned by the rows of $A$;

$\mathcal{L}_c(A) \subseteq \mathbb{Z}^d$ denotes the sublattice spanned by the columns of $A$;

$A_I$, $A^I$ for any $I \subseteq \{1, \ldots, m\}$, the former is the submatrix of $A$ given by the columns indexed by I and the latter is the submatrix of $A$ whose columns are indexed by the complementary subset $\{1, \ldots, m\} \backslash I$;

Given a fan matrix $V = (\mathbf{v}_1, \ldots, \mathbf{v}_{n+r}) \in \mathbf{M}(n, n+r; \mathbb{Z})$ then

$\langle V \rangle = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{n+r} \rangle \subseteq N_{\mathbb{R}}$ denotes the cone generated by the columns of $V$;

$\mathcal{SF}(V) = \mathcal{SF}(\mathbf{v}_1, \ldots, \mathbf{v}_{n+r})$ is the set of all rational simplicial and complete fans $\Sigma$ such that $\Sigma(1) = \{\langle \mathbf{v}_1 \rangle, \ldots, \langle \mathbf{v}_{n+r} \rangle\} \subset N_{\mathbb{R}}$ (see [7, Def. 1.3]).

Given a fan $\Sigma \in \mathcal{SF}(V)$ we put

$$\mathcal{I}^\Sigma = \{I \subseteq \{1, \ldots, n+r\} : \left\langle V^I \right\rangle \in \Sigma(n)\}.$$

### 3.2 A sufficient condition

This section is aimed to give a sufficient condition for a $\mathbb{Q}$-factorial complete toric variety to be a pure variety. Let us, first of all, outline some equivalent facts.

**Proposition 4** *Let $X(\Sigma)$ be a $\mathbb{Q}$-factorial complete toric variety, $Y(\widehat{\Sigma}) \twoheadrightarrow X(\Sigma)$ be its universal 1-covering, $V$ and $\widehat{V}$ be fan matrices of $X$ and $Y$, respectively. Assuming notation as in diagram (3), the following are equivalent:*

(a) *$X$ is a pure variety;*
(b) *there is a decomposition $\mathcal{W}_T(X) = \mathcal{L}_r(\widehat{V}) \oplus F$ such that $\mathcal{C}_T(X) \subseteq \mathcal{L}_r(V) \oplus F$;*
(c) *for every prime $p$ there exists a $\mathbb{Z}_{(p)}$-module $F_p$ and a decomposition*

$$\mathcal{W}_T(X)_{(p)} = \mathcal{L}_r(\widehat{V})_{(p)} \oplus F_p$$

*such that $\mathcal{C}_T(X)_{(p)} \subseteq \mathcal{L}_r(V)_{(p)} \oplus F_p$.*

***Proof*** a) $\Rightarrow$ b): If $X$ is a pure variety, let $\mathrm{Cl}(X) = L \oplus T$ be a decomposition such that $L$ is a free part and $\mathrm{Pic}(X) \subseteq L$. We can identify $L$ with $\mathbb{Z}^r$ in the first row of diagram (3). Let $s : L \to \mathcal{W}_T(X)$ be any section (i.e., $Q \circ s = id_L$ ) and put $F = s(L)$. If $x \in \mathcal{W}_T(X)$, write $d_X(x) = a + b$, with $a \in L$ and $b \in T$. Then, $Q(x - s(a)) = 0$ so that $x - s(a) \in \mathcal{L}_r(\widehat{V})$; this proves that $\mathcal{W}_T(X) = \mathcal{L}_r(\widehat{V}) \oplus F$. If $x \in \mathcal{C}_T(X)$, then write $x = a + b$ with $a \in \mathcal{L}_r(\widehat{V})$ and $b \in F$; since $d_X(x) \in L$, we have $\Gamma \cdot x = \Gamma \cdot a = 0$, so that $a \in \mathcal{L}_r(V)$.

b) $\Rightarrow$ c) is obvious.

c) $\Rightarrow$ a): Let $p$ be a prime and put $F'_p = F_p \cap \mathcal{C}_T(X)_{(p)}$. We have

$$\mathcal{C}_T(X)_{(p)} = \mathcal{L}_r(V)_{(p)} \oplus F'_p$$

so that

$$\begin{aligned}\mathrm{Cl}(X)_{(p)}/\operatorname{Pic}(X)_{(p)} = \mathcal{W}_T(X)_{(p)}/\mathcal{C}_T(X)_{(p)} &\cong \left(\mathcal{L}_r(\widehat{V})_{(p)}/\mathcal{L}_r(V)_{(p)}\right) \oplus \left(F_p/F'_p\right) \\ &\cong T_{(p)} \oplus \left(F_p/F'_p\right)\end{aligned}$$

Then, we see that the image of $T_{(p)}$ is a direct summand in $\mathrm{Cl}(X)_{(p)}/\operatorname{Pic}(X)_{(p)}$, so that $\operatorname{Pic}(X)_{(p)}$ is contained in a free part of $\mathrm{Cl}(X)_{(p)}$ by Proposition 2 b). Since this holds for every $p$ we can apply Proposition 3 and deduce that $\mathrm{Pic}(X)$ is contained in a free part of $\mathrm{Cl}(X)$, so that $X$ is pure.

□

**Definition 4** Let $\Sigma$ be a fan in $\mathbb{R}^n$. For every simplicial cone $\sigma \in \Sigma$, let $\mathbf{w}_1, \ldots, \mathbf{w}_k \in \mathbb{Z}^n$ be the set of minimal generators of $\sigma$. Let $\mathcal{V}$ be the subspace of $\mathbb{R}^n$ generated by $\sigma$, and $L = \mathcal{V} \cap \mathbb{Z}^n$. The *multiplicity* of $\sigma$ is the index

$$\mathrm{mult}(\sigma) = [L : \mathbb{Z}\mathbf{w}_1 \oplus \ldots \oplus \mathbb{Z}\mathbf{w}_k].$$

If $\Sigma$ is a simplicial fan we put

$$m_\Sigma = \gcd\{\mathrm{mult}(\sigma) \mid \sigma \text{ is a maximal cone in } \Sigma\}.$$

Set, once for all, the following notation:

$$\forall\, I \subseteq \{1, \ldots, n+r\} \quad E_I := \{\mathbf{x} = (x_1, \ldots, x_{n+r}) \in \mathbb{Z}^{n+r} \mid x_i = 0, \forall i \notin I\}. \quad (4)$$

We are now in a position to state and prove the main result of the present paper.

**Theorem 2** *Let $X = X(\Sigma)$ be a complete $\mathbb{Q}$-factorial toric variety and $Y = Y(\widehat{\Sigma})$ be its universal 1-covering; let $\widehat{V}$ be a fan matrix associated with $Y$, and $V = \beta \cdot \widehat{V}$ be a fan matrix associated with $X$. Suppose that* $(\det(\beta), m_{\widehat{\Sigma}}) = 1$. *Then, $X$ is a pure variety.*

***Proof*** By Proposition 4, it suffices to show that for every prime $p$ there exists a $\mathbb{Z}_{(p)}$-module $F_p$ and a decomposition $\mathcal{W}_T(X)_{(p)} = \mathcal{L}_r(\widehat{V})_{(p)} \oplus F_p$ such that $\mathcal{C}_T(X)_{(p)} \subseteq \mathcal{L}_r(V)_{(p)} \oplus F_p$. If $p \not| \det(\beta)$, then $\mathcal{L}_r(V)_{(p)} = \mathcal{L}_r(\widehat{V})_{(p)}$ and we are done. Assume that $p | \det(\beta)$; by hypothesis there exists a maximal cone $\widehat{\sigma} = \widehat{\sigma}^I \in \widehat{\Sigma}$ such that $p \not| \mathrm{mult}(\widehat{\sigma}) = \det(Q_I)$. Put $F_p = E_{I,(p)}$, where $E_I$ is defined in (4). By definition $\mathcal{C}_T(X) \subseteq \mathcal{L}_r(V) \oplus E_I$, so that $\mathcal{C}_T(X)_{(p)} \subseteq \mathcal{L}_r(V)_{(p)} \oplus F_p$. We claim that $\mathbb{Z}^{n+r}_{(p)} = \mathcal{L}_r(\widehat{V})_{(p)} \oplus F_p$. The inclusion $\supseteq$ being obvious, assume that $\mathbf{x} \in \mathbb{Z}^{n+r}_{(p)}$. Since $\det(Q_I)$ is invertible in $\mathbb{Z}_{(p)}$, there exists $\mathbf{y} \in E_I$ such that $Q\mathbf{x} = Q\mathbf{y}$, that is, $\mathbf{x} - \mathbf{y} \in \ker(Q) = \mathcal{L}_r(\widehat{V})$. □

**Corollary 1** *Let $Y = Y(\widehat{\Sigma})$ be a poly weighted projective space such that $m_{\widehat{\Sigma}} = 1$. Then, every $\mathbb{Q}$-factorial complete toric variety having $Y$ as universal* 1*-covering is pure.*

***Remark 2*** Geometrically the previous Theorem 2 translates precisely in Theorem 1 stated in the introduction. In fact a $\mathbb{Q}$-factorial complete toric variety is an orbifold (see [2, Thm. 3.1.19 (b)]) whose $n$-skeleton parameterizes a covering by affine charts. In particular $Y$ has only finite quotient singularities whose order is necessarily a divisor of some multiplicity $\mathrm{mult}(\widehat{\sigma})$, for $\widehat{\sigma} \in \widehat{\Sigma}(n)$. Moreover, the affine chart $U_{\widehat{\sigma}}$ has always a quotient singularity of maximum order $\mathrm{mult}(\widehat{\sigma})$. Hence Theorem 1 follows.

In particular the previous Corollary 1 gives the following

**Corollary 2** *Let $Y$ be a $n$-dimensional, $\mathbb{Q}$-factorial, complete toric variety admitting a torus invariant, Zariski open subset $U \subseteq Y$, biregular to $\mathbb{C}^n$. Then, $Y$ is a PWS and every $\mathbb{Q}$-factorial complete toric variety having $Y$ as universal 1-covering is pure.*

### 3.3 Examples

The present section is devoted to give some examples of pure and impure $\mathbb{Q}$-factorial complete toric varieties.

***Example 1*** Consider the fan matrix

$$\widehat{V} = \begin{pmatrix} 1 & -1 & 2 & -3 & -1 \\ 1 & -1 & -1 & 2 & -1 \\ 1 & 1 & 1 & 1 & -5 \end{pmatrix}$$

The corresponding weight matrix is

$$Q = \begin{pmatrix} 3 & 1 & 10 & 6 & 4 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

One can check that $|\mathcal{SF}(\widehat{V})| = 2$. These two fans are given by taking all the faces of the following lists of maximal cones:

$$\widehat{\Sigma}_1 = \{\langle 1,2,3\rangle, \langle 1,2,4\rangle, \langle 2,4,5\rangle, \langle 1,4,5\rangle, \langle 2,3,5\rangle, \langle 1,3,5\rangle\}$$
$$\widehat{\Sigma}_2 = \{\langle 1,3,4\rangle, \langle 2,3,4\rangle, \langle 2,4,5\rangle, \langle 1,4,5\rangle, \langle 2,3,5\rangle, \langle 1,3,5\rangle\}$$

We denote by $\langle i,j,k\rangle$ the cone generated by the columns $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$ of the matrix $\widehat{V}$. The list of multiplicities of maximal cones for the two fans is, respectively,

$$6, 10, 30, 20, 18, 12 \text{ and } 7, 9, 30, 20, 18, 12,$$

so that

$$m_{\widehat{\Sigma}_1} = 2, \qquad m_{\widehat{\Sigma}_2} = 1.$$

Define

$$\beta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$V := \beta \cdot \widehat{V} = \begin{pmatrix} 1 & -1 & 2 & -3 & -1 \\ 1 & -1 & -1 & 2 & -1 \\ 2 & 2 & 2 & 2 & -10 \end{pmatrix}.$$

A torsion matrix $\Gamma$ with entries in $\mathbb{Z}/2\mathbb{Z}$ such that $Q \oplus \Gamma$ represents the morphism assigning to each divisor its class, as in the previous diagram (3), is given by

$$\Gamma = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let $\Sigma_1$ be the fan in $\mathcal{SF}(V)$ corresponding to $\widehat{\Sigma}_1$. We show that $X(\Sigma_1)$ is an impure variety. Using methods explained in [9, Thm. 3.2 (2)], we obtain that a basis of $\mathcal{C}_T(X)$ is given by the rows of the following matrix

$$C_X = \begin{pmatrix} 40 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -24 & -24 & 0 & 1 & 0 \\ -9 & -47 & -2 & 0 & 1 \end{pmatrix}.$$

Then,

$$Q \cdot C_X^{\mathrm{T}} = \begin{pmatrix} 120 & 60 & 30 & -90 & -90 \\ 120 & 120 & 0 & -120 & -120 \end{pmatrix}, \qquad \Gamma \cdot C_X^T = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Then, we see that $\mathrm{Pic}(X)$ is generated in $\mathrm{Cl}(X) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ by elements

$$(120, 120), (60, 120), (30, 0) + [1]_2, (90, 120) + [1]_2.$$

the first and the last of them are obviously generated by the remaining two elements, so that $\mathrm{Pic}(X)$ is generated by $(60, 120)$ and $(30, 0) + [1]_2$. Every free part of $\mathrm{Cl}(X)$ contains an element $z$ of the form $(15, 0) + [a]_2$ for some $a \in \{0, 1\}$; therefore, it must contain $2z = (30, 0)$; then, $(30, 0) + [1]_2$ cannot belong to any free part, meaning that $X(\Sigma_1)$ is impure.

Notice that purity is a property depending on the fan choice. In fact $\Sigma_2$ satisfies hypothesis of Theorem 2, as $m_{\Sigma_2} = 1$. Then, $X(\Sigma_2)$ is pure.

The following is a counterexample showing that a converse of Theorem 2 cannot hold.

***Example 2*** Let $\widehat{V}$ be the fan matrix of Example 1. Consider the matrix

$$\beta' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and put

$$V' := \beta' \cdot \widehat{V} = \begin{pmatrix} 1 & -1 & 2 & -3 & -1 \\ 2 & -2 & -2 & 4 & -2 \\ 1 & 1 & 1 & 1 & -5 \end{pmatrix}.$$

A torsion matrix $\Gamma'$ with entries in $\mathbb{Z}/2\mathbb{Z}$ such that $Q \oplus \Gamma'$ represents the morphism assigning to each divisor its class is given by

$$\Gamma' = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{5}$$

Let $\Sigma_1'$ be the fan in $\mathcal{SF}(V)$ corresponding to $\widehat{\Sigma}_1$ and $X' = X(\Sigma_1')$. In this case $X'$ is a pure variety. In fact, a basis of $\mathcal{C}_T(X')$ is given by the rows of the following matrix

$$C_{X'} = \begin{pmatrix} 40 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 \\ -20 & -30 & 3 & 0 & 0 \\ -8 & -48 & 0 & 2 & 0 \\ 15 & 37 & -2 & -1 & 1 \end{pmatrix}.$$

Then,

$$Q \cdot C_{X'}^T = \begin{pmatrix} 120 & 60 & -60 & -60 & 60 \\ 120 & 120 & -120 & -120 & 120 \end{pmatrix}, \qquad \Gamma' \cdot C_{X'}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, we see that $\mathrm{Pic}(X')$ is generated in $\mathrm{Cl}(X') \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ by the elements

$$(120, 120), (60, 120)$$

so that $X'$ is pure. On the other hand $m_{\widehat{\Sigma}_1} = 2 = \det(\beta')$, so proving that a converse of Theorem 2 cannot hold.

### 3.4 The case $|\mathcal{SF}(V)| = 1$

The aim of this section is to exhibit a large class of pure toric varieties, by establishing the purity of every $\mathbb{Q}$-factorial complete toric variety $X = X(\Sigma)$ whose fan matrix $V$ admits a unique simplicial and complete fan given by $\Sigma$ itself. Geometrically, this property means that a small $\mathbb{Q}$-factorial modification of $X$ is necessarily an isomorphism, as explained in the Introduction.

We need a few preliminary lemmas. If $V$ is an $F$-matrix we put

$$\begin{aligned} \mathcal{I}_{V,\mathrm{tot}} &= \{I \subseteq \{1, \dots, n+r\} \mid |I| = r \text{ and } \det(V^I) \neq 0\} \\ \mathcal{I}_{V,\min} &= \{I \in \mathcal{I}_{V,\mathrm{tot}} \mid \langle V^I \rangle \text{ does not contain any column of } V \\ &\qquad\qquad \text{apart from its generators }\}. \end{aligned}$$

**Lemma 3** *Put*

$$\begin{aligned} m_{V,\mathrm{tot}} &= \gcd\{\det(V^I) \mid I \in \mathcal{I}_{V,\mathrm{tot}}\} \\ m_{V,\min} &= \gcd\{\det(V^I) \mid I \in \mathcal{I}_{V,\min}\}; \end{aligned}$$

*then* $m_{V,\min} = m_{V,\mathrm{tot}}$.

***Proof*** Since $\mathcal{I}_{V,\min} \subseteq \mathcal{I}_{V,\mathrm{tot}}$ we have $m_{V,\mathrm{tot}} | m_{V,\min}$ We firstly show that the assertion is true when $m_{V,\mathrm{tot}} = 1$. Otherwise, there would exist a prime number $p$ dividing $\det(V^I)$ for every $I \in \mathcal{I}_{V,\min}$; and there would exist $I_0 \in \mathcal{I}_{\mathrm{tot}}$ such that $p \nmid \det(I_0)$. We choose such an $I_0$ with the property that the number $n_0$ of columns of $V$ belonging to

$\langle V^{I_0}\rangle$ is minimum. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the columns of $V^{I_0}$ and let $\mathbf{v}^* \in \langle V^{I_0}\rangle$ be a column of $V$ different from $\mathbf{v}_i$ for every $i$; then, we can write $\mathbf{v}^* = \sum_{i=1}^n \frac{a_i}{b}\mathbf{v}_i$ with $a_i, b \in \mathbb{Z}$ and $(a_1, \ldots, a_n, b) = 1$. For $i = 1, \ldots, n$ let $\sigma_i = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}^*, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\rangle$. Then, $|\det(\sigma_i)| = \left(\frac{a_i}{b}\right)^n |\det(V^{I_0})|$ and $p$ divides $\det(\sigma_i)$ by the minimality hypothesis on $I_0$. It follows that $p$ divides $a_i$ for $i = 1, \ldots, n$; therefore, $\mathbf{v}^* \in p\mathbb{Z}^n$ and this is a contradiction because $V$ is a fan matrix, hence reduced (see [7, Def. 3.13]).
Suppose now that $m_{V,\mathrm{tot}} \neq 1$. Then, by [7, Prop. 3.1 (3)] there exist a $CF$-matrix $\widehat{V}$ such that $V = \beta\widehat{V}$ for some $\beta \in \mathbf{M}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$; and $m_{\widehat{V},\mathrm{tot}} = 1$ by [7, Prop. 2.6], so that we can apply the first part of the proof to $\widehat{V}$ and deduce that $m_{\widehat{V},\min} = 1$. Notice that $\mathcal{I}_{V,\mathrm{tot}} = \mathcal{I}_{\widehat{V},\mathrm{tot}}$, $\mathcal{I}_{V,\min} = \mathcal{I}_{\widehat{V},\min}$ and $\det(V^I) = \det(\beta)\det(\widehat{V}^I)$ for every $I \in \mathcal{I}_{V,\mathrm{tot}}$, so that $m_{V,\min} = \det(\beta)m_{\widehat{V},\min}$ and $m_{V,\mathrm{tot}} = \det(\beta)m_{\widehat{V},\mathrm{tot}}$. It follows that $m_{V,\min} = m_{V,\mathrm{tot}} = \det(\beta)$. □

**Lemma 4** *Let $\Sigma_0$ be a simplicial fan in $\mathbb{R}^n$ such that $\sigma = |\Sigma_0|$ is a full dimensional convex cone. Let $\mathbf{w}_1, \ldots, \mathbf{w}_k \in \mathbb{R}^n$ be such that $\mathbf{w}_i \notin \sigma$ for $i = 1, \ldots, k$. There exists a simplicial fan $\Sigma$ in $\mathbb{R}^n$ such that*

(a) $|\Sigma| = \sigma + \langle \mathbf{w}_1, \ldots, \mathbf{w}_k\rangle$;
(b) $\Sigma(1) = \Sigma_0(1) \cup \{\langle \mathbf{w}_1\rangle, \ldots, \langle \mathbf{w}_k\rangle\}$;
(c) $\Sigma_0 \subseteq \Sigma$.

***Proof*** By induction on $k$. For the case $k = 0$, we take $\Sigma = \Sigma_0$. Assume that the result holds true for $k-1$. Let $\mathcal{W}' = \sigma + \langle \mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\rangle$, $\mathcal{W} = \sigma + \langle \mathbf{w}_1, \ldots, \mathbf{w}_k\rangle$. By inductive hypothesis there exists a simplicial fan $\Sigma'$ such that $|\Sigma'| = \mathcal{W}'$, $\Sigma'(1) = \Sigma_0(1) \cup \{\langle \mathbf{w}_1\rangle, \ldots, \langle \mathbf{w}_{k-1}\rangle\}$ and $\Sigma_0 \subseteq \Sigma'$. We distinguish two cases:

- **Case 1:** $\mathbf{w}_k \in \mathcal{W}'$, so that $\mathcal{W} = \mathcal{W}'$; let $\tau$ be the minimal cone in $\Sigma'$ containing $\mathbf{w}_k$. We take $\Sigma = s(\mathbf{w}_k, \tau)\Sigma'$, the stellar subdivision of $\Sigma'$ in direction $\mathbf{w}_k$ (see [3, Def. III.2.1]). Concretely, every $m$-dimensional cone $\mu = \langle \mathbf{x}_1, \ldots, \mathbf{x}_m\rangle \in \Sigma$ containing $\mathbf{w}_k$ is replaced by the set of the $m$-dimensional cones of the form $\langle \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{w}_k, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_m\rangle$. Conditions $a$) and $b$) are immediately verified. For condition $c$) notice that, since $\mathbf{w}_k \notin \sigma$, $\tau$ is not a face of any cone in $\Sigma_0$; therefore, $\Sigma_0 \subseteq \Sigma$.
- **Case 2:** $\mathbf{w}_k \notin \mathcal{W}'$, so that $\mathcal{W}' \subsetneq \mathcal{W}$. Let $\mathcal{F}$ be the set of facets $f$ in $\Sigma'(n-1)$ which are cut out by an hyperplane strictly separating $\mathcal{W}'$ and $\mathbf{w}_k$; that is $f \subseteq \partial\mathcal{W}'$, $f \nsubseteq \partial\mathcal{W}$ and the cone $\tau_f = \langle f, \mathbf{w}_k\rangle$ is $n$ dimensional. Notice that $\mathcal{F} \neq \emptyset$: In fact $\mathcal{W}'$ is a convex polyhedral cone and $\mathbf{w}_k \notin \mathcal{W}'$; then, there is an hyperplane $H$ cutting a facet $\varphi$ of $\mathcal{W}'$ and strictly separating $\mathcal{W}'$ and $\mathbf{w}_k$; let $f$ be a facet of $\Sigma'$ contained in $\varphi$; then, $f \in \mathcal{F}$.

Consider the set of simplicial cones

$$\Sigma = \Sigma' \cup \{\tau \mid \tau \preceq \tau_f \text{ for some } f \in \mathcal{F}\}.$$

We claim that $\Sigma$ is a fan. By construction it is closed by faces, so that it suffices to show that $\tau_1 \cap \tau_2$ is a face of both $\tau_1$ and $\tau_2$, whenever $\tau_1, \tau_2 \in \Sigma$. Let $\tau_1, \tau_2 \in \Sigma$. If they are both in $\Sigma'$, then $\tau_1 \cap \tau_2$ is a face of $\tau_1, \tau_2$ because $\Sigma'$ is a fan. Assume that $\tau_1 \in \Sigma'$ and $\tau_2 \notin \Sigma'$; then, $\tau_2 = \langle \tau, \mathbf{w}_k\rangle$, where $\tau$ is a face of some $f \in \mathcal{F}$. Let $H$ be the hyperplane

cutting $f$; then, $\mathbf{w}_k$ lies on the other side of $H$ with respect to $\mathcal{W}'$, so that $\tau_1 \cap \tau_2 \subseteq f$. Therefore, $\tau_1 \cap \tau_2 = \tau_1 \cap \tau \in \Sigma'$, so that it is a face of both $\tau_1$ and $\tau$ by induction hypothesis; but $\tau \preceq \tau_1$ so that $\tau_1 \cap \tau_2 \preceq \tau_2$. Finally, assume that both $\tau_1$ and $\tau_2$ are not in $\Sigma'$. This means that there are facets $f_1, f_2$ in $\mathcal{F}$ and faces $\mu_1 \preceq f_1, \mu_2 \preceq f_2$ such that $\tau_1 = \langle \mu_1, \mathbf{w}_k \rangle$ and $\tau_2 = \langle \mu_2, \mathbf{w}_k \rangle$. We show that $\tau_1 \cap \tau_2 = \langle \mu_1 \cap \mu_2, \mathbf{w}_k \rangle \in \Sigma$. Let $\mathbf{x} \in \tau_1 \cap \tau_2$: Then, we can write $\mathbf{x} = \mathbf{y}_1 + \lambda_1 \mathbf{w}_k = \mathbf{y}_2 + \lambda_2 \mathbf{w}_k$, where $\mathbf{y}_1 \in \mu_1, \mathbf{y}_2 \in \mu_2$ and $\lambda_1, \lambda_2 \geq 0$. Without loss of generality we can assume $\lambda_1 \geq \lambda_2$; put $\lambda = \lambda_1 - \lambda_2$; then, $\mathbf{y}_1 + \lambda \mathbf{w}_k = \mathbf{y}_2$. Let $H$ be the hyperplane cutting $f_1$; since $f_1 \in \mathcal{F}$, $\mathbf{w}_k \notin H$, so that there exists a vector $\mathbf{n}$ be a normal to $H$ such that $\mathbf{n} \cdot \mathbf{x} \leq 0$ for every $\mathbf{x} \in \mathcal{W}'$ and $\mathbf{n} \cdot \mathbf{w}_k > 0$. Then, $\mathbf{n} \cdot \mathbf{y}_2 \leq 0$ and $\mathbf{n} \cdot (\mathbf{y}_1 + \lambda \mathbf{w}_k) = \mathbf{n} \cdot \lambda \mathbf{w}_k \geq 0$, so that $\lambda = 0$; this implies $\mathbf{y}_1 = \mathbf{y}_2 \in \mu_1 \cap \mu_2$ and $\mathbf{x} \in \langle \mu_1 \cap \mu_2, \mathbf{w}_k \rangle$.
Now we show that condition *a*) holds for $\Sigma$. By construction $|\Sigma| = |\Sigma'| \cup \bigcup_{f \in \mathcal{F}} \tau_f \subseteq \mathcal{W}' + \langle \mathbf{w}_k \rangle = \mathcal{W}$; conversely, let $\mathbf{x} \in \mathcal{W}$; if $\mathbf{x} \in \mathcal{W}'$, then $\mathbf{x} \in |\Sigma'| \subseteq |\Sigma|$; if $\mathbf{x} \notin \mathcal{W}'$, then $\mathbf{x} = \mathbf{y} + \lambda \mathbf{w}_k$ for some $\mathbf{y} \in \mathcal{W}'$ and $\lambda > 0$; up to replacing $\mathbf{y}$ by $\mathbf{y} + \mu \mathbf{w}_k$ for some $0 \leq \mu < \lambda$ we can assume that $\mathbf{y} + \epsilon \mathbf{w}_k \notin \mathcal{W}'$ if $\epsilon > 0$. Then, for every $\epsilon$ there exists an hyperplane $H_\epsilon$ cutting a facet $\varphi_\epsilon$ of $\mathcal{W}'$ which separates $\mathcal{W}'$ and $\mathbf{y} + \epsilon \mathbf{w}_k$; since the facets of $\mathcal{W}'$ are finitely many, by the pigeonhole principle $H_\epsilon, \varphi_\epsilon$ do not depend on $\epsilon$ for $\epsilon \to 0$; call them $H, \varphi$, respectively. Let $\mathbf{n}$ be a normal vector to $H$ such that $\mathbf{n} \cdot \mathbf{y} \leq 0$ and $\mathbf{n} \cdot (\mathbf{y} + \epsilon \mathbf{w}_k) > 0$ for $\epsilon \to 0$; the existence of such $\mathbf{n}$ implies $\mathbf{n} \cdot \mathbf{y} = 0$ and $\mathbf{n} \cdot \mathbf{w}_k > 0$, so that $\mathbf{y} \in \varphi$ and $\mathbf{w}_k \notin H$. Then, there is a facet $f \in \mathcal{F}$ such that $\mathbf{y} \in f$; therefore, $\mathbf{x} \in \tau_f$, and *a*) is proved. We showed that $\Sigma \backslash \Sigma' \neq \emptyset$; and every cone in $\Sigma \backslash \Sigma'$ has $\mathbf{w}_k$ as a vertex and all other vertices in $\Sigma'(1)$. Then, condition *b*) is verified. Condition *c*) is obvious since $\Sigma_0 \subseteq \Sigma' \subseteq \Sigma$. □

**Corollary 3** *Let $V$ be a fan matrix. Then, for every $I \in \mathcal{I}_{V,\min}$ the cone $\langle V^I \rangle$ belongs to a fan in $\mathcal{SF}(V)$.*

***Proof*** It suffices to apply Lemma 4 in the case $\Sigma_0 = \{\tau \mid \tau \preceq \langle V^I \rangle\}$ and $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are the columns of $V_I$. □

Corollary 3 has the following immediate consequence:

**Corollary 4** *Let $V$ be a fan matrix such that $\mathcal{SF}(V)$ contains a unique fan $\Sigma$. Then, for every $I \in \mathcal{I}_{V,\min}$ the cone $\langle V^I \rangle$ belongs to $\Sigma$.*

We are now in position to prove our purity condition:

**Proposition 5** *Let $X$ be a $\mathbb{Q}$-factorial complete toric variety and let $V$ be a fan matrix of $X$. Assume that $\mathcal{SF}(V)$ contains a unique fan. Then, $X$ is pure.*

***Proof*** Let $Y = Y(\widehat{\Sigma})$ be the universal 1-covering of $X$ and let $\widehat{V}$ be a fan matrix associated with $Y$. Then, $\widehat{V}$ is a $CF$-matrix, so that $m_{\widehat{V},\mathrm{tot}} = 1$ by [7, Prop. 2.6 and Def. 2.7]. By Corollary 4, $\mathcal{I}^{\Sigma} = \mathcal{I}^{\widehat{\Sigma}} = \mathcal{I}_{\widehat{V},\min}$ so that $m_{\widehat{\Sigma}} = m_{\widehat{V},\min}$ and, by Lemma 3, the latter is equal to $m_{\widehat{V},\mathrm{tot}} = 1$. Then, $X$ is pure by Corollary 1. □

***Remark 3*** Proposition 5 implies that the following toric varieties are pure:

– two-dimensional $\mathbb{Q}$-factorial complete toric varieties

– toric varieties whose universal 1-covering is a product of weighted projective spaces.

***Remark 4*** In the case $|\mathcal{SF}(V)| = 1$ the unique complete and $\mathbb{Q}$-factorial toric variety $X$ whose fan matrix is $V$ is necessarily projective. This is a consequence of the fact that $\mathrm{Nef}(X) = \overline{\mathrm{Mov}}(X)$, recalling that the latter is a full dimensional cone, by [1, Thm. 2.2.2.6].

### 3.4.1 An application to completions of fans

Lemma 4 can be applied to give a complete refinement $\Sigma$ of a given fan $\Sigma_0$ satisfying the further additional hypothesis:

(*) *assume that* $|\Sigma| = \Sigma_0 + \langle \mathbf{w}_1, \dots, \mathbf{w}_k \rangle = \mathbb{R}^n$.

In particular, if we consider the fan $\Sigma' = \Sigma_0 \cup \{\langle \mathbf{w}_1 \rangle \ \dots, \langle \mathbf{w}_k \rangle\}$, then Lemma 4 *gives a completion* $\Sigma$ *of* $\Sigma'$ *without adding any new ray.*

The latter seems to us an original result. In fact, it is actually well known that every fan $\Sigma'$ can be refined to a complete fan $\Sigma$ (see [3, Thm. III.2.8], [4] and the more recent [6]). Anyway, in general the known completion procedures need the addition of some new ray, so giving $\Sigma'(1) \subsetneq \Sigma(1)$. As observed in the Remark following the proof of [3, Thm. III.2.8], just for $n = 3$ "completion without additional 1-cones can be found," but this fact does no more hold for $n \geq 4$: At this purpose, Ewald refers the reader to the Appendix to section III, where he is further referred to a number of references. Unfortunately we were not able to recover, from those references and, more generally, from the current literature, as far as we know, an explicit example of a four-dimensional fan which cannot be completed without adding some new ray. For this reason, we believe that the following example may fill up a lack in the literature on these topics.

***Example 3*** Consider the fan matrix

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

and consider the fan $\Sigma$ given by taking all the faces of the following three maximal cones generated by columns of $V$

$$\Sigma(4) = \{\langle 2, 3, 4, 6 \rangle, \langle 2, 4, 5, 7 \rangle, \langle 1, 4, 5, 6 \rangle\}$$

The fact that $\Sigma$ is a fan follows immediately by easily checking that

$$\begin{aligned} \langle 2, 3, 4, 6 \rangle \cap \langle 2, 4, 5, 7 \rangle &= \langle 2, 4 \rangle \\ \langle 2, 3, 4, 6 \rangle \cap \langle 1, 4, 5, 6 \rangle &= \langle 4, 6 \rangle \\ \langle 1, 4, 5, 6 \rangle \cap \langle 2, 4, 5, 7 \rangle &= \langle 4, 5 \rangle. \end{aligned}$$

Notice that $\Sigma$ is not a complete fan since, e.g., the three-dimensional cone

$$\langle 2,3,6\rangle = \left\langle \begin{matrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{matrix} \right\rangle$$

is a facet of the unique cone

$$\langle 2,3,4,6\rangle = \left\langle \begin{matrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{matrix} \right\rangle \in \Sigma(4).$$

Moreover, it cannot be completed since every further maximal cone admitting $\langle 2,3,6\rangle$ as a facet does not intersect correctly the remaining cones in $\Sigma(4)$. In fact

– $\langle 1,2,3,6\rangle \cap \langle 2,4,5,7\rangle \supsetneq \langle 2\rangle$ : Consider, e.g.,

$$\mathbf{v} = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T \in \langle 1,2\rangle \cap \langle 5,7\rangle$$

– $\langle 2,3,5,6\rangle \cap \langle 1,4,5,6\rangle \supsetneq \langle 5,6\rangle$ : Consider, e.g.,

$$\mathbf{w} = \begin{pmatrix} 0 & -2 & -1 & -2 \end{pmatrix}^T \in \langle 3,5\rangle \cap \langle 1,5,6\rangle \quad \text{but} \quad \mathbf{w} \notin \langle 5,6\rangle$$

– $\langle 2,3,6,7\rangle$ is not a maximal cone.

## 4 A characterization of Pic($X$) for some pure toric variety

Let $X = X(\Sigma)$ be a complete $\mathbb{Q}$-factorial toric variety having $V$ as a fan matrix; let $Y$ be its universal 1-covering, $\widehat{V}$ be a fan matrix associated with $Y$ and $V = \beta\widehat{V}$. Recall that a Weil divisor $L = \sum_{j=1}^{n+r} a_j D_j$ is a Cartier divisor if it is locally principal, that is,

$$\forall I \in \mathcal{I}^\Sigma \; \exists \mathbf{m}_I \in M \text{ such that } \mathbf{m}_I \cdot \mathbf{v}_j = a_j, \forall j \notin I.$$

Let $\mathcal{C}_T(X)$ be the group of torus invariant Cartier divisors of $X$. Then,

$$\mathcal{C}_T(X) = \bigcap_{I \in \mathcal{I}^\Sigma} \mathcal{L}_r(V^I) = \bigcap_{I \in \mathcal{I}^\Sigma} (\mathcal{L}_r(V) \oplus E_I)$$

recalling notation (4). The Picard group Pic($X$) of $X$ is the image of $\mathcal{C}_T(X)$ in Cl($X$), via morphism $d_X$ (recall here and in the following, notation introduced in diagram (3)).

In [7, Thm. 2.9.2] we showed that if $Y$ is a PWS, then we can identify

$$\operatorname{Pic}(Y) = \bigcap_{I \in \mathcal{I}^\Sigma} \mathcal{L}_c(Q_I) \subseteq \mathbb{Z}^r. \tag{6}$$

Let $\mathbf{x} \in \operatorname{Pic}(Y)$. For $I \in \mathcal{I}^\Sigma$ we can write $\mathbf{x} = Q \cdot \mathbf{a}_I$ where $\mathbf{a}_I \in E_I$. If $I, J \in \mathcal{I}^\Sigma$ put

$$\mathbf{u}_{IJ} = \mathbf{a}_I - \mathbf{a}_J \in \ker(Q) = \mathcal{L}_r(\widehat{V}). \tag{7}$$

Let $\mathbf{z} \in \mathcal{C}_T(Y)$ such that $Q \cdot \mathbf{z} = \mathbf{x}$. By definition, for every $I \in \mathcal{I}^\Sigma$ there is a unique decomposition $\mathbf{z} = \mathbf{t}(I) + \mathbf{a}_I$ with $\mathbf{t}(I) \in \mathcal{L}_r(\widehat{V})$. Moreover,

$$\mathbf{z} \in \mathcal{C}_T(X) \Leftrightarrow \mathbf{t}(I) \in \mathcal{L}_r(V), \forall I \in \mathcal{I}^\Sigma. \tag{8}$$

**Proposition 6** $\mathbf{x} \in \overline{\alpha}(\operatorname{Pic}(X))$ *if and only if* $\mathbf{x} \in \operatorname{Pic}(Y)$ *and* $\mathbf{u}_{IJ} \in \mathcal{L}_r(V)$*, for every* $I, J \in \mathcal{I}^\Sigma$*, where* $\mathbf{u}_{IJ}$ *is defined by* (7).

***Proof*** Suppose that $\mathbf{x} \in \overline{\alpha}(\operatorname{Pic}(X))$. Then, there exists $\mathbf{z} \in \mathcal{C}_T(X)$ such that $Q \cdot \mathbf{z} = \mathbf{x}$. For every $I \in \mathcal{I}^\Sigma$ consider the decomposition $\mathbf{z} = \mathbf{t}(I) + \mathbf{a}_I$ with $\mathbf{t}(I) \in \mathcal{L}_r(V)$. Then, $\mathbf{u}_{IJ} = \mathbf{a}_I - \mathbf{a}_J = \mathbf{t}(J) - \mathbf{t}(I) \in \mathcal{L}_r(V)$ for every $I, J \in I^\Sigma$. Conversely, suppose that $\mathbf{u}_{IJ} \in \mathcal{L}_r(V)$ for every $I, J \in \mathcal{I}^\Sigma$. Let $\mathbf{z}' \in \mathcal{C}_T(Y)$ be such that $Q \cdot \mathbf{z}' = \mathbf{x}$. For every $I \in \mathcal{I}^\Sigma$ there is a decomposition $\mathbf{z}' = \mathbf{t}'(I) + \mathbf{a}_I$ with $\mathbf{t}'(I) \in \mathcal{L}_r(\widehat{V})$. Fix $I_0 \in \mathcal{I}^\Sigma$ and put $\mathbf{z} = \mathbf{z}' - \mathbf{t}'(I_0)$. We claim that $\mathbf{z} \in \mathcal{C}_T(X)$. Indeed, let $I \in \mathcal{I}^\Sigma$ and decompose $\mathbf{z} = \mathbf{t}(I) + \mathbf{a}_I$ with $\mathbf{t}(I) \in \mathcal{L}_r(\widehat{V})$ and $\mathbf{t}(I_0) = \mathbf{0}$. It follows that for every $I \in \mathcal{I}^\Sigma$

$$\mathbf{t}(I) = \mathbf{t}(I) - \mathbf{t}(I_0) = \mathbf{a}_{I_0} - \mathbf{a}_I = \mathbf{u}_{I_0 I} \in \mathcal{L}_r(V).$$

□

**Theorem 3** *Let $X$ be a pure $\mathbb{Q}$-factorial complete toric variety and choose an isomorphism* $\operatorname{Cl}(X) \cong \mathbb{Z}^r \oplus T$ *such that* $\operatorname{Pic}(X)$ *is mapped in* $\mathbb{Z}^r$*. Then, the following characterization of* $\operatorname{Pic}(X)$ *holds:*

$$\mathbf{x} \in \operatorname{Pic}(X) \;\Leftrightarrow\; \forall\, I, J \in \mathcal{I}^\Sigma \quad \mathbf{x} \in \bigcap_{I \in \mathcal{I}^\Sigma} \mathcal{L}_c(Q_I) \text{ and } \mathbf{u}_{IJ} \in \mathcal{L}_r(V)$$

*where* $\mathbf{u}_{IJ}$ *is defined by* (7).

***Proof*** Define $s : \mathbb{Z}^r \to \mathbb{Z}^r \oplus T$ by $s(a) = (a, 0)$. Then, $\overline{\alpha} \circ s = id_{\mathbb{Z}_r}$ and $s \circ \overline{\alpha}|_{\operatorname{Pic}(X)} = id_{\operatorname{Pic}(X)}$. Then, we have for every $\mathbf{x} \in \mathbb{Z}^r$

$$\mathbf{x} \in \overline{\alpha}(\operatorname{Pic}(X)) \Leftrightarrow s(\mathbf{x}) \in \operatorname{Pic}(X).$$

The result follows from Proposition 6 by identifying $\mathbf{x}$ and $s(\mathbf{x})$. □

***Example 4*** Let $\Sigma'_1$ be the fan defined in Example 2. Then,

$$\mathcal{I}^{\Sigma'_1} = \mathcal{I}^{\widehat{\Sigma}_1} = \{\{1,3\},\{2,3\},\{3,5\},\{1,4\},\{2,4\},\{4,5\}\}.$$

so that

$$\bigcap_{I \in \mathcal{I}^{\Sigma'_1}} \mathcal{L}_c(Q_I) = \mathbb{Z}(30,0) \oplus \mathbb{Z}(0,60).$$

Let $\mathbf{x} = (30x, 60y) \in \bigcap_{I \in \mathcal{I}^{\Sigma'_1}} \mathcal{L}_c(Q_I)$, with $x, y \in \mathbb{Z}$. For $I \in \mathcal{I}^{\Sigma'_1}$ we can write $\mathbf{x} = Q \cdot \mathbf{a}_I$ where $\mathbf{a}_I \in E_I$. The $\mathbf{a}_I$'s are easily calculated:

$$\begin{aligned}
\mathbf{a}_{\{1,3\}} &= (\ 20y \quad 0 \quad 3x-6y \quad 0 \quad 0\ )\\
\mathbf{a}_{\{2,3\}} &= (\ 0 \quad 30y \quad 3x-3y \quad 0 \quad 0\ )\\
\mathbf{a}_{\{3,5\}} &= (\ 0 \quad 0 \quad 3x-24y \quad 0 \quad 60y\ )\\
\mathbf{a}_{\{1,4\}} &= (\ 20y \quad 0 \quad 0 \quad 5x-10y \quad 0\ )\\
\mathbf{a}_{\{2,4\}} &= (\ 0 \quad 30y \quad 0 \quad 5x-5y \quad 0\ )\\
\mathbf{a}_{\{4,5\}} &= (\ 0 \quad 0 \quad 0 \quad 5x-40y \quad 60y\ )
\end{aligned}$$

Notice that, with the notation of (7), $\mathbf{u}_{IJ} = \mathbf{u}_{IK} - \mathbf{u}_{KJ}$; then, in order to calculate $\mathbf{u}_{IJ}$ for every $I, J \in \mathcal{I}^{\Sigma'_1}$ it suffices to compute $\mathbf{u}_{I_j I_{j+1}}$ for a sequence $I_1, \ldots, I_s$ such that $\langle Q_{I_j}\rangle$ and $\langle Q_{I_{j+1}}\rangle$ have a common facet and $\mathcal{I}^{\Sigma'_1} = \{I_1, \ldots, I_s\}$; in this way, we obtain vectors having at most $r + 1 = 3$ nonzero components:

$$\begin{aligned}
\mathbf{u}_{\{1,3\}\{2,3\}} &= (\ 20y \quad -30y \quad -3y \quad 0 \quad 0\ )\\
\mathbf{u}_{\{2,3\}\{3,5\}} &= (\ 0 \quad 30y \quad 21y \quad 0 \quad -60y\ )\\
\mathbf{u}_{\{3,5\}\{4,5\}} &= (\ 0 \quad 0 \quad 3x-24y \quad -5x+40y \quad 0\ )\\
\mathbf{u}_{\{4,5\}\{1,4\}} &= (\ -20y \quad 0 \quad 0 \quad -30y \quad 60y\ )\\
\mathbf{u}_{\{1,4\}\{2,4\}} &= (\ 20y \quad -30y \quad 0 \quad -5y \quad 0\ )
\end{aligned}$$

Multiplying by the matrix $\Gamma'$ found in (5) we obtain

$$\Gamma' \cdot \mathbf{u}^T_{\{1,3\}\{2,3\}} = 0; \quad \Gamma' \cdot \mathbf{u}^T_{\{2,3\}\{3,5\}} = -60y; \quad \Gamma' \cdot \mathbf{u}^T_{\{3,5\}\{4,5\}} = -5x + 40y;$$
$$\Gamma' \cdot \mathbf{u}^T_{\{4,5\}\{1,4\}} = 30y; \qquad \Gamma' \cdot \mathbf{u}^T_{\{1,4\}\{2,4\}} = -5y.$$

Recall that $\varGamma'$ takes values in $\mathbb{Z}/2\mathbb{Z}$ and that for every $\mathbf{u} \in \mathcal{L}_r(\widehat{V})$

$$\Gamma' \cdot \mathbf{u}^T = 0 \text{ if and only if } \mathbf{u} \in \mathcal{L}_r(V');$$

then, we see that $\mathbf{u}_{IJ} \in \ker(\Gamma')$ for every $I, J \in \mathcal{I}^{\Sigma'_1}$ if and only if $x, y \in 2\mathbb{Z}$, that is, if and only if $\mathbf{x} \in \mathbb{Z}(60,0) \oplus \mathbb{Z}(0,120)$. By Theorem 3, $\mathrm{Pic}(X')$ can be identified with the subgroup $\mathbb{Z}(60,0) \oplus \mathbb{Z}(0,120)$ in $\mathrm{Cl}(X') \simeq \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$, according to what we established in Example 2.

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